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Bounds for the Betti numbers of successive stellar subdivisions of a simplex

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Abstract. We give a bound for the Betti numbers of the Stanley-Reisner ring of a stellar subdivision of a Gorenstein* simplicial complex by applying unprojection theory. From this we derive a bound for the Betti numbers of iterated stellar subdivisions of the boundary complex of a simplex. The bound depends only on the number of subdivisions, and we construct examples which prove that it is sharp.

Key words: Stanley-Reisner rings, Betti numbers, Resolutions, Stellar subdivisions, Simplices.

1. Introduction

Consider the class of simplicial complexes obtained from the boundary complex of a simplex with $q+1$ vertices by any sequence of $c-1$ stellar subdivisions. We give bounds for the (total) Betti numbers of the minimal resolution of the associated Stanley-Reisner rings. The bounds depend only on c and not on q. Our main tool is the relation of stellar subdivisions of Gorenstein* simplicial complexes with the Kustin-Miller complex construction obtained in [1], which gives an easy way to control the Betti numbers of a stellar subdivision. By constructing a specific class of examples, we prove that for fixed c our bounds are attained for q sufficiently large.

There are bounds in the literature for various classes of simplicial complexes. If we only subdivide facets starting from a simplex the process will yield a stacked polytope. In this case, there is an explicit formula for the Betti numbers due to Terai and Hibi [16]. See also [7] for a combinatorial proof, and [9, Theorem 3.3], [1] for the construction of the resolutions. In [9, Theorem 2.1, Proposition 3.4], Herzog and Li Marzi consider bounds for

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a more general class than Gorenstein, leading to a less sharp bound in our setting. Migliore and Nagel discuss in [11, Proposition 9.5] a bound for fixed h-vector. The bounds of Römer [14] apply for arbitrary ideals with a fixed number of generators and linear resolution.

To state our results, for $c \geq 1$ and $q \geq 2$ denote by $\mathcal{D}_{q,c}$ the set of simplicial complexes D on $q+c$ vertices which are obtained by $c-1$ iterated stellar subdivisions of faces of positive dimension, starting from the boundary complex of a q-simplex. We call a simplicial complex D generalized stacked if there exist $c \geq 1$ and $q \geq 2$ such that $D \in \mathcal{D}_{q,c}$. If k is any field, we denote by $k[D]$ the Stanley-Reisner ring of D. Note, that $k[D]$ is the quotient of a polynomial ring by a Gorenstein ideal with codimension c. Define inductively $l_c = (l_{c,0}, l_{c,1}, \ldots, l_{c,c}) \in \mathbb{Z}^{c+1}$ by $l_1 = (1,1)$ and

$$l_c = 2(l_{c-1}, 0) + 2(0, l_{c-1}) - (1,1,0,\ldots,0) - (0,\ldots,0,1,1) \in \mathbb{Z}^{c+1}$$

for $c \geq 2$. For example $l_2 = 2(1,1,0) + 2(0,1,1) - (1,1,0) - (0,1,1) = (1,2,1)$, $l_3 = (1,5,5,1)$, and $l_4 = (1,11,20,11,1)$. The main result of the paper is the following theorem giving an upper bound for the Betti numbers of $k[D]$ when D is a generalized stacked simplicial complex. The bound follows immediately from the stronger Theorem 3.7, and that it is sharp from Proposition 5.2.

Theorem 1.1 *Suppose $c \geq 1, q \geq 2$ and $D \in \mathcal{D}_{q,c}$ is a generalized stacked simplicial complex. Then for the Betti numbers of $k[D]$ it holds that*

$$b_i(k[D]) \leq l_{c,i} \tag{1.1}$$

for all $0 \leq i \leq c$. Moreover, the bound is sharp in the following sense: Given $c \geq 1$, there exists $q \geq 2$ and $F \in \mathcal{D}_{q,c}$ with $b_i(k[F]) = l_{c,i}$ for all $0 \leq i \leq c$.

In Section 3 we focus on bounding the Betti numbers of stellar subdivisions. The first result is Proposition 3.2, which gives a bound for the Betti numbers of the Stanley-Reisner ring of a stellar subdivision of a Gorenstein* simplicial complex D with respect to a face τ in terms of those of D and of the link of τ. The proof of this proposition uses Proposition 3.1, which is a generalization of [1, Theorem 1.1], and the Kustin-Miller complex construction (see [10] and Section 4). To prove Theorem 1.1 by induction on the codimension c, we have to enlarge the class of complexes $\mathcal{D}_{q,c}$ by including also the links of faces. We give a bound for their Betti numbers in Proposition 3.6. According to the combinatorial Lemma 3.4 there are three types of links to consider.

Focussing on proving that the bound of Theorem 1.1 is sharp, we first analyze in Section 4 the Kustin-Miller complex construction in the setting of stellar subdivisions. In particular, we prove in Proposition 4.2 a sufficient condition for the minimality of the Kustin-Miller complex. In Section 5 we construct for any c a generalized stacked simplicial complex $F \in \mathcal{D}_{q,c}$ (for suitable q), and using Proposition 4.2 we show that the inequalities (1.1) are in fact equalities for F. For an implementation of the construction see our package BettiBounds [5] for the computer algebra system Macaulay2 [8]. Using the minimality of the Kustin-Miller complex, we provide in the package a function which efficiently produces the graded Betti numbers of the extremal examples without the use of Gröbner bases.

2. Notation

For an ideal I of a ring R and $u \in R$ write $(I : u) = \{r \in R \mid ru \in I\}$ for the ideal quotient. Denote by $\mathbb{N}$ the set of strictly positive integer numbers. For $n \in \mathbb{N}$ we set $[n] = \{1, 2, \ldots, n\}$. Assume $A \subset \mathbb{N}$ is a finite subset. We set 2^A to be the simplex with vertex set A, by definition it is the set of all subsets of A. A simplicial subcomplex $D \subset 2^A$ is a subset with the property that if $\tau \in D$ and $\sigma \subset \tau$ then $\sigma \in D$. The elements of D are also called faces of D, and the dimension of a face τ of D is one less than the cardinality of τ. We define the support of D to be

$$\operatorname{supp} D = \{i \in A \mid \{i\} \in D\}.$$

We fix a field k. Denote by R_A the polynomial ring $k[x_a \mid a \in A]$ with the degrees of all variables x_a equal to 1. For a finitely generated graded R_A-module M we denote by $b_i(M)$ the i-th Betti number of M, by definition $b_i(M) = \dim_{R_A/m} \operatorname{Tor}_i^{R_A}(R_A/m, M)$, where $m = (x_a \mid a \in A)$ is the maximal homogeneous ideal of R_A. It is well-known that if we ignore shifts the minimal graded free resolution of M as R_A-module has the shape

$$M \leftarrow R_A^{b_0(M)} \leftarrow R_A^{b_1(M)} \leftarrow R_A^{b_2(M)} \leftarrow \cdots$$

For a simplicial subcomplex $D \subset 2^A$ we define the Stanley-Reisner ideal $I_{D,A} \subset R_A$ to be the ideal generated by the square-free monomials $x_{i_1}x_{i_2}\ldots x_{i_p}$ where $\{i_1, i_2, \ldots, i_p\}$ is not a face of D. In particular, $I_{D,A}$ contains linear terms if and only if $\operatorname{supp} D \neq A$. The Stanley-Reisner ring $k[D,A]$ is defined by $k[D,A] = R_A/I_{D,A}$. Taking into account that $\dim R_A = \#A$, we define the codimension of $k[D,A]$ by $\operatorname{codim} k[D,A] = \#A - \dim k[D,A]$. For a nonempty face σ of D we set $x_\sigma = \prod_{i\in\sigma} x_i \in k[D,A]$. We denote by $b_i(k[D,A])$ the i-th Betti number of $k[D,A]$ considered as R_A-module. In the following, when the set A is clear we will sometimes simplify the notations $I_{D,A}$ to I_D and $k[D,A]$ to $k[D]$. In some situations, however, it will be convenient to consider Stanley-Reisner ideals containing variables.

For a nonempty subset $A \subset \mathbb{N}$, we set $\partial A = 2^A \setminus \{A\} \subset 2^A$ to be the boundary complex of the simplex 2^A. For the Stanley-Reisner ring of ∂A we have $k[\partial A, A] = R_A/(\prod_{a\in A} x_a)$.

Assume that, for $i = 1, 2$, $D_i \subset 2^{A_i}$ is a subcomplex and the subsets A_1, A_2 of $\mathbb{N}$ are disjoint. By the join $D_1 * D_2$ of D_1 and D_2 we mean the subcomplex $D_1 * D_2 \subset 2^{A_1\cup A_2}$ defined by

$$D_1 * D_2 = \{\alpha_1 \cup \alpha_2 \mid \alpha_1 \in D_1, \alpha_2 \in D_2\}.$$

By [6, p. 221, Exerc. 5.1.20] we have

$$k[D_1 * D_2, A_1 \cup A_2] = k[D_1, A_1] \otimes_k k[D_2, A_2].$$

As a consequence, using the well-known fact that the tensor product of the minimal resolutions of two modules is a minimal resolution of the tensor product of the modules we get that

$$b_i(k[D_1 * D_2]) = \sum_{t=0}^{i} b_t(k[D_1])b_{i-t}(k[D_2]) \tag{2.1}$$

for all $i \geq 0$.

If σ is a face of $D \subset 2^A$ define the link of σ in D to be the subcomplex

$$\operatorname{lk}_D \sigma = \{\alpha \in D \mid \alpha \cap \sigma = \emptyset \text{ and } \alpha \cup \sigma \in D\} \subset 2^{A\setminus\sigma}.$$

It is clear that the Stanley-Reisner ideal of $\operatorname{lk}_D \sigma$ is equal to the intersection of the ideal $(I_{D,A} : x_\sigma)$ with the subring $R_{A\setminus\sigma}$ of R_A. In other words, it is the ideal of $R_{A\setminus\sigma}$ generated by the minimal monomial generating set of $(I_{D,A} : x_\sigma)$. Furthermore, define the star of σ in D to be the subcomplex

$$\operatorname{star}_D \sigma = \{\alpha \in D \mid \alpha \cup \sigma \in D\} \subset 2^A.$$

If τ is a nonempty face of $D \subset 2^A$ and $j \in \mathbb{N} \setminus A$, we define the stellar subdivision D_τ with new vertex j to be the subcomplex

$$D_\tau = (D \setminus \operatorname{star}_D \tau) \cup \big(2^{\{j\}} * \operatorname{lk}_D \tau * \partial\tau\big) \subset 2^{A\cup\{j\}}.$$

Note that D_τ consists of the following faces:

(1) All faces of D which do not contain τ.
(2) For each face $\beta \in D$ with $\tau \subset \beta$ the faces $(\beta \setminus \rho) \cup \{j\}$ for all nonempty subsets ρ of τ.

It is easy to see that

$$k\big[D_\tau, 2^{A\cup\{j\}}\big] = R_{A\cup\{j\}}/(I_{D,A}, x_\tau, x_j u \mid u \in (I_{D,A} : x_\tau)). \tag{2.2}$$

Following [15, p. 67], we say that a subcomplex $D \subset 2^A$ is Gorenstein* over k if $A = \operatorname{supp} D$, $k[D]$ is Gorenstein, and for every $i \in A$ there exists $\sigma \in D$ with $\sigma \cup \{i\}$ not a face of D. The last condition combinatorially means that D is not a join of the form $2^{\{i\}} * D_1$, and algebraically that x_i divides at least one element of the minimal monomial generating set of $I_{D,A}$. We say that $D \subset 2^A$ is generalized Gorenstein* over k if $D \subset 2^{\operatorname{supp} D}$ is Gorenstein* over k. When there is no ambiguity about the field k we will just say Gorenstein* and generalized Gorenstein*. It is well-known (cf. [15, Section II.5]) that if $D \subset 2^A$ is Gorenstein* and $\sigma \in D$ is a face then $\operatorname{lk}_D \sigma \subset 2^{A\setminus\sigma}$ is also Gorenstein*. It follows that if $D \subset 2^A$ is generalized Gorenstein* and $\sigma \in D$ then $\operatorname{lk}_D \sigma \subset 2^{A\setminus\sigma}$ is also generalized Gorenstein*.

Recall also from [13, Definition 1.2] that if $I = (f_1, \dots, f_r) \subset R$ is a homogeneous codimension 1 ideal of a graded Gorenstein ring R such that the quotient R/I is Gorenstein, then there exists $\psi \in \operatorname{Hom}_R(I, R)$ such that ψ together with the inclusion $I \hookrightarrow R$ generate $\operatorname{Hom}_R(I, R)$ as an R-module. The Kustin–Miller unprojection ring of the pair $I \subset R$ is defined as the quotient of $R[T]$ by the ideal generated by the elements $Tf_i - \psi(f_i)$, where T is a new variable.

3. Bounds for the Betti numbers of successive stellar subdivisions

The main result of this section is Theorem 3.7, which gives bounds for the Betti numbers of generalized stacked simplicial complexes and links thereof.

In the following, let $D \subset 2^A$ be a generalized Gorenstein* simplicial complex, $\tau \in D$ a nonempty face and $D_\tau \subset 2^{A \cup \{j\}}$ the corresponding stellar subdivision with new vertex $j \in \mathbb{N} \setminus A$. For simplicity set $\mathcal{R} = R_A[z]/(I_{D,A}) = k[D,A][z]$, where z is a new variable.

In [1] we showed that a stellar subdivision of a face of a Gorenstein* simplicial complex corresponds on the level of Stanley-Reisner rings to a certain Kustin–Miller unprojection. In the following proposition we generalize this statement for generalized Gorenstein* simplicial complexes.

Proposition 3.1 *Assume that* $\dim \tau \geq 1$. *Consider the ideal* $Q = (I_{D,A} : x_\tau, z) \subset R_A[z]$, *and set*

$$M = \operatorname{Hom}_{\mathcal{R}}(Q/(I_{D,A}), \mathcal{R}).$$

Then M *is generated, as* $\mathcal{R}$*-module, by the inclusion homomorphism together with the map* ψ *that sends* $(I_{D,A} : x_\tau)$ *to* 0 *and* z *to* x_τ. *Denote by* S *the Kustin–Miller unprojection ring of the pair* $Q/(I_{D,A}) \subset \mathcal{R}$ *associated to the map* ψ. *We have that* z *is* S*-regular and* $S/(z) \cong k[D_\tau, A \cup \{j\}]$.

Proof. If $A = \operatorname{supp} D$ then the statement is [1, Theorem 1.1(b)]. Now assume that $\operatorname{supp} D$ is a proper subset of A. Consider $P = \{x_a \mid a \in A \setminus \operatorname{supp} D\} \subset R_A$. We have

$$I_{D,A} = (I_{D,\operatorname{supp} D}) + (P), \quad Q = (I_{D,\operatorname{supp} D} : x_\tau, z) + (P)$$

and

$$I_{D_\tau, A \cup \{j\}} = (I_{D_\tau, \operatorname{supp} D \cup \{j\}}) + (P).$$

The arguments in the proof of [1, Theorem 1.1] also prove that M is generated by the inclusion together with the map ψ that sends $(I_{D,A} : x_\tau)$ to 0 and z to x_τ. They also prove that z is S-regular and that $S/(z) \cong k[D_\tau, A \cup \{j\}]$. □

We will now study the Betti numbers b_i of $k[D_\tau, A \cup \{j\}]$ as $R_{A \cup \{j\}}$-

module in terms of the Betti numbers of $k[D, A]$ as R_A-module and the Betti numbers of $k[\mathrm{lk}_D(\tau), A \setminus \tau]$ as $R_{A\setminus\tau}$-module.

Proposition 3.2 *Denote by $L = \mathrm{lk}_D(\tau) \subset 2^{A\setminus\tau}$ the link of the face τ of D. We then have*

$$b_1(k[D_\tau]) \leq b_1(k[D]) + b_1(k[L]) + 1$$

and that, for $2 \leq i \leq \operatorname{codim} k[D_\tau] - 2$,

$$b_i(k[D_\tau]) \leq b_{i-1}(k[D]) + b_i(k[D]) + b_{i-1}(k[L]) + b_i(k[L]).$$

Proof. If $\dim \tau = 0$, say $\tau = \{p\}$, then

$$I_{D_\tau, A\cup\{j\}} = (G, x_p),$$

where G is the finite set obtained by substituting x_j for x_p in the minimal monomial generating set of $I_{D,A}$. Hence

$$b_i(k[D_\tau]) = b_{i-1}(k[D]) + b_i(k[D])$$

for all $i \geq 1$. If $2 \leq i \leq \operatorname{codim} k[D_\tau] - 2$ then the inequality we want to prove follows. If $i = 1$, then since $b_0(k[D]) = 1$ the inequality for the first Betti numbers b_1 also follows.

Now assume that $\dim \tau \geq 1$. Using the notations of Proposition 3.1, we have that S is the Kustin–Miller unprojection of the pair $Q/(I_{D,A}) \subset \mathcal{R}$ and that $b_i(k[D_\tau]) = b_i[S]$ for all i.

We denote by C_U the graded free resolution of S obtained by the Kustin–Miller complex construction, cf. Section 4 and [3, Section 2], with initial data the minimal graded free resolutions of $\mathcal{R} = R_A[z]/(I_{D,A})$ and $R_A[z]/Q$ over $R_A[z]$. Since C_U is a graded free resolution of S we have $b_i(S) \leq b_i(C_U)$ for all i, where $b_i(C_U)$ denotes the rank of the finitely generated free $R_A[z]$-module $(C_U)_i$. The variable z does not appear in the minimal monomial generating set of $I_{D,A}$, as a consequence $b_i(\mathcal{R}) = b_i(k[D])$ for all i. Since $Q = (I_{D,A} : x_\tau, z)$ and the variable z does not appear in the minimal generating set of $(I_{D,A} : x_\tau)$ we have for all i

$$\begin{aligned} b_i(R_A[z]/Q) &= b_{i-1}(R_A/(I_{D,A} : x_\tau)) + b_i(R_A/(I_{D,A} : x_\tau)) \\ &= b_{i-1}(k[L]) + b_i(k[L]). \end{aligned} \tag{3.1}$$

Moreover, by the formulas in Section 4 expressing $(C_U)_i$ in terms of A'_j and B'_j we have

$$b_1(C_U) \leq b_1(k[D]) + b_1(R_A[z]/Q) = b_1(k[D]) + b_1(k[L]) + 1$$

and, for $2 \leq i \leq \operatorname{codim} k[D_\tau] - 2$, that

$$b_i(C_U) \leq b_{i-1}(k[D]) + b_i(R_A[z]/Q) + b_i(k[D]).$$

Hence

$$b_i(k[D_\tau]) = b_i[S] \leq b_i(C_U) \leq b_{i-1}(k[D]) + b_i(R_A[z]/Q) + b_i(k[D])$$

which combined with Equality(3.1) finishes the proof. $\square$

Remark 3.3 It may be interesting to investigate, perhaps with the use of Hochster's formula or a generalization of the Kustin-Miller complex technique, whether the inequalities of Proposition 3.2 hold in a more general setting than Gorenstein* property.

For the proof of Proposition 3.6 we will need the following combinatorial lemma which relates a link of a stellar subdivision with links of the original simplicial complex:

Lemma 3.4 *If σ is a nonempty face of D_τ the following hold*:

(1) (*Case I*) *Assume $j \notin \sigma$ and $\tau \cup \sigma \in D$. Then $\tau \setminus \sigma$ is a nonempty face of $\operatorname{lk}_D \sigma$ and*

$$\operatorname{lk}_{D_\tau} \sigma = (\operatorname{lk}_D \sigma)_{\tau \setminus \sigma}$$

that is, $\operatorname{lk}_{D_\tau} \sigma$ is the stellar subdivision of $\operatorname{lk}_D \sigma$ with respect to $\tau \setminus \sigma$.

(2) (*Case II*) *Assume that $j \notin \sigma$ and $\tau \cup \sigma \notin D$. Then $\operatorname{lk}_{D_\tau} \sigma$ is equal to $\operatorname{lk}_D \sigma$ considered as a subcomplex of $2^{(A \cup \{j\}) \setminus \sigma}$.*

(3) (*Case III*) *Assume $j \in \sigma$. Then $\tau \cup \sigma \setminus \{j\}$ is a face of D, $\tau \setminus \sigma$ is nonempty and*

$$\mathrm{lk}_{D_\tau}\,\sigma = \mathrm{lk}_D(\tau \cup \sigma \setminus \{j\}) * \partial(\tau \setminus \sigma)$$

that is, $\mathrm{lk}_{D_\tau}\,\sigma$ *is equal to the join of* $\mathrm{lk}_D(\tau \cup \sigma \setminus \{j\})$ *with* $\partial(\tau \setminus \sigma)$.

For the straightforward proof see [4, Section 3.1].

Remark 3.5 Case II corresponds to faces σ of $D \setminus \mathrm{star}_D\,\tau$, while Cases I and III to faces of $2^{\{j\}} * \mathrm{lk}_D\,\tau * \partial\tau$.

The next proposition gives bounds on the Betti numbers of links of a stellar subdivision in terms of links of the original complex.

Proposition 3.6 *Let* σ *be a face of* D_τ, *and set* $L = \mathrm{lk}_{D_\tau}\,\sigma \subset 2^{(A\cup\{j\})\setminus\sigma}$.

(1) (*Case I*) *If* $j \notin \sigma$ *and* $\tau \cup \sigma$ *is a face of* D *then we have that*

$$b_1(k[L]) \le b_1(k[L_1]) + b_1(k[L_2]) + 1$$

and that for $2 \le i \le \operatorname{codim} k[L] - 2$

$$b_i(k[L]) \le b_{i-1}(k[L_1]) + b_i(k[L_1]) + b_{i-1}(k[L_2]) + b_i(k[L_2]),$$

where $L_1 = \mathrm{lk}_D\,\sigma \subset 2^{A\setminus\sigma}$ *and* $L_2 = \mathrm{lk}_D(\tau\cup\sigma) \subset 2^{A\setminus(\tau\cup\sigma)}$.

(2) (*Case II*) *If* $j \notin \sigma$ *and* $\tau \cup \sigma$ *is not a face of* D *then we have that for all* i

$$b_i(k[L]) = b_{i-1}(k[L_1]) + b_i(k[L_1]).$$

(3) (*Case III*) *Assume* $j \in \sigma$. *Then* $\tau \cup \sigma \setminus \{j\}$ *is a face of* D *and we have that for all* i

$$b_i(k[L]) = b_{i-1}(k[L_3]) + b_i(k[L_3]),$$

where $L_3 = \mathrm{lk}_D(\tau \cup \sigma \setminus \{j\}) \subset 2^{A\setminus(\tau\cup\sigma)}$.

Proof. Assume first we are in Case I, that is $j \notin \sigma$ and $\tau \cup \sigma$ is a face of D. By part (1) of Lemma 3.4 we have $L = (L_1)_{\tau\setminus\sigma}$. Furthermore, a straightforward calculation shows that $\mathrm{lk}_D(\tau\cup\sigma) = \mathrm{lk}_{L_1}(\tau\setminus\sigma)$. The result follows from Proposition 3.2 applied to the stellar subdivision of the face $\tau\setminus\sigma$ of L_1.

Assume now we are in Case II, that is $j \notin \sigma$ and $\tau \cup \sigma$ is not a face of

D. By part (2) of Lemma 3.4 we have

$$I_{L,(A\cup\{j\})\setminus\sigma} = (I_{L_1,A\setminus\sigma}) + (x_j) \subset R_{(A\cup\{j\})\setminus\sigma}$$

and the result is clear. In Case III, that is, $j \in \sigma$, we have by part (3) of Lemma 3.4 that $L = L_3 * \partial(\tau \setminus \sigma)$. Since $k[\partial(\tau \setminus \sigma)]$ is the quotient of a polynomial ring by a single equation, hence has nonzero Betti numbers only $b_0 = b_1 = 1$, the result follows by Equation (2.1). $\square$

For $c \geq 1$ and $q \geq 2$ recall that we defined $\mathcal{D}_{q,c}$ as the set of simplicial subcomplexes $D \subset 2^{[q+c]}$ such that there exists a sequence of simplicial complexes

$$D_1, D_2, \ldots, D_{c-1}, D_c = D$$

with the property that $D_1 = \partial([q+1]) \subset 2^{[q+1]}$ is the boundary complex of the simplex on $q+1$ vertices, and, for $0 \leq i \leq c-1$, $D_{i+1} \subset 2^{[q+i+1]}$ is obtained from $D_i \subset 2^{[q+i]}$ by a stellar subdivision of a face of D_i of dimension at least 1 with new vertex $q+i+1$. It is clear that $\operatorname{supp} D_i = [q+i]$ and $\operatorname{codim} k[D_i] = i$ for all i.

Assume $D \in \mathcal{D}_{q,c}$ and consider the Stanley–Reisner ring $k[D] = R_{[q+c]}/I_D$. By [6, Corollary 5.6.5] D is Gorenstein*. As a consequence, since $\operatorname{codim} k[D] = c$ the only nonzero Betti numbers b_i of $k[D]$ are $1 = b_0, b_1, \ldots, b_{c-1}, b_c = 1$ and $b_i = b_{c-i}$ for all i.

To prove Theorem 1.1 we need to enlarge the class of ideals we consider by including the ideals of links. For $q \geq 2$ and $c \geq 1$ we define

$$\begin{aligned}\mathcal{I}_{q,c} = &\{I_D \mid D \in \mathcal{D}_{q,c}\}\\ &\cup \{(I_D : x_\sigma) \subset k[x_1, \ldots, x_{q+c}] \mid D \in \mathcal{D}_{q,c},\ \sigma \in D \text{ a nonempty face}\}.\end{aligned}$$

The following theorem is the key technical result.

Theorem 3.7 *Suppose $c \geq 1$, $q \geq 2$, $R = k[x_1, \ldots, x_{q+c}]$ and $I \in \mathcal{I}_{q,c}$. Then for the Betti numbers $b_i(R/I)$ of the minimal resolution of R/I as R-module it holds that*

$$b_i(R/I) \leq l_{c,i}$$

for all $0 \leq i \leq c$.

Proof. First note that from the definition of the bounding sequence l_c it is clear that $l_{c,i} = l_{c,c-i}$ for all $0 \leq i \leq c$; that $l_{c,0} = l_{c,c} = 1$ for all $c \geq 1$; that $l_{c+1,1} = 2l_{c,1} + 1$ for all $c \geq 2$; and that $l_{c+1,i} = 2l_{c,i-1} + 2l_{c,i}$ for $2 \leq i \leq (c+1) - 2$.

Assume the claim is not true. Then there exist $c \geq 1$, $q \geq 2$ and an ideal $I \in \mathcal{I}_{q,c}$ with $I \notin \mathcal{W}_c$, where by definition

$$\mathcal{W}_c = \{I \in \mathcal{I}_{p,c} \mid p \geq 2 \text{ and } b_i(R/I) \leq l_{c,i} \text{ for all } i\}.$$

We fix such an ideal I with c the least possible, and we will get a contradiction. Since Gorenstein codimension 1 or 2 implies complete intersection, we necessarily have $c \geq 3$.

The first case is that $I = I_{D_1}$ for some $D_1 \in \mathcal{D}_{q,c}$, so there exist $D \in \mathcal{D}_{q,c-1}$ and a face τ of D of dimension at least 1 such that $D_1 = D_\tau$. Since c has been chosen to be the smallest possible, we have that $I_D \in \mathcal{W}_{c-1}$ and $(I_D : x_\tau) \in \mathcal{W}_{c-1}$. Using the properties of l_c mentioned above, it follows by Proposition 3.2 that $I \in \mathcal{W}_c$, which is a contradiction.

Assume now that $I = (I_{D_1} : x_\sigma)$ for some $D_1 \in \mathcal{D}_{q,c}$ and face σ of D_1. Write $D_1 = D_\tau$, for some $D \in \mathcal{D}_{q,c-1}$ and face τ of D of dimension at least 1. The new vertex j of D_τ is $q+c$. In the remaining of the proof we will use the simplicial complexes L and L_i, with $1 \leq i \leq 3$, defined in Proposition 3.6. We have three cases. For all of them we will show that $I \in \mathcal{W}_c$, which is a contradiction.

Assume we are in Case I, that is $j \notin \sigma$ and $\tau \cup \sigma$ is a face of D. Since by the minimality of c we have that both ideals I_L and I_{L_1} are in $\mathcal{W}_{c-1}$, it follows by Case I of Proposition 3.6 that $I \in \mathcal{W}_c$. Assume now we are in Case II, that is $j \notin \sigma$ and $\tau \cup \sigma$ is not a face of D. Again by the minimality of c we have $I_{L_1} \in \mathcal{W}_{c-1}$, so using Case II of Proposition 3.6 it follows that $I \in \mathcal{W}_c$. Finally, assume we are in Case III, that is $j \in \sigma$. By the minimality of c we have $I_{L_3} \in \mathcal{W}_{c-1}$, so using Case III of Proposition 3.6 it follows that $I \in \mathcal{W}_c$. This finishes the proof. $\square$

Remark 3.8 Combining Proposition 3.6 with Theorem 3.7 it is not hard to show that for fixed $q \geq 2$, there exists $c_0 \geq 1$ such that $b_i(k[D]) < l_{c,i}$ for all $c \geq c_0, D \in \mathcal{D}_{q,c}$ and $1 \leq i \leq c-1$. So if we fix q, then for c sufficiently large the Betti bound in Theorem 1.1 is not sharp. We leave the details to the interested reader.

4. The structure of the Kustin–Miller complex in the stellar subdivision case

Kustin and Miller introduced in [10] the Kustin–Miller complex construction which produces a projective resolution of the Kustin–Miller unprojection ring in terms of projective resolutions of the initial data. In Proposition 4.2 we prove a criterion for the minimality of the resolution, which will be used in Section 5. For that, we analyze the additional structure of the construction in the case of stellar subdivisions.

We will use the graded version of the Kustin–Miller complex construction as described in [3, Section 2]. Note, that there is an implementation of the construction available for the computer algebra system MACAULAY2, see [loc. cit.].

In this section $D \subset 2^A$ will be a generalized Gorenstein* simplicial complex, $\tau \in D$ a face of positive dimension and $D_\tau \subset 2^{A \cup \{j\}}$ the corresponding stellar subdivision with new vertex $j \in \mathbb{N} \setminus A$.

Let $R = R_A[z]$ with the following grading: $\deg x_a = 1$ for $a \in A$ and $\deg z = \dim \tau$. Write $I \subset R$ for the ideal generated by $I_{D,A}$ and set $J = (I_{D,A} : x_\tau, z) \subset R$. Denote by

$$C_J : R/J \leftarrow A_0 \overset{a_1}{\leftarrow} A_1 \overset{a_2}{\leftarrow} \cdots \overset{a_{g-1}}{\longleftarrow} A_{g-1} \overset{a_g}{\leftarrow} A_g \leftarrow 0$$

$$C_I : R/I \leftarrow B_0 \overset{b_1}{\leftarrow} B_1 \overset{b_2}{\leftarrow} \cdots \overset{b_{g-1}}{\longleftarrow} B_{g-1} \leftarrow 0$$

the minimal graded free resolutions of R/J and R/I respectively.

By Proposition 3.1 $\operatorname{Hom}_{R/I}(J/I, R/I)$ is generated as an R/I-module by the inclusion homomorphism together with the map ψ that sends $(I_{D,A} : x_\tau)$ to 0 and z to x_τ. By the Kustin-Miller complex construction we obtain the unprojection ideal $U \subset R[T]$ of the pair $J/I \subset R/I$ defined by ψ with new variable T, and a, in general non-minimal, graded free resolution C_U of $R[T]/U$ as $R[T]$-module. For more details see [loc. cit.].

Clearly, the k-algebra S defined in Proposition 3.1 is isomorphic to $R[T]/U$, since it is obtained from $R[T]/U$ by substituting T with x_j. By the same proposition z is $R[T]/U$-regular and $(R[T]/U)/(z) \cong k[D_\tau]$.

We denote by P the ideal $(I_{D,A} : x_\tau)$ of R_A, and by

$$C_P : R_A/P \leftarrow P_0 \overset{p_1}{\leftarrow} P_1 \overset{p_2}{\leftarrow} \cdots \overset{p_{g-1}}{\longleftarrow} P_{g-1} \leftarrow 0$$

the minimal graded free resolution of R/P as R_A-module. Moreover, we denote by

$$C_z : k[z]/(z) \leftarrow k[z] \leftarrow k[z] \leftarrow 0$$

the minimal graded free resolution of $k[z]/(z)$ as $k[z]$-module. Since $J = (P, z)$ we have that C_J is the tensor product (over k) of the complexes C_P and C_z. Hence $A_0 = P_0^a$, $A_g = P_{g-1}^a$ and

$$A_i = P_{i-1}^a \oplus P_i^a \tag{4.1}$$

for all $1 \leq i \leq g-1$, where $P_i^a = P_i \otimes_k k[z]$ considered as R-module. Moreover, using this decomposition, we have that

$$a_1 = (p_1 \quad z), \quad a_g = \begin{pmatrix} -z \\ p_{g-1} \end{pmatrix}, \quad \text{and} \quad a_i = \begin{pmatrix} p_i & -zE \\ 0 & p_{i-1} \end{pmatrix}$$

for $2 \leq i \leq g-1$, where E denotes the identity matrix of size equal to the rank of P_{i-1}.

Recall from [loc. cit.] that the construction of C_U involves chain maps $\alpha : C_I \to C_J$, $\beta : C_J \to C_I[-1]$ and a homotopy map $h : C_I \to C_I$, given by maps $\alpha_i : B_i \to A_i$, $\beta_i : A_i \to B_{i-1}$ and $h_i : B_i \to B_i$ for all i. We will use that α_0 is an invertible element of R, that $h_0 = h_g = 0$, and that the h_i satisfy the defining property

$$\beta_i \alpha_i = h_{i-1} b_i + b_i h_i \tag{4.2}$$

for all i. We will also use that if we do not write the twists we have $(C_U)_0 = B_0'$, $(C_U)_1 = B_1' \oplus A_1'$, $(C_U)_i = B_i' \oplus A_i' \oplus B_{i-1}'$ for $2 \leq i \leq g-2$, $(C_U)_{g-1} = A_{g-1}' \oplus B_{g-2}'$, and $(C_U)_g = B_{g-1}'$, where for an R-module M we set $M' = M \otimes_R R[T]$.

Using the decomposition (4.1), we can write, for $1 \leq i \leq g-1$

$$\alpha_i = \begin{pmatrix} \alpha_{i,1} \\ \alpha_{i,2} \end{pmatrix}, \quad \beta_i = (\beta_{i,1} \quad \beta_{i,2}).$$

Proposition 4.1 *We can choose α_i, β_i and h_i in the following way*:

(1) *α_i, β_i do not involve z for all i,*

(2) $\alpha_{i,2} = \beta_{i,1} = 0$ *for* $1 \leq i \leq g-1$, *and*
(3) $h_i = 0$ *for all* i.

Proof. For the maps α_i the arguments are as follows. Since α_0 is an invertible element of R it does not involve z. Assume now that $i = 1$. Using that α is a chain map, we have $\alpha_0 b_1 = a_1 \alpha_1$, hence

$$\alpha_0 b_1 = \begin{pmatrix} p_1 & z \end{pmatrix} \begin{pmatrix} \alpha_{1,1} \\ \alpha_{1,2} \end{pmatrix} = p_1 \alpha_{1,1} + z\alpha_{1,2}.$$

Since z does not appear in the product $\alpha_0 b_1$ or in p_1 we can assume $\alpha_{1,2} = 0$ and that z does not appear in $\alpha_{1,1}$. Assume now that $\alpha_{i,2} = 0$ and $\alpha_{i,1}$ does not involve the variable z and we will show that we can choose α_{i+1} with $\alpha_{i+1,2} = 0$ and that z does not appear in $\alpha_{i+1,1}$. Indeed, since α is a chain map, we have $\alpha_i b_{i+1} = a_{i+1}\alpha_{i+1}$, so

$$\begin{pmatrix} \alpha_{i,1} \\ 0 \end{pmatrix} b_{i+1} = \begin{pmatrix} p_{i+1} & -zE \\ 0 & p_i \end{pmatrix} \begin{pmatrix} \alpha_{i+1,1} \\ \alpha_{i+1,2} \end{pmatrix}.$$

Hence we get the equations

$$\alpha_{i,1} b_{i+1} = p_{i+1}\alpha_{i+1,1} - z\alpha_{i+1,2}, \quad 0 = p_i \alpha_{i+1,2}. \tag{4.3}$$

Write $\alpha_{i+1,1} = q_1 + zq_2$ with z not appearing in q_1. Equation (4.3) implies that $\alpha_{i,1} b_{i+1} = p_{i+1} q_1$. As a consequence, we can assume that $\alpha_{i+1,2} = 0$ and that $\alpha_{i+1,1} = q_1$, hence $\alpha_{i+1,1}$ does not involve z.

For the maps β_i the argument is as follows. Since $\psi(u) = 0$ for all $u \in P$ and $\psi(z) = x_\tau$, we have by [3, Section 2] that $\beta_1 = (0 \ \ldots \ 0 \ x_\tau)$, hence $\beta_{1,1} = 0$ and z does not appear in $\beta_{1,2}$. Assume now $\beta_{i,1} = 0$ and z does not appear in $\beta_{i,2}$ and we will show that we can choose β_{i+1} with $\beta_{i+1,1} = 0$ and z not appearing in $\beta_{i+1,2}$. Indeed, since β is a chain map, we have $b_i \beta_{i+1} = \beta_i a_{i+1}$, hence

$$b_i \begin{pmatrix} \beta_{i+1,1} & \beta_{i+1,2} \end{pmatrix} = \begin{pmatrix} 0 & \beta_{i,2} \end{pmatrix} \begin{pmatrix} p_{i+1} & -zE \\ 0 & p_i \end{pmatrix}.$$

Hence we get the equations

$$b_i \beta_{i+1,1} = 0, \qquad b_i \beta_{i+1,2} = \beta_{i,2} p_i$$

so we can assume that $\beta_{i+1,1} = 0$ and that z does not appear in $\beta_{i+1,2}$.

We will now prove the statement for the maps h_i. Since, as proved above, we can assume that $\alpha_{i,2} = \beta_{i,1} = 0$, we have

$$\beta_i \alpha_i = (0 \quad \beta_{i,2}) \begin{pmatrix} \alpha_{i,1} \\ 0 \end{pmatrix} = 0.$$

As a consequence, Equation (4.2) can be satisfied by taking $h_i = 0$ for all i. □

In what follows, we will assume α_i, β_i and h_i are chosen as in Proposition 4.1.

Proposition 4.2 *Assume that the face τ ofD has the following property: every minimal non-face of D contains at least one vertex of τ (algebraically it means that for every minimal monomial generator v of I there exists $p \in \tau$ such that x_p divides v). Then C_U is a minimal complex. As a consequence, we have that $C_U \otimes_R R/(z)$ is, after substituting T with x_j, the minimal graded free resolution of $k[D_\tau, 2^{A \cup \{j\}}]$.*

Proof. We first show the minimality of C_U. Since we have $h_i = 0$ for all i, it is enough to show that, for $1 \leq i \leq g-1$ the chain maps α_i and β_i are minimal, in the sense that no nonzero constants appear in the corresponding matrix representations. It follows by the defining properties of the chain maps α and β in [3, Section 2] that β_i is minimal if and only if α_{g-i} is. So it is enough to prove that the map $\alpha_i \colon B_i \to A_i$ is minimal for $1 \leq i \leq g-1$. Denote by M the monoid of exponent vectors on the variables of R.

Since the ideals I and J of R are monomial, there exist, for $1 \leq i \leq g-1$, positive integers $q_{1,i}, q_{2,i}$ and multidegrees $\bar{a}_{i,j_1}, \bar{b}_{i,j_2} \in M$ with $1 \leq j_1 \leq q_{1,i}$ and $1 \leq j_2 \leq q_{2,i}$ such that

$$A_i = \bigoplus_{1 \leq j_1 \leq q_{1,i}} R(-\bar{a}_{i,j_1}) \quad \text{and} \quad B_i = \bigoplus_{1 \leq j_2 \leq q_{2,i}} R(-\bar{b}_{i,j_2}).$$

For the minimality of α_i it is enough to show (compare [12, Remark 8.30]) that given i with $1 \leq i \leq g-1$ there are no j_1, j_2 with $1 \leq j_1 \leq q_{1,i}$, $1 \leq j_2 \leq q_{2,i}$ and $\bar{a}_{i,j_1} = \bar{b}_{i,j_2}$, which we will now prove. By the assumptions, given v in the minimal monomial generating set of I

there exists $p \in \tau$ with x_p dividing v in the polynomial ring R. Hence, given j_2 with $1 \leq j_2 \leq q_{2,i}$ there is a nonzero coordinate of $\bar{b}_{1,j_2}$ corresponding to a variable x_p with $p \in \tau$. This implies that the same is true for every $\bar{b}_{i,j_2}$ with $i \geq 1$ and $1 \leq j_2 \leq q_{2,i}$. On the other hand, no variable x_p with $p \in \tau$ appears in any minimal monomial generators of J, hence the same is true for the coordinates of every $\bar{a}_{i,j_1}$ with $i \geq 1$ and $1 \leq j_1 \leq q_{1,i}$. So $\bar{a}_{i,j_1} = \bar{b}_{i,j_2}$ is impossible for $i \geq 1$. This finishes the proof that C_U is a minimal complex.

By Proposition 3.1 z is S-regular and $S/(z) \cong k[D_\tau]$. Hence using [6, Proposition 1.1.5], since C_U is minimal, the complex $C_U \otimes_R R/(z)$ is, after substituting T with x_j, the minimal graded free resolution of $k[D_\tau]$. □

Remark 4.3 We give an example where the condition for τ in the statement Proposition 4.2 is not satisfied but C_U is still minimal. Let D be the simplicial complex triangulating the 1-dimensional sphere S^1 having n vertices with $n \geq 4$, and suppose τ is a 1-face of D. Since $n \geq 4$ there exist minimal non-faces of D with vertex set disjoint from τ. On the other hand C_U is minimal, see, for example, [1, Section 5.2].

5. Champions

5.1. Construction

Assume a positive integer $c \geq 1$ is given. We will define a positive integer q and construct a generalized stacked simplicial complex $F_c \in \mathcal{D}_{q-1,c}$ such that the inequalities of Theorem 1.1 are equalities. First note that for $c = 1$ we can take the boundary complex of any simplex, and for $c = 2$ any single stellar subdivision of that.

For $c \geq 3$ we define inductively positive integers d_t, for $0 \leq t \leq c-1$, by $d_0 = 0$ and $d_{t+1} = d_t + (c-t)$, and set $q = d_{c-1}$. We also define inductively, for $1 \leq t \leq c-1$, subsets $\sigma_t \subset [q]$ of cardinality c by $\sigma_1 = \{1, \ldots, d_1 = c\}$ and

$$\sigma_{t+1} = \{(\sigma_1)_t, (\sigma_2)_t, \ldots, (\sigma_t)_t\} \cup \{i \mid d_t + 1 \leq i \leq d_{t+1}\},$$

where $(\sigma_i)_p$ denotes the p-th element of σ_i with respect to the usual ordering of $\mathbb{N}$. The main properties are that $\#(\sigma_i \cap \sigma_j) = 1$ for all $i \neq j$, every three distinct σ_i have empty intersection, and the last element d_i of σ_i is not in σ_j for $j \neq i$.

Example 5.1 For $c = 4$ we have $(d_1, d_2, d_3) = (4, 7, 9)$, $q = 9$, $\sigma_1 = \{1, 2, 3, 4\}$, $\sigma_2 = \{1, 5, 6, 7\}$ and $\sigma_3 = \{2, 5, 8, 9\}$. For $c = 5$ we have $(d_1, \dots, d_4) = (5, 9, 12, 14)$, $q = 14$, $\sigma_1 = \{1, 2, 3, 4, 5\}$, $\sigma_2 = \{1, 6, 7, 8, 9\}$, $\sigma_3 = \{2, 6, 10, 11, 12\}$ and $\sigma_4 = \{3, 7, 10, 13, 14\}$.

We define inductively simplicial subcomplexes $F_t \subset 2^{[q+t-1]}$ for $1 \le t \le c$. Since σ_i is not a subset of σ_j for $i \neq j$ we will be able to apply the elementary observation that if σ, τ are two faces of a simplicial complex D then τ not a subset of σ implies that σ is also a face of the stellar subdivision D_τ. First set $F_1 = \partial([q]) \subset 2^{[q]}$ to be the boundary complex of the simplex on q vertices $1, \dots, q$. Clearly σ_i, for $1 \le i \le c-1$, is a face of F_1. Set F_2 to be the stellar subdivision of F_1 with respect to σ_1 with new vertex $q+1$. Suppose $1 \le t \le c-1$ and F_t has been constructed. Since σ_i is a face of F_t for $i \ge t$, we can continue inductively and define F_{t+1} to be the stellar subdivision of F_t with respect to σ_t with new vertex $q+t$.

The Stanley-Reisner ring of F_c has the maximal possible Betti numbers among all elements in $\bigcup_{p \ge 2} \mathcal{D}_{p,c}$:

Proposition 5.2 *For all t with $1 \le t \le c$ and all $i \ge 0$ we have*

$$b_i(R_{[q+t-1]}/I_{F_t}) = l_{t,i}.$$

We will give the proof in Subsection 5.2.

Remark 5.3 Note that boundary complexes of stacked polytopes do not, in general, reach the bounds.

Remark 5.4 In the Macaulay2 package BettiBounds [5] we provide an implementation of the construction of F_t. Using the minimality of the Kustin-Miller complex, we also provide a function which produces their graded Betti numbers. This works far beyond the range which is accessible by computing the minimal free resolution via Gröbner bases.

Example 5.5 We use the implementation to produce F_4:

```
i1:  loadPackage "BettiBounds";

i2:  F4 = champion 4;

i3:  I4 = ideal F4
```

```
o3:  ideal(x_1x_2x_3x_4,x_1x_5x_6x_7,x_2x_5x_8x_9,x_5x_6x_7x_8x_9x_10,x_2x_3x_4x_11,
            x_8x_9x_10x_11,x_1x_3x_4x_12,x_1x_6x_7x_12,
            x_6x_7x_10x_12,x_3x_4x_11x_12,x_10x_11x_12)

i4:  betti res I4

               0   1   2   3  4

o4:  total:    1  11  20  11  1
         0:    1   .   .   .  .
         1:    .   .   .   .  .
         2:    .   1   .   .  .
         3:    .   9   9   1  .
         4:    .   .   2   .  .
         5:    .   1   9   9  .
         6:    .   .   .   1  .
         7:    .   .   .   .  .
         8:    .   .   .   .  1
```

The command `gradedBettiChampion 20`, will produce the Betti table of the minimal free resolution of $I_{F_{20}}$ with projective dimension 20 and regularity 208 in 0.7 seconds[1]. For more examples, see the documentation of BettiBounds.

5.2. Proof of Proposition 5.2

The main idea of the proof is that when passing from F_t to F_{t+1} by subdividing σ_t, the ideals I_{F_t} and $(I_{F_t} : x_{\sigma_t})$ have the same total Betti numbers (Proposition 5.9) and the Kustin-Miller complex construction yields a minimal free resolution (Lemma 5.8).

It is convenient to introduce the following notations, which will be used only in the present subsection. For nonzero monomials $v = \prod_{i=1}^{l} x_i^{a_i}$ and $w = \prod_{i=1}^{l} x_i^{b_i}$ in $R_{[l]}$ we set

[1]On a singe core of an Intel i7-2640M at 3.4 GHz.

$$\frac{v}{w} = \prod_{i=1}^{l} x_i^{c_i}, \quad \text{with } c_i = \max(a_i - b_i, 0),$$

and for a set S of monomials we set $S/w = \{(v/w) \mid v \in S\}$. Clearly $\frac{v}{w}$ is the monomial generator of the ideal quotient $((v) : (w))$.

For simplicity of notation write $\mathcal{T} = R_{[q+t-1]}$. We will now study in more detail the Stanley–Reisner ideal $I_{F_t} \subset \mathcal{T}$ of F_t. We set $u_1 = \prod_{i=1}^{q} x_i$, $u_2 = x_{q+1}u_1/x_{\sigma_1}$ and inductively define finite subsets $S_t \subset I_{F_t}$ by $S_1 = \{u_1\}, S_2 = \{u_2, x_{\sigma_1}\}$ and, for $t \geq 2$,

$$S_{t+1} = S_t \cup \frac{x_{q+t}S_t}{x_{\sigma_t}} \cup \{x_{\sigma_t}\}. \tag{5.1}$$

Clearly S_1 (resp. S_2) is the minimal monomial generating set of I_{F_1} (resp. I_{F_2}). Moreover, an easy induction on t using Equation (2.2) shows that S_t is a set of monomials generating I_{F_t} for all $1 \leq t \leq c$. In Proposition 5.9 we will show that S_t is actually the minimal monomial generating set of I_{F_t} for all t.

Equation (5.1) and induction imply that given an element v of S_{t+1} there exists $e_v \in \{u_2, x_{\sigma_1}, \ldots, x_{\sigma_t}\}$ such that either $v = e_v$ or $v = w_1 e_v/w_2$, with $w_1 = \prod_{j=1}^{l} x_{q+r_j}$ and $w_2 = \prod_{j=1}^{l} x_{\sigma_{r_j}}$ for some $l \geq 1$ and $r_1 < r_2 < \cdots < r_l \leq t$. Moreover, if $e_v = u_2$ we have $2 \leq r_1$, while if $e_v = x_{\sigma_p}$ we have $p + 1 \leq r_1$. A priori e_v may not be uniquely determined and we fix one of them and call it the original source of v. One can actually show that in our setting e_v is uniquely determined by v but we do not prove it and do not use it in the following.

Example 5.6 We have

$$S_3 = \left\{u_2,\ \frac{x_{q+2}u_2}{x_{\sigma_2}}\right\} \cup \left\{x_{\sigma_1},\ \frac{x_{q+2}x_{\sigma_1}}{x_{\sigma_2}}\right\} \cup \{x_{\sigma_2}\}$$

and

$$\begin{aligned} S_4 = &\left\{u_2,\ \frac{x_{q+2}u_2}{x_{\sigma_2}},\ \frac{x_{q+3}u_2}{x_{\sigma_3}},\ \frac{x_{q+2}x_{q+3}u_2}{x_{\sigma_2}x_{\sigma_3}}\right\} \\ &\cup \left\{x_{\sigma_1},\ \frac{x_{q+2}x_{\sigma_1}}{x_{\sigma_2}},\ \frac{x_{q+3}x_{\sigma_1}}{x_{\sigma_3}},\ \frac{x_{q+2}x_{q+3}x_{\sigma_1}}{x_{\sigma_2}x_{\sigma_3}}\right\} \end{aligned}$$

$$\cup \left\{ x_{\sigma_2}, \ \frac{x_{q+3}x_{\sigma_2}}{x_{\sigma_3}} \right\} \cup \{x_{\sigma_3}\}.$$

We now fix t with $t \leq c-1$. Part (1) of the following combinatorial lemma will be used in Lemma 5.8 for the proof of the minimality of the Kustin–Miller complex construction, while part (2) will be used in Proposition 5.9 for the proof of the equality of the corresponding Betti numbers of $\mathcal{T}/I_{F_t}$ and $\mathcal{T}/(I_{F_t} : x_{\sigma_t})$.

Lemma 5.7 (1) *For every $v \in S_t$ there exists $a \in \sigma_t$ such that x_a divides v.*

(2) *We can recover S_t from S_t/x_{σ_t} in the following way: S_t is the set obtained from S_t/x_{σ_t} by substituting, for $p = 1, 2, \ldots, t-1$, the variable $x_{(\sigma_p)_t}$ by the product $x_{(\sigma_p)_t}x_{(\sigma_p)_t-1}$, and substituting the variable x_{d_t+1} by the product $\prod_{r=d_{t-1}+1}^{d_t+1} x_r$.*

Proof. Let $v \in S_t$ and consider the original source $e_v \in \{u_2, x_{\sigma_1}, \ldots, x_{\sigma_{t-1}}\}$ of v. Write

$$v = \frac{w_1 e_v}{w_2}, \tag{5.2}$$

with either $(w_1 = w_2 = 1)$ or $w_1 = \prod_{j=1}^{l} x_{q+r_j}$ and $w_2 = \prod_{j=1}^{l} x_{\sigma_{r_j}}$ for some $l \geq 1$ and $r_1 < r_2 < \cdots < r_l \leq t-1$. Moreover, if $e_v = u_2$ we have $2 \leq r_1$, while if $e_v = x_{\sigma_p}$ we have $p+1 \leq r_1$.

We first prove (1). If $e_v = u_2$, we set $a = d_t \in \sigma_t$ and observe that x_a divides e_v. Since d_t is not in any σ_i for $i < t$ we have that x_a does not divide w_2, hence it follows by (5.2) that x_a divides v. Assume now that $e_v = \sigma_p$ for some p with $1 \leq p \leq t-1$. We set $a = (\sigma_p)_{t-1}$. By the definition of the sets σ_r, we have that a is in the intersection of σ_p with σ_t and in no other σ_r. Hence x_a divides e_v but not w_2, hence it follows by (5.2) that x_a divides v.

We will now prove (2). We first fix $p \in \{1, 2, \ldots, t-1\}$, set $m = (\sigma_p)_t$, assume x_m divides v, and prove that x_{m-1} also divides v. The assumption that x_m divides v implies that, when $v \neq e_v$, in the expression (5.2) we have $r_i \neq p$ for $1 \leq i \leq l$. Taking into account that m is not in σ_j for $1 \leq j \leq t-1$ and $j \neq p$ we get that $e_v = u_2$ or $e_v = \sigma_p$. Since $p < t$ we have $m - 1 = (\sigma_p)_{t-1}$. This, together with $e_v \in \{u_2, \sigma_p\}$ implies that x_{m-1} divides e_v. It also implies that $m-1$ is not in any σ_j for $1 \leq j \leq t-1$

and $j \neq p$. Hence, x_{m-1} does not divide w_2 and since it divides e_v if follows from (5.2) that it also divides v.

We now assume x_{d_t+1} divides v and will show that $\prod_{r=d_{t-1}+1}^{d_t+1} x_r$ also divides v. Since d_t+1 is not in σ_i for $1 \leq i \leq t-1$, we have that $e_v = u_2$. Fix r with $d_{t-1}+1 \leq r \leq d_t$. Then r is not an element of σ_j for $1 \leq j \leq t-1$. Hence x_r does not divide w_2 and since it divides u_2 if follows from (5.2) that it also divides v. Taking into account that m and d_t+1 are not in σ_t, this completes the proof of (2). □

Lemma 5.8 *Fix t with $2 \leq t \leq c-1$. Then the Kustin–Miller complex construction related to the unprojection pair $(I_{F_t} : x_{\sigma_t}, z) \subset \mathcal{T}[z]/(I_{F_t})$ and using as initial data the minimal graded free resolutions of $\mathcal{T}[z]/(I_{F_t})$ and $\mathcal{T}[z]/(I_{F_t} : x_{\sigma_t}, z)$ gives a minimal complex.*

Proof. The minimal monomial generating set of I_{F_t} is a subset, say $\tilde{S}_t$, of S_t. By part (1) of Lemma 5.7 given $v \in \tilde{S}_t$, there is an $a \in \sigma_t$ with x_a dividing v. As a consequence, the result follows from Proposition 4.2. □

Proposition 5.9 *Fix t with $2 \leq t \leq c$. Then*

(1) *The set S_t is the minimal monomial generating set of I_{F_t}.*
(2) *The corresponding Betti numbers of $\mathcal{T}/I_{F_t}$ and $\mathcal{T}/(I_{F_t} : x_{\sigma_t})$ are equal, that is*

$$b_i(\mathcal{T}/(I_{F_t} : x_{\sigma_t})) = b_i(\mathcal{T}/I_{F_t})$$

for all i. In particular, the set S_t/x_{σ_t} has the same cardinality as S_t and is the minimal monomial generating set of $(I_{F_t} : x_{\sigma_t})$.

Proof. We use induction on t. For $t = 2$ we have that both I_{F_t} and $(I_{F_t} : x_{\sigma_t})$ are codimension 2 complete intersections, so both (1) and (2) are obvious. Assume that (1) and (2) are true for a value $t < c-1$ and we will show that they are true also for the value $t+1$. By Lemma 5.8 the Kustin–Miller complex construction related to the unprojection pair $(I_{F_t} : x_{\sigma_t}, z) \subset \mathcal{T}[z]/I_{F_t}$ and using as input data the minimal graded free resolutions of $\mathcal{T}[z]/I_{F_t}$ and $\mathcal{T}[z]/(I_{F_t} : x_{\sigma_t}, z)$ gives a minimal complex. In particular, this implies that S_{t+1} is the minimal monomial generating set of $I_{F_{t+1}}$.

We now look more carefully the substitutions in part (2) of Lemma 5.7. Assume $p \leq t$ and set $m = (\sigma_p)_{t+1}$. Since $p < t+1$ we have by the construction of σ_p that $m-1=(\sigma_p)_t$, so $m-1$ is an element of σ_{t+1}. Consequently x_{m-1} does not appear as variable in $S_{t+1}/x_{\sigma_{t+1}}$. Similarly, for each r with $d_t+1\le r\le d_{t+1}$ we have $r\in\sigma_{t+1}$, so x_r does not appear as variable in $S_{t+1}/x_{\sigma_{t+1}}$. Using these facts, the equality of Betti numbers in part (2) follows by arguing as in the proof of [2, Proposition 6.5]. Since we have shown that S_{t+1} is the minimal monomial generating set of $I_{F_{t+1}}$, and $S_{t+1}/x_{\sigma_{t+1}}$ contains the minimal monomial generating set of $(I_{F_t}:x_{\sigma_t})$, the equality of Betti numbers we just showed implies, for $i=1$, that $S_{t+1}/x_{\sigma_{t+1}}$ has the same cardinality as S_{t+1} and is the minimal monomial generating set of the ideal $(I_{F_{t+1}}:x_{\sigma_{t+1}})$. □

We now give the proof of Proposition 5.2.

Proof. The proof is by induction on t. For $t=1,2$ the result is clear. Assume that the result is true for some value $2\le t\le c-1$ and we will show it is true for $t+1$. We set for simplicity $A_1=\mathcal{T}/I_{F_t}$ and $A_2=\mathcal{T}/(I_{F_t}:x_\sigma)$.

By the inductive hypothesis $b_i(A_1)=l_{t,i}$ and by part (2) of Proposition 5.9 $b_i(A_2)=b_i(A_1)$, hence $b_i(A_2)=b_i(A_1)=l_{t,i}$ for all i. Since by Lemma 5.8 the corresponding Kustin–Miller construction is minimal, we get that

$$b_1(R_{[q+t]}/I_{F_{t+1}})=b_1(A_1)+b_1(A_2)+1=2l_{t,1}+1=l_{t+1,1}$$

and that for i with $2\le i\le \operatorname{codim} R_{[q+t]}/(I_{F_{t+1}})-2$

$$\begin{aligned} b_i(R_{[q+t]}/I_{F_{t+1}}) &= b_{i-1}(A_1)+b_i(A_1)+b_{i-1}(A_2)+b_i(A_2)\\ &= 2l_{t,i-1}+2l_{t,i}=l_{t+1,i}\end{aligned}$$

which finishes the proof. □

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