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Some properties of the algebra $H^{\infty}(m)$

By Kazuo KISHI

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§ 1. Introduction.

For a complex commutative Banach algebra B, let $M(B)$ be the maximal ideal space of B endowed with the Gelfand topology, let $\hat{f}$ and $\hat{B}$ be the Gelfand transform of $f(\in B)$ and B respectively, and let $\Gamma(B)$ be the Shilov boundary of B.

Let A be a uniform algebra on a compact Hausdorff space X. We suppose that $m\in M(A)$ has a unique representing measure m on X, and that the Gleason part P of m for A is nontrivial. We denote by $H^{\infty}(m)$ the w^{*} (*i.e.*, weak-star) closure of A in $L^{\infty}(dm)$, and define $\tilde{m}\in M(H^{\infty}(m))$ by $\tilde{m}(f)=\int f\,dm$, $f\in H^{\infty}(m)$. Then it is known that $\hat{H}^{\infty}(m)$ is a logmodular algebra on $\tilde{X}=M(L^{\infty}(dm))$ and hence $\Gamma(H^{\infty}(m))=\tilde{X}$, and the Gleason part $\mathscr{P}$ of $\tilde{m}$ for $\hat{H}^{\infty}(m)$ is nontrivial. We put $I^{\infty}=\{f\in H^{\infty}(m):\phi(f)=0$ for all ϕ in $\mathscr{P}\}$. Let Z be the Wermer's embedding function (see § 2), and let $\mathscr{L}^{\infty}$ be the w^{*} closure of the polynomials in Z and $\bar{Z}$ in $L^{\infty}(dm)$. Then $M(\mathscr{L}^{\infty})$ can be identified with the Shilov boundary Y of the algebra $\hat{H}^{\infty}(m)|\bar{\mathscr{P}}$, where $\bar{\mathscr{P}}$ is the closure of $\mathscr{P}$ in $M(H^{\infty}(m))$ (see § 2). If M is a closed subspace of $L^{1}(dm)$, we define the *support set* of M (denoted by $E(M)$) as the complement of a set of maximal measure on which all $f\in M$ are null. A function $f\in H^{\infty}(m)$ with $|f|=1$ a. e. (dm) is called an *inner function*.

In § 3 we shall prove the following.

LEMMA. $\tilde{X}\cap\bar{\mathscr{P}}=\tilde{X}\cap Y$.

THEOREM A. *Let $E=E(I^{\infty})$ and let $F=\tilde{X}\cap Y$. Then we have the following.*

(i) *The characteristic function χ_{E} of E belongs to $\mathscr{L}^{\infty}$.*

(ii) $\{\phi\in Y:\hat{\chi}_{E}(\phi)=1\}=Y\backslash F$.

(iii) $\{\tilde{x}\in\tilde{X}:\hat{\chi}_{E}(\tilde{x})=1\}=\tilde{X}\backslash F$.

COROLLARY 1. $\tilde{\pi}(\tilde{X}\backslash F)=Y\backslash F$ *(for $\tilde{\pi}$ see* § 2).

COROLLARY 2. *$\tilde{X}\supset Y$ if and only if $H^{\infty}(m)$ is maximal as a w^{*} closed subalgebra of $L^{\infty}(dm)$.*

COROLLARY 3. *$\tilde{X}\cap Y=\phi$ if and only if there is an inner function h*

in I^{∞}.

THEOREM B. (i) *The space* $M(H^{\infty}(m))\backslash\mathscr{P}$ *is connected.*

(ii) *If* $I^{\infty}\neq\{0\}$, *then* $M(H^{\infty}(m))\backslash\bar{\mathscr{P}}$ *is disconnected, and hence* $M(I^{\infty})$ *is disconnected.*

Theorem B, (i) is a generalization of Hoffman's theorem (cf. Hoffman [4], Theorem 3).

In § 4 we shall construct a certain algebra $H^{\infty}(m)$ with $\tilde{X}\cap Y\neq\phi$ and $I^{\infty}\neq\{0\}$. In our example $\tilde{X}\cap Y$ can be any clopen (*i. e.*, closed and open) set in Y. In § 2 some preliminaries are given.

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§ 2. Preliminaries.

Let A be a uniform algebra on a compact Hausdorff space X. When $\phi\in M(A)$ has a unique representing measure, sometimes we use the same symbol ϕ to denote its representing measure. (In other places, the representing measure is denoted as μ_{ϕ}.) *Hereafter we suppose that* m *(fixed) in* $M(A)$ *has a unique representing measure* m, *and that the Gleason part* $P=P(m)$ *of* m *is nontrivial.* There is a probability measure $\tilde{m}$ on $\tilde{X}=M(L^{\infty}(dm))$ such that

$$\int_{X} f\,dm=\int_{\tilde{X}} f\,d\tilde{m}\,, \qquad f\in L^{\infty}(dm)\,.$$

This measure $\tilde{m}$ is called the Radonization of m (cf. Srinivasan and Wang [11], p. 222). It is known that $\phi\in M(H^{\infty}(m))$ belongs to $\tilde{X}$ if and only if $|\phi(f)|=1$ for every inner function f in $H^{\infty}(m)$ (cf. Douglas and Rudin [2], p. 318).

An inner function Z known as *Wermer's embedding function* satisfies $ZH^{\infty}(m)=\{f\in H^{\infty}(m):\int f\,dm=0\}$, and $\phi\mapsto\hat{Z}(\phi)=\int Z\,d\phi$ is a one-to-one map of $P(m)$ onto the open unit disk D. The inverse map τ of $\hat{Z}$ is a one-to-one continuous map of D onto $P(m)$, and for every f in $H^{\infty}(m)$ the composition $f\circ\tau$ is analytic in D (Wermer's embedding theorem, cf. Leibowitz [7], p. 143).

Let $\mathscr{H}^{p}$ be the closure in $L^{p}(dm)$ norm of the polynomials in Z, and let $\mathscr{L}^{p}$ be the closure in $L^{p}(dm)$ norm of the polynomials in Z and $\bar{Z}$. (For $p=\infty$, the closure is taken in the w^{*} topology.) Let σ be the normalized Lebesgue measure on the unit circle ∂D in the complex plane, and let $H^{\infty}(d\sigma)$ be the classical Hardy space on ∂D. By Fatou's theorem, $H^{\infty}(d\sigma)$ is identified with the Banach algebra $H^{\infty}(D)$ of all bounded analytic functions in D.

The correspondence

$$(2.1)\qquad T:\ Z\longmapsto e^{i\theta}$$

induces an isometric *-isomorphism of $\mathscr{L}^p$ onto $L^p(d\sigma)$, for $1\leqq p\leqq\infty$. This map is also an isometric isomorphism of $\mathscr{H}^\infty$ onto $H^\infty(d\sigma)\,(=H^\infty(D))$. Therefore the adjoint T^* of T is a homeomorphism of $M(L^\infty(d\sigma))$ and $M(H^\infty(D))$ onto $M(\mathscr{L}^\infty)$ and $M(\mathscr{H}^\infty)$ respectively.

For $1\leqq p\leqq\infty$, if we set

$$I^p=\left\{f\in H^p(m):\ \int \bar{Z}^n f\,dm=0,\quad n=0,1,2,\cdots\right\}$$

and

$$N^p=\left\{f\in L^p(dm):\ \int Z^n f\,dm=0,\quad n=0,\pm1,\pm2,\cdots\right\}$$

then we have

$$(2.2)\qquad H^p(m)=\mathscr{H}^p\oplus I^p \text{ and } L^p(dm)=\mathscr{L}^p\oplus N^p\,,$$

where $\oplus$ denotes algebraic direct sum. The set N^p is the closure of $\bar{I^p}\oplus I^p$ in $L^p(dm)$ (norm closure for $1\leqq p<\infty$; w^* closure for $p=\infty$) and we have $I^p=\{f\in H^p(m):\ \int f d\phi=0 \text{ for all } \phi\in P(m)\}$. (Cf. Merrill and Lal [9].) Here we shall state a consequence of Nakazi [10].

Nakazi's Theorem. *Let $E=E(I^\infty)$ be the support set of I^∞. Then there is a function $h\in I^\infty$ with $|h|=\chi_E$, where χ_E is the characteristic function of E.*

We shall collect some results in Kishi [6] which will be needed in §§ 3 and 4. (Sometimes we do not distinguish between a Banach algebra B and its Gelfand transform $\hat{B}$.) The set I^∞ is an ideal of $H^\infty(m)$, and, by the map $S:\ f+I^\infty\mapsto f,\ f\in\mathscr{H}^\infty$, the quotient Banach algebra $H^\infty(m)/I^\infty$ is isometrically isomorphic to $\mathscr{H}^\infty$. Hence, under the adjoint Σ of S, the space $M(\mathscr{H}^\infty)$ can be identified with hull $(I^\infty)\,(\subset M(H^\infty(m)))$. Since $M(H^\infty(D))=\bar{D}$ (Carleson's corona theorem) and $\Sigma(T^*(D))=\{\phi\in \text{hull}(I^\infty):\ |\phi(Z)|<1\}=\mathscr{P}$ (cf. Kishi [5], p. 469), we have $\Sigma(T^*(\bar{D}))=\Sigma(M(\mathscr{H}^\infty))=\bar{\mathscr{P}}$. Hence we have

$$(2.3)\qquad \Sigma\Big(M(\mathscr{H}^\infty)\Big)=\text{hull}(I^\infty)=\bar{\mathscr{P}}\,.$$

If we put $Y=\Sigma(\Gamma(\mathscr{H}^\infty))=\Sigma(M(\mathscr{L}^\infty))$, then, as functions on Y, we have $\log|(\mathscr{H}^\infty)^{-1}|=C_R(Y)$. By using the map $\Sigma\circ T^*$ we see that the space Y is stonian (*i.e.*, if U is open in Y, then $\bar{U}$ is also open), and that a unique representing measure $\lambda_{\tilde{m}}$ on Y of $\tilde{m}\in\mathscr{P}$ $(\subset\Sigma(M(\mathscr{H}^\infty)))$ for $\mathscr{H}^\infty$ is a normal measure. We have

$$\tilde{m}(f)=\int_Y f\,d\lambda_{\tilde{m}}, \qquad f\in\mathscr{H}^\infty.$$

If $\phi\in M(I^\infty)$, then there is some $h\in I^\infty$ such that $\phi(h)=1$. We define $\Phi\in M(H^\infty(m))$ by $\Phi(f)=\phi(fh)$, $f\in H^\infty(m)$. By well known fact, the map

$$\prod : \phi\longmapsto\Phi$$

is a homeomorphism of $M(I^\infty)$ onto $M(H^\infty(m))\backslash\mathscr{P}$, and under Π the space $M(I^\infty)$ can be identified with $M(H^\infty(m))\backslash\mathscr{P}$. On the other hand the algebraic direct sum $B=\mathscr{L}^\infty\oplus I^\infty$ of $\mathscr{L}^\infty$ and I^∞ is a Banach algebra, and I^∞ is an ideal of B. We define $\Phi'\in M(B)$ by $\Phi'(f)=\phi(fh)$, $f\in B$. The map $\phi\mapsto\Phi'$ is a homeomorphism of $M(I^\infty)$ onto $M(B)\backslash\mathrm{hull}(I^\infty)$, and $M(I^\infty)$ can be identified with $M(B)\backslash\mathrm{hull}(I^\infty)$. Since $\log|(\mathscr{H}^\infty)^{-1}|=\mathscr{L}^\infty_R$, $\Phi'|\mathscr{L}^\infty\in M(\mathscr{L}^\infty)$ and $\Phi|\mathscr{H}^\infty=\Phi'|\mathscr{H}^\infty$, $\Phi|\mathscr{H}^\infty$ can be identified with a complex homomorphism of $\mathscr{L}^\infty$. Now we define a continuous map π_1 of $M(H^\infty(m))\backslash\mathscr{P}$ into $Y(\subset\mathscr{P})$ by

$$\pi_1(\Phi)=\Sigma(\Phi|\mathscr{H}^\infty), \qquad \Phi\in M\big(H^\infty(m)\big)\backslash\mathscr{P}.$$

Further we define a continuous map $\tilde{\pi}$ of $\tilde{X}=M(L^\infty(dm))$ onto Y by

$$\tilde{\pi}(\tilde{x})=\begin{cases}\Sigma(\tilde{x}|\mathscr{L}^\infty) & \text{if } \tilde{x}\in\tilde{X}\backslash\mathscr{P}\\ \tilde{x} & \text{if } \tilde{x}\in\tilde{X}\cap\mathscr{P}.\end{cases}$$

If $\phi\in Y$, then for every f in $\mathscr{L}^\infty$, $\hat{f}$ is a constant $(=\phi(f))$ on the closed support $(=\mathrm{supp}\,\mu_\phi)$ of the representing measure μ_ϕ for ϕ.

§ 3. Proofs of the results.

PROOF OF LEMMA. If $\phi\in\tilde{X}\cap\mathscr{P}$, then we have $|\phi(f)|=1$ for every inner function f in $\mathscr{H}^\infty(\subset H^\infty(m))$. Hence, by Kishi [6], Lemma 2. 3m ϕ belongs to $\tilde{X}\cap Y$.

PROOF OF THEOREM A. (i) Let $E=E(I^\infty)$ and $E^c=X\backslash E$. Then we have $\int\chi_{E^c}f\,dm=0$, $f\in I^\infty$, and hence $\int\chi_{E^c}f\,dm=0$, $f\in I^2+\overline{I^2}$. So, by (2. 2), $\chi_{E^c}\in\mathscr{L}^2\cap L^\infty(dm)=\mathscr{L}^\infty$, and hence we have $\chi_E=1-\chi_{E^c}\in\mathscr{L}^\infty$.

(ii) Let $F=\tilde{X}\cap Y$. By Nakazi's Theorem there is a function $h\in I^\infty$ with $|h|=\chi_E$. Then we have $\hat{\chi}_E=\widehat{|h|}=|\hat{h}|=0$ on F, and hence we have $F\subset\{\phi\in Y:\ \hat{\chi}_E(\phi)=0\}$.

If $\phi_0\in Y\backslash F$, then there is an inner function $f\in H^\infty(m)$ such that $|\phi_0(f)|<1$. If $f=g+h$, where $g\in\mathscr{H}^\infty$ and $h\in I^\infty$, then we have $|\phi_0(g)|<c<1$ for some constant c. Let $V(\phi_0)$ be a clopen neighborhood of ϕ_0 in Y such that $V(\phi_0)\subset\{\phi\in Y: |\phi(g)|<c\}$. Then there is a $\chi_G\in\mathscr{L}^\infty$ with $V(\phi_0)=\{\phi\in Y;\ \hat{\chi}_G(\phi)=1\}$. If $\tilde{x}\in\tilde{\pi}^{-1}(V(\phi_0))$ and $\tilde{\pi}(\tilde{x})=\phi$, then we have $|\tilde{x}(h)|\geqq|\tilde{x}(f)|-|\tilde{x}(g)|=1-|\phi(g)|>1-c>0$. Hence we have $|\hat{h}|>1-c$ on $\tilde{\pi}^{-1}(V(\phi_0))=\{\tilde{x}\in\tilde{X}: \hat{\chi}_G(\tilde{x})=1\}$. So we have $\hat{\chi}_{G^c}+\hat{\chi}_G|\hat{h}|>1-c$ on $\tilde{X}$, and hence $\chi_{G^c}+\chi_G|h|\geqq 1-c$ a. e. (dm). Thus we have $G\subset E$, and hence $V(\phi_0)\subset\{\phi\in Y: \hat{\chi}_E(\phi)=1\}$. Therefore we have $Y\backslash F\subset\{\phi\in Y: \hat{\chi}_E(\phi)=1\}$, and obtain (ii).

(iii) If $\tilde{x}\in\tilde{X}\backslash F=\tilde{X}\backslash\bar{\mathscr{P}}$, then there are a clopen neiborhood $V(\tilde{x})$ of $\tilde{x}$ in $\tilde{X}$ and a function $h\in I^\infty$ such that $|\hat{h}|\geqq c>0$ on $V(\tilde{x})$, where c is a constant with $0<c<1$. Then, by the same method as (ii), we obtain $\tilde{X}\backslash F\subset\{\tilde{x}\in\tilde{X}: \hat{\chi}_E(\tilde{x})=1\}$. Further we have $F\subset\{\tilde{x}\in\tilde{X}: \hat{\chi}_E(\tilde{x})=0\}$. Hence we obtain $\tilde{X}\backslash F=\{\tilde{x}\in\tilde{X}: \hat{\chi}_E(\tilde{x})=1\}$.

PROOF OF COROLLARY 1. Let $E=E(I^\infty)$ and $F=\tilde{X}\cap Y$. Then, by Theorem A, we have $1=\hat{\chi}_E(\tilde{x})=\hat{\chi}_E(\tilde{\pi}(\tilde{x}))$ for $\tilde{x}\in\tilde{X}\backslash F$, so we obtain $\tilde{\pi}(\tilde{X}\backslash F)\subset Y\backslash F$. Further we have $\tilde{\pi}(\tilde{X})=Y$ and $\tilde{\pi}(F)=F$. Hence we obtain $\tilde{\pi}(\tilde{X}\backslash F)=Y\backslash F$.

PROOF OF COROLLARY 2. If $\tilde{X}\supset Y$, then, by Corollary 1, we have $\tilde{\pi}(\tilde{X}\backslash Y)=\tilde{\pi}(\tilde{X}\backslash(\tilde{X}\cap Y))=Y\backslash(\tilde{X}\cap Y)=\phi$, and hence we obtain $\tilde{X}=Y$. Hence we have $I^\infty=\{0\}$, and hence $H^\infty(m)$ is maximal as a w^* closed subalgebra of $L^\infty(dm)$ (cf. Merrill [8]). Conversely, if $H^\infty(m)$ is maximal as a w^* closed subalgebra of $L^\infty(dm)$, then $I^\infty=\{0\}$ (cf. Merrill [8]), and hence $\tilde{X}=Y$.

PROOF OF COROLLARY 3. If $\tilde{X}\cap Y=\phi$, then, by Theorem A, (iii) and Nakazi's theorem, there is an inner function h in I^∞. Conversely if there is an inner function h in I^∞, then we have $\hat{h}=0$ on Y and $|\hat{h}|=1$ on $\tilde{X}$, and hence we obtain $\tilde{X}\cap Y=\phi$.

PROOF OF THEOREM B. (i) Let Z be Wermer's embedding function, let $S=\{1, Z, Z^2, \cdots\}$, and let $\mathscr{A}$ be the norm closure in $L^\infty(dm)$ of the set $\{\bar{s}f: s\in S, f\in H^\infty(m)\}$. Then $\mathscr{A}$ is a Banach algebra, and $M(\mathscr{A})$ can be identified with the set $M(H^\infty(m))\backslash\mathscr{P}$ (cf. Douglas and Rudin [2], p. 317 and Kishi [5], p. 469). If we put $B=\mathscr{L}^\infty\oplus I^\infty$, then B is a Banach algebra which contains $\mathscr{A}$ and $M(B)$ can be identified with set $M(H^\infty(m))\backslash(\bar{\mathscr{P}}\backslash Y)$ (cf. Kishi [6], Theorem 3. 5). If $f(=g+h)$ in $\mathscr{A}$ vanishes on $\partial\mathscr{P}=\bar{\mathscr{P}}\backslash\mathscr{P}$, where $g\in\mathscr{L}^\infty$ and $h\in I^\infty$, then $f\in I^\infty$. In fact, since $f=0$ on $Y(\subset\partial\mathscr{P})$ and $h=0$ on Y, we have $g=0$ on Y. And, remembering that Y can be identified with $M(\mathscr{L}^\infty)$, we have $g=0$, and hence we have $f=h\in I^\infty$.

Suppose that $M(H^\infty(m))\backslash\mathscr{P}$ is not connected. Then there is a nontrivial clopen set V in $M(H^\infty(m))\backslash\mathscr{P}$. Since $\mathscr{P}$ is open in $M(H^\infty(m))$, V is closed in $M(H^\infty(m))$.

If $V\cap\bar{\mathscr{P}}=\phi$, then V is open in $M(H^\infty(m))\backslash\bar{\mathscr{P}}$, and hence V is open in $M(H^\infty(m))$. Thus V is a nontrivial clopen set in $M(H^\infty(m))$. This is absurd (cf. Leibowitz [7], p. 167).

Suppose $V\cap\bar{\mathscr{P}}\neq\phi$. Then $V\cap\bar{\mathscr{P}}=V\cap\partial\mathscr{P}$ is a non-empty clopen set in $\partial\mathscr{P}$. Since $\partial\mathscr{P}=(\Sigma\circ T^*)(\bar{D}\backslash D)$ (see (2.1) and (2.3)) and $\bar{D}\backslash D$ is connected (Hoffman [4], Theorem 3), $\partial\mathscr{P}$ is connected. Hence we have $\partial\mathscr{P}\subset V\subsetneqq M(\mathscr{A})$. By Shilov idempotent theorem (cf. Leibowitz [7], p. 167) there is an element $f\in\mathscr{A}$ such that $f=0$ on V and $f=1$ on $M(\mathscr{A})\backslash V$. Hence $f=0$ on $\partial\mathscr{P}$, and we have $f\in I^\infty$. Therefore $f=0$ on $V\cup\bar{\mathscr{P}}(\subset M(H^\infty(m)))$ and $f=1$ on $M(H^\infty(m))\backslash(V\cup\bar{\mathscr{P}})$. This is absurd.

(ii) If we put $F=\tilde{X}\cap Y$, then F is a clopen set in Y such that $F\subsetneqq Y$. In fact, if $\tilde{X}\cap Y=Y$, then $\tilde{X}\supset Y$ and, by Corollary 2, we have $I^\infty=\{0\}$. For any element $\tilde{x}$ in $\tilde{X}\backslash F$, we have $\tilde{\pi}(\tilde{x})=\pi_1(\tilde{x})$. Now, since Y is a stonian space, there is a nonempty clopen set U such that $U\subsetneqq Y\backslash F$. Then $\pi_1^{-1}(\mathrm{U})$ is a nontrivial clopen set in $M(H^\infty(m))\backslash\bar{\mathscr{P}}$, as this set does not contain all of $\tilde{\pi}^{-1}(Y\backslash F)$. Hence the subspace $M(H^\infty(m))\backslash\bar{\mathscr{P}}$ is disconnected. And, since $\Pi^{-1}(\pi_1^{-1}(V))$ is a nontrivial clopen set in $M(I^\infty)$, the space $M(I^\infty)$ is disconnected.

§ 4. An example.

Let $H^\infty(m)$ be any Banach algebra with $\tilde{X}\cap Y=\phi$. (Two examples in Kishi [6], § 5 satisfy such a condition.) Let χ be any function in $\mathscr{L}^\infty$ with $\chi^2=\chi$ and $\chi\neq 0, 1$, and let $A_1=\mathscr{A}^\infty\oplus\chi I^\infty$. Let $X_1=(\tilde{X}\backslash E)\cup U$, where $U=\{\phi\in Y:\ \chi(\phi)=0\}$ and $E=\tilde{\pi}^{-1}(U)$, and let m_1 be a probability measure on X_1 such that $m_1=\tilde{m}$ on $\tilde{X}\backslash E$ and $m_1=\lambda_{\tilde{m}}$ on U (for $\tilde{m}$ and $\lambda_{\tilde{m}}$ see § 2). Then we have the following.

(i) A_1 is a w^* Dirichlet algebra in $L^\infty(dm_1)$ and $A_1=H^\infty(m_1)$, where $H^\infty(m_1)$ is the w^* closure of A_1 in $L^\infty(dm_1)$.

(ii) $M(L^\infty(dm_1))=X_1$, and $M(A_1)$ can be identified with $M(H^\infty(m))\backslash(\pi_1^{-1}(U)\backslash U)(\subset M(H^\infty(m)))$, and $\mathscr{P}$ is the Gleason part of m_1 for A_1. Hence we have $\bar{\mathscr{P}}\cap M(L^\infty(dm_1))=U$. Thus A_1 is a logmodular algebra on X_1 and $H^\infty(m_1)$ is an example which has the properties of $\tilde{X}\cap Y\neq\phi$ and $I^\infty\neq\{0\}$.

Indeed, since both $\tilde{X}$ and Y are stonian spaces, $X_1=(\tilde{X}\backslash E)\cup U$ is a compact stonian space. Let ξ be a continuous map of $\tilde{X}$ onto X_1 defined by

$$\xi(\tilde{x})=\begin{cases}\tilde{x} & \text{if } \tilde{x}\in\tilde{X}\backslash E\\ \tilde{\pi}(\tilde{x}) & \text{if } \tilde{x}\in E.\end{cases}$$

Then there is a linear transformation ρ induced by ξ of the dual space of $C(\tilde{X})$ onto the dual space of $C(X_1)$. If $m_1=\rho(\tilde{m})$, then we have $m_1=\tilde{m}$ on $\tilde{X}\backslash E$ and $m_1=\lambda_{\tilde{m}}$ on U (cf. Kishi [6], p. 489). Hence m_1 is a normal probability measure on X_1 such that $\operatorname{supp} m_1 = X_1$, and hence the natural injection $C(X_1) \subset L^\infty(dm_1)$ is an isometric isomorphism of $C(X_1)$ and $L^\infty(dm_1)$ (cf. Bade [1], Lemma 8.16). Thus we have $M(L^\infty(m_1)) = X_1$.

The measure m_1 is multiplicative on a subalgebra A_1 of $L^\infty(dm_1)$. In fact, if $f = g + \chi h \in A_1$, where $g \in \mathscr{A}^\infty$ and $\chi h \in \chi I^\infty$, then we have

$$\int_{X_1} g\, dm_1 = \int_{X_1} g\, d\big(\rho(\tilde{m})\big) = \int_{\tilde{X}} g \circ \xi\, d\tilde{m} = \int_{\tilde{X}} g\, d\tilde{m}$$

and, by using $\chi = \chi_{\tilde{X} \setminus E}$ on $\tilde{X}$,

$$\begin{aligned}\int_{X_1} \chi h\, dm_1 &= \int_{\tilde{X}\setminus E} \chi h\, d\big(\rho(\tilde{m})\big) + \int_U \chi h\, d\big(\rho(\tilde{m})\big) \\ &= \int_{\tilde{X}\setminus E} \chi h\, d\tilde{m} = \int_{\tilde{X}} \chi h\, d\tilde{m} (=0)\,,\end{aligned}$$

and hence we obtain

$$\int_{X_1} f\, dm_1 = \int_{\tilde{X}} f\, d\tilde{m}\,, \qquad f \in A_1\,.$$

The set $\mathscr{A}^\infty$ is w^* closed in $L^\infty(dm_1)$. In fact, $\mathscr{A}^\infty$ is a convex set in $L^\infty(\mathrm{dm}_1)$. Let $\{f_n\}$ be a sequence in $\mathscr{A}^\infty$ such that $\|f_n\| \leqq M$, where M is a constant, $f_n \to f$ a.e.(dm_1). Then, since $(1-\chi)\,\mathrm{f}_n \in C(Y)\,(=\hat{\mathscr{L}}^\infty)$, $\|(1-\chi) f_n\| \leqq M$ and $(1-\chi) f_n \to (1-\chi) f$ a.e.$(d\lambda_{\tilde{m}})$, we have $(1-\chi) f \in L^\infty(d\lambda_{\tilde{m}})$. But, since $C(Y)$ is isometrically isomorphic to $L^\infty(d\lambda_{\tilde{m}})$, $(1-\chi) f$ can be identified with a function in $C(Y)$. There is a subset N of U such that $\lambda_{\tilde{m}}(N) = 0$ and $(1-\chi) f_n \to (1-\chi) f$ on $Y \setminus N$. Then $\tilde{m}(\tilde{\pi}^{-1}(N)) = \lambda_{\tilde{m}}(N) = 0$ and $(1-\chi) f_n \to (1-\chi) f$ on $\tilde{\pi}^{-1}(Y \setminus N)$, and hence $(1-\chi) f_n \to (1-\chi) f$ a.e.$(d\tilde{m})$. Of course $\chi f_n \to \chi f$ a.e.$(d\tilde{m})$, so $f_n \to f$ a.e.$(d\tilde{m})$. Since $\mathscr{A}^\infty$ is w^* closed in $L^\infty(d\tilde{m})$, $f \in \mathscr{A}^\infty$. Therefore, by Gamelin and Lumer [3], Lemma 3.5, $\mathscr{A}^\infty$ is w^* closed in $L^\infty(dm_1)$.

It is easy to see that χI^∞ is w^* closed in $L^\infty(dm_1)$. Hence A_1 is w^* closed in $L^\infty(dm_1)$ (see the proof of Kishi [6], Theorem 3.5). Hence we have $A_1 = H^\infty(m_1)$. And, we easily see that A_1 is a w^* Dirichlet algebra in $L^\infty(dm_1)$. Further we note that χI^∞ is an ideal of A_1 and $A_1/\chi I^\infty$ is isometrically isomorphic to $\mathscr{A}^\infty$.

$M(A_1)$ can be identified with $\mathscr{P} \cup M_1$, where $M_1 = M(H^\infty(m)) \setminus (\mathscr{P} \cup \pi_1^{-1}(U))$. In fact, since $A_1/\chi I^\infty$ and $H^\infty(m)/I^\infty$ are isometrically isomorphic, $\operatorname{hull}(\chi I^\infty)$ can be identified with $\operatorname{hull}(I^\infty) = \mathscr{P}$. The map $\Lambda: f \mapsto \chi f$, $f \in I^\infty$ is a homomorphism of I^∞ onto χI^∞, and the kernel $\Lambda^{-1}(0)$ of Λ is $\{(1-\chi) f : f \in I^\infty\}$. Hence $M(\chi I^\infty)$ can be identified with $\operatorname{hull}(\Lambda^{-1}(0))$, *i.e.*, we have

$$M(\chi I^\infty) = \{\phi \in M(I^\infty) : \phi(h) = \phi(\chi h) \text{ for all } h \in I^\infty\}$$

If $\phi\in M(\chi I^\infty)$, there is some $h\in I^\infty$ such that $\phi(h)=\phi(\chi h)=1$. We define $\Phi\in M(H^\infty(m))$ by $\Phi(f)=\phi(fh)$, $f\in H^\infty(m)$. As we stated in § 2, $\Phi|\mathscr{L}^\infty$ can be identified with a complex homomorphism of $\mathscr{L}^\infty$. Since $\Phi(\chi)=\phi(\chi h)=1$, we have $\pi_1(\Phi)\in Y\backslash U$. Hence Φ belongs to M_1. Conversely, if $\Phi\in M_1$ and $\phi=\Pi^{-1}(\Phi)$, then Φ is defined by $\Phi(f)=\phi(hf)$ for $f\in H^\infty m$, where h is a function $h\in I^\infty$ with $\phi(h)=1$. Since $\pi_1(\Phi)$ is in $Y\backslash U$, $\pi_1(\Phi)(\chi)=1$. Hence $\Phi(\chi)=1$, which means $\phi(\chi h)=1$, and ϕ is a nonzero complex homomorphism of χI^∞. Thus $M(\chi I^\infty)$ can be identified with M_1. Therefore $M(A_1)$ can be identified with $\bar{\mathscr{P}}\cup M_1$.

For Wermer's embedding function Z we have $\mathscr{P}=\{\phi\in M(A_1): |\phi(Z)|<1\}$ (cf. Kishi [5], p. 469). Hence $\mathscr{P}$ is a nontrivial Gleason part of m_1 for A_1.

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Kazuo KISHI
Faculty of Education
Wakayama University
Wakayama, Japan