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Some considerations on various curvature tensors

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K. Yano has introduced the notion of complex conformal connections in Kählerian spaces and showed

THEOREM A ([12]). *In a Kählerian space of real dimension* $\geqq 4$, *if there exists a complex conformal connection with zero curvature, then the Bochner curvature tensor of the space vanishes.*

K. Yano has also introduced the notion of contact conformal connections in Sasakian spaces corresponding to complex conformal connections in Kählerian spaces and had

THEOREM B ([13]). *In a Sasakian space of dimension* $\geqq 5$, *if there exists a contact conformal connection with zero curvature, then the contact Bochner curvature tensor of the space vanishes.*

In the present paper, we consider the converses of Theorem A and Theorem B.

We give algebraic preliminaries and notations in §§1 and 2. §3 is devoted to the proof of Theorem 1, which asserts that if there exists a non-constant solution of a certain partial differential equation, the converse of Theorem A is true. In §4, from a viewpoint of the notions of K-curvature and F-invariant curvature tensors, we define the Bochner curvature tensor of a K-space. Theorem 2 gives a characterization of the vanishing of the Bochner curvature tensor of a K-space. Lemma 10 shows that the converse of Theorem B is true, if there exists a non-constant function satisfying a certain system of partial differential equations.

We remark that a Sasakian space satisfying the assumptions in Lemma 10 admits another Sasakian structure of constant ϕ-holomorphic sectional -3. The latter part of §5 is devoted to the study of a system of partial differential equations in Lemma 10. Theorem 3 and Theorem 4 give a characterization of the Sasakian structure of constant ϕ-holomorphic sectional curvature -3.

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Throughout this paper, our arguments are local and sometimes pointwise.

§ 1. Algebraic considerations.

Let V be a d-dimensional real vector space with an inner product g. A tensor L of type $(1, 3)$ over V can be considered as a bilinear mapping

$$(x, y) \in V \times V \longrightarrow L(x, y) \in \mathrm{Hom}(V, V).$$

Such a tensor L is called a curvature tensor on V ([6]) if it has the following properties;

(a) $L(x, y) = -L(y, x)$;

(b) $g\big(L(x, y)\, z, w\big) + g\big(L(x, y)\, w, z\big) = 0$;

(c) $L(x, y)\, z + L(y, z)\, x + L(z, x)\, y = 0$ (*the first Bianchi identity*).

We denote by $\mathscr{L}_0(V)$ the vector space of all curvature tensors on V. It is a subspace of the tensor space of type $(1, 3)$ over V and has a natural inner·product induced from that in V.

For $L \in \mathscr{L}_0(V)$, the Ricci tensor S_L of type $(1, 1)$ is a symmetric endomorphism of V given by

$S_L(x)$ = trace of the bilinear map: $(y, z) \in V \times V \rightarrow L(x, y)\, z \in V$.

1.1. *K*-curvature tensors.

Consider the case when d is even, say $2n$, and V has a complex structure F and Hermitian inner product g. Then, a tensor $L \in \mathscr{L}_0(V)$ is said to be F-invariant ([11]) if it satisfies

(d) $g\big(L(x, y)\, z, w\big) = g\big(L(Fx, Fy)\, Fz, Fw\big)$,

and L is called a K-curvature tensor ([5]) if it satisfies

(e) $L(x, y) \circ F = F \circ L(x, y)$.

And we call an F-invariant curvature tensor L a K^*-curvature one if it satisfies

$$g\big(L(Fx, y)\, z, w\big) + g\big(L(x, Fy)\, z, w\big) + g\big(L(x, y)\, Fz, w\big) + g\big(L(x, y)\, z, Fw\big) = 0.$$

If we denote by $\mathscr{L}(V)^*$, $\mathscr{L}^*(V)$ and $\mathscr{L}(V)$ the vector spaces of all F-invariant curvature, of all K^*-curvature and of all K-curvature tensors, respectively, then we have $\mathscr{L}_0(V) \supset \mathscr{L}(V)^* \supset \mathscr{L}^*(V) \supset \mathscr{L}(V)$. The projections of $\mathscr{L}_0(V)$ onto $\mathscr{L}(V)^*$, of $\mathscr{L}(V)^*$ onto $\mathscr{L}^*(V)$ and of $\mathscr{L}^*(V)$ onto $\mathscr{L}(V)$ are respectively given by

$$L_0 \in \mathscr{L}_0(V) \longrightarrow L^* \in \mathscr{L}(V)^*,$$

where $2g\big(L^*(x,y)\,z,\,w\big)=g\big(L_0(x,y)\,z,\,w\big)+g\big(L_0(Fx,Fy)\,Fz,\,Fw\big)$,

$$L^*\in\mathscr{L}(V)^*\longrightarrow {}^*L\in\mathscr{L}^*(V)\,,$$

where
$$4g\big({}^*L(x,y)\,z,\,w\big)=3g\big(L^*(x,y)\,z,\,w\big)+3g\big(L^*(x,y)\,Fz,\,Fw\big)$$
$$+g\big(L^*(x,Fy)\,z,\,Fw\big)+g\big(L^*(x,Fy)\,Fz,\,w\big)\,,$$

and
$$ {}^*L\in\mathscr{L}^*(V)\longrightarrow L\in\mathscr{L}(V)\,,$$

where
$$4g\big(L(x,y)\,z,\,w\big)=g\big({}^*L(x,y)\,z,\,w\big)+2g\big({}^*L(x,y)\,Fz,\,Fw\big)$$
$$-g\big({}^*L(x,w)\,Fy,\,Fz\big)+g\big({}^*L(y,w)\,Fx,\,Fz\big)\,.$$

For an $L\in\mathscr{L}(V)^*$, its Ricci tensor commutes with F.

Let $\{x,y\}$ be an orthonormal basis for a 2-dimensional subspace P of V. Then for $L\in\mathscr{L}_0(V)$ we put

$$H(P)=H_L(P)=H_L(x,y)=g\big(L(x,y)\,y,\,x\big)$$

and call $H_L(P)$ the sectional curvature of L for P. It is well-known that $H_L(P)$ is independent on the choice of x and y in P. In particular, if the 2-dimensional subspace P is holomorphic, *i.e.*, invariant by the complex structure F and x is a unit vector in P, then $\{x,Fx\}$ is an orthonormal basis for P and for L we have

$$H_L(P)=g\big(L(x,Fx)\,Fx,\,x\big)\,.$$

We call such $H_L(P)$ the holomorphic sectional curvature of L for holomorphic plane P.

For $x,\ y\in V$, we denote by $x\wedge y$ the skew-symmetric endomorphism of V defined by

$$(x\wedge y)\,z=g(y,z)\,x-g(x,z)\,y\,.$$

Let A and B be two symmetric endomorphisms of V which commute with F. We define $L=L_{A,B}$ by

$$L(x,y)=Ax\wedge By+Bx\wedge Ay+FAx\wedge FBy+FBx\wedge FAy$$
$$+2g(Ax,Fy)\,FB-2g(Fx,By)\,FA\,.$$

Then the L is a K-curvature tensor.

In particular, if we take $A=cI/8$ and $B=I$, where I is the identity transformation of V and c is a constant, then the L becomes

$$L(x,y)=c/4\cdot\big(x\wedge y+Fx\wedge Fy+2g(x,Fy)\,F\big)\,.$$

In this case the holomorphic sectional curvature $H_L(P)$ for every holomorphic plane P in V is identically equal to c. It is well-known that if $L\in\mathscr{L}(V)$ has constant holomorphic sectional curvature, say c, then it is of the above form. Hence, if $L\in\mathscr{L}(V)$ has constant holomorphic sectional curvature 0, then we have $L=0$.

Now, we define the Bochner tensor L_B associated to $L\in\mathscr{L}(V)$ ([5]) by

$$L_B=L-\frac{I}{2(n+2)}\left(L_{S_L,I}-\frac{trS_L}{4(n+1)}L_{I,I}\right).$$

LEMMA 1. *Two K-curvature tensors L_1 and L_2 have the same Bochner tensor if and only if there exists a symmetric endomorphism A which commutes with F and satisfies*

$$L_1-L_2=L_{A,I}\,.$$

We call an orthonormal basis $\{e_1, \cdots, e_n, Fe_1, \cdots, Fe_n\}$ for V an F-basis.

LEMMA 2. *If $n\geqq 2$ and $L_B=0$ for an $L\in\mathscr{L}(V)$, then we have*

$$H_L(e_i, Fe_i)+H_L(e_j, Fe_j)=8H_L(e_i, e_j)\,, \qquad (i\neq j)$$

for every F-basis for V, where $i, j=1, 2, \cdots, n$.

For the proofs of Lemmas 1 and 2, see [5] and [4], respectively.

LEMMA 3. *Let $L_B=0$ for $L\in\mathscr{L}(V)$ and $n\geqq 2$. If there exists on V a nonzero vector v such that $L(v, x)=0$ for all $x\in V$, then we have $L=0$.*

PROOF. Let e_1 be a unit vector in the direction of v. Then by means of properties of L and our assumption we have

$$L(e_1, x)=L(x, e_1)=L(Fe_1, x)=L(x, Fe_1)=0$$

for all $x\in V$. For any unit vector x of V, take an F-basis $\{e_i, Fe_i\}$ in such a way that

$$x=ae_1+bFe_1+ce_2\,,$$

where a, b and c are constants. Then, by virtue of Lemma 2, we have

$$H_L(e_1, Fe_1)+H_L(e_i, Fe_i)=8H_L(e_1, e_i)\,, \qquad (i\neq 1)$$

and hence

$$H_L(e_i, Fe_i)=0\,, \qquad (i=2, \cdots, n)\,.$$

Therefore we have

$$H_L(x, Fx)=c^4H_L(e_2, Fe_2)=0\,.$$

Since x is arbitrarily taken, L has constant holomorphic sectional curvature 0 and we have $L=0$.

LEMMA 4. *The Bochner tensor associated to a K-curvature tensor L vanishes if and only if there exists a symmetric endomorphism A of V which commutes with F and satisfies*

$$H_L(x, Fx) = -8g(Ax, x),$$

for any unit vector $x \in V$.

For the proof, see [2].

For $L \in \mathscr{L}(V)^*$, the Bochner tensor associated to L means $(\pi L)_B$, where π is the projection map of $\mathscr{L}(V)^*$ onto $\mathscr{L}(V)$.

LEMMA 5. *The Bochner tensor associated to an F-invariant curvature tensor L vanishes if and only if there exists a symmetric endomorphism A which commutes with F and satisfies*

$$H_L(x, Fx) = -8g(Ax, x),$$

for any unit vector $x \in V$.

PROOF. Taking account of Lemma 4, we can get Lemma 5 by

$$H_L(x, Fx) = H_{\pi L}(x, Fx).$$

1.2. S-curvature-like tensors.

Consider the case when d is odd and V has a (ϕ, ξ, η, g)-structure. An $L_0 \in \mathscr{L}_0(V)$ is called an S-curvature tensor over V ([15]) if it has the properties;

$$\begin{aligned} L_0(x, y)\,\phi z &= \phi L_0(x, y)\,z + g(\phi x, z)\,y - g(\phi y, z)\,x - g(y, z)\,\phi x \\ &\quad + g(x, z)\phi y\,; \\ L_0(\xi, x)\,y &= g(x, y)\,\xi - \eta(y)\,x\,. \end{aligned}$$

A curvature tensor L is called an S-curvature-like tensor over V ([15]) if it satisfies

$$\begin{aligned} L(x, y)\circ\phi &= \phi\circ L(x, y)\,; \\ L(\xi, x) &= 0\,. \end{aligned}$$

To each S-curvature tensor L_0, we assign an S-curvature-like tensor L by the relation

$$(\alpha) \qquad \begin{aligned} L(x, y)\,z = L_0(x, y)\,z &+ \eta(x)\big(\eta(z)\,y - g(y, z)\,\xi\big) \\ &- \eta(y)\big(\eta(z)\,x - g(x, z)\,\xi - g(\phi x, z)\,\phi y \\ &+ g(\phi y, z)\,\phi x - 2g(\phi x, y)\,\phi z\,. \end{aligned}$$

We define an even dimensional subspace D of V by

$$\eta=0\,,$$

and put $g_0(x,y)=g(x,y)-\eta(x)\eta(y)$. Then (ϕ, g_0) can be considered as the Hermitian structure on D. When we restrict ourselves to D, every S-curvature-like tensor over V can be regarded as a K-curvature tensor on D with respect to the Hermitian structure (ϕ, g_0). Hence, we can introduce some quantities corresponding to those in 1.1. In particular, we can define the contact Bochner tensor L_B associated to each S-curvature-like tensor L. L_B may be regarded as the contact Bochner tensor associated to an S-curvature tensor L_0 if L and L_0 are related by the relation (α). We have a ϕ-holomorphic sectional curvature corresponding to the holomorphic sectional curvature.

Then we easily see

LEMMA 6 (cf. [7]). *Let L and L_0 be an S-curvature-like tensor and an S-curvature tensor, respectively, which are related by (α). Then $L=0$ if and only if L_0 is of constant ϕ-holomorphic sectional curvature* -3.

§ 2. Notations.

Let M be a d-dimensional Riemannian manifold with a positive definite Riemannian metric g. Since the tangent vector space $T_m(M)$ of M at each point m of M has an inner product $g(m)$, we may consider curvature tensors over $T_m(M)$. A (differentiable) tensor field L of type (1, 3) on M is called a generalized curvature tensor ([6]) if for each m the tensor $L(m)$ is a curvature tensor over $T_m(M)$. Similarly, we define a generalized K-curvature tensor, etc., in an almost Hermitian space, and a generalized S-curvature-like tensor, etc., in an almost contact metric space. For details, see [6], [5] and [15].

Let $\{U;\ (x^i)\}$ be a system of coordinate neighborhoods in M, where and in the sequel the Latin indices run over the range $\{1, 2, \cdots, d\}$. We denote by $\left\{ {h \atop j\ i} \right\}$, ∇_j, $K_{kji}{}^h$, K_{ji} and K the Riemann-Christoffel symbols, the covariant differentiation, the Riemannian curvature tensor, the Ricci tensor and the scalar curvature with respect to the metric $g=(g_{ji})$, respectively. Then $K_{kji}{}^h$ is a generalized curvature tensor.

§ 3. Complex conformal connections.

In §§ 3 and 4, we consider the case when d is even, say $2n$, and M has an almost Hermitian structure $(F_i{}^h, g_{ji})$.

Let M be a Kählerian space of real dimension $\geqq 4$. In a Kählerian space M, the Riemannian curvature tensor $K_{kji}{}^h$ is a generalized K-curvature tensor and we call the Bochner tensor associated to $K_{kji}{}^h$ the Bochner curvature tensor of M.

An affine connection $\bar{\nabla}$ is called a complex conformal connection ([13]) if its components $\Gamma_{ji}{}^h$ is given by

$$\Gamma_{ji}{}^h = \left\{ {h \atop j\ i} \right\} + \delta_j^h p_i + \delta_i^h p_j - g_{ji} p^h + F_j{}^h q_i + F_i{}^h q_j - F_{ji} q^h ,$$

where $(p_i) = dp$ for a certain function p on M, and $q_i = -F_i{}^j p_j$. The curvature tensor of the complex conformal connection is given by

$$\begin{aligned} R_{kji}{}^h = K_{kji}{}^h &- \delta_k^h P_{ji} + \delta_j^h P_{ki} - P_k{}^h g_{ji} + P_j{}^h g_{ki} - F_k{}^h Q_{ji} \\ &+ F_j{}^h Q_{ki} - Q_k{}^h F_{ji} + Q_j{}^h F_{ji} + Q_j{}^h F_{ki} + (\nabla_k q_j - \nabla_j q_k) F_i{}^h \\ &- 2F_{kj}(p_i q^h - q_i p^h) , \end{aligned}$$

where

$$\begin{aligned} P_{ji} &= \nabla_j p_i - p_j p_i + q_j q_i + 1/2 \cdot \lambda g_{ji} , \\ Q_{ji} &= -P_{jt} F_i{}^t , \end{aligned}$$

and $\lambda = g^{ji} p_j p_i$.

Since we have

$$\begin{aligned} R_{kji}{}^h + R_{jik}{}^h + R_{ikj}{}^h &= 2F_{ij}(\nabla_k q^h - 2q_k p^h + \lambda F_k{}^h) \\ &+ 2F_{ki}(\nabla_j q^h - 2q_j p^h + \lambda F_j{}^h) + 2F_{jk}(\nabla_i q^h - 2q_i p^h + \lambda F_i{}^h) , \end{aligned}$$

in order that

$$R_{kji}{}^h + R_{jik}{}^h + R_{ikj}{}^h = 0 ,$$

it is necessary and sufficient that

$$\nabla_j q_i - 2q_j p_i + \lambda F_{ji} = 0 ,$$

or

$$(*) \qquad \nabla_j p_i + 2q_j q_i + \lambda g_{ji} = 0 .$$

In particular, if we get $R_{kji}{}^h = 0$, then we have (*) and by Theorem A the Bochner curvature tensor of M vanishes.

We consider the converse.

We suppose the existence of a nonconstant function p satisfying (*). Then a complex conformal connection $\bar{\nabla}$ is defined corresponding to the function p and its curvature tensor $R_{kji}{}^h$ is a generalized K-curvature tensor. Since it is easily verified that the two generalized K-curvature tensors $R_{kji}{}^h$

and $R_{kji}{}^{h}$ are related by the relation stated in Lemma 1, we have

LEMMA 7. *If there exists a nonconstant function p satisfying* (*), *then two generalized K-curvature tensors $K_{kji}{}^{h}$ and $R_{kji}{}^{h}$ have the same Bochner tensor. In particular, the Bochner tensor associated to $K_{kji}{}^{h}$ vanishes if and only if that to $R_{kji}{}^{h}$ vanishes.*

Let $\bar{\nabla}_j$ denote the covariant differentiation for the complex conformal connection $\bar{\nabla}$. Then from (*), we have

$$\bar{\nabla}_j p_i = -p_j p_i .$$

The Ricci identity for $\bar{\nabla}$ gives

$$\bar{\nabla}_k \bar{\nabla}_j p_i - \bar{\nabla}_j \bar{\nabla}_k p_i = -R_{kji}{}^{h} p_h - T_{kj}{}^{h} \bar{\nabla}_h p_i ,$$

where $T_{kj}{}^{h}$ is the torsion tensor of $\bar{\nabla}$, from which we obtain

$$R_{kji}{}^{h} p_h = 0 .$$

Taking account of Lemmas 3 and 7, we have

THEOREM 1. *In a Kählerian space of real dimension $\geqq 4$, with the vanishing Bochner curvature tensor, if there exists a nonconstant function p satisfying* (*), *then we have a complex conformal connection with zero curvature.*

The Bochner curvature tensor $B_{kji}{}^{h}$ of M is rewritten as follows ([9]);

$$(\beta) \qquad \begin{aligned} B_{kji}{}^{h} = K_{kji}{}^{h} &+ \delta_k^h L_{ji} - \delta_j^h L_{ki} + L_k{}^{h} g_{ji} - L_j{}^{h} g_{ki} \\ &+ F_k{}^{h} M_{ji} - F_j{}^{h} M_{ki} + M_k{}^{h} F_{ji} - M_j{}^{h} F_{ki} \\ &- 2(M_{kj} F_i{}^{h} + F_{kj} M_i{}^{h}) \end{aligned}$$

where $L_{ji} = -\dfrac{1}{2(n+2)}\left(K_{ji} - \dfrac{1}{4(n+1)} K g_{ji}\right)$,

$M_{ji} = -L_{jt} F_i{}^{t}$.

§ 4. Bochner curvature tensor in a K-space.

M is a K-space if the associated almost Hermitian structure $(F_i{}^{h}, g_{ji})$ satisfies

$$\nabla_j F_i{}^{h} + \nabla_i F_j{}^{h} = 0 .$$

PROPOSITION 1. *In a K-space, if the tensor $B_{kji}{}^{h}$ given by* (β) *vanishes, the space is Kählerian.*

PROOF. In general, we have

$$B_{tji}{}^{t} = \frac{3}{n+4}(K_{ji} - K_{ts} F_j{}^{t} F_i{}^{s}) ,$$

and

$$\begin{aligned}\nabla_k\nabla_j F_i^{\ h}-\nabla_j\nabla_k F_i^{\ h} &= K_{kjt}{}^h F_i^{\ t}-K_{kji}{}^t F_t^{\ h}\\ &= B_{kjt}{}^h F_i^{\ t}-B_{kji}{}^t F_t^{\ h}\\ &\quad -\frac{2}{n+4}F_{kj}(K_{ia}-K_{ts}F_i^{\ t}F_a^{\ s})\,g^{ah}\,.\end{aligned}$$

Hence, if $B_{kji}{}^h=0$, we get

$$(\gamma)\quad \nabla_k\nabla_j F_i^{\ h}-\nabla_j\nabla_k F_i^{\ h}=0\,.$$

It is well-known that if a K-space has the property (γ), then the space is Kählerian.

Proposition 1 tells us that it is meanless to consider a K-space with $B_{kji}{}^h=0$. Therefore we should go to the meanful direction.

LEMMA 8 ([3]). *In a K-space, we have*

$$K_{kjih}=K_{dcba}F_k^{\ d}F_j^{\ c}F_i^{\ b}F_h^{\ a}\,,$$

that is, $K_{kji}{}^h$ *is F-invariant.*

By the argument in 1.1 and Lemma 8, we have the Bochner curbature tensor $B^*_{kji}{}^h$ of a K-space, that is, the Bochner tensor associated to $K_{kji}{}^h$. However, since by Lemma 9 below we see that $K_{kji}{}^h$ in a K-space is a generalized K^*-curvature tensor we here give the Bochner one in this sence;

$$\begin{aligned}B^*_{kji}{}^h=1/4\cdot(&K_{kji}{}^h-2K_{kjt}{}^sF_i^{\ t}F_s^{\ h}-K_k{}^h{}_{ts}F_j^{\ t}F_i^{\ s}+K_j{}^h{}_{ts}F_k^{\ t}F_i^{\ s})\\ &+\delta_k^h L^*_{ji}-\delta_j^h L^*_{ki}+L_k^{*h}g_{ji}-L_j^{*h}g_{ki}+F_k^{\ h}M^*_{ji}-F_j^{\ h}M^*_{ki}\\ &+M_k^{*h}F_{ji}-M_j^{*h}F_{ki}-2(M^*_{kj}F_i^{\ h}+F_{kj}M_i^{*h})\,,\end{aligned}$$

where $\displaystyle L^*_{ji}=-\frac{1}{8(n+2)}\left(K_{ji}+3K^*_{ji}-\frac{1}{4(n+1)}(K+3K^*)\,g_{ji}\right),$

$$M^*_{ji}=-L^*_{jt}F_i^{\ t}\,,$$

and $K^*_{ji}=-1/2\cdot K_{jcba}F_i^{\ c}F^{ba}$, $K^*=g^{ji}K^*_{ji}$. $B^*_{kji}{}^h$ is the Bochner tensor associated to a generalized K^*-curvature tensor $K_{kji}{}^h$, and in a K-space we can rewrite $B^*_{kji}{}^h$ more precisely. To do this we need

LEMMA 9 ([3]). *In a K-space we have*

$$K_{kjih}-K_{kjts}F_i^{\ t}F_h^{\ s}=-\nabla_k F_{jt}\nabla_i F_h^{\ t}\,.$$

(Therefore, $K_{kji}{}^h$ *in a K-space is a generalized* K^**-curvature tensor.)*

By Lemma 9 we get

$$\begin{aligned}&1/4\cdot(K_{kjih}+2K_{kjts}F_i^{\ t}F_h^{\ s}-K_{khts}F_j^{\ t}F_i^{\ s}+K_{jhts}F_k^{\ t}F_i^{\ s})\\ &\quad=K_{kjih}-1/4\cdot(3K_{kjih}-2K_{kjts}F_i^{\ t}F_h^{\ s}+K_{khts}F_j^{\ t}F_i^{\ s}-K_{jhts}F_k^{\ t}F_i^{\ s})\end{aligned}$$

$$= K_{kjih} - 1/4 \cdot \Big(2(K_{kjih} - K_{kjts} F_i{}^t F_h{}^s) - (K_{khji} - K_{khts} F_j{}^t F_i{}^s) \\ + (K_{jhki} - K_{jhts} F_k{}^t F_i{}^s) \Big) \\ = K_{kjih} - 1/4 \cdot (\nabla_k F_{ht} \nabla_j F_i{}^t - \nabla_k F_{it} \nabla_j F_h{}^t - 2\nabla_k F_{jt} \nabla_i F_h{}^t) \,,$$

by virtue of the first Bianchi identity for $K_{kji}{}^h$, and

$$B^*_{kji}{}^h = \delta_k^h L^*_{ji} - \delta_j^h L^*_{ki} + L^{*h}_k g_{ji} - L^{*h}_j g_{ki} + F_k{}^h M^*_{ji} - F_j{}^h M^*_{ki} \\ + M^{*h}_k F_{ji} - M^{*h}_j F_{ki} - 2(M^*_{kj} F_i{}^h + F_{kj} M^{*h}_i) + K_{kji}{}^h \\ + 1/4 \cdot (\nabla_k F_t{}^h \nabla_j F_i{}^t - \nabla_k F_i{}^t \nabla_j F_t{}^h - 2\nabla_k F_j{}^t \nabla_i F_t{}^h) \,.$$

Thus we obtain

THEOREM 2 (cf. [2], [8]). *Let M be a K-space. M has the vanishing Bochner curvature tensor if and only if there exists a symmetric and hybrid* (0, 2)*-tensor A_{ji} satisfying*

$$H(x^h) = -8A_{ji} x^j x^i \,,$$

for any unit vector x^h, where H denotes the holomorphic sectional curvature for $K_{kji}{}^h$. In this case we have

$$A_{ji} = L^*_{ji} \,.$$

PROOF. By Lemma 5 the proof is easy.

§ 5. Contact conformal connections.

In this section, we consider the case when d is odd, and M has a Sasakian structure.

5. 1. Sasakian structure.

Let (ϕ, ξ, η, g) be an almost contact metric structure on M. Then the following four tensors are well-known;

$$N^{(1)}(x, y) = \phi^2[x, y] + [\phi x, \phi y] - \phi[\phi x, y] + \phi[\phi x, \phi y] + d\eta(x, y)\xi \,,$$

$$N^{(2)}(x, y) = (\mathcal{L}_{\varphi x}\phi)\, y - (\mathcal{L}_{\varphi y}\phi)\, x \,.$$

$$N^{(3)}(x) = (\mathcal{L}_\xi \phi)\, x \,,$$

$$N^{(4)}(x) = (\mathcal{L}_\xi \eta)(x) \,,$$

for any vectors x and y on M, where $\mathcal{L}$ denotes the Lie differentiation. We define a distribution D on M by the equation $\eta=0$.

(ϕ, ξ, η, g) is normal if $N^{(1)}=0$ (then we also have $N^{(2)}=N^{(3)}=N^{(4)}=0$), and contact if $g(\phi x, y)=1/2 \cdot d\eta(x, y)$ for all x and y. A normal and contact structure is called a Sasakian structure. For a Sasakian structure (ϕ, ξ, η, g) we have

$$
(1) \qquad \begin{aligned} &(\nabla_x \phi) y = \eta(y)\, x - g(x, y)\, \xi\,, \\ &\nabla_x \xi = \phi x\,. \end{aligned}
$$

In a Sasakian space, the Riemannian curvature tensor is a generalized S-curvature tensor, The contact Bochner tensor associated to the Riemannian curvature tensor is called a contact Bochner curvature tensor of the space.

PROPOSITION 2 ([10]). *In a Sasakian space, there exists a unique affine connection* $\overset{\circ}{\nabla}$ *satisfying the following conditions*:

(i) $\overset{\circ}{\nabla}\phi = \overset{\circ}{\nabla}\xi = \overset{\circ}{\nabla}\eta = \overset{\circ}{\nabla}g = 0$;

(ii) $\overset{\circ}{T}(x, y) = d\eta(x, y)\, \xi$,

$\overset{\circ}{T}(\phi x, \xi) = -\phi \overset{\circ}{T}(x, \xi)$, *for all* $x, y \in D$,

where $\overset{\circ}{T}$ *is the torsion tensor of* $\overset{\circ}{\nabla}$. *(The connection* $\overset{\circ}{\nabla}$ *is called the canonical affine connection of* (ϕ, ξ, η, g).)

If we denote by $\overset{\circ}{\Gamma}{}_{ji}{}^h$ the components of $\overset{\circ}{\nabla}$, then we have

$$
\overset{\circ}{\Gamma}{}_{ji}{}^h = \left\{ {h \atop j\ i} \right\} - \phi_j{}^h \eta_i - \phi_i{}^h \eta_j + \phi_{ji} \xi^h\,.
$$

In this case, the curvature tensor for $\overset{\circ}{\nabla}$ is a generalized S-curvature-like tensor and corresponds to the generalized S-curvature tensor $K_{kji}{}^h$ under the relation (α).

5.2. Contact conformal connections.

In the sequel, we consider a Sasakian space $M(\phi_i{}^h, \xi^h, \eta_i, g_{ji})$.

An affine connection $\bar{\nabla}$ is called a contact conformal connection ([14]) if its components $\Gamma_{ji}{}^h$ is given by

$$
\begin{aligned} \Gamma_{ji}{}^h = \left\{ {h \atop j\ i} \right\} &+ (\delta_j^h - \eta_j \xi^h)\, p_i + (\delta_i^h - \eta_i \xi^h)\, p_j - (g_{ji} - \eta_j \eta_i)\, p_h \\ &+ \phi_j{}^h (q_i - \eta_i) + \phi_i{}^h (q_j - \eta_j) - \phi_{ji} (q^h - \xi^h)\,, \end{aligned}
$$

where $(p_i) = dp$ for a certain function p on M satisfying $\mathcal{L}_\xi p = 0$, and $q_i = \phi_i{}^j p_j$.

We consider the following:

$$
(**) \quad \nabla_j p_i + 2 q_j q_i - q_j \eta_i - \eta_j q_i + \lambda (g_{ji} - \eta_j \eta_i) = 0\,,
$$

where $\lambda = g^{ji} p_j p_i$. Then corresponding to theorem 1, we have

LEMMA 10. *In a Sasakian space of dimension* $\geqq 5$, *with the vanishing contact Bochner curvature tensor, if there exists a nonconstant function* p

satisfying (**) *and* $\mathcal{L}_\xi p=0$, *then we have a contact conformal connection with zero curvature.*

We assume the existence of a function p satisfying (**) and $\mathcal{L}_\xi p=0$. Then, if we set

$$(\delta)\qquad \begin{aligned} &{}^*\eta_i=e^{2p}\eta_i,\quad {}^*\xi^h=e^{-2p}(\xi^h-q^h),\quad {}^*\phi_i^h=\phi_i^h-\eta_i p^h,\\ &{}^*g_{ji}=e^{2p}(g_{ji}+(e^{2p}-1+\lambda)\,\eta_j\eta_i+q_j\eta_i+\eta_j q_i)\,, \end{aligned}$$

then the structure $({}^*\phi, {}^*\xi, {}^*\eta, {}^*g)$ is another almost contact metric structure on M. Furthermore we can be easily verify that

$$(\varepsilon)\qquad \begin{aligned} &d{}^*\eta(x,y)=d{}^*\eta(\phi x,\phi y)\,,\\ &[{}^*\phi x, {}^*\phi y]-[x,y]-{}^*\phi[{}^*\phi x,y]-{}^*\phi[x,{}^*\phi y]=0\,, \end{aligned}$$

for all $x, y\in {}^*D=D$. The second equation of (ε) is equivalent to

$${}^*N^{(1)}(x,y)+{}^*\eta(x){}^*\phi\,{}^*N^{(3)}(y)-{}^*\eta(y){}^*\phi\,{}^*N^{(3)}(x)=0\,,$$

for all vectors x and y on M, under the first equation of (ε).

On the other hand, substituting (δ) into

$$({}^*N^{(3)})_i^h=\mathcal{L}_{{}^*\xi}{}^*\phi_i^h={}^*\xi^t\nabla_t{}^*\phi_i^h+{}^*\phi_t^h\nabla_i{}^*\xi^t-{}^*\phi_i^t\nabla_t{}^*\xi^h\,,$$

and using (1), we have ${}^*N^{(3)}=0$. Consequently, the structure $({}^*\phi, {}^*\xi, {}^*\eta, {}^*g)$ is normal. Since it is very easy to verify that the structure $({}^*\phi, {}^*\xi, {}^*\eta, {}^*g)$ is contact, we have another Sasakian structure $({}^*\phi, {}^*\xi, {}^*\eta, {}^*g)$. By the direct calculation, we also see that the curvature tensor for the canonical affine connection ${}^*\mathring{\nabla}$ of the structure $({}^*\phi, {}^*\xi, {}^*\eta, {}^*g)$ coincides with that for $\bar{\nabla}$. Hence, by Lemmas 6 and 10, we have

THEOREM 3. *In a Sasakian space of dimension* $\geqq 5$, *with the vanishing contact Bochner curvature tensor, if there exists a nonconstant function* p *satisfying* (**) *and* $\mathcal{L}_\xi p=0$, *then we have another Sasakian structure of constant* ϕ*-holomorphic sectional curvature* -3.

Lastly, we consider a Sasakian structure $({}^*\phi, {}^*\xi, {}^*\eta, {}^*g)$ of constant ϕ-holomorphic sectional curvature -3. Then the Riemannian curvature tensor ${}^*K_{kji}{}^h$ has the form ([7])

$$(2)\qquad \begin{aligned} {}^*K_{kji}{}^h=\;&\delta_k^h{}^*\eta_j{}^*\eta_i-\delta_j^h{}^*\eta_k{}^*\eta_i+{}^*\eta_k{}^*\xi^h{}^*g_{ji}\\ &-{}^*\eta_j{}^*\xi^h{}^*g_{ki}-{}^*\phi_k^h{}^*\phi_{ji}+{}^*\phi_j^h{}^*\phi_{ki}+2{}^*\phi_{kj}{}^*\phi_i^h\,. \end{aligned}$$

We consider a system of partial differential equations:

$$(***)\qquad {}^*\nabla_j p_i=-2p_jp_i+{}^*q_j{}^*\eta_i+{}^*q_i{}^*\eta_j\,,$$

where ${}^*q_i=-{}^*\phi_i{}^j p_j$ and ${}^*\nabla_j$ is the covariant differentiation for ${}^*g$. The integrability condition of (***),

$$*\nabla_k *\nabla_j p_i - *\nabla_j *\nabla_k p_i = -*K_{kji}{}^h p_h ,$$

is satisfied by any p_i satisfying (***) by virtue of the form of $*K_{kji}{}^h$. Then. transvecting (***) with $*\xi^i$ and taking account of $*\nabla_j *\xi^i = *\phi_j{}^i$, we have

$$*\nabla_j(*\xi^i p_i) = -2*\xi^i p_i p_j ,$$

and

$$*\nabla_j(*\xi^i p_i e^{2p}) = 0 ,$$

that is, $*\xi^i p_i e^{2p}$ is a constant, say c, where p is a (local) function satisfying $*\nabla_j p = p_j$. If we give an initial condition for p_i such that $*\xi^i p_i = 0$ at a point m of M, then we get $*\xi^i p_i = ce^{-2p} = 0$ at m and $c=0$, that is,

$$\mathcal{L}_{*\xi} p = *\xi^i p_i = 0 .$$

By the similar argument about $*g^{ji} p_j p_i$ we see that there exists a (local) non-constant function p satisfying $\mathcal{L}_{*\xi} p = 0$ and (***), $(p_i) = dp$.

Then we can easily verify that there exists a (local) Sasakian structure (ϕ, ξ, η, g) with the vanishing contact Bochner curvature tensor, which is related to the structure $(*\phi, *\xi, *\eta, *g)$ by (δ). The function p satisfies (**) and $\mathcal{L}_\xi p = 0$. Hence we have proved

THEOREM 4. *Given a Sasakian structure of constant ϕ-holomorphic sectional curvature -3, then there exists a (local) Sasakian structure with the vanishing contact Bochner curvature tensor and a (local) non-constant function p satisfying* (**) *and $\mathcal{L}_\xi p = 0$.*

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