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Some remarks in a qualitative theory of similarity pseudogroups

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1. Introduction

The qualitative theory of foliations has been developed in the last two decades and many results were obtained. For example, see Hector [Hec2] and Cantwell-Conlon [C-C]. But the developments were obtained essentially only for codimension one C^2 foliations. In Nishimori [Nis1], the author began a study of the qualitative properties of foliations of higher codimension and obtained some experimental results in the form of a qualitative theory of similarity pseudogroups. After [Nis1], this area had two papers: Matsuda [Mat] and Matsuda-Minakawa [M-M]. The setting taken in these works is as follows.

In order to set our goal clearly, we take a classical theorem of the qualitative theory of codimension one foliations as a model for our intended theory and try to find an analogy of this theorem. The theorem taken is the following.

Theorem A (Sacksteder's Theorem, see Sacksteder [Sac]). *Let $\mathscr{F}$ be a codimension one C^2 foliation of a closed manifold M, and $\mathscr{M}\subset M$ an exceptional minimal set with respect to $\mathscr{F}$. Then there exists a leaf F of $\mathscr{F}$ contained in $\mathscr{M}$ such that F has a contracting element in its linear holonomy group* $\mathrm{LHol}(F)$.

Dynamical systems can be considered as foliations of higher codimension and they are known to show generally chaotic behaviour. In the present stage, we want to avoid such complications. So we decide to treat only foliations with transverse similarity structures. For the simplicity, we consider (holonomy) pseudogroups instead of foliations. In this setting, the author obtained an analogy of Sacksteder's Theorem. Then Matsuda and Minakawa got a stronger result in the same setting, which is an analogy of the following.

Theorem B (Hector [Hec1] in the analytic case, Duminy in the general case). *In the same situation as in Theorem A, a semi-proper leaf can be* *taken as F in the conclusion of Theorem A.*

The purpose of this paper is to improve the formulations in the papers [Nis1], [Mat] and [M-M] and to give some remarks obtained after those papers. This paper is organized as follows. In §2, we introduce a notion *Sacksteder system* which is an analogue of the exceptional minimal set in codimension one foliations. We study fundamental properties of Sacksteder's systems, and we propose a fundamental conjecture which is analogous to the above Sacksteder's Theorem. We give some affirmative evidences for the conjecture. In §3, we generalize the notion *orbits with bubbles* in [Nis1] to *orbits almost with bubbles*, and prove a theorem analogous to Sacksteder's Theorem. In §4, we study how the points in a Sacksteder system are scattered in the euclidean space. We have a new phenomenon which appears only in higher dimension (that is, dimension greater than one). In §5, we introduce a class of Sacksteder systems (called *convexly self-similar*), which prove to be affirmative examples for a conjecture proposed in §4.

2. Formulations and the main conjecture

We want to work in the simplest setting where an analogy of Sacksteder's Theorem could exist. So we introduce the following definitions. Denote by $\Gamma_{q,+}^{\mathrm{sim},*}$ the set of orientation preserving homeomorphisms f between nonempty bounded convex open subsets of $\boldsymbol{R}^q$ which is the restriction of a similarity transformation $\hat{f}$ of $\boldsymbol{R}^q$. For a map f, we denote by $D(f)$ (respectively $\boldsymbol{R}(f)$) the domain (respectively the range) of f. Put $\Gamma_{q,+}^{\mathrm{sim}}=\Gamma_{q,+}^{\mathrm{sim},*}\cup\{\mathrm{id}_{\boldsymbol{R}^q}, \mathrm{id}_\emptyset\}$, where $\mathrm{id}_{\boldsymbol{R}^q}$ is the identity map of $\boldsymbol{R}^q$ and $\mathrm{id}_\emptyset$ is the unique transformation of the empty set $\emptyset$.

Definition 2.1 A subset $\Gamma_0\subset\Gamma_{q,+}^{\mathrm{sim},*}$ is called *symmetric* if, for each $f: U\to V$ belonging to Γ_0, the inverse $f^{-1}: V\to U$ belongs to Γ_0.

Definition 2.2 A subset $\Gamma\subset\Gamma_{q,+}^{\mathrm{sim}}$ is called a *pseudogroup* if (a) $\mathrm{id}_{\boldsymbol{R}^q}$ belongs to Γ, (b) for each $f, g\in\Gamma$ the composition $fg: g^{-1}(D(f)\cap R(g))\to f(D(f)\cap R(g))$ belongs to Γ and (c) for each $f\in\Gamma$ the inverse f^{-1} belongs to Γ. For a subset $\Gamma_0\subset\Gamma_{q,+}^{\mathrm{sim},*}$, the *pseudogroup generated by* Γ_0 means the smallest one containing Γ_0 and is denoted by $\langle\Gamma_0\rangle$.

Definition 2.3 Fix a pseudogroup $\Gamma\subset\Gamma_{q,+}^{\mathrm{sim}}$. For $x\in\boldsymbol{R}^q$, the subset $\Gamma(x):=\{f(x): f\in\Gamma, x\in D(f)\}$ is called an *orbit* of x. A subset $A\subset\boldsymbol{R}^q$ is called *invariant* if for each $x\in A$ the orbit $\Gamma(x)$ is contained in A.

Naturally we have the usual properties in the qualitative theory: *the* *complement, the interior and the closure of an invariant subset are invariant, and the union and the intersection of a family of invariant subsets are invariant,* etc.

Now we formulate our object precisely, which is an analogue of an *exceptional minimal set* in codimension one foliations.

Definition 2.4 A *Sacksteder system* is a pair $\mathscr{M}=(\Gamma_0, \mathscr{S})$ of an infinite compact nowhere dense subset $\mathscr{M}$ of $\boldsymbol{R}^q$ and a finite symmetric subset $\Gamma_0 \subset \Gamma_{q,+}^{\mathrm{sim},*}$ satisfying

(a) $\mathscr{M} \subset \Omega := \bigcup_{h \in \Gamma_0} D(h)$,
(b) $\mathscr{M} \cap \delta D(h)=\emptyset$ for each $h \in \Gamma_0$,
(c) $h(\mathscr{M} \cap D(h))=\mathscr{M} \cap R(h)$ for each $h \in \Gamma_0$,
(d) for each $x \in \mathscr{M}$ the orbit $\Gamma(x)$ is dense in $\mathscr{M}$,

where we put $\delta A := \bar{A} - \mathrm{Int}\, A$ for a subset $A \subset \boldsymbol{R}^q$ and Γ is the pseudogroup generated by Γ_0. We call $\boldsymbol{R}^q$ the *ambient space* of $\mathscr{S}$.

Remark 2.5. In the above situation, the condition (d) implies that $\mathscr{M}$ is a *minimal* set of the pseudogroup Γ: that is, $\mathscr{M}$ is a minimal element of the set of nonempty closed invariant subsets of $\boldsymbol{R}^q$ partially ordered by the inclusion $\subset$. In the case $q>1$, the pseudogroup Γ may have minimal sets of various Hausdorff dimensions between 0 and q. (In the case $q=1$, an *exceptional* minimal set is defined as a nowhere dense minimal set containing more than one orbit.) Here we consider a *nowhere dense* minimal set as a rough generalization of an *exceptional* minimal set in the case $q=1$. (As we see later, we need a further condition to obtain an analogy of Sacksteder's Theorem). Since an invariant subset of a pseudogroup corresponding to an exceptional minimal set of a codimension one foliation is known to be a Cantor set, the assumption in Definition 2.4 that $\mathscr{M}$ is infinite compact nowhere dense is natural for our purpose. The other conditions (a), ... , (d) can be justified in the similar way. In the case $q=1$, the underlying set $\mathscr{M}$ of a Sacksteder system $\mathscr{S}$ is a Cantor set as we see later.

As a substitute of the *holonomy group* of a leaf of a foliation, we consider the stabilizer defined in the following.

Definition 2.6 Let $\mathscr{S}=(\Gamma_0, \mathscr{M})$ be a Sacksteder system and denote by Γ the pseudogroup generated by Γ_0. For a point $x \in \mathscr{M}$, put

$$\mathrm{Stab}(x) := \{f \in \Gamma : x \in D(f) \text{ and } f(x)=x\},$$

and we call it the *stabilizer* of Γ at x. A point $x \in \mathscr{M}$ is called a *focus* of the system $\mathscr{S}$ if Stab(x) contains a contraction.

Before we go ahead, we give an observation on the topology of the underlying subset $\mathscr{M}$ of a Sacksteder system $\mathscr{S}$.

Proposition 2.7 *Let $\mathscr{S}=(\Gamma_0, \mathscr{M})$ be a Sacksteder system. Then*
(1) *The underling set $\mathscr{M}$ is perfect.*
(2) *If $q=1$ or $\mathscr{M}$ is totally disconnected, then $\mathscr{M}$ is a Cantor set.*
(3) *If $x\in\mathscr{M}$ is a focus of the system $\mathscr{S}$, then the singleton $\{x\}$ is a connected component of $\mathscr{M}$.*

Proof. (1) Since $\mathscr{M}$ is infinite and compact, there exists a sequence $x_1, x_2, x_3, \ldots$ of pairwise distinct points in $\mathscr{M}$ which converges to some point $a\in\mathscr{M}$. We may suppose that $a\neq x_n$ for all $n\in\boldsymbol{N}$. This implies that $a=\lim_{n\to\infty}x_n\in\overline{\mathscr{M}-\{a\}}$. Take a point $y=f(a)\in\Gamma(a)$, where $f\in\Gamma$ and $a\in D(f)$. For sufficiently large $n\in\boldsymbol{N}$ the point x_n is contained in $D(f)$ and the points $f(x_n)$ converges to $f(a)$, which implies that $y=f(a)$ belongs to $\overline{\mathscr{M}-\{y\}}$. For a point $z\in\mathscr{M}-\Gamma(a)$, we see that

$$z\in\overline{\Gamma(a)}\subset\overline{\mathscr{M}-\{z\}}$$

since $\Gamma(a)$ is dense in $\mathscr{M}$. Therefore $\mathscr{M}$ is perfect.

(2) If $q=1$, then $\mathscr{M}$ is totally disconnected since $\mathscr{M}$ is a nowhere dense (that is, not locally dense) subset of $\boldsymbol{R}$. If $\mathscr{M}$ is totally disconnected, then $\mathscr{M}$ is a Cantor set since $\mathscr{M}$ is also compact and perfect (see [Wil, Theorem 30.3]).

(3) Let $x\in\mathscr{M}$ be a focus of $\mathscr{S}$, and C the connected component of $\mathscr{M}$ containing the point x. By definition, we have a contracting element $f\in\Gamma$ such that $x\in D(f)$ and $f(x)=x$. It is easy to see that $\mathscr{M}\cap\delta D(f)=\emptyset$, and furthermore that for every $n\in\boldsymbol{N}$ the subset $\delta f^n(D(f))$ does not intersect $\mathscr{M}$. If follows that $C\subset f^n(D(f))$ for every $n\in\boldsymbol{N}$. Since f is a contraction, the intersection $\bigcap_{n=1}^{\infty}f^n(D(f))$ equals to the singleton $\{x\}$. Therefore $C=\{x\}$. □

The main conjecture in our theory is the following.

Conjecture 2.8 *A Sacksteder system $\mathscr{S}=(\Gamma_0, \mathscr{M})$ has a focus if the underlying subset $\mathscr{M}$ is a Cantor set.*

Remark 2.9. (1) Consider the case $q=1$. The C^2 pseudogroup version of the original Sacksteder's Theorem implies that every Sacksteder system $\mathscr{S}$ has a focus. Thus Conjecture 2.8 is true in this case.

(2) Concerning the connectivity of $\mathscr{M}$, we know the following facts. In [Nis1, Example 3.1(2)], we construct a Sacksteder system $\mathscr{S}=(\Gamma_0, \mathscr{M})$

such that $\mathscr{S}$ has no focus and $\mathscr{M}$ is connected (actually $q=2$ and $\mathscr{M}$ is a circle). This means that, in order to prove that a Sacksteder system $\mathscr{S}$ has a focus, we must suppose some type of disconnectedness on $\mathscr{M}$. Although we suppose here that $\mathscr{M}$ is a Cantor set, Proposition 2.7(3) may suggest that the right condition can happen to be that $\mathscr{M}$ has a connected component consisting of one point.

We have examples supporting Conjecture 2.8: Williams [Wil], Hutchinson [Hut] and Hata [Hat] imply the following.

Proposition 2.10 *Let $\mathscr{S}=(\Gamma_0, \mathscr{M})$ be a Sacksteder system. Suppose that there exists a subset $\Gamma_1 \subset \Gamma_0$ satisfying that* (a) *each $h \in \Gamma_1$ is a contraction,* (b) *$\mathscr{M} \subset D(h)$ for each $h \in \Gamma_1$ and* (c) *$\mathscr{M} \subset \bigcup_{h \in \Gamma_1} h(\mathscr{M})$. Then*

$$\mathscr{M} = \mathrm{Cl}\Big(\bigcup_{n=1}^{\infty} \bigcup_{h_1, \cdots, h_n \in \Gamma_1} \mathrm{Fix}(h_1 \ldots h_n)\Big),$$

which implies that the system $\mathscr{S}$ has a focus.

It is easy to construct a Sacksteder system satisfying the condition in Proposition 2.10.

Example 2.11. Take a closed disk D in $\boldsymbol{R}^q$ and a finite number of disjoint closed disks $D_1, \ldots, D_n \subset \mathrm{Int}(D)$. For each $i=1, \ldots, n$, take a similarity transformation $\bar{h}_i$ with $\bar{h}_i(D)=D_i$ and put $h_i=\bar{h}_i|_{\mathrm{Int}(D)}: \mathrm{Int}(D) \to \mathrm{Int}(D_i)$. Let $\Gamma_1=\{h_1, \ldots, h_n\}$, $\Gamma_0=\{h_1, \ldots, h_n, h_1^{-1}, \ldots, h_n^{-1}\}$ and $\mathscr{M} = \mathrm{Cl}(\bigcup_{k=1}^{\infty} \bigcup_{g_1, \cdots, g_k \in \Gamma_1} \mathrm{Fix}(g_1 \ldots g_k))$. In the case $n \geq 2$ the pair $\mathscr{S}=(\Gamma_0, \mathscr{M})$ is a Sacksteder system having the desired property.

3. Orbits almost with bubbles

We are going to introduce a notion of *orbits almost with bubbles* as a generalization of the notion of *orbits with bubbles* in Nishimori [Nis1], which causes the existence of a focus. This means that we will replace the condition in Conjecture 2.8 that $\mathscr{M}$ is a Cantor set by the existence of a focus and prove an analogy of Sacksteder's Theorem. We consider that the result cast light on Conjecture 2.8 from a side.

Throughout this section, we fix a Sacksteder system $\mathscr{S}=(\Gamma_0, \mathscr{M})$ and denote by Γ the pseudogroup generated by Γ_0.

Definition 3.1 For a point $x \in \mathscr{M}$, denote by $G(x)$ the graph whose vertices are the points of the orbit $\Gamma(x)$ and whose edges between two vertices $y, z \in \Gamma(x)$ are triplets $e=(y, h, z)$ such that $h \in \Gamma_0$, $y \in D(h)$ and $h(y) = z$. We identify naturally the edges $e=(y, h, z)$ and $e^{-1}=(z, h^{-1}, y)$ in the topological space $G(x)$. We call $G(x)$ the *Cayley graph* of the system

$\mathscr{S}$ at x.

Definition 3.2 For a point $x \in \mathscr{M}$, the orbit $\Gamma(x)$ is called *almost with bubbles* if there exists a compact subset $K \subset G(x)$ and for each $y \in \Gamma(x) - K$ there exists a nonempty bounded convex open subset B_y (called a *bubble* at y) of $\Omega := \bigcup_{h \in \Gamma_0} D(h)$ satisfying the following conditions:

(a) $y \in \delta B_y$ where $\delta B_y := \overline{B_y} - \mathrm{Int}(B_y)$,
(b) $B_y \cap B_z = \emptyset$ if $y \neq z$,
(c) if $e = (y, h, z)$ is an edge contained in $G(x) - K$ with $y \neq z$, then $B_y \subset D(h)$ and $h(B_y) = B_z$.

If the empty set can be taken as K, the orbit $\Gamma(x)$ is called *with bubbles.*

The papers Nishimori [Nis1], Matsuda [Mat] and Matsuda-Minakawa [M-M] treated orbits with bubbles and obtained almost thorough results on them. The purpose of this section is to generalize the main result on orbits with bubbles in [Nis1] and to prove the following.

Theorem 3.3 *Let $\mathscr{S} = (\Gamma_0, \mathscr{M})$ be a Sacksteder system. Suppose that for some point $x \in \mathscr{M}$ the orbit $\Gamma(x)$ is almost with bubbles. Then the system $\mathscr{S}$ has a focus.*

We make some preparations for the proof of Theorem 3.3.
Herefter let $\mathscr{S} = (\Gamma_0, \mathscr{M})$ be a Sacksteder system.

Definition 3.4 (1) Denote by $W(\Gamma_0)$ the set of words with Γ_0 as the alphabet. In order to distinguish a word from a composition, we prefer to write a word $w \in W(\Gamma_0)$ in such a way as $w = (h_m, \dots, h_1)$ rather than $w = h_m \cdots h_1$. In this way, we identify $W(\Gamma_0)$ with the disjoint union $\coprod_{m=0}^{\infty} (\Gamma_0)^m$, where $(\Gamma_0)^m$ denotes the product of m-copies of Γ_0 and $(\Gamma_0)^0$ is the singleton consisting of the empty word ( ).

(2) For $w = (h_m, \dots, h_1) \in (\Gamma_0)^m \subset W(\Gamma_0)$, let $g_w = h_m \cdots h_1$. For the empty word ( ), let $g_{(\ )} = \mathrm{id}_{R^q}$.

The following proposition gives a description of elements of the pseudogroup Γ generated by the symmetric subset Γ_0.

Proposition 3.5 (1) *If $w \in W(\Gamma_0)$, then $g_w \in \Gamma$.*
(2) *The map $\Phi : W(\Gamma_0) \to \Gamma$ defined by*

$$\Phi(w) = g_w \quad \textit{for all } w \in W(\Gamma_0)$$

is surjective.

Proof. (1) follows from the definition of pseudogroups. (2) follows from the assumption that Γ_0 is symmetric. □

Definition 3.6 (1) For a word $w \in W(\Gamma_0)$, denote by $|w|$ the word length of w; that is, $|w|=m$ if $w=(h_m, \dots, h_1) \in (\Gamma_0)^m$, and $|w|=0$ if w is the empty word $(\ ) \in (\Gamma_0)^0$.

(2) For $x, y \in \boldsymbol{R}^q$ with $y \in \Gamma(x)$, put

$$d_G(x, y) = \min\{|w| : w \in W(\Gamma_0), x \in D(g_w) \text{ and } g_w(x)=y\}.$$

Distinguish $d_G(x, y)$ from the Euclidean distance $\|x-y\|$ in $\boldsymbol{R}^q$. Consider the Cayley graph $G(x)$ as a metric space such that each edges has length 1. Then $d_G(x, y)$ coincides with the distance of x and y in this metric space $G(x)$.

Definition 3.7 Let $x, y \in \boldsymbol{R}^q$. A word $w \in W(\Gamma_0)$ is called a *path* from x to y if $x \in D(g_w)$ and $g_w(x)=y$. If w satisfies $|w|=d_G(x, y)$ in addition, then w is called a *short-cut* from x to y.

Let $w=(h_m, \dots, h_1) \in W(\Gamma_0)$ be a path from x to y. Then the triplet (x, w, y) is considered as an edge path in the Cayley graph $G(x)$: that is,

$$(x, w, y) = (x, h_1, x_1)(x_1, h_2, x_2)\cdots(x_{m-1}, h_m, y)$$

where $x_i := h_i h_{i-1} \cdots h_1(x)$. If w is a short-cut from x to y in addition, then the edge path (x, w, y) is the shortest one from x to y in $G(x)$ and the points $x, x_1, \cdots, x_{m-1}, y$ are pairwise distinct.

Notation 3.8 For a point $x \in \boldsymbol{R}^q$ and a nonnegative integer n, denote by $D_G(x\,;\,n)$ (respectively $C_G(x\,;\,n)$, $S_G(x\,;\,n)$) the subgraph of $G(x)$ consisting of the vertices y of $G(x)$ with $d_G(x, y) \leqq n$ (respectively $d_G(x, y) \geqq n$, $d_G(x, y)=n$) and edges of $G(x)$ connecting such vertices.

Note that $D_G(x\,;\,n)$ and $S_G(x\,;\,n)$ are finite subgraphs, $C_G(x\,;\,n)$ is an infinite subgraph and they satisfy

$$D_G(x\,;\,n) \cup C_G(x\,;\,n) = G(x), \qquad D_G(x\,;\,n) \cap C_G(x\,;\,n) = S_G(x\,;\,n).$$

Hereafter suppose that for some point $a \in \mathscr{M}$ the orbit $\Gamma(a)$ is almost with bubbles.

For sufficiently large $N \in \boldsymbol{N}$, we can take $D_G(a\,;\,N-1)$ as K in Definition 3.2. This means that for each vertex x in $C_G(a\,;\,N)$ there exists a bubble B_x satisfying the conditions in Definition 3.2. For example, for each edge $e=(x, h, y)$ in $C_G(a\,;\,N)$ with $x \neq y$, we have $B_x \subset D(h)$ and $h(B_x)=B_y$.

Lemma 3.9 *There exists a sequence* $\{\mu_n\}_{n=N}^{\infty}$ *of positive numbers such that*

(1) $\mu_n > \mu_{n+1}$ *for all* $n \geqq N$,

(2) $\lim_{n\to\infty} \mu_n = 0$,

(3) *if* x *is a vertex of* $S_G(a\,;\,n)$ $(n \geqq N)$, *then* $\operatorname{diam} B_x < \mu_n$.

Proof. Since the bubbles B_x are disjoint and their union is contained in the bounded subset $\Omega := \bigcup_{h\in\Gamma_0} D(h)$ of $\boldsymbol{R}^q$, it follows that $\sum_{x\in C_G(x)} \operatorname{vol}(B_x) < \infty$. Note that every bubble B_x is similar to one of the finite bubbles $\{B_y\}_{y\in S_G(a;N)}$ because every vertex $x \in C_G(a\,;\,N)$ is connected to some vertex $y \in S_G(a\,;\,N)$ by an edge path in $C_G(a\,;\,N)$. It follows that

$$\sum_{x\in C_G(a\,;\,N)} (\operatorname{diam} B_x)^q < \infty.$$

For $n \geqq N$, let

$$\delta_n = \sup\{\operatorname{diam} B_x :\ x \in C_G(a\,;\,n)\},$$

which is not infinity because

$$(\delta_n)^q \leqq \sum_{x\in C_G(a\,;\,N)} (\operatorname{diam} B_x)^q < \infty.$$

Since the sequence $\{\delta_n\}_{n\geqq N}$ is weakly decreasing and has a lower bound 0, there exists a limit $\delta_\infty := \lim_{n\to\infty} \delta_n$. If $\delta_\infty > 0$, then there exists an infinite number of $x \in C_G(a : N)$ with $\operatorname{diam} B_x > \delta_\infty/2$, which contradicts the above inequality. Hence $\delta_\infty = 0$. Now put $\mu_n := \delta_n + 1/n$ for all $n \geqq N$. It is easy to see that the sequence $\{\mu_n\}_{n\geqq N}$ satisfies the desired conditions. $\square$

The main point in our arguments is that we must always worry about the domains of elements in the pseudogroup Γ, since the domains of composed elements become usually the smaller if the more elements of Γ_0 are composed. We consider this problem now. First note that there is a positive number ε such that for all $x \in \mathscr{M}$ and for all $h \in \Gamma_0$ with $x \in D(h)$ the ε-neighborhood $U(x\,;\,\varepsilon) := \{z \in \boldsymbol{R}^q : \|z - x\| < \varepsilon\}$ is contained in the domain $D(h)$. This follows from the condition (b) in Definition 2.4.

The following two lemmas are the key step to prove Theorem 3.3. Let $\Delta := \sup\{\operatorname{diam} B_x :\ x \in C_G(a\,;\,N)\}$.

Lemma 3.10 *Let* $x \in S_G(a\,;\,N)$. *If* $w \in W(\Gamma_0)$ *is a short-cut from* x *in* $C_G(a\,;\,N)$, *then*

$$U(x\,;\,\varepsilon\cdot\frac{\operatorname{diam} B_x}{\Delta}) \subset D(g_w).$$

Proof. We proceed by an induction on $m = |w|$.

(I) If $m=1$, then $h:=g_w$ is an element in Γ_0. Since $\operatorname{diam} B_x \leqq \Delta$ and $x \in D(h)$, the above remark implies that

$$U(x\,;\ \varepsilon \cdot \frac{\operatorname{diam} B_x}{\Delta}) \subset U(x\,;\ \varepsilon) \subset D(h) = D(g_w).$$

(II) Suppose that Lemma 3.10 is satisfied for shortcuts of word-length less than m. For a short-cut $w=(h_m, \cdots, h_1) \in W(\Gamma_0)$ from x, let $\hat{w} := (h_{m-1}, \ldots, h_1)$ and $\hat{g} := g_{\hat{w}}$. Note that $\hat{w}$ is also a short-cut from x and that $\hat{g}(x) \in \mathscr{M} \cap D(h_m)$, which implies that $U(\hat{g}(x)\,;\ \varepsilon) \subset D(h_m)$. Note also that $\hat{g}(x) \neq h_m(\hat{g}_{(x)})(=g_w(x))$, which implies that $h_m(B_{\hat{g}(x)}) = B_{g_w(x)}$ and the similitude ratio of h_m equals to $\operatorname{diam} B_{g_w(x)} / \operatorname{diam} B_{\hat{g}(x)}$. By the induction hypothesis, it follows that $U(x\,;\ \varepsilon \operatorname{diam} B_x/\Delta) \subset D(\hat{g})$ and the following computation has the meaning:

$$\begin{aligned}\hat{g}(U(x\,;\ \varepsilon \cdot \frac{\operatorname{diam} B_x}{\Delta})) &= U(\hat{g}(x)\,;\ \varepsilon \cdot \frac{\operatorname{diam} B_x}{\Delta} \cdot \frac{\operatorname{diam} B_{\hat{g}(x)}}{\operatorname{diam} B_x}) \\ &\subset U(\hat{g}(x)\,;\ \varepsilon) \\ &\subset D(h_m).\end{aligned}$$

This implies that

$$U(x\,;\ \varepsilon \cdot \frac{\operatorname{diam} B_x}{\Delta}) \subset D(h_m \hat{g}) = D(g_w). \quad \square$$

Lemma 3.11 *There exists a positive number* ε_0 *such that for all short-cut* $w \in W(\Gamma_0)$ *from* a *to a point in* $G_G(a\,;\ N)$,

$$U(a\,;\ \varepsilon_0) \subset D(g_w).$$

Proof. Denote by $\mathscr{S}$ the set of all the short-cuts $w \in W(\Gamma_0)$ from a to a point in $S_G(a\,;\ N)$. Then $\mathscr{S}$ is a finite set. Since the intersection

$$V := \bigcap_{w \in \mathscr{S}} g_w^{-1}(U(g_w(a)\,;\ \varepsilon \cdot \frac{\operatorname{diam} B_{g_w(a)}}{\Delta}))$$

is an open subset in $\boldsymbol{R}^q$ containing a, there exists a positive number ε_0 such that $V_0 = U(a\,;\ \varepsilon_0) \subset V$. Now let $w \in W(\Gamma_0)$ be a short-cut from a to a point in $C_G(a\,;\ N)$. We can divide w to a composition of a path v from a to a point $x \in S_G(a\,;\ N)$ and a path u from x: $w=uv$. Note that v is a short-cut from a and u is a short-cut from x. It follows that $U_x := U(x\,;\ \varepsilon \operatorname{diam} B_x/\Delta) \subset D(g_u)$ and $V_0 \subset V \subset g_v^{-1}(U_x)$, hence that $g_v(V_0) \subset U_x \subset D(g_u)$. Therefore

$$U(a\,;\ \varepsilon_0) = V_0 \subset D(g_u g_v) = D(g_{uv}) = D(g_w). \quad \square$$

Proof of Theorem 3. 3. Let $\sigma := \max\{\mathrm{SR}(g_v) : v \in \mathscr{P}\}$ and $\delta := \min\{\operatorname{diam} B_x : x \in S_G(a\,;\,N)\}$, where $\mathrm{SR}(g)$ means the similitude ratio of $g \in \Gamma$. Take $n \geqq N$ in such a way that

$$\sigma \cdot \frac{\mu_n}{\delta} < \frac{1}{3}.$$

Since the orbit $\Gamma(a)$ is dense in $\mathscr{M}$, there exists a point $x \in \Gamma(a)$ such that $x \in U(a\,;\,1/3)$ and $d_G(a, x) \geqq n$. Take a short-cut $w \in W(\Gamma_0)$ from a to x. Divide w to a composition of a short-cut v from a to a point $y \in S_G(a\,;\,N)$ and a short-cut u from y to x. Since $\operatorname{diam} B_y \geqq \delta$ and $\operatorname{diam} B_x < \mu_n$, it follows that

$$\begin{aligned}\mathrm{SR}(g_w) = \mathrm{SR}(g_u) \cdot \mathrm{SR}(g_v) &= \frac{\operatorname{diam} B_x}{\operatorname{diam} B_y} \cdot \mathrm{SR}(g_v) \\ &\leqq \frac{\mu_n}{\delta} \cdot \sigma \leqq \frac{1}{3}.\end{aligned}$$

This implies that

$$g_w(U(a\,;\,\varepsilon_0)) \subset U(g_w(a)\,;\,\frac{\varepsilon_0}{3}) \subset U(a\,;\,\frac{2}{3} \cdot \varepsilon_0).$$

Hence according to the Brouwer fixed point theorem, there exists a point $z \in U(a\,;\,2\varepsilon_0/3)$ fixed by g_w. Since $\mathrm{SR}(g_w) < 1/3$, the element g_w is a contraction. This implies that

$$z = \lim_{k \to \infty} (g_w)^k(a) \in \overline{\Gamma(a)} = \mathscr{M},$$

hence z is a focus of the Sacksteder system $\mathscr{S}$. This completes the proof of Theorem 3. 3. $\square$

4. Distribution dimensions of Sacksteder systems

In this section, for a Sacksteder system $\mathscr{S} = (\Gamma_0, \mathscr{M})$, we study how the points of $\mathscr{M}$ are scattered in the ambient space $\boldsymbol{R}^q$.

Definition 4. 1 Let A be an infinite subset of $\boldsymbol{R}^q$ and x a point belonging to A. (1) Denote by $\mathrm{ldd}(A, x)$ the minimum of natural numbers n such that there exists an n-dimensional affine subspace P and a neighborhood V of x in $\boldsymbol{R}^q$ with $A \cap V \subset P$, and call it the *local distribution dimension* of A at x. (2) Denote by $\mathrm{add}(A, x)$ the minimum of natural numbers n such that there exists an n-dimensional linear subspace S of $\boldsymbol{R}^q$ satisfying the condition: if $\{x_i\}_{i=1}^{\infty} \subset A - \{x\}$ is an infinite sequence converging to x and the sequence $\{(x_i - x)/\|x_i - x\|\}_{i=1}^{\infty}$ converges to a vector $v \in \boldsymbol{R}^q$, then $v \in S$. We call it the *asymptotic distribution dimension* of A at x. (Clearly $\mathrm{add}(A, x) \leqq \mathrm{ldd}(A, x)$.)

The first fundamental property of distribution dimensions is the following.

Proposition 4.2 *Let $\mathscr{S} = (\Gamma_0, \mathscr{M})$ be a Sacksteder system. Then the local distribution dimension $\mathrm{ldd}(\mathscr{M}, x)$ does not depend on $x \in \mathscr{M}$.*

Proof. Let $x, y \in \mathscr{M}$. Put $n := \mathrm{ldd}(\mathscr{M}, x)$. By definition, there exists an n-dimensional affine plane P and an open subset W containing x in $\boldsymbol{R}^q$ with $\mathscr{M} \cap W \subset P$. Since the orbit $\Gamma(y)$ is dense in $\mathscr{M}$, there exists an element $g \in \Gamma$ such that $y \in D(g)$ and $z := g(y) \in \mathscr{M} \cap W$. It follows that

$$y \in \mathscr{M} \cap g^{-1}(W) = g^{-1}(\mathscr{M} \cap W) \subset g^{-1}(P).$$

Since $g^{-1}(P)$ is contained in an n-dimensional affine plane, it follows that $\mathrm{ldd}(\mathscr{M}, y) \leqq n = \mathrm{ldd}(\mathscr{M}, x)$. By exchanging the roles of x and y, we have $\mathrm{ldd}(\mathscr{M}, x) \leqq \mathrm{ldd}(\mathscr{M}, y)$. This completes the proof. $\square$

This proposition motivates the following.

Definition 4.3 For a Sacksteder system $\mathscr{S} = (\Gamma_0, \mathscr{M})$, we denote by $\mathrm{ldd}(\mathscr{S})$ the constant number $\mathrm{ldd}(\mathscr{M}, x)$ where $x \in \mathscr{M}$, and we call it the *local distribution dimension* of $\mathscr{S}$. We call the system $\mathscr{S}$ *irreducible* if the local distribution dimension $\mathrm{ldd}(\mathscr{S})$ coincides with the dimension q of the ambient space $\boldsymbol{R}^q$ of $\mathscr{S}$.

If a Sacksteder system $\mathscr{S}$ is not irreducible, it is natural to study $\mathscr{S}$ by taking locally $\mathrm{ldd}(\mathscr{S})$-dimensional affine planes in place of the ambient space. By this reason, we introduce a *reduction* of a Sacksteder system as follows.

Definition 4.4 We say that a Sacksteder system $\mathscr{S} = (\Gamma_0, \mathscr{M})$ with ambient space $\boldsymbol{R}^q$ is *reducible* to a Sacksteder system $\widehat{\mathscr{S}} = (\widehat{\Gamma}_0, \widehat{\mathscr{M}})$ with ambient space $\boldsymbol{R}^{\widehat{q}}$ if the dimensions of the ambient spaces satisfy $q > \widehat{q}$ and there exist a decomposition $\mathscr{M} = \mathscr{M}_1 \cup \cdots \cup \mathscr{M}_k$, disjoint open sets W_i containing each $\mathscr{M}_i$ in $\boldsymbol{R}^q$, and affine maps $\varphi_i : \boldsymbol{R}^q \to \boldsymbol{R}^{\widehat{q}}$ satisfying the following conditions:

(a) the images $\widehat{W}_1 := \varphi_1(W_1), \dots, W_k := \varphi_k(\widehat{W}_k)$ are disjoint open sets in $\boldsymbol{R}^{\widehat{q}}$,
(b) $\widehat{\mathscr{M}} = \widehat{\mathscr{M}}_1 \cup \cdots \cup \widehat{\mathscr{M}}_k$ where $\widehat{\mathscr{M}}_i := \varphi_i(\mathscr{M}_i)$,
(c) each φ_i maps the affine space Q_i spanned by $\mathscr{M}_i$ isometrically to

the affine space $\hat{Q}_i$ spanned by $\hat{\mathcal{M}}_i$,

(d) for each $h\in\Gamma_0$ and $\mathcal{M}_i$, $\mathcal{M}_j$ with $D(h)\cap\mathcal{M}_i\neq\emptyset$ and $R(h)\cap\mathcal{M}_j\neq\emptyset$, there exists an open set $W_{hji}\subset D(h)$ containing $D(h)\cap\mathcal{M}_i$ in $\boldsymbol{R}^q$ and an element $\hat{h}_{hji}\in\hat{\Gamma}_0$ such that $\hat{h}_{hji}\varphi_i=\varphi_j h$ on W_{hji} (thus we have the following commutative diagram:

$$\begin{array}{ccccccc}
\mathcal{M}_i\cap D(h) & \xrightarrow{\subset} & W_{hji} & \xrightarrow{h} & W_{h^{-1}ij} & \xrightarrow{\subset} & \boldsymbol{R}^q \\
 & & \varphi_i\big\downarrow & & \varphi_j\big\downarrow & & \\
\hat{\mathcal{M}}_i\cap D(\hat{h}_{hji}) & \xrightarrow{\subset} & \hat{W}_{hji} & \xrightarrow{\hat{h}_{hji}} & \hat{W}_{h^{-1}ij} & \xrightarrow{\subset} & \boldsymbol{R}^{\hat{q}}
\end{array}$$

where $\hat{W}_{hji}:=\varphi_i(W_{hji})$ and $\hat{W}_{h^{-1}ij}:=\varphi_j(W_{h^{-1}ij})$),

(e) $\hat{\Gamma}_0$ consists of such elements $\hat{h}_{hji}$ as in (d).

We call the map $\varphi:=\varphi|_{W_1}\cup\cdots\cup\varphi|_{W_k}:W:=W_1\cup\cdots\cup W_k\to\boldsymbol{R}^{\hat{q}}$ a *reduction* of $\mathcal{S}$ to $\hat{\mathcal{S}}$.

Proposition 4.5 (1) *If a Sacksteder sytem $\mathcal{S}=(\Gamma_0,\mathcal{M})$ is reducible by a reduction φ to a Sacksteder system $\hat{\mathcal{S}}=(\hat{\Gamma}_0,\hat{\mathcal{M}})$, then* $\mathrm{ldd}(\hat{\mathcal{S}})=\mathrm{ldd}(\mathcal{S})$ *and* $\mathrm{add}(\hat{\mathcal{M}},\varphi(x))=\mathrm{add}(\mathcal{M},x)$ *for all $x\in\mathcal{M}$.*

(2) *Let $\mathcal{S}=(\Gamma_0,\mathcal{M})$ be a Sacksteder system with ambient space $\boldsymbol{R}^q$ and put $\hat{q}=\mathrm{ldd}(\mathcal{S})$. Then $\mathcal{S}$ is reducible to an irreducible Sacksteder system $\hat{\mathcal{S}}=(\hat{\Gamma}_0,\hat{\mathcal{M}})$ (that is, with ambient space $\boldsymbol{R}^{\hat{q}}$).*

Proof. (1) We use the notations in Definition 4.4. Let $x\in\mathcal{M}_i\subset\mathcal{M}$. The condition (c) in Definition 4.4 implies that φ_i: $Q_i\to\hat{Q}_i$ is an isomorphism of affine spaces. For each open set $V\subset W_i$ containing x, the image $\hat{V}:=\varphi_i(V)$ is an open set containing $\hat{x}:=\varphi_i(x)$ in $\boldsymbol{R}^{\hat{q}}$ by the condition (a) in Definition 4.3 (which implies $\mathrm{rank}(\varphi_i)=\hat{q}$). These facts imply that $\mathrm{ldd}(\hat{\mathcal{M}}_i,\hat{x})=\mathrm{ldd}(\mathcal{M}_i,x)$ and $\mathrm{add}(\hat{\mathcal{M}}_i,\hat{x})=\mathrm{add}(\mathcal{M}_i,x)$. It follows that

$$\mathrm{ldd}(\hat{\mathcal{S}})=\mathrm{ldd}(\hat{\mathcal{M}}_i,\hat{x})=\mathrm{ldd}(\mathcal{M}_i,x)=\mathrm{ldd}(\mathcal{S}),$$
$$\mathrm{add}(\hat{\mathcal{M}},\hat{x})=\mathrm{add}(\hat{\mathcal{M}}_i,\hat{x})=\mathrm{add}(\mathcal{M}_i,x)=\mathrm{add}(\mathcal{M},x).$$

(2) Put $\hat{q}=\mathrm{ldd}(\mathcal{S})$. For each $x\in\mathcal{M}$, take a convex open set V_x containing x and a $\hat{q}$-dimensional affine plane P_x in $\boldsymbol{R}^q$ with $\mathcal{M}\cap V_x\subset P_x$. Since $\mathcal{M}$ is compact and $\{V_x\}_{x\in\mathcal{M}}$ is an open covering of $\mathcal{M}$, there exist a finite number of points $x(1),\dots,x(k)\in\mathcal{M}$ such that $\mathcal{M}\subset V_{x(1)}\cup\cdots\cup V_{x(k)}$. Let $W_i:=V_{x(i)}$, $\mathcal{M}_i:=\mathcal{M}\cap W_i$ and $Q_i:=P_{x(i)}$. Take affine isometries ψ_i:

$Q_i \to \boldsymbol{R}^{\hat{q}}$ in such a way that $\varphi_1(W_1), \ldots, \varphi_k(W_k)$ are pairwise disjoint, where φ_i is the composition of ψ_i and the orthogonal projection of $\boldsymbol{R}^q$ to Q_i. Let $\widehat{\mathscr{M}}_i := \varphi_i(\mathscr{M}_i)$ and $\widehat{\mathscr{M}} := \widehat{\mathscr{M}}_1 \cup \cdots \cup \widehat{\mathscr{M}}_k$. For each element $h \in \Gamma_0$ and $\mathscr{M}_i$, $\mathscr{M}_j$ with $D(h) \cap \mathscr{M}_i \neq \emptyset$ and $R(h) \cap \mathscr{M}_j \neq \emptyset$, take a sufficiently small convex open set $\widehat{W}_{hji}$ containing $\varphi_1(D(h) \cap \mathscr{M}_i)$ and put $\hat{h}_{hji} := \psi_j h \psi_i^{-1}$: $\widehat{W}_{hji} \to \boldsymbol{R}^{\hat{q}}$. (After we made the above construction for $h \in \Gamma_0$, we take $(\hat{h}_{hji})^{-1}$ as $\hat{h}_{h^{-1}ij}$ for the inverse $h^{-1} \in \Gamma_0$.) Denote by $\hat{\Gamma}_0$ the set of such $\hat{h}_{hji}$'s. Then the pair $\widehat{\mathscr{S}} := (\hat{\Gamma}_0, \widehat{\mathscr{M}})$ is a Sacksteder system with ambient space $\boldsymbol{R}^{\hat{q}}$ and the given Sacksteder system $\mathscr{S}$ is reducible to $\widehat{\mathscr{S}}$ by the reduction $\varphi := \varphi_1 \cup \cdots \cup \varphi_k$. $\square$

Note that if a Sacksteder system $\mathscr{S}$ is reducible to a Sacksteder system $\widehat{\mathscr{S}}$ then the qualitative properties of $\mathscr{S}$ and $\widehat{\mathscr{S}}$ are completely the same.

The main result in this section is the following.

Theorem 4.6 . *Les $\mathscr{S} = (\Gamma_0, \mathscr{M})$ be a Sacksteder system. If $\mathscr{S}$ has a focus $x_0 \in \mathscr{M}$, then* $\operatorname{add}(\mathscr{M}, x_0) = \operatorname{ldd}(\mathscr{S})$.

Proof. It is sufficient to prove that $\operatorname{add}(\mathscr{M}, x_0) \geqq \operatorname{ldd}(\mathscr{S})$. By the hypothesis, there exists a contraction $g \in \operatorname{Stab}(x_0)$. Take an open set V containing x_0 in $\boldsymbol{R}^q$ and a subspace S of dimenesion $n := \operatorname{add}(\mathscr{M}, x_0)$ in the tangent space $T_{x_0}\boldsymbol{R}^q = \boldsymbol{R}^q$ such that $V \subset D(g)$ and that, if $\{x_i\}_{i=1}^{\infty} \subset \mathscr{M} - \{x_0\}$ is an infinite sequence converging to x_0 and the sequence $\{(x_i - x_0)/\|x_i - x_0\|\}_{i=1}^{\infty}$ converges to a vector $v \in \boldsymbol{R}^q$, then $v \in S$. Consider the affine plane $P := S + x_0$.

We claim that $\mathscr{M} \cap V \subset P$. We prove this by absurdity. Suppose that there exists a point $y \in (\mathscr{M} \cap V) - P$. Since g is a contraction and $y \in V \subset D(g)$, we can define a sequence $\{y_j\}_{j=1}^{\infty} \subset \mathscr{M} \cap D(g)$ by $y_i = g^j(y)$. Clearly the sequence $\{y_j\}_{j=1}^{\infty}$ converges to x_0. Note that for all j the angle between the line containing x_0, y_j and the affine plane P coincides with the angle θ between the line containing x_0, y and P. This implies that $y_j \in \mathscr{M} \cap D(g) - P$ for all j. Denote by C the set of unit tangent vectors $u \in T_{x_0}\boldsymbol{R}^q = \boldsymbol{R}^q$ such that the angle between u and S is θ. Since C is compact and $(y_j - x_0)/\|y_j - x_0\| \in C$ for all j, there exists a subsequence $\{x_i\}_{j=1}^{\infty}$ of $\{y_j\}_{j=1}^{\infty}$ such that the sequence $(x_i - x_0)/\|x_i - x_0\|$ converges to a vector $v \in C$. Since $C \cap S = \emptyset$, it follows that $v \notin S$. This contradicts the choice of the subspace $S \subset T_{x_0}\boldsymbol{R}^q$.

The above claim implies that

$$\operatorname{ldd}(\mathscr{S}) = \operatorname{ldd}(\mathscr{M}, x_0) \leqq \dim P = n = \operatorname{add}(\mathscr{M}, x_0).$$

This completes the proof of Theorem 4.6. □

The following is an application of Theorem 4.6.

Theorem 4.7 *Let $\mathscr{S}=(\Gamma_0, \mathscr{M})$ be a Sacksteder system. If $\mathscr{S}$ has a focus $x_0 \in \mathscr{M}$, then $\mathscr{M}$ cannot be contained in any submanifold of the ambient space $\boldsymbol{R}^q$ with positive curvature.*

Proof. We prove this theorem by absuridity. Suppose that $\mathscr{M}$ is contained in a submanifold $\mathscr{M}$ of $\boldsymbol{R}^q$ with positive curvature. Fix a focus $x_0 \in \mathscr{M}$ and put $n := \mathrm{ldd}(\mathscr{M})$. There exist an open set V containing x_0 and an n-dimensional affine plane P in $\boldsymbol{R}^q$ such that $\mathscr{M} \cap V \subset P$. The intersection $N := M \cap V \cap P$ is a submanifold of the affine plane P. Since $\mathscr{M} \cap V \subset N$, the manifold N has an accumulation point. Hence $\dim N \geqq 1$. If $\dim N = 1$, the manifold N is a curve with non vanishing curvature. If $\dim N \geqq 2$, then N is a Riemannian submanifold with positive curvature. In the both cases, we have $\dim N < \dim P$. Consider an sequence $\{x_i\}_{j=1}^{\infty} \subset \mathscr{M} - \{x_0\}$ converging to x_0 such that the sequence $\{(x_i - x_0)/\|x_i - x_0\|\}_{j=1}^{\infty}$ converges to a vector $v \in T_{x_0}\boldsymbol{R}^q = \boldsymbol{R}^q$. Since $x_i \in N$ for all i, the vector v belongs to the tangent space $T_{x_0}N$. Hence we can take $T_{x_0}N$ as the subspace S in Definition 4.1, which implies that

$$\mathrm{add}(\mathscr{M}, x_0) \leqq \dim T_{x_0}N = \dim N < \dim P = n = \mathrm{ldd}(\mathscr{S}).$$

This contradicts Theorem 4.6. □

This theorem suggests the following.

Conjecture 4.8 *If $\mathscr{S}=(\Gamma_0, \mathscr{M})$ is an irreducible Sacksteder system with a focus, then $\mathscr{M}$ cannot be contained in the frontier of any convex domain in the ambient space $\boldsymbol{R}^q$.*

In the next section, we give an affirmative example to this conjecture.

5. Convexly self-similar Sacksteder systems

We introduce a class of Sacksteder sysems.

Definition 5.1 A subset $A \subset \boldsymbol{R}^q$ is called *convexly self-similar* if A is compact and infinite and, for all $x \in A$ and for all open set V containing x, there exist a convex open subset U containing A and a similarity transformation γ of $\boldsymbol{R}^q$ such that $x \in W := \gamma(U) \subset V$ and $\gamma(A) = A \cap W$. (For the simplicity, we consider only orientation-preserving similarity transformations.)

Note that the subset $\mathscr{M}$ in Example 2.11 is a convexly self-similar set. We give a fundamental property of convexly self-similar sets.

Proposition 5.2 *Let $A \subset \boldsymbol{R}^q$ be a convexly self-similar set. Then* (1) *A is a Cantor set.* (2) *There exists a finite number of similarity transformations $\gamma_1, \dots, \gamma_k$ of $\boldsymbol{R}^q$ such that every γ_i is a contraction and $A = \gamma_1(A) \cup \cdots \cup \gamma_k(A)$.* (3) *Put $\mathscr{M} := A$ and $\Gamma_0 := \{h_1, \dots, h_k, h_1^{-1}, \dots, h_k^{-1}\}$ where h_i is the restriction of γ_i in* (2) *to a suitable convex open subset containing A. Then the pair $\mathscr{S} = (\Gamma_0, \mathscr{M})$ is a Sacksteder system with a focus.*

Proof. (1) First we see that A is perfect. Let $x \in A$ and $\varepsilon > 0$. Then there exist a convex open set U containing A and a similarity transformation γ of $\boldsymbol{R}^q$ such that $x \in W := \gamma(U) \subset V := U(x\,;\,\varepsilon)$ and $\gamma(A) = A \cap \mathrm{W}$. Since A is an infinite set, so is $A \cap W$. Hence the intersection $U(x\,;\,\varepsilon) \cap (A - \{x\})$ is not empty, which implies that A is perfect. Second we see that A is not totally disconnected. Let $x \in A$ and $\varepsilon > 0$. Consider such convex open set U and such similarity transformation γ as above. Take an open set U_1 such that $A \subset U_1 \subset \bar{U}_1 \subset U$. Note that the frontier of $\gamma(U_1)$ doses not intersect A. Hence the connected component C_x of A containing x must be contained in $\gamma(U_1)(\subset U(x\,;\,\varepsilon))$. It follows that $C_x \subset \bigcap_{\varepsilon>0} U(x\,;\,\varepsilon) = \{x\}$. Therefore A is totally disconnected. Since $A \subset \boldsymbol{R}^q$ is compact perfect totally disconnected, A is a Cantor set.

(2) Fix a positive number $\varepsilon < \mathrm{diam} A/3$. For each $x \in A$, denote by U_x, W_x and γ_x the above U, W and γ. Then $\{\mathrm{W}_x\}$ is an open covering of A. Since A is compact, there exist a finite number of points $x(1), \dots, x(k) \in A$ such that $A \subset W_{x(1)} \cup \cdots \cup W_{x(k)}$. It follows that

$$A = (A \cap W_{x(1)}) \cup \cdots \cup (A \cap W_{x(k)}) = \gamma_{x(1)}(A) \cup \cdots \cup \gamma_{x(k)}(A).$$

Note that $\gamma_{x(i)}$ has the similitude ratio smaller than $2/3$. Hence $\gamma_{x(i)}$ is a contraction.

(3) We have already known that the subset $\mathscr{M}$ is infinite nowhere dense and Γ_0 is a finite symmetric subset of $\Gamma_{q,+}^{\mathrm{sin},*}$ satisfying (a), (b) and (c) in Definition 2.4. The theory of self-similar sets in [Wil], [Hut] and [Hat] implies that $\mathscr{M}$ coincides with the closure of the set of the fixed points of $h_{i(1)} \cdots h_{i(n)}$ where $n \in \boldsymbol{N}$ and $i(1), \dots, i(n) \in \{1, 2, \dots, k\}$. We see that for all $x \in \mathscr{M}$ the orbit $\Gamma(x)$ is dense in $\mathscr{M}$ where $\Gamma = \langle \Gamma_0 \rangle$. Let $y \in \mathscr{M}$ and $\varepsilon > 0$. Then $U(y\,;\,\varepsilon/2)$ contains the fixed point z of some $g := h_{i(1)} \cdots h_{i(n)}$ as above. Since $\mathscr{M}$ is contained in $D(g)$ and g is a contraction, there exists $m \in \boldsymbol{N}$ such that $g^m(\mathscr{M}) \subset U(z\,;\,\varepsilon/2)$. This implies that $U(y\,;\,\varepsilon) \cap \Gamma(x) \neq \emptyset$. Hence the condition (d) in Definition 2.4 is verified. Clearly the Sacksteder system $\mathscr{S}$ has a focus. $\square$

Definition 5.3 A Sacksteder system $\mathscr{S}=(\Gamma_0, \mathscr{M})$ is called *convexly self-similar* if it is obtained from a convexly self-similar set A in the same way as in (3) of Proposition 5.2.

Concerning Conjecture 4.11, we have the following.

Theorem 5.4 *Let $\mathscr{S}=(\Gamma_0, \mathscr{M})$ be a convexly self-similar irreducible Sacksteder system with ambient space $\boldsymbol{R}^q$. If $q=2$, then $\mathscr{M}$ cannot be contained in the frontier of any convex domain of $\boldsymbol{R}^q$.*

Proof. We prove this theorem by absurdity. Suppose that $\mathscr{M}$ is contained in the frontier of a convex domain Ω of $\boldsymbol{R}^q$.

First we see that any three points of $\mathscr{M}$ are not on a line. Suppose that there exist three point x, y, z on a line $l\subset\boldsymbol{R}^2$ in this order. By the irreducibility, we have independent three points $a, b, c\in\mathscr{M}$. Then a, b, c form a triangle. The points x and z form a triangle T with one of a, b, c, say a. Since Ω is convex, the domain $\mathrm{Int}(T)$ surrounded by T is contained in Ω. Take a positive number $\varepsilon<\min\{\|x-y\|, \|z-y\|, \|a-y\|\}/9$. There exists an element $g\in\Gamma$ such that $\mathscr{M}\subset D(g)$ and $g(\mathscr{M})\subset U(y\,;\,\varepsilon)$ as in the proof of (3) in Proposition 5.2. The triangle Δ formed by $g(a)$, $g(b), g(c)\in U(y, \varepsilon)$ cannot intersect $\mathrm{Int}(T)$ because $g(a), g(b), g(c)\notin\Omega$. Hence $U(y\,;\,\varepsilon)-T$ contains at least one of $g(a), g(b), g(c)$, say $g(b)$. This means that y belongs to the interior of the convex hull of x, z, a, $g(b)$. It follows that $y\in\Omega$, which contradicts the hypothesis $\mathscr{M}\subset\delta\Omega(=\bar{\Omega}-\Omega)$. Second we see that for a given positive number θ any convex n-gon has at most $[2\pi/(\pi-\theta)]$ vertices with inner angle$\leqq\theta$. This follows from the fact that the sum of all the inner angle is $(n-2)\pi$.

Now take three points $a, b, c\in\mathscr{M}$. Denote by α the maximum of the inner angles of the triangle Δ formed by a, b, c and take θ such that $(\pi+\alpha)/2<\theta<\pi$. Take 100 $[2\pi/(\pi-\theta)]$ points from $\mathscr{M}$. They form a convex polygon P. The above second result implies that P has successive three vertices p, q, r with inner angle$>\theta$. Denote by o the vertex before p and by s the vertex after r. Denote by x the intersecting point of the lines $\overline{op}$ and $\overline{rq}$ and denote by y the intersecting point of the lines $\overline{pq}$ and $\overline{sr}$. Put $\varepsilon:=\min(\|x-q\|, \|y-q\|, \min\{\|t-q\|: t\neq q \text{ is a vertex of } P\})/100$. As above, there exists $g\in\Gamma$ such that $\mathscr{M}\subset D(g)$ and $g(\mathscr{M})\subset U(q\,;\,\varepsilon)$. The lines $\overline{py}$ and $\overline{rx}$ divide $U(q\,;\,\varepsilon)$ into four sectors. If $g(\mathscr{M})$ intersects the interior of the sector contained in $\mathrm{Int}(P)$ or that of the sector opposite to $\mathrm{Int}(P)$, we have a contradiction immediately. The lines $\overline{py}$ and $\overline{rx}$ can contain at most two points of $\mathscr{M}$ by the above first result. Therefore $g(\mathscr{M})$ intersects one of the remained two sectors. We may suppose that

$g(\mathscr{M})$ intersects the interior of the sector S contained in the triangle formed by p, q, x. Take a point $q_1 \in \mathrm{Int}(S) \cap g(\mathscr{M})$. By considering a small neighborhood of y_1, we obtain $g_1 \in \Gamma$ such that $g_1(\Delta) \subset \mathrm{Int}(S)$. We may suppose that $g_1(a)$ is the nearest vertex of $g_1(\Delta)$ to p and that $g_1(b)$ is the nearest vertex of $g_1(\Delta)$ to q. Then $g_1(a)$ and $g_1(b)$ cannot belong to the triangle T formed by $p, q, g_1(c)$ and the line segment between $g_1(a)$ and $g_1(b)$ must intersect the triangle T. This implies that $\angle p g_1(c) q < \angle g_1(a) g_1(c) g_1(b) = \angle acb < \alpha$. On the other hand, we have

$$\angle p q_1(c) q > \angle pxq = \angle opq + \angle pqr - \pi > 2\theta - \pi > (\pi + \alpha) - \pi = \alpha.$$

This contradiction completes the proof of Theorem 5. 4.

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