Hokkaido Mathematical Journal Vol. 24 (1995) p. 105-111

Note on H-separable Frobenius extensions

Kozo SUGANO
(Received April 28, 1994)

Throughout this paper A will be a ring with the identity element 1, B a subring of A containing 1, C the center of A and D will be the centralizer of B in A. We will use the same notation as the author's previous paper [9]. For any subset X of A and any A-A-module M we will write

$$V_A(X)=\{a\ \varepsilon\ A \mid ax=xa \text{ for any } x\ \varepsilon\ X\}, \text{ and}$$
$$M^A=\{m\ \varepsilon\ M \mid am=ma \text{ for any } a\ \varepsilon\ A\}$$

respectively. Thus we have $D=A^B=V_A(B)$ and $(A\otimes_B A)^A=\{\sum a_i\otimes b_i\ \varepsilon\ A\otimes_B A \mid \sum xa_i\otimes b_i=\sum a_i\otimes b_i x \text{ for any } x\ \varepsilon\ A\}$.

A is said to be a Frobenius extension of B in the case where A is left B-$f.g.$ (finitely generated) projective and there exists a left A and right B-isomorphism of A to $\mathrm{Hom}({}_BA, {}_BB)$. This is the case if and only if there exist finite x_k, $y_k\ \varepsilon\ A$ and $h\ \varepsilon\ \mathrm{Hom}({}_BA_B, {}_BB_B)$ such that $x=\sum h(xx_k)y_k=\sum x_k h(y_k x)$ hold for each $x\ \varepsilon\ A$. In this case we call the set $\{x_k, y_k, h\}$ a Frobenius system, and the map h a Frobenius homomorphism, of $A|B$ respectively.

Now for any $h\ \varepsilon\ \mathrm{Hom}({}_BA_B, {}_BB_B)$ we can define a multiplication among the elements of $A\otimes_B A$ by $(a\otimes b)(c\otimes d)=ah(bc)\otimes d$ for any a, b, c, $d\ \varepsilon\ A$ (See Proposition 4.1 [2]). This multiplication is well defined, and by this definition we can make $A\otimes_B A$ an associative ring which does not always have the identity element. On the other hand we can define the following maps

$$\phi_r:\ A\otimes_B A\longrightarrow \mathrm{Hom}(A_B, A_B)$$
$$\phi_l:\ A\otimes_B A\longrightarrow \mathrm{Hom}({}_BA, {}_BA)$$

by $\phi_r(a\otimes b)(x)=ah(bx)$ and $\phi_l(a\otimes b)(x)=h(xa)b$ for any a, b, $x\ \varepsilon\ A$. Direct calculation shows that ϕ_r and ϕ_l are ring, and opposite ring, homomorphisms respectively. Now we have

Lemma 1 *For an $h\ \varepsilon\ \mathrm{Hom}({}_BA_B, {}_BB_B)$ the following conditions are equivalent;*

(i) *A is a Frobenius extension of B with h a Frobenius homomorphism*

(ii) *ϕ_r defined as above is an isomorphism*
(iii) *ϕ_l defined as above is an isomorphism*
(vi) *$A\otimes_B A$ has the identity element as a ring defined as above.*

If $\sum x_k\otimes y_k$ is the identity of $A\otimes_B A$, then $\{x_k, y_k, h\}$ is a Frobenius system of $A|B$.

Proof. (i)$\mapsto$(ii) and (i)$\mapsto$(iii) are well known (See e.g. the proof of Theorem 1 on page 94 [5]), and (ii)$\mapsto$(vi) and (iii)$\mapsto$(vi) are obvious. Let $\sum x_k\otimes y_k$ be the identity of $A\otimes_B A$. Then for any $x \varepsilon A$ we have $1\otimes x=(1\otimes x)(\sum x_k\otimes y_k)=\sum h(xx_k)\otimes y_k$ and $x\otimes 1=(\sum x_k\otimes y_k)(x\otimes 1)=\sum x_k h(y_k x)\otimes 1$. Then we have $x=\sum h(xx_k)y_k=\sum x_k h(y_k x)$. Thus we have proved (vi)$\mapsto$(i) and the last assertion.

There are some special maps as follows

$$\begin{array}{lll} \eta: & A\otimes_B A\longrightarrow \mathrm{Hom}({}_C D, {}_C A) & \eta(a\otimes b)(x)=axb \\ \eta_l: & D\otimes_C A\longrightarrow \mathrm{Hom}({}_B A, {}_B A) & \eta_l(d\otimes a)(x)=dxa \\ \eta_t: & D\otimes_C D\longrightarrow \mathrm{Hom}({}_B A_B, {}_B A_B) & \eta_t(d\otimes e)(x)=dxe \end{array}$$

for $a, b, x \varepsilon A$ and $d, e \varepsilon D$. $\eta_r: A\otimes_C D\to \mathrm{Hom}(A_B, A_B)$ is defined similarly. η, η_l and η_r are A-A-maps and η_t is a D-D-map. A is an H-separable extension of B if and only if η is an isomorphism and D is C-$f.g.$ projective. This is the case if and only if $1\otimes 1=\sum d_i\sum x_{ij}\otimes y_{ij}$ for some $d_i \varepsilon D$ and $\sum x_{ij}\otimes y_{ij} \varepsilon (A\otimes_B A)^A$. We call such set $\{d_i, \sum x_{ij}\otimes y_{ij}\}$ an H-system of $A|B$. In the case where A is an H-separable extension of B all the above maps are isomorphisms. The next lemma is an immediate consequence of Corollary 3 [7].

Lemma 2 *In the case where A is left B-f.g. projective the following conditions are equivalent:*
(i) *A is an H-separable extension of B*
(ii) *η_l is an isomorphism and D is C-f.g. projective.*
(iii) *There exists an A-A-split epimorphism of finite direct sum of copies of A to Hom $({}_B A, {}_B A)$.*

Let $\{x_k, y_k, h\}$ be a Frobenius system of $A|B$ and $\sum a_j\otimes b_j$ an arbitrary in $(A\otimes_B A)^A$. For any $x \varepsilon A$ we have $\sum xa_j h(b_j)=\sum a_j h(b_j x)$ and for any $b \varepsilon B$ $\sum ba_j h(b_j)=\sum a_j h(b_j b)=\sum a_j h(b_j)b$. Thus we have $\sum a_j h(b_j) \varepsilon D$, and $\sum a_j\otimes b_j=\sum x_k h(y_k a_j)\otimes h(b_j x_l)y_l=\sum x_k h(y_k a_j)h(b_j x_l)\otimes y_l=\sum x_k h(y_k a_j h(b_j x_l))\otimes y_l=\sum a_j h(b_j x_l)\otimes y_l=\sum x_l a_i h(b_j)\otimes y_l \ \varepsilon \ \sum x_l D\otimes y_l$. Thus we have $(A\otimes_B A)^A\subset\sum x_k D\otimes y_k$. Since $\sum x_k\otimes y_k \varepsilon (A\otimes_B A)^A$, the converse inclusion is clear. Therefore we have $(A\otimes_B A)^A=\sum x_k D\otimes y_k$ (See page 370 [1]). By this equality we have

Theorem 1 *Let A be a Frobenius extension of B with a Frobenius system $\{x_k, y_k, \tilde{h}\}$. Then the following conditions are equivalent;*

(i) *A is an H-separable extension of B.*

(ii) *There exist finite d_i, $e_i \varepsilon D$ such that $1\otimes 1=\sum d_i x_k e_i \otimes y_k$ in $A\otimes_B A$.*

If there exist d_i, $e_j \varepsilon D$ which satisfy the condition of (ii), then we have the following assertions

(1) *$\sum d_i\otimes e_i \varepsilon (D\otimes_C D)^D$, and we have $\tilde{h}(x)=\sum d_i x e_i$ for each $x \varepsilon A$.*

(2) *We can obtain a Frobenius system $\{d_i, e_i, h\}$ of $D|C$, where h is defined by $h(d)=\sum x_k d y_k$ for each d in D.*

Proof. Since A is an H-separable extension of B if and only if $1\otimes 1 \varepsilon D(A\otimes_B A)^A$, we have the equivalence of (i) and (ii) immediately by $(A\otimes_B A)^A=\sum x_k D\otimes y_k$. Assume (ii). Then $\sum x_k e_i\otimes y_k \varepsilon (A\otimes_B A)^A$, and for each x in A we have $1\otimes x=\sum d_i x_k e_i\otimes y_k x=\sum d_i x x_k e_i\otimes y_k$, and $1\otimes \tilde{h}(x)=\sum d_i x x_k e_i\otimes \tilde{h}(y_k)$. But $\tilde{h}(y_k) \varepsilon B$ and $e_i \varepsilon D$. Hence we have $\tilde{h}(x)=\sum d_i x x_k e_i \tilde{h}(y_k)=\sum d_i x x_k \tilde{h}(y_k) e_i=\sum d_i x e_i \varepsilon B$. Now for each $d \varepsilon D$ we have $\sum d d_i x e_i=\sum d_i x e_i d$. Then since η_t is an isomorphism by (i), we have $\sum d d_i\otimes e_i=\sum d_i\otimes e_i d$ in $D\otimes_C D$. Thus we have proved (1). Next we will prove (2). The map h defined in (2) is in $\mathrm{Hom}({}_C D, {}_C C)$, since $\sum x_k D y_k \subset C$. Then since $\sum d_i x e_i \varepsilon V_A(D)$ for each $x \varepsilon A$, we have $\sum d_i h(e_i d)=\sum d_i x_k e_i d y_k=\sum d d_i x_k e_i y_k=d\sum \tilde{h}(x_k) y_k=d$. Similarly we have $\sum \tilde{h}(d d_i) e_i=d$. Thus $\{d_i, e_i, h\}$ is a Frobenius system of $D|C$.

Theorem 2 *Assume $B=V_A(D)$, and let D be a Frobenius C-algebra with a Frobenius system $\{d_i, e_i, h\}$. Then the following three conditions are equivalent*

(i) *A is an H-separable extension of B.*

(ii) *There exist finite x_k, $y_k \varepsilon A$ such that $1\otimes 1=\sum d_i\otimes x_k e_i y_k$ in $D\otimes_C A$.*

(iii) *There exists $\sum x_k\otimes y_k \varepsilon (A\otimes_B A)^A$ such that $1\otimes 1=\sum d_i x_k e_i\otimes y_k$ in $A\otimes_B A$.*

In the case where there exist x_k, $y_k \varepsilon A$ which satisfy the condition of (ii), we have the following assertions;

(1) *$\sum x_k\otimes y_k \varepsilon (A\otimes_B A)^A$ and $h(d)=\sum x_k d y_k$ holds for any $d \varepsilon D$.*

(2) *A is a Frobenius extension of B with a Frobenius system $\{x_k, y_k, \tilde{h}\}$, where $\tilde{h}$ is defined by $\tilde{h}(x)=\sum d_i x e_i$ for each $x \varepsilon A$.*

Proof. (iii)$\mapsto$(i) is obvious, since $\sum x_k e_i\otimes y_k$ is contained in $(A\otimes_B A)^A$. Assume A is an H-separable extension of B, and consider the isomorphism η introduced above. η induces the isomorphism $(A\otimes_B A)^A\cong$

Hom $({}_CD, {}_CC)$. Therefore there exists $\sum x_k\otimes y_k \varepsilon (A\otimes_B A)^A$ such that $\eta(\sum x_k\otimes y_k)=h$. Then since $1=\sum d_ih(e_i)=\sum d_ix_ke_iy_k$ and $\sum d_ix_ke_i \varepsilon V_A(D)=B$, we have in $A\otimes_B A$ that $1\otimes 1=1\otimes\sum d_ix_ke_iy_k=\sum d_ix_ke_i\otimes y_k$, while in $D\otimes_C A$ we have $1\otimes 1=\sum d_ix_ke_iy_k\otimes 1=\sum d_i\otimes x_ke_iy_k$, since $\sum x_ke_iy_k \varepsilon C$. Thus we have (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii). Now assume (ii). Since $\sum d_i\otimes e_i \varepsilon (D\otimes_C D)^D$, we have also $\sum d_iax_ke_i \varepsilon B$ for each $a \varepsilon A$ and k. Hence we can obtain left B-homomorphisms f_k of A to B defined by $f_k(a)=\sum d_iax_ke_i$ for ecah $a \varepsilon A$, which satisfy $a=\sum d_iax_ke_iy_k=\sum f_k(a)y_k$ for each $a \varepsilon A$. Thus $\{y_k, f_k\}$ forms a dual basis for ${}_BA$. Now consider the map η_l of $D\otimes_C A$ to $\mathrm{Hom}({}_BA, {}_BA)$ introduced above. For any $f \varepsilon \mathrm{Hom}({}_BA, {}_BA)$ and $a \varepsilon A$ we have $\eta_l(\sum d_i\otimes x_ke_if(y_k))(a)=\sum d_iax_ke_if(y_k)=\sum f_k(a)f(y_k)=f(\sum f_k(a)y_k)=f(a)$. Hence we have $\eta_l(\sum d_i\otimes x_ke_if(y_k))=f$, which means that η_l is an epimorphism. Next suppose $\sum c_j\otimes a_j \varepsilon \mathrm{Ker}\ \eta_l$. Then $\sum c_jy_ka_j=0$ and $\sum d_i\otimes x_ke_i \varepsilon (D\otimes_C A)^D$ for each k, and we have $\sum c_j\otimes a_j=\sum c_jd_i\otimes x_ke_iy_ka_j=\sum d_i\otimes x_ke_ic_jy_ka_j=0$. Thus η_l is a monomorphism, and we see that η_l is an isomorphism. But D is C-$f.g.$ projective, and A is left B-$f.g.$ projective. Therefore A is an H-separable extension of B by Lemma 2. Thus we have proved (ii)$\Rightarrow$(i). Now we will prove (1) and (2) of the second assertion under the condition of (ii). For any $d \varepsilon D$ we have $d\otimes 1=\sum dd_i\otimes x_ke_iy_k$, and $h(d)=\sum h(dd_i)x_ke_iy_k=\sum x_kh(dd_i)e_iy_k=\sum x_kdy_k \varepsilon C$. Then since $\sum x_kDy_k\subset C$, we have $h=\eta(\sum x_k\otimes y_k) \varepsilon \mathrm{Hom}({}_CD, {}_CC)\cong(A\otimes_B A)^A$. Therefore we have $\sum x_k\otimes y_k \varepsilon (A\otimes_B A)^A$. Since $\sum d_i\otimes e_i \varepsilon (D\otimes_C D)^D$, we can define the map $\tilde{h}$ of A to B $(=V_A(D))$ by $\tilde{h}(x)=\sum d_ixe_i$ for each $x \varepsilon A$. Then $\sum\tilde{h}(ax_k)y_k=\sum d_iax_ke_iy_k=\sum d_ix_ke_iy_ka=a$. Similarly we have $\sum x_k\tilde{h}(y_ka)=a$. Thus $\{x_k, y_k, \tilde{h}\}$ is a Frobenius system of $A|B$.

Proposition 1 *Let A be a Frobenius extension of B with a Frobenius system $\{x_k, y_k, \tilde{h}\}$. Consider the following two conditions;*

(i) *A is an H-separable extension of B.*

(ii) *There exists $\sum d_i\otimes e_i \varepsilon(D\otimes_C D)^D$ with $1\otimes 1=\sum d_i\otimes x_ke_iy_k$ in $D\otimes_C A$*

(i) *always implies* (ii). *If $V_A(D)=B$,* (i) *and* (ii) *are equivalent.*

Proof. Assume (i). By Theorem 1 there exists $\sum d_i\otimes e_i \varepsilon (D\otimes_C D)^D$ such that $1\otimes 1=\sum d_ix_ke_i\otimes y_k$ holds in $A\otimes_B A$. Then since $\sum x_ke_iy_k \varepsilon C$, we have $\sum d_i\otimes x_ke_iy_k=\sum d_ix_ke_iy_k\otimes 1=1\otimes 1$ in $D\otimes_C A$. Thus we have (i)$\Rightarrow$(ii). Next let $B=V_A(D)$, and assume (ii). Then for the completely same reason as the proof of (ii)$\Rightarrow$(i) of Theorem 2 we see that the map η_l of $D\otimes_C A$ to $\mathrm{End}({}_BA)$ is an isomorphism. On the other hand we have $d=\sum dd_ix_ke_iy_k=\sum d_ix_ke_idy_k$ for each $d \varepsilon D$, since $1=\sum d_ix_ke_iy_k$ and $\sum d_ix_ke_i$

$\varepsilon\ V_A(D)$. But $\sum x_k e_i d y_k\ \varepsilon\ C$, since $\sum x_k \otimes y_k\ \varepsilon\ (A\otimes_B A)^A$. Therefore if we define maps g_i by $g_i(d)=\sum x_k e_i d y_k$ for each $d\ \varepsilon\ D$ and each i, we have $g_i\ \varepsilon\ \mathrm{Hom}({}_CD, {}_CC)$ and $d=\sum d_i g_i(d)$ for each $d\ \varepsilon\ D$. Hence D is C-$f.g.$ projective. Then by Lemma 2 we have (i).

As is introduced in [9] when we write $\{A/B,\ S/T\}$, we mean that S is a ring containing A as subring with the common identity, and T is a subring of S containing B. In this case we will always write $\tilde{D}=V_S(T)$ and $\tilde{C}=V_S(S)$, the center of S. There exists also the canonical homomorphism $\tilde{\eta}$ of $S\otimes_T S$ to $\mathrm{Hom}({}_{\tilde{C}}\tilde{D}, {}_{\tilde{C}}S)$ defined by $\tilde{\eta}\ (s\otimes t)(\tilde{d})=s\tilde{d}t$ for $s,\ t\ \varepsilon\ S$ and $\tilde{d}\ \varepsilon\ \tilde{D}$. $\{A/B,\ S/T\}$ is said to have the centralizer property in the case where $V_S(A)=\tilde{C}$, $V_S(B)=\tilde{D}$ and $T=V_S(\tilde{D})$ hold. By the same argument as is stated on page 602 [8] we have the next lemma

Lemma 3 *Let $\{A/B,\ S/T\}$ have the centralizer property, and assume that A is an H-separable extension of B. Then we have*

(1) *The canonical map $i\otimes i$ of $A\otimes_B A$ to $S\otimes_T S$ is a monomorphism, where i is the inclusion map of A to S.*

(2) *If $\tilde{\eta}$ is a monomorphism, then S is an H-separable extension of T, and we have $(i\otimes i)[(A\otimes_B A)^A]\subset(S\otimes_T S)^S$.*

Proof. We will give the proof very briefly following the same lines as page 602 [8]. Since $D\otimes_C\tilde{C}\cong\tilde{D}$ via $d\otimes\tilde{c}\to d\tilde{c}$ for $d\ \varepsilon\ D$ and $\tilde{c}\ \varepsilon\ \tilde{C}$ we have the natural isomorphism ϕ of $\mathrm{Hom}({}_{\tilde{C}}\tilde{D}, {}_{\tilde{C}}S)$ to $\mathrm{Hom}({}_CD, {}_CS)$ such that $\phi\ (f)(d)=f(d)$ for $f\ \varepsilon\ \mathrm{Hom}({}_{\tilde{C}}\tilde{D}, {}_{\tilde{C}}S)$ and $d\ \varepsilon\ D$ and the following commutaive diagram;

$$\begin{array}{ccccc}
A\otimes_B A & & \xrightarrow{\quad\quad\eta\quad\quad} & & \mathrm{Hom}({}_CD, {}_CA) \\
\downarrow i\otimes i & & & & \downarrow i_*=\mathrm{Hom}(D, i) \\
S\otimes_T S & \xrightarrow{\tilde{\eta}} & \mathrm{Hom}({}_{\tilde{C}}\tilde{D}, {}_{\tilde{C}}S) & \xrightarrow{\phi} & \mathrm{Hom}({}_CD, {}_CS)
\end{array}$$

Since $i_*\eta$ is a monomorphism, so is $i\otimes i$. On the other hand we have

$$\begin{aligned}
i_*\eta\ [(A\otimes_B A)^A] &= i_*[(\mathrm{Hom}({}_CD, {}_CA))^A]\subset[\mathrm{Hom}({}_CD, {}_CS)]^A \\
&=\mathrm{Hom}({}_CD, {}_CV_S(A))=\mathrm{Hom}({}_CD, {}_C\tilde{C})=\phi[\mathrm{Hom}({}_{\tilde{C}}\tilde{D}, {}_{\tilde{C}}\tilde{C})].
\end{aligned}$$

Therefore if $\tilde{\eta}$ is a monomorphism, we have $(i\otimes i)[A\otimes_B A)^A]\subset(S\otimes_T S)^S$. Then since $D\subset\tilde{D}$, each H-system of A/B is an H-system of S/T. Thus we have (2).

Theorem 3 *Let $\{A/B, S/T\}$ have the centralizer property, and assume that A is an H-separable Frobenius extension of B with a Frobenius system*

$\{x_k, y_k, \tilde{h}\}$. *Then S is also an H-separable Frobenius extension of T with a Frobenius system* $\{x_k, y_k, \tilde{h}^*\}$ *such that* $\tilde{h}^*|A=\tilde{h}$.

Proof. Since A is left (and right) B-*f. g.* projective, S is an H-separable extension of T by Theorem 1.1 [9]. By Theorem 1 there exists $\sum d_i \otimes e_i \ \varepsilon (D \otimes_C D)^D$ such that $1 \otimes 1 = \sum d_i x_k e_i \otimes y_k$ in $A \otimes_B A$, and $\{d_i, e_i, h\}$ is a Frobenius system of $D|C$, where h is defined by $h(d)=\sum x_k d y_k$ for $d \ \varepsilon \ D$. Then since $\tilde{D}=D\tilde{C}$ and $\sum x_k \otimes y_k \ \varepsilon \ (A \otimes_B A)^A \subset (S \otimes_T S)^S$ by Lemma 3, if we define a map h^* by $h^*(\tilde{d})=\sum x_k \tilde{d} y_k$ for $\tilde{d} \ \varepsilon \ \tilde{D}$, we have $h^* \ \varepsilon$ $\mathrm{Hom}({}_{\tilde{C}}\tilde{D}, {}_{\tilde{C}}\tilde{C})$ with $h^*|D=h$, and $\{d_i, e_i, h^*\}$ forms a Frobenius system of $\tilde{D}|\tilde{C}$. On the other hand we have $1 \otimes 1 = \sum d_i \otimes x_k e_i y_k$ in $D \otimes_C A$ by the proof of (i)$\Rightarrow$(ii) Proposition 1. The same equality holds also in $\tilde{D} \otimes_{\tilde{C}} S$. Then by Theorem 2 we see that $\{x_k, y_k, \tilde{h}^*\}$ is a Frobenius system of $S|T$, where $\tilde{h}^*(x)=\sum d_i x e_i$ for any $x \ \varepsilon \ S$. We have $\tilde{h}^*|A=\tilde{h}$, since $\tilde{h}(x)=\sum d_i x e_i$ for $x \ \varepsilon \ A$ by Theorem 1 (2).

The next proposition which is the improvement of Theorems 4 and 5 [6] is a modification of Theorems 1 and 2 [4] which were proved by using H-system.

Proposition 2 *Let A be an H-separable extension of B. Then we have*

(1) *Assume that A is a Frobenius extension of B with a Frobenius system* $\{x_k, y_k, \tilde{h}\}$, *and define a map h by* $h(d)=\sum x_k d y_k$ *for any* $d \ \varepsilon \ D$. *Then for any* $d_i, e_i \ \varepsilon \ D$ *such that* $\eta_t(\sum d_i \otimes e_i)=\tilde{h}$ $\{d_i, e_i, h\}$ *is a Frobenius system of* $D|C$.

(2) *Assume that D is a Frobenius C-algebra with a Frobenius system* $\{d_i, e_i, h\}$, *and define a map* $\tilde{h}$ *by* $\tilde{h}(x)=\sum d_i x e_i$ *for each* $x \ \varepsilon \ A$. *Then for any* $x_k, y_k \ \varepsilon \ A$ *such that* $\eta\ (\sum x_k \otimes y_k)=h$, $\{x_k, y_k, \tilde{h}\}$ *is a Frobenius system of* $A|B'$, *where* $B'=V_A(V_A(B))$.

Proof. (1). Let $d \ \varepsilon \ D$. We have $\sum d_i h(e_i d)=\sum d_i x_k e_i d y_k=\sum d d_i x_k e_i y_k=\sum d\tilde{h}(x_k)y_k=d$. Similarly we have $d=\sum h(dd_i)e_i$. (2). Let $x \ \varepsilon \ A$. Then we have $\sum x_k \tilde{h}(y_k x)=\sum x_k d_i y_k x e_i=\sum x x_k d_i y_k e_i=\sum x h(d_i) e_i = x$. Similarly we have $x=\sum \tilde{h}(x x_k) y_k$. Obviously $\tilde{h}$ is a B'-B'-map of A to B'.

References

[1] K. HIRATA and K. SUGANO; On semisimple extensions and separable extensions over non comutative rings, J. Math. Soc. Japan, **18** (1966), 360-373.

[2] L. KADISON; Global dimension, tower of algebras, and Jones index of split separable subalgebras with unitality condition, preprint.

[3] Y. MIYASHITA; On Galois extensions and crossed products, Journal Fac. Sci. Hokkaido Univ., **21** (1970), 97-121.

[4] T. NAKAMOTO; On QF-extensions in an H-separable exension, Proc. Japan Accad., **50** (1974), 440-443.
[5] T. ONODERA; Some studies on projective Frobenius extensions, J. Fac. Sci. Hokkaido University, **18** (1964), 89-107.
[6] K. SUGANO; Separable extensions and Frobenius extensions, Osaka J. Math., **7** (1970), 291-299.
[7] K. SUGANO; Note on separability of endomorphism rings, J. Fac. Sci. Hokkaido University, **21** (1971), 196-208.
[8] K. SUGANO; On bicommutators of modules over H-separable extension rings II, Hokkaido Math. J., **20** (1991), 601-608.
[9] K. SUGANO; On bicommutators of modules over H-separable extension rings III, Hokkaido Math. J., **23** (1994), 277-289.

Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060
Japan