

## Quaternionic CR geometry

*Dedicated to Professor Seiki Nishikawa on his sixtieth birthday*

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**Abstract.** Modelled on a real hypersurface in a quaternionic manifold, we introduce a quaternionic analogue of CR structure, called quaternionic CR structure. We define the strong pseudoconvexity of this structure as well as the notion of quaternionic pseudohermitian structure. Following the construction of the Tanaka-Webster connection in complex CR geometry, we construct a canonical connection associated with a quaternionic pseudohermitian structure, when the underlying quaternionic CR structure satisfies the ultra-pseudoconvexity which is stronger than the strong pseudoconvexity. Comparison to Biquard's quaternionic contact structure [4] is also made.

*Key words:* hyper CR structure, quaternionic CR structure, pseudohermitian structure, ultra-pseudoconvex, canonical connection.

### Introduction

A CR structure is a corank one subbundle of the tangent bundle of an odd dimensional manifold, equipped with a complex structure. Such a structure typically arises on a real hypersurface of a complex manifold. Assuming that the CR structure is strongly pseudoconvex, the underlying subbundle defines a contact structure on the manifold, and a strongly pseudoconvex CR structure together with a choice of contact form is called a pseudohermitian structure. Associated with a pseudohermitian structure, there is a hermitian metric on the subbundle, called the Levi form. It is the simplest and most important example of Carnot-Carathéodory metric. In pseudohermitian geometry the so-called Tanaka-Webster connection [17], [19] plays the role of the Levi-Civita connection in Riemannian geometry. Multiplying the contact form by a nowhere vanishing function gives another pseudohermitian structure, and accordingly the Levi form changes confor-

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mally, being multiplied by the same function. CR geometry thereby has a nature of conformal geometry, and in particular, CR invariants can be computed as those pseudohermitian invariants which are independent of the choice of contact form.

In this paper, we shall introduce quaternionic analogues of CR and pseudohermitian structures and lay the foundation of the geometry of these structures. We introduce two kinds of quaternionic analogues of CR structure, with one refining the other. An almost hyper CR structure is defined on a manifold of dimension  $4n + 3$ , as a pair of almost CR structures whose underlying subbundles are transversal to each other and whose complex structures are anti-commuting in an appropriate sense. Then the third almost CR structure can be defined, and a corank three subbundle is defined as the intersection of the three corank one subbundles. The three complex structures leave this bundle invariant, and satisfy the quaternion relations there. An almost hyper CR structure which satisfies a certain integrability condition is called a hyper CR structure. Analogously to the complex CR case, any real hypersurface of a hypercomplex manifold has a natural hyper CR structure (satisfying a stronger integrability condition). A hyper CR structure exists also on a real hypersurface of a quaternionic manifold, but only locally. In order to have a global structure on any real hypersurface of a quaternionic manifold, we refine the notion of hyper CR structure. A quaternionic CR structure is a covering of a manifold by local hyper CR structures, satisfying a certain gluing condition on each domain where two such local structures overlap.

Associated with a hyper CR structure, there is a distinguished  $\mathbb{R}^3$ -valued one-form, unique up to multiplication by nowhere-vanishing real-valued functions. For a choice of such one-form, the corresponding Levi form is a quaternionic hermitian form on the corank three subbundle. The strong pseudoconvexity of a hyper CR structure and the pseudohermitian structure are then defined exactly as in the complex CR case. On the other hand, the definition of the Levi form itself is not quite similar to that in the complex CR case, and this is the point where the integrability of the hyper CR structure does play the crucial role. These notions of Levi form, strong pseudoconvexity and pseudohermitian structure introduced for the hyper CR structure extend to the quaternionic CR structure.

With these structures at hand, our main concern is whether there exists a quaternionic analogue of the Tanaka-Webster connection, associated with

a quaternionic pseudohermitian structure. In the hyper pseudohermitian case, our search for such a canonical connection proceeds as follows. There is a distinguished family of three-plane fields transverse to the corank three subbundle and parametrized by sections of the corank three subbundle. We choose such a three-plane field, and use it to define a one-parameter family of Riemannian metrics on the manifold, extending the Levi form on the corank three subbundle. We then construct an affine connection, characterized by the property that it has the smallest torsion among those affine connections with respect to which the above Riemannian metrics are all parallel.

What remains to be done is to determine the transverse three-plane field so that the corresponding connection be best adapted to the structure under consideration in an appropriate sense. When a hyper pseudohermitian structure is given, the corank three subbundle associated with the underlying hyper CR structure comes equipped with an  $Sp(n)$ -structure. We will, however, be moderate by regarding the bundle as an  $Sp(n) \cdot Sp(1)$ -bundle, for the sake of later generalization of the construction to the quaternionic CR case. Then the best possible one can expect is that the connection restricts to an  $Sp(n) \cdot Sp(1)$ -connection on the bundle. Since this turns out not to be possible in general, we will be contented by requiring that the connection be “as close to an  $Sp(n) \cdot Sp(1)$ -connection as possible.” The last expression will be made explicit by using the representation theory for  $Sp(n) \cdot Sp(1)$ . Note that for this strategy to work, we must primarily assume that  $n \geq 2$ , that is, the dimension of the underlying manifold is greater than seven; when  $n = 1$ , since  $Sp(1) \cdot Sp(1) = SO(4)$ , any orthogonal connection on the oriented corank three bundle should necessarily be an  $Sp(1) \cdot Sp(1)$ -connection. It turns out that we must also assume that the hyper CR structure is what we call ultra-pseudoconvex. When these assumptions are satisfied, the above strategy completely works and thereby gives a connection in search. Note that the class of hyper CR structures which are ultra-pseudoconvex contains all strictly convex real hypersurfaces in  $\mathbb{H}^{n+1}$ . The notion of ultra-pseudoconvexity and the construction of the canonical connection extend to the quaternionic CR case. It remains to see whether a canonical connection can be constructed when the dimension of the underlying manifold is seven. We will address this problem in a future work.

It should be mentioned that several quaternionic analogues of CR structures other than those in this paper have been introduced and studied

by Hernandez [7], Biquard [4], Alekseevsky-Kamishima [1], [2] and others. Among them, Biquard's quaternionic contact structure is most influential and extensively studied. We therefore compare our quaternionic CR structure to the quaternionic contact structure. We observe that while a quaternionic contact structure can always be "extended" to a quaternionic CR structure, the quaternionic contact structure is more restrictive than the quaternionic CR structure. Indeed, we characterize, in terms of the Levi forms, a quaternionic CR structure whose underlying corank three subbundle has compatible quaternionic contact structure. We also give explicit examples of quaternionic CR manifolds which do not satisfy the characterizing condition.

More recently, Duchemin [6] introduced the notion of weakly quaternionic contact structure, generalizing that of quaternionic contact structure. He showed that a real hypersurface in a quaternionic manifold admitted a canonical weakly quaternionic contact structure. More generally, one easily verifies that a quaternionic CR structure naturally produces a weakly quaternionic contact structure. As mentioned in [6, Section 6], the construction of a canonical connection for a weakly quaternionic contact structure, generalizing the so-called Biquard connection for a quaternionic contact structure, remains to be done.

This paper is organized as follows: In Section 1, we introduce the definitions of hyper and quaternionic CR structures as well as hyper and quaternionic pseudohermitian structures. In Section 2, we give examples of hyper and quaternionic CR manifolds. In Section 3, we construct a canonical connection associated with a hyper/quaternionic pseudohermitian structure, when the underlying hyper/quaternionic CR structure is ultra-pseudoconvex. The proof of a technical lemma is postponed to Section 4. Comparison to Biquard's quaternionic contact structure is made in Section 5. In Appendix, we give proofs of some fundamental facts which are stated in Section 1.

The main contents of an earlier version of this paper were announced in [13]. However, some significant changes have been made in the present manuscript. Among others, we modified the definition of the integrability of hyper CR structure. The former definition required for each of the three CR structures constituting a hyper CR structure to be integrable as a CR structure. As pointed out by the referee, this definition had the following demerit: if one has a candidate for a quaternionic CR structure,

e.g., a real hypersurface in a quaternionic manifold, one cannot tell whether there are integrable choices of local hyper CR structures inducing it, unless the quaternionic manifold is e.g., a hypercomplex manifold. As mentioned above, under the new definition of integrability, any real hypersurface in a quaternionic manifold has a natural quaternionic CR structure.

Some words on notation. Throughout this paper, the triple of indices  $(a, b, c)$  always stands for a cyclic permutation of  $(1, 2, 3)$ , unless otherwise stated.

### 1. Hyper & quaternionic CR structures and strong pseudoconvexity

We start with a brief review of CR structure. Let  $M$  be an orientable manifold of real dimension  $2n + 1$ . An *almost CR structure* on  $M$  is given by a corank one subbundle  $Q$  of  $TM$ , the tangent bundle of  $M$ , together with a complex structure  $J : Q \rightarrow Q$ . Let  $Q^{1,0} = \{Z \in Q \otimes \mathbb{C} \mid JZ = \sqrt{-1}Z\}$ ; it is a complex rank  $n$  subbundle of  $TM \otimes \mathbb{C}$  satisfying  $Q^{1,0} \cap \overline{Q^{1,0}} = \{0\}$ . The bundle  $Q^{1,0}$  recovers  $Q$  and  $J$  by  $Q = \text{Re}(Q^{1,0} \oplus \overline{Q^{1,0}})$  and  $J(Z + \overline{Z}) = \sqrt{-1}(Z - \overline{Z})$  for  $Z \in Q^{1,0}$ , respectively. A *CR structure* is an almost CR structure satisfying the integrability condition  $[\Gamma(Q^{1,0}), \Gamma(Q^{1,0})] \subset \Gamma(Q^{1,0})$ , or equivalently

$$[X, Y] - [JX, JY], \quad [X, JY] + [JX, Y] \in \Gamma(Q) \tag{1.1}$$

and

$$J([X, Y] - [JX, JY]) = [X, JY] + [JX, Y] \tag{1.2}$$

for all  $X, Y \in \Gamma(Q)$ . An almost CR structure is said to be *partially integrable* if it satisfies (1.1), which is equivalent to the condition  $[\Gamma(Q^{1,0}), \Gamma(Q^{1,0})] \subset \Gamma(Q \otimes \mathbb{C})$ .

Let  $M$  be an almost CR manifold, and  $\theta$  a one-form on  $M$  whose kernel is the bundle of hyperplanes  $Q$ . Such a  $\theta$  exists globally, since we assume  $M$  is orientable, and  $Q$  is oriented by its complex structure. Associated with  $\theta$  is a form  $\text{Levi}_\theta$  on  $Q$ , defined by

$$\text{Levi}_\theta(X, Y) = d\theta(X, JY), \quad X, Y \in Q,$$

and called the *Levi form* of  $\theta$ . If the almost CR structure is partially integrable, then  $\text{Levi}_\theta$  is symmetric and  $J$ -invariant. If  $\theta$  is replaced by  $\theta' = \lambda\theta$  for a function  $\lambda \neq 0$ , then  $\text{Levi}_\theta$  changes conformally by  $\text{Levi}_{\theta'} = \lambda \text{Levi}_\theta$ . An almost CR structure is said to be *strongly pseudoconvex* if it is partially integrable and  $\text{Levi}_\theta$  is positive or negative definite for some (hence any) choice of  $\theta$ . In this case,  $Q$  gives a contact structure on  $M$ , and  $\theta$  is a contact form.

A CR structure typically arises on a real hypersurface  $M$  of a complex manifold (of complex dimension  $n + 1$ ). In this case  $Q = TM \cap \mathcal{J}(TM)$  and  $J = \mathcal{J}|_Q$ , where  $\mathcal{J}$  is the complex structure of the ambient complex manifold. If  $\rho$  is a defining function for  $M$ , then  $\theta = -\mathcal{J}(d\rho)/2$  annihilates  $Q$ .

A *pseudohermitian structure* on  $M$  is a strongly pseudoconvex almost CR structure together with a choice of  $\theta$  such that  $\text{Levi}_\theta$  is positive definite. As  $\theta$  is a contact form, it is accompanied by the corresponding Reeb field  $T$ , determined by the equations  $\theta(T) = 1$  and  $d\theta(T, \cdot) = 0$ .

We now introduce a quaternionic analogue of CR structure.

**Definition 1.1** Let  $M$  be a connected, orientable manifold of dimension  $4n+3$ . An *almost hyper CR structure* on  $M$  is a pair of almost CR structures  $(Q_1, I)$  and  $(Q_2, J)$  which satisfies the following conditions:

- (i)  $Q_1$  and  $Q_2$  are transversal to each other;
- (ii) the relation  $IJ = -JI$  holds on  $I(Q_1 \cap Q_2) \cap J(Q_1 \cap Q_2)$ , the maximal domain on which the both sides make sense.

We define the third almost CR structure  $(Q_3, K)$  as follows. Set

$$Q_3 = I(Q_1 \cap Q_2) + J(Q_1 \cap Q_2) \quad \text{and} \quad K = \begin{cases} -JI & \text{on } I(Q_1 \cap Q_2), \\ IJ & \text{on } J(Q_1 \cap Q_2). \end{cases}$$

Then  $Q_3$  is a corank one subbundle of  $TM$ , and  $K$  is well-defined and satisfies the equation  $K^2 = -\text{Id}$ . By the condition (ii),  $Q_3$  is transversal to both  $Q_1$  and  $Q_2$ . Moreover, the following relations hold:

$$\begin{aligned} I(Q_1 \cap Q_2) &= Q_1 \cap Q_3, & J(Q_2 \cap Q_3) &= Q_2 \cap Q_1, & K(Q_3 \cap Q_1) &= Q_3 \cap Q_2; \\ IJ &= K \text{ on } Q_2 \cap Q_3, & JI &= -K \text{ on } Q_1 \cap Q_3, & JK &= I \text{ on } Q_3 \cap Q_1, \\ KJ &= -I \text{ on } Q_2 \cap Q_1, & KI &= J \text{ on } Q_1 \cap Q_2, & IK &= -J \text{ on } Q_3 \cap Q_2. \end{aligned}$$

Set  $Q = \cap_{a=1}^3 Q_a$ . It is a corank three subbundle of  $TM$ , and has three complex structures  $I, J, K$  satisfying the quaternion relations. Henceforth, we shall write  $I_1 = I, I_2 = J$  and  $I_3 = K$  when appropriate.

**Definition 1.2** A triple  $(T_1, T_2, T_3)$  of vector fields transverse to the subbundle  $Q$  is called an *admissible triple* if it satisfies the following conditions:

$$(i) T_a \in \Gamma(Q_b \cap Q_c); \quad (ii) I_a T_b = T_c.$$

We have

$$Q_a = Q \oplus \mathbb{R}T_b \oplus \mathbb{R}T_c,$$

$$TM = Q_a \oplus \mathbb{R}T_a = Q \oplus \mathbb{R}T_1 \oplus \mathbb{R}T_2 \oplus \mathbb{R}T_3.$$

We call  $Q^\perp = \oplus_{a=1}^3 \mathbb{R}T_a$  an *admissible three-plane field*.

Note that an admissible triple  $(T_1, T_2, T_3)$  certainly exists. Indeed, take  $T_1 \in \Gamma(Q_2 \cap Q_3)$  such that  $(T_1)_q \notin Q_q$  ( $\Leftrightarrow (T_1)_q \notin (Q_1)_q$ ) for all  $q \in M$ . Such a  $T_1$  exists globally since  $Q_2 \cap Q_3$  is orientable and  $Q$  is oriented by its complex structures. Now it suffices to set  $T_2 = KT_1$  and  $T_3 = IT_2$ .

We shall next define an almost CR structure  $(Q_{\mathbf{v}}, I_{\mathbf{v}})$  for each unit vector  $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ . Roughly speaking,  $I_{\mathbf{v}}$  is defined to be  $v_1I + v_2J + v_3K$ , which, however, makes sense only on  $Q$ . We rectify this defect by proceeding as follows. Let  $(T_1, T_2, T_3)$  be an admissible triple, and extend  $I_1, I_2, I_3$  to endomorphisms  $\tilde{I}_a: TM \rightarrow TM$  by setting  $\tilde{I}_a T_a = 0$ , and define  $(Q_{\mathbf{v}}, I_{\mathbf{v}})$  by

$$Q_{\mathbf{v}} = Q \oplus \left\{ x_1 T_1 + x_2 T_2 + x_3 T_3 \mid x_1, x_2, x_3 \in \mathbb{R}, \sum_{a=1}^3 x_a v_a = 0 \right\}, \quad (1.3)$$

$$I_{\mathbf{v}} = (v_1 \tilde{I}_1 + v_2 \tilde{I}_2 + v_3 \tilde{I}_3)|_{Q_{\mathbf{v}}}. \quad (1.4)$$

It is easy to verify that  $I_{\mathbf{v}}$  indeed preserves  $Q_{\mathbf{v}}$ , satisfies the equation  $I_{\mathbf{v}}^2 = -\text{Id}$ , and  $(Q_{\mathbf{v}}, I_{\mathbf{v}})$  is independent of the particular choice of admissible triple. Thus, associated with an almost hyper CR structure, there is a canonical family of almost CR structures parametrized by the unit sphere  $S^2$ .

Note that the above construction of the almost CR structure  $(Q_{\mathbf{v}}, I_{\mathbf{v}})$  may be performed pointwise. Therefore, as  $\mathbf{v}$ , we can also take a variable function with values in  $S^2$ .

There are two possible ways to define when a diffeomorphism between two almost hyper CR manifolds is an isomorphism. One way is to require that the diffeomorphism preserves each of the two almost CR structures constituting the almost hyper CR structure. The other is to require that it preserves the  $S^2$ -family of almost CR structures constructed above. We will find the effect of this difference when we investigate the automorphisms of the sphere  $S^{4n+3}$  (Section 2, Example 1).

**Definition 1.3** An  $\mathbb{R}^3$ -valued one-form  $\theta = (\theta_1, \theta_2, \theta_3)$  on an almost hyper CR manifold  $M$  is said to be *compatible* with the almost hyper CR structure if it satisfies

$$\begin{aligned} \ker \theta_a &= Q_a, \quad a = 1, 2, 3, \\ \theta_3 \circ I &= \theta_2 \text{ on } Q_1, \quad \theta_1 \circ J = \theta_3 \text{ on } Q_2, \quad \theta_2 \circ K = \theta_1 \text{ on } Q_3. \end{aligned} \quad (1.5)$$

Note that such a  $\theta$  exists; it is enough to take an admissible triple  $(T_1, T_2, T_3)$  and choose  $\theta_a$  annihilating  $Q_a$  so that  $\theta_a(T_a)$  are nonzero and equal to each other (e.g.,  $\theta_a(T_a) = 1$ ). It is unique up to multiplication by a nowhere vanishing, real-valued function.

In order to define a quaternionic analogue of Levi form, we require that our almost hyper CR structure should satisfy some sort of integrability condition.

**Definition 1.4** An almost hyper CR structure is said to be *integrable* if it satisfies the following conditions for  $a = 1, 2, 3$  and for all  $X, Y \in \Gamma(Q)$ :

$$[X, Y] - [I_a X, I_a Y] \in \Gamma(Q_a); \quad (1.6)$$

$$I_a([X, Y] - [I_a X, I_a Y]) - [X, I_a Y] - [I_a X, Y] \in \Gamma(Q). \quad (1.7)$$

Henceforth, we shall assume throughout that our almost hyper CR structure is integrable and refer to it as a *hyper CR structure*.

**Remark 1** The integrability conditions (1.6), (1.7) are natural ones, as they are satisfied by the local hyper CR structure of any real hypersurface in a quaternionic manifold. See Section 2 for details.

When a hyper CR structure is given, we can show that for any  $S^2$ -valued function  $\mathbf{v}$ , the almost CR structure  $(Q_{\mathbf{v}}, I_{\mathbf{v}})$  defined above satisfies (appropriately modified versions of) (1.6), (1.7) (see Proposition 6.1 in

Appendix 6.1).

Let  $M$  be a hyper CR manifold and  $\theta = (\theta_1, \theta_2, \theta_3)$  a compatible  $\mathbb{R}^3$ -valued one-form on  $M$ . Note that by (1.6) we have

$$d\theta_a(X, Y) = d\theta_a(I_a X, I_a Y) \tag{1.8}$$

for  $a = 1, 2, 3$  and for all  $X, Y \in \Gamma(Q)$ . Moreover, we have the following identity for  $X, Y \in Q$ :

$$\begin{aligned} d\theta_1(X, IY) + d\theta_1(JX, KY) &= d\theta_2(X, JY) + d\theta_2(KX, IY) \\ &= d\theta_3(X, KY) + d\theta_3(IX, JY). \end{aligned} \tag{1.9}$$

Indeed, by plugging the both sides of (1.7) with  $a = 1$  in  $\theta_3$  and using (1.5), we obtain

$$\theta_2([X, Y] - [IX, IY]) = \theta_3([X, IY] + [IX, Y]),$$

where  $X, Y$  are extended to sections of  $Q$ . Therefore,

$$d\theta_2(X, Y) - d\theta_2(IX, IY) = d\theta_3(X, IY) + d\theta_3(IX, Y).$$

Replacing  $Y$  by  $JY$  and using (1.8), we obtain the second equality of (1.9). We now define  $\text{Levi}_\theta(X, Y)$ , the *Levi form* of  $\theta$ , to be the half of this common quantity:

$$\begin{aligned} \text{Levi}_\theta(X, Y) &= \frac{1}{2}(d\theta_1(X, IY) + d\theta_1(JX, KY)) \\ &= \frac{1}{2}(d\theta_2(X, JY) + d\theta_2(KX, IY)) \\ &= \frac{1}{2}(d\theta_3(X, KY) + d\theta_3(IX, JY)). \end{aligned}$$

Note that  $\text{Levi}_\theta$  is nothing but the quaternion-hermtian (that is, symmetric and invariant under  $I, J, K$ ) part of the “complex” Levi form  $\text{Levi}_{\theta_a} = d\theta_a(\cdot, I_a \cdot)$  restricted to  $Q$ . If  $\theta$  is replaced by  $\theta' = \lambda\theta$ ,  $\lambda \neq 0$ , then  $\text{Levi}_\theta$  changes conformally by  $\text{Levi}_{\theta'} = \lambda \text{Levi}_\theta$ .

**Definition 1.5** We say that a hyper CR structure is *strongly pseudoconvex*

if  $\text{Levi}_\theta$  is positive or negative definite for some (hence any) choice of  $\theta$ . A *hyper pseudohermitian structure* is a strongly pseudoconvex hyper CR structure together with a choice of  $\theta$  such that  $\text{Levi}_\theta$  is positive definite. We also call, by abuse, such a  $\theta$  a *pseudohermitian structure*.

We now introduce another quaternionic analogue of CR structure.

**Definition 1.6** A *quaternionic CR structure* on  $M$  is a covering of  $M$  by local hyper CR structures which satisfies the following condition: let  $\{(Q_a, I_a)\}_{a=1,2,3}$  and  $\{(Q'_a, I'_a)\}_{a=1,2,3}$  be two such local structures defined on open subsets  $U$  and  $U'$  respectively. If  $U \cap U' \neq \emptyset$ , there is an  $SO(3)$ -valued function  $S = S_{UU'}: U \cap U' \rightarrow SO(3)$  such that

$$Q'_\mathbf{v} = Q_{S\mathbf{v}}, \quad I'_\mathbf{v} = I_{S\mathbf{v}}, \quad \mathbf{v} \in S^2, \quad (1.10)$$

where the notation is as in (1.3), (1.4). (Note that  $S\mathbf{v}$  is a variable function of  $q \in U \cap U'$ .)

There is a double covering  $Sp(1) \rightarrow SO(3)$ , and if  $S$  can be lifted to an  $Sp(1)$ -valued function  $\sigma: U \cap U' \rightarrow Sp(1)$ , which is the case when  $U \cap U'$  is simply-connected, then (1.10) may be written as

$$Q'_\mathbf{v} = Q_{\sigma^{-1}\mathbf{v}\sigma}, \quad I'_\mathbf{v} = I_{\sigma^{-1}\mathbf{v}\sigma}, \quad \mathbf{v} \in S^2. \quad (1.11)$$

Here,  $\sigma^{-1}\mathbf{v}\sigma$  is computed by regarding  $\mathbf{v}$  as an imaginary quaternion via the identification  $\mathbb{R}^3 = \text{Im } \mathbb{H}$ . We adopt the convention that the atlas defining a quaternionic CR structure is extended to a maximal one. In particular, any (global) hyper CR structure canonically determines a quaternionic CR structure. Henceforth, we shall regard a hyper CR manifold as equipped with this quaternionic CR structure.

Given a quaternionic CR structure, there are local corank three bundles  $Q_U$  associated with the local hyper CR structures. But  $Q_U = Q_{U'}$  on  $U \cap U'$ , and they give rise to a bundle  $Q$  defined globally on  $M$ .

Let  $\Theta = \{\theta_U\}$  be a collection of local  $\mathbb{R}^3$ -valued one-forms compatible with the local hyper CR structures such that

$$(\theta_{U'})_a = \sum_{b=1}^3 s_{ab}(\theta_U)_b, \quad a = 1, 2, 3 \quad (1.12)$$

on  $U \cap U'$ , where  $S = (s_{ab})$  is the  $SO(3)$ -valued function as in the definition above. Such a collection exists, and it is unique up to multiplication by a nowhere vanishing, real-valued function. Associated with  $\theta_U$  are the local Levi forms  $\text{Levi}_{\theta_U}$ , for which we have the following

**Proposition 1.7** *Let  $M$  be a quaternionic CR manifold, and  $\Theta = \{\theta_U\}$  a collection of local  $\mathbb{R}^3$ -valued one-forms on  $M$  as above. Then the local Levi forms  $\text{Levi}_{\theta_U}$  and  $\text{Levi}_{\theta_{U'}}$  coincide on  $U \cap U'$ .*

The proof of this proposition will be given in Appendix 6.2. By Proposition 1.7, we obtain a globally defined symmetric bilinear form, denoted by  $\text{Levi}_\Theta$ , and call it the *Levi form* of  $\Theta$ . Using this we define the *strong pseudoconvexity* of a quaternionic CR structure as before. A *quaternionic pseudohermitian structure* is a strongly pseudoconvex quaternionic CR structure together with a choice of a collection  $\Theta$  such that  $\text{Levi}_\Theta$  is positive definite. Again, by abuse, such a collection  $\Theta$  is called a *pseudohermitian structure*.

Each fibre of  $Q$  has a family of complex structures parametrized by the two-sphere  $S^2$  with no preferred choice of triple satisfying the quaternion relations. This amounts to saying that the bundle  $Q$  has a  $GL(n, \mathbb{H}) \cdot \mathbb{H}^*$ -structure, where  $GL(n, \mathbb{H}) \cdot \mathbb{H}^* = GL(n, \mathbb{H}) \times Sp(1)/\{\pm I_{n+1}\}$ . A choice of a pseudohermitian structure  $\Theta = \{\theta_U\}$  gives  $Q$  a fibre metric  $\text{Levi}_\Theta$ , and it is invariant under any of the complex structures on the fibre. Thus the choice of  $\Theta$  reduces the structure group of  $Q$  from  $GL(n, \mathbb{H}) \cdot \mathbb{H}^*$  to  $Sp(n) \cdot Sp(1) = Sp(n) \times Sp(1)/\{\pm I_{n+1}\}$ .

## 2. Real hypersurface and examples

### 2.1. Real hypersurface

Let  $\mathcal{N}$  be a quaternionic manifold of dimension  $4n + 4$ ; thus  $\mathcal{N}$  admits a torsion-free  $GL(n + 1, \mathbb{H}) \cdot \mathbb{H}^*$ -affine connection  $\mathcal{D}$ . Then, in a neighborhood  $\mathcal{U}$  of any point of  $\mathcal{N}$ , there exist almost complex structures  $\mathcal{I}_a$ ,  $a = 1, 2, 3$ , which satisfy  $\mathcal{D}\mathcal{I}_a = \sum_{b=1}^3 \gamma_{ab} \otimes \mathcal{I}_b$ , where  $\gamma_{ab}$  are one-forms on  $\mathcal{U}$  satisfying  $\gamma_{ab} = -\gamma_{ba}$ . Let  $M$  be a real hypersurface in  $\mathcal{N}$ . Then  $M$  comes equipped with a quaternionic CR structure in a canonical manner, by setting  $U = M \cap \mathcal{U}$  for each  $\mathcal{U}$  as above and defining  $Q_a = TU \cap \mathcal{I}_a(TU)$  and  $I_a = \mathcal{I}_a|_{Q_a}$ . Thus the corank three subbundle  $Q$  is given by  $Q|_U = TU \cap \mathcal{I}_1(TU) \cap \mathcal{I}_2(TU) \cap \mathcal{I}_3(TU)$ . It remains to verify that the integrability conditions (1.6), (1.7) hold for all  $X, Y \in \Gamma(Q|_U)$ . To see that (1.6) holds, it suffices to

show

$$\mathcal{I}_a([X, Y] - [I_a X, I_a Y]) \in \Gamma(TU). \quad (2.1)$$

For this, let  $\mathcal{X}, \mathcal{Y} \in \Gamma(TU)$  be local extensions of  $X, Y$  respectively, and compute

$$\begin{aligned} & \mathcal{I}_a([\mathcal{X}, \mathcal{Y}] - [\mathcal{I}_a \mathcal{X}, \mathcal{I}_a \mathcal{Y}]) \\ &= \mathcal{I}_a(\mathcal{D}_{\mathcal{X}} \mathcal{Y} - \mathcal{D}_{\mathcal{Y}} \mathcal{X} - \mathcal{D}_{\mathcal{I}_a \mathcal{X}} \mathcal{I}_a \mathcal{Y} + \mathcal{D}_{\mathcal{I}_a \mathcal{Y}} \mathcal{I}_a \mathcal{X}) \\ &= \mathcal{D}_{\mathcal{X}} \mathcal{I}_a \mathcal{Y} - \sum_b \gamma_{ab}(\mathcal{X}) \mathcal{I}_b \mathcal{Y} - \mathcal{D}_{\mathcal{Y}} \mathcal{I}_a \mathcal{X} + \sum_b \gamma_{ab}(\mathcal{Y}) \mathcal{I}_b \mathcal{X} \\ &\quad + \mathcal{D}_{\mathcal{I}_a \mathcal{X}} \mathcal{Y} - \sum_b \gamma_{ab}(\mathcal{I}_a \mathcal{X}) \mathcal{I}_a \mathcal{I}_b \mathcal{Y} - \mathcal{D}_{\mathcal{I}_a \mathcal{Y}} \mathcal{X} + \sum_b \gamma_{ab}(\mathcal{I}_a \mathcal{Y}) \mathcal{I}_a \mathcal{I}_b \mathcal{X} \\ &= [\mathcal{X}, \mathcal{I}_a \mathcal{Y}] + [\mathcal{I}_a \mathcal{X}, \mathcal{Y}] - \sum_b \gamma_{ab}(\mathcal{X}) \mathcal{I}_b \mathcal{Y} + \sum_b \gamma_{ab}(\mathcal{Y}) \mathcal{I}_b \mathcal{X} \\ &\quad - \sum_b \gamma_{ab}(\mathcal{I}_a \mathcal{X}) \mathcal{I}_a \mathcal{I}_b \mathcal{Y} + \sum_b \gamma_{ab}(\mathcal{I}_a \mathcal{Y}) \mathcal{I}_a \mathcal{I}_b \mathcal{X}. \end{aligned}$$

Restricted to  $U$ , this shows

$$\mathcal{I}_a([X, Y] - [I_a X, I_a Y]) = [X, I_a Y] + [I_a X, Y] \quad \text{mod } \Gamma(Q|_U).$$

Therefore, (2.1) holds, and the second condition (1.7) for integrability is also verified.

We now restrict ourselves to the case that the quaternionic manifold  $\mathcal{N}$  is a quaternionic affine space  $\mathbb{H}^{n+1}$ . Let  $M$  be a (local) real hypersurface in  $\mathbb{H}^{n+1}$ , and  $\rho$  a defining function for  $M$ :  $M = \rho^{-1}(0)$ ,  $d\rho \neq 0$  along  $M$ . Note that the tangent spaces of  $M$  are given by  $T_q M = \{X \in \mathbb{H}^{n+1} \mid d\rho_q(X) = 0\}$ ,  $q \in M$ . Here and throughout, the tangent spaces  $T_q \mathbb{H}^{n+1}$  are identified with  $\mathbb{H}^{n+1}$  in the standard manner. For each unit imaginary quaternion  $\mathbf{v}$ , a parallel complex structure  $\mathcal{I}_{\mathbf{v}}$  on  $\mathbb{H}^{n+1}$  is defined by  $\mathcal{I}_{\mathbf{v}} X = X \mathbf{v}^{-1}$ ,  $X \in \mathbb{H}^{n+1}$ . Thus there is a family of complex structures on  $\mathbb{H}^{n+1}$  parametrized by  $S^2$ , the unit sphere in  $\text{Im } \mathbb{H} = \mathbb{R}^3$ . Each of these complex structures,  $\mathcal{I}_{\mathbf{v}}$ , determines an (integrable) CR structure  $(Q_{\mathbf{v}}, I_{\mathbf{v}})$  on  $M$ , where  $Q_{\mathbf{v}} = TM \cap \mathcal{I}_{\mathbf{v}}(TM)$  and  $I_{\mathbf{v}} = \mathcal{I}_{\mathbf{v}}|_{Q_{\mathbf{v}}}$ . In particular, the CR structures  $(Q_1, I) = (Q_{\mathbf{i}}, I_{\mathbf{i}})$ ,  $(Q_2, J) = (Q_{\mathbf{j}}, I_{\mathbf{j}})$  and  $(Q_3, K) = (Q_{\mathbf{k}}, I_{\mathbf{k}})$  define a hyper CR structure on

$M$ . Note that the condition that the three CR structures are all integrable is stronger than the integrability conditions (1.6), (1.7).

For the computation below, it is convenient to introduce the complex coordinates  $z_h = x_h^0 + \sqrt{-1}x_h^1$ ,  $w_h = x_h^2 + \sqrt{-1}x_h^3$ ,  $h = 1, \dots, n + 1$ , where  $q = (q_1, \dots, q_{n+1})$  and  $q_h = x_h^0 - x_h^1 \mathbf{i} - x_h^2 \mathbf{j} - x_h^3 \mathbf{k}$ . We have

$$\begin{aligned} \mathcal{I}(dz_h) &= \sqrt{-1}dz_h, \quad \mathcal{I}(dw_h) = \sqrt{-1}dw_h; \\ \mathcal{J}(dz_h) &= -d\bar{w}_h, \quad \mathcal{J}(dw_h) = d\bar{z}_h; \\ \mathcal{K}(dz_h) &= -\sqrt{-1}d\bar{w}_h, \quad \mathcal{K}(dw_h) = \sqrt{-1}d\bar{z}_h. \end{aligned} \tag{2.2}$$

As in the complex CR case, let  $\theta_a = -\mathcal{I}_a(d\rho)/2$ ,  $a = 1, 2, 3$ . Then  $\theta = (\theta_1, \theta_2, \theta_3)$  is compatible with the hyper CR structure of  $M$ . In the complex coordinates,

$$\begin{aligned} \theta_1 &= \frac{\sqrt{-1}}{2} \sum_{h=1}^{n+1} \left( -\frac{\partial \rho}{\partial z_h} dz_h + \frac{\partial \rho}{\partial \bar{z}_h} d\bar{z}_h - \frac{\partial \rho}{\partial w_h} dw_h + \frac{\partial \rho}{\partial \bar{w}_h} d\bar{w}_h \right), \\ \theta_2 + \sqrt{-1}\theta_3 &= \sum_{h=1}^{n+1} \left( -\frac{\partial \rho}{\partial w_h} dz_h + \frac{\partial \rho}{\partial \bar{z}_h} dw_h \right). \end{aligned}$$

One can verify that the Levi form of  $\theta$  is given by

$$\begin{aligned} \text{Levi}_\theta &= \sum_{h,l=1}^{n+1} \left[ \left( \frac{\partial^2 \rho}{\partial z_h \partial \bar{z}_l} + \frac{\partial^2 \rho}{\partial \bar{w}_h \partial w_l} \right) (dz_h \cdot d\bar{z}_l + d\bar{w}_h \cdot dw_l) \right. \\ &\quad \left. + \frac{\partial^2 \rho}{\partial z_h \partial \bar{w}_l} (dz_h \cdot d\bar{w}_l - d\bar{w}_h \cdot dz_l) \right. \\ &\quad \left. + \frac{\partial^2 \rho}{\partial \bar{z}_h \partial w_l} (d\bar{z}_h \cdot dw_l - dw_h \cdot d\bar{z}_l) \right]. \end{aligned} \tag{2.3}$$

**Example 1** Let  $S^{4n+3} = \{q \in \mathbb{H}^{n+1} \mid |q|^2 = \bar{q} \cdot q = 1\}$  be the unit sphere in  $\mathbb{H}^{n+1}$ , where  $v \cdot w = \sum_{h=1}^{n+1} v_h w_h$  for  $v = (v_1, \dots, v_{n+1})$ ,  $w = (w_1, \dots, w_{n+1}) \in \mathbb{H}^{n+1}$ . As a real hypersurface in  $\mathbb{H}^{n+1}$ ,  $S^{4n+3}$  is endowed with a hyper CR structure, whose underlying corank three bundle  $Q$  is given by  $Q_q = \{X \in \mathbb{H}^{n+1} \mid \bar{q} \cdot X = 0\}$ ,  $q \in S^{4n+3}$ . A standard choice of defining function for  $S^{4n+3}$  is  $\rho(q) = |q|^2 - 1$ , and then the corresponding

$\mathbb{R}^3$ -valued one-form is given by  $\theta_S = \sum_{h=1}^{n+1} (d\bar{q}_h q_h - \bar{q}_h dq_h)/2$ , where we identify  $\mathbb{R}^3 = \text{Im } \mathbb{H}$ . By (2.3), we see that the Levi form  $\text{Levi}_{\theta_S}$  is twice the standard Riemannian metric of  $S^{4n+3}$  restricted to  $Q$ . In particular, the hyper CR structure of  $S^{4n+3}$  is strongly pseudoconvex. We shall refer to  $\theta_S$  as the *standard pseudohermitian structure* of  $S^{4n+3}$ .

The sphere  $S^{4n+3}$  may be regarded as the boundary at infinity of quaternionic hyperbolic space  $H_{\mathbb{H}}^{n+1}$ . The isometry group  $G$  of  $H_{\mathbb{H}}^{n+1}$  is given by  $G = Sp(n+1, 1)/\{\pm I_{n+2}\}$ . As  $G$  acts on  $H_{\mathbb{H}}^{n+1}$  transitively, we have a representation of  $H_{\mathbb{H}}^{n+1}$  as a coset space  $G/K$ , where  $K = Sp(n+1) \cdot Sp(1) = Sp(n+1) \times Sp(1)/\{\pm I_{n+2}\}$ , a maximal compact subgroup of  $G$ . The action of  $G$  extends to  $S^{4n+3}$ , and we shall examine how it transforms the hyper CR structure of  $S^{4n+3}$ . We represent  $\gamma \in G$  by a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(n+1, 1),$$

where  $a^*a - c^*c = I_{n+1}$ ,  $b^*b - d^*d = -1$ ,  $a^*b - c^*d = 0$ . The action of  $\gamma$  on  $S^{4n+3}$  is then given by  $q = {}^t(q_1, \dots, q_{n+1}) \mapsto (aq + b)(cq + d)^{-1}$ . By direct calculation, we obtain

$$\gamma^* \theta_S = \frac{1}{|cq + d|^2} \left( \frac{cq + d}{|cq + d|} \right) \theta_S \left( \frac{cq + d}{|cq + d|} \right)^{-1}. \quad (2.4)$$

This formula means that for each unit imaginary quaternion  $\mathbf{v} \in S^2$ , the CR structure  $(Q_{\mathbf{v}}, I_{\mathbf{v}})$  is transformed as  $\gamma^* Q_{\mathbf{v}} = Q_{\sigma_{\gamma}^{-1} \mathbf{v} \sigma_{\gamma}}$ ,  $\gamma^* I_{\mathbf{v}} = I_{\sigma_{\gamma}^{-1} \mathbf{v} \sigma_{\gamma}}$ , where  $\sigma_{\gamma}$  is the function of  $q$  given by  $\sigma_{\gamma}(q) = (cq + d)/|cq + d|$ . Thus  $G$  preserves the quaternionic CR structure of  $S^{4n+3}$ . Likewise, if  $\gamma \in K$ , then  $c = 0$  and  $\sigma_{\gamma}$  is constant. Therefore,  $K$  preserves the canonical  $S^2$ -family of CR structures associated with the hyper CR structure of  $S^{4n+3}$ . If  $\gamma \in Sp(n)$  further, then  $d = \pm 1$  and  $\sigma_{\gamma} = \pm 1$ . Therefore,  $Sp(n)$  preserves each of the two CR structures constituting the hyper CR structure.

For  $\mathbf{v} \in S^2$ , let  $T_{\mathbf{v}}$  be the vector field on  $S^{4n+3}$  defined by  $(T_{\mathbf{v}})_q = q\mathbf{v}^{-1}$ ,  $q \in S^{4n+3}$ , and let  $T_1 = T_{\mathbf{i}}$ ,  $T_2 = T_{\mathbf{j}}$  and  $T_3 = T_{\mathbf{k}}$ . One can check that  $T_a$  satisfies  $\theta_a(T_a) = 1$  and  $d\theta_a(T_a, X) = 0$  for all  $X \in Q$ . The action of  $\gamma \in G$  transforms  $T_{\mathbf{v}}$  as

$$\gamma^* T_{\mathbf{v}} = (\gamma^{-1})_* T_{\mathbf{v}} = e^{-2f} [T_{\sigma_{\gamma}^{-1} \mathbf{v} \sigma_{\gamma}} - 2I_{\sigma_{\gamma}^{-1} \mathbf{v} \sigma_{\gamma}} dbf^{\#}], \quad (2.5)$$

where  $f = -\log |cq + d|$ ,  $d_b f$  denotes the restriction of  $df$  to  $Q$  and  $d_b f^\#$  is the section of  $Q$  dual to  $d_b f$  with respect to  $\text{Levi}_{\theta_S}$  (cf. Section 3, Remark 4).

**Example 2** Let  $E$  be the real ellipsoid

$$E : \sum_{i=1}^{n+1} (a_i(z_i^2 + \bar{z}_i^2) + b_i z_i \bar{z}_i + c_i(w_i^2 + \bar{w}_i^2) + d_i w_i \bar{w}_i) - 1 = 0,$$

where  $a_i, \dots, d_i$  are real,  $b_i, d_i > 0$  and we are using the complex coordinates  $z_i, w_i$  of  $\mathbb{H}^{n+1}$  to write the defining equation. Let  $\theta = (\theta_1, \theta_2, \theta_3)$  be the  $\mathbb{R}^3$ -valued one-form compatible with the hyper CR structure of  $E$ , corresponding to the defining function chosen to be the left-hand side of the defining equation. By (2.3), the Levi form is

$$\text{Levi}_\theta = \sum_{i=1}^{n+1} (b_i + d_i)(dz_i \cdot d\bar{z}_i + d\bar{w}_i \cdot dw_i)$$

restricted to  $Q$ , and therefore the hyper CR structure of  $E$  is strongly pseudoconvex.

On the other hand, the complex Levi forms are

$$\text{Levi}_{\theta_1} = 2 \sum_{i=1}^{n+1} (b_i dz_i \cdot d\bar{z}_i + d_i d\bar{w}_i \cdot dw_i),$$

$$\begin{aligned} \text{Levi}_{\theta_2} = \text{Re} \sum_{i=1}^{n+1} & [(b_i + d_i)(dz_i \cdot d\bar{z}_i + d\bar{w}_i \cdot dw_i) \\ & + 2(a_i + c_i)(dz_i \cdot dz_i + d\bar{w}_i \cdot d\bar{w}_i)], \end{aligned}$$

$$\begin{aligned} \text{Levi}_{\theta_3} = \text{Re} \sum_{i=1}^{n+1} & [(b_i + d_i)(dz_i \cdot d\bar{z}_i + d\bar{w}_i \cdot dw_i) \\ & + 2(a_i - c_i)(dz_i \cdot dz_i - d\bar{w}_i \cdot d\bar{w}_i)], \end{aligned}$$

and these almost never coincide on  $Q$ ; indeed, they coincide on  $Q$  if and only if  $b_i = d_i$  and  $a_i = c_i = 0$  hold for  $i = 1, \dots, n + 1$ , that is,  $E$  should be a “quaternionic ellipsoid”. We will revisit this example in Section 5, where we

compare the quaternionic CR structure to Biquard's quaternionic contact structure [4].

## 2.2. Quaternionic Heisenberg group and its deformation.

**Example 3** The *quaternionic Heisenberg group*  $\mathcal{H}^{4n+3}$  is the Lie group whose underlying manifold is  $\mathbb{H}^n \times \text{Im } \mathbb{H}$  with coordinates  $(p, \tau) = (p_1, \dots, p_n, \tau)$  and whose group law is given by

$$(p, \tau) \cdot (p', \tau') = (p + p', \tau + \tau' + (\bar{p} \cdot p' - \bar{p}' \cdot p)).$$

(There is a concise treatment of the complex Heisenberg group as a CR manifold in [12].)

Write  $p_\alpha = x_\alpha^0 - x_\alpha^1 \mathbf{i} - x_\alpha^2 \mathbf{j} - x_\alpha^3 \mathbf{k}$  and  $\tau = t_1 \mathbf{i} + t_2 \mathbf{j} + t_3 \mathbf{k}$ . The vector fields

$$\begin{aligned} X_\alpha^0 &= \frac{\partial}{\partial x_\alpha^0} + 2x_\alpha^1 \frac{\partial}{\partial t_1} + 2x_\alpha^2 \frac{\partial}{\partial t_2} + 2x_\alpha^3 \frac{\partial}{\partial t_3}, \\ X_\alpha^1 &= \frac{\partial}{\partial x_\alpha^1} - 2x_\alpha^0 \frac{\partial}{\partial t_1} - 2x_\alpha^3 \frac{\partial}{\partial t_2} + 2x_\alpha^2 \frac{\partial}{\partial t_3}, \\ X_\alpha^2 &= \frac{\partial}{\partial x_\alpha^2} + 2x_\alpha^3 \frac{\partial}{\partial t_1} - 2x_\alpha^0 \frac{\partial}{\partial t_2} - 2x_\alpha^1 \frac{\partial}{\partial t_3}, \\ X_\alpha^3 &= \frac{\partial}{\partial x_\alpha^3} - 2x_\alpha^2 \frac{\partial}{\partial t_1} + 2x_\alpha^1 \frac{\partial}{\partial t_2} - 2x_\alpha^0 \frac{\partial}{\partial t_3}, \\ T_1 &= 2 \frac{\partial}{\partial t_1}, \quad T_2 = 2 \frac{\partial}{\partial t_2}, \quad T_3 = 2 \frac{\partial}{\partial t_3} \end{aligned}$$

are left-invariant. Let

$$Q = \text{span}\{X_\alpha^a\}_{1 \leq \alpha \leq n, 0 \leq a \leq 3}, \quad Q_a = Q \oplus \mathbb{R}T_b \oplus \mathbb{R}T_c,$$

and define complex structures  $I_a$  on  $Q_a$  by

$$I_a X_\alpha^0 = X_\alpha^a, \quad I_a X_\alpha^b = X_\alpha^c, \quad I_a T_b = T_c.$$

Then the triple of (integrable) CR structures  $(Q_a, I_a)$  gives a left-invariant hyper CR structure on  $\mathcal{H}^{4n+3}$ . The  $\text{Im } \mathbb{H}$ -valued one-form

$$\theta_H = \frac{1}{2} \left[ d\tau + \sum_{\alpha=1}^n (d\bar{p}_\alpha p_\alpha - \bar{p}_\alpha dp_\alpha) \right]$$

is left-invariant and compatible with the hyper CR structure. The Levi form of  $\theta_H$  is given by  $\text{Levi}_{\theta_H}(X_\alpha^a, X_\beta^b) = 2\delta_{\alpha\beta}\delta_{ab}$ . Hence the hyper CR structure of  $\mathcal{H}^{4n+3}$  is strongly pseudoconvex. We shall refer to  $\theta_H$  as the *standard pseudohermitian structure* of  $\mathcal{H}^{4n+3}$ . It is also worthwhile to mention that in this example the CR structures  $(Q_a, I_a)$  are *not* strongly pseudoconvex.

**Remark 2** The quaternionic CR and pseudohermitian structures of  $S^{4n+3}$  and  $\mathcal{H}^{4n+3}$  are related as follows. Let  $\mathfrak{P} = \{(p', p_{n+1}) \in \mathbb{H}^{n+1} \mid \text{Re } p_{n+1} = |p'|^2\}$ . The mapping

$$\begin{aligned} (q_1, \dots, q_{n+1}) &\in S^{4n+3} \setminus \{(0, \dots, 0, -1)\} \\ \mapsto (q_1, \dots, q_n, 1 - q_{n+1})(1 + q_{n+1})^{-1} &\in \mathfrak{P} \end{aligned}$$

is a quaternionic analogue of Cayley transform. This mapping composed with  $(p', p_{n+1}) \mapsto (p', p_{n+1} - |p'|^2)$  gives the equivalence mapping

$$\begin{aligned} F : (q_1, \dots, q_{n+1}) &\in S^{4n+3} \setminus \{(0, \dots, 0, -1)\} \\ \mapsto (q_1(1 + q_{n+1})^{-1}, \dots, q_n(1 + q_{n+1})^{-1}, \\ &(1 + q_{n+1})^{-1} - (1 + \bar{q}_{n+1})^{-1}) \in \mathcal{H}^{4n+3} \end{aligned}$$

between quaternionic CR manifolds. Hence  $F^*\theta_H$  is a pseudohermitian structure (singular at  $(0, \dots, 0, -1)$ ) for the quaternionic CR structure of  $S^{4n+3}$ , and thus has the form  $\lambda\sigma\theta_S\sigma^{-1}$ , where  $\lambda$  and  $\sigma$  are respectively positive and  $Sp(1)$ -valued functions. Explicitly, we have  $\lambda = 1/|1 + q_{n+1}|^2$ ,  $\sigma = (1 + q_{n+1})/|1 + q_{n+1}|$ . The quaternionic Cayley transform is utilized extensively in [8], [9], [10], [11] in connection with problems such as the quaternionic contact Yamabe problem.

**Example 4** The hyper pseudohermitian structure of quaternionic Heisenberg group can be deformed by changing the definition of vector fields  $X_\alpha^a$  as follows:

$$X_\alpha^0 = \frac{\partial}{\partial x_\alpha^0} + A_\alpha^1 x_\alpha^1 \frac{\partial}{\partial t_1} + A_\alpha^2 x_\alpha^2 \frac{\partial}{\partial t_2} + A_\alpha^3 x_\alpha^3 \frac{\partial}{\partial t_3},$$

$$\begin{aligned}
X_\alpha^1 &= \frac{\partial}{\partial x_\alpha^1} - B_\alpha^1 x_\alpha^0 \frac{\partial}{\partial t_1} - B_\alpha^2 x_\alpha^3 \frac{\partial}{\partial t_2} + B_\alpha^3 x_\alpha^2 \frac{\partial}{\partial t_3}, \\
X_\alpha^2 &= \frac{\partial}{\partial x_\alpha^2} + C_\alpha^1 x_\alpha^3 \frac{\partial}{\partial t_1} - C_\alpha^2 x_\alpha^0 \frac{\partial}{\partial t_2} - C_\alpha^3 x_\alpha^1 \frac{\partial}{\partial t_3}, \\
X_\alpha^3 &= \frac{\partial}{\partial x_\alpha^3} - D_\alpha^1 x_\alpha^2 \frac{\partial}{\partial t_1} + D_\alpha^2 x_\alpha^1 \frac{\partial}{\partial t_2} - D_\alpha^3 x_\alpha^0 \frac{\partial}{\partial t_3},
\end{aligned}$$

where  $A_\alpha^a, B_\alpha^a, C_\alpha^a, D_\alpha^a$  are real constants. We define vector fields  $T_a$ , bundles  $Q, Q_a$  and complex structures  $I_a$  on  $Q_a$  as in Example 3. Then the triple of almost CR structures  $(Q_a, I_a)$  gives an almost hyper CR structure on  $\mathbb{R}^{4n} \times \mathbb{R}^3$ , and always satisfies (1.6). It satisfies (1.7) (or more strongly,  $(Q_a, I_a)$  are CR structures) if and only if  $A_\alpha^a + B_\alpha^a + C_\alpha^a + D_\alpha^a$  does not depend on  $a$  (though may depend on  $\alpha$ ). Henceforth we assume this condition is satisfied, and therefore, we obtain a hyper CR structure. Then the  $\mathbb{R}^3$ -valued one-form  $\theta = (\theta_1, \theta_2, \theta_3)$  given by

$$\begin{aligned}
\theta_1 &= \frac{1}{2} \left[ dt_1 + \sum_\alpha \left( -A_\alpha^1 x_\alpha^1 dx_\alpha^0 + B_\alpha^1 x_\alpha^0 dx_\alpha^1 - C_\alpha^1 x_\alpha^3 dx_\alpha^2 + D_\alpha^1 x_\alpha^2 dx_\alpha^3 \right) \right], \\
\theta_2 &= \frac{1}{2} \left[ dt_2 + \sum_\alpha \left( -A_\alpha^2 x_\alpha^2 dx_\alpha^0 + B_\alpha^2 x_\alpha^3 dx_\alpha^1 + C_\alpha^2 x_\alpha^0 dx_\alpha^2 - D_\alpha^2 x_\alpha^1 dx_\alpha^3 \right) \right], \\
\theta_3 &= \frac{1}{2} \left[ dt_3 + \sum_\alpha \left( -A_\alpha^3 x_\alpha^3 dx_\alpha^0 - B_\alpha^3 x_\alpha^2 dx_\alpha^1 + C_\alpha^3 x_\alpha^1 dx_\alpha^2 + D_\alpha^3 x_\alpha^0 dx_\alpha^3 \right) \right]
\end{aligned}$$

is compatible with the hyper CR structure and satisfies  $\theta_a(T_b) = \delta_{ab}$ . The complex Levi forms are

$$\begin{aligned}
\text{Levi}_{\theta_1} &= \frac{1}{2} \sum_\alpha \left\{ (A_\alpha^1 + B_\alpha^1) \left( (dx_\alpha^0)^2 + (dx_\alpha^1)^2 \right) \right. \\
&\quad \left. + (C_\alpha^1 + D_\alpha^1) \left( (dx_\alpha^2)^2 + (dx_\alpha^3)^2 \right) \right\}, \\
\text{Levi}_{\theta_2} &= \frac{1}{2} \sum_\alpha \left\{ (A_\alpha^2 + C_\alpha^2) \left( (dx_\alpha^0)^2 + (dx_\alpha^2)^2 \right) \right. \\
&\quad \left. + (B_\alpha^2 + D_\alpha^2) \left( (dx_\alpha^3)^2 + (dx_\alpha^1)^2 \right) \right\}, \\
\text{Levi}_{\theta_3} &= \frac{1}{2} \sum_\alpha \left\{ (A_\alpha^3 + D_\alpha^3) \left( (dx_\alpha^0)^2 + (dx_\alpha^3)^2 \right) \right. \\
&\quad \left. + (B_\alpha^3 + C_\alpha^3) \left( (dx_\alpha^1)^2 + (dx_\alpha^2)^2 \right) \right\},
\end{aligned}$$

and the Levi form is

$$\text{Levi}_\theta = \frac{1}{4} \sum_{\alpha} \Lambda_{\alpha} ((dx_{\alpha}^0)^2 + (dx_{\alpha}^1)^2 + (dx_{\alpha}^2)^2 + (dx_{\alpha}^3)^2),$$

where we set  $\Lambda_{\alpha} = A_{\alpha}^a + B_{\alpha}^a + C_{\alpha}^a + D_{\alpha}^a$ . Suppose now that  $\Lambda_{\alpha} > 0$  for all  $\alpha$ , so that the hyper CR structure is strongly pseudoconvex. We will revisit this example in Section 3, Section 5.

**2.3. Principal bundle over a hypercomplex manifold**

Let  $(N, I, J, K)$  be a hypercomplex manifold of real dimension  $4n$ , that is,  $I, J, K$  are (integrable) complex structures on  $N$  satisfying  $IJ = -JI = K$ . Write  $I_1 = I, I_2 = J$  and  $I_3 = K$ . Let  $G$  be a Lie group of dimension three, and let  $\pi : M \rightarrow N$  be a principal  $G$ -bundle over  $N$  with connection form  $\theta$  and the corresponding curvature form  $\Omega$ . Via an identification of the Lie algebra  $\mathfrak{g}$  of  $G$  with  $\mathbb{R}^3$  as vector spaces, we write  $\theta$  and  $\Omega$  as  $\theta = (\theta_1, \theta_2, \theta_3)$  and  $\Omega = (\pi^*\Omega_1, \pi^*\Omega_2, \pi^*\Omega_3)$ , respectively. Assume the following two conditions:

- (i) The two-forms  $\Omega_a$  on  $N$  are invariant by  $I_a$ :

$$\Omega_a(I_a X, I_a Y) = \Omega_a(X, Y), \quad X, Y \in TN.$$

- (ii) A hyperhermitian metric  $g$  on  $N$  is chosen so that the fundamental two-forms  $F_a = g(I_a \cdot, \cdot)$  are given by

$$F_a(X, Y) = \frac{1}{2}(\Omega_a(X, Y) - \Omega_a(I_b X, I_b Y)), \quad X, Y \in TN.$$

Here, the indices  $a, b$  are so that  $(a, b, c)$  is a cyclic permutation of  $(1, 2, 3)$ . Let  $Q_a$  be the kernel of  $\theta_a$ . Then  $Q = \cap_{a=1}^3 Q_a$  is the horizontal distribution for the connection form  $\theta$ . Take a triple of vertical vector fields  $(T_1, T_2, T_3)$  on  $M$  satisfying  $\theta_a(T_b) = \delta_{ab}$ . Then  $Q_a$  is expressed as  $Q_a = Q \oplus \mathbb{R}T_b \oplus \mathbb{R}T_c$ . We define complex structures  $I_a$  on  $Q_a$  by  $I_a \tilde{X} = \tilde{I}_a \tilde{X}$  and  $I_a T_b = T_c$ , where  $\tilde{X}$  denotes the horizontal lift of a vector field  $X$  on  $N$ . Then it is straightforward to verify that the almost CR structures  $(Q_a, I_a)$  are integrable, and in particular, they satisfy (1.6), (1.7).

Thus we obtain the following

**Proposition 2.1** *Let  $G$  be a Lie group of dimension three. Let  $\pi : M \rightarrow N$*

be a principal  $G$ -bundle over a hyperhermitian manifold  $N$  with connection form  $\theta$  and the corresponding curvature form  $\Omega$ , satisfying the conditions (i), (ii) as above. Then  $\{(Q_a, I_a)\}_{a=1,2,3}$  defined above is a hyper CR structure on the total space  $M$ , and  $\theta$  is compatible with it. The Levi form  $\text{Levi}_\theta$  of  $\theta$  is given by the pull-back of  $g$ , restricted to  $Q : \text{Levi}_\theta = (\pi^*g)|_{Q \times Q}$ . In particular, the hyper CR structure of  $M$  is strongly pseudoconvex, and together with  $\theta$ , gives a hyper pseudohermitian structure on  $M$ .

As a concrete example, the standard hyper pseudohermitian structure of the quaternionic Heisenberg group  $\mathcal{H}^{4n+3}$ , which is an  $\text{Im } \mathbb{H}$ -bundle over the hypercomplex manifold  $\mathbb{H}^n$ , can be understood by the bundle construction as above.

An example with compact total space follows.

**Example 5** ( $T^3$ -bundle over  $S^1 \times S^3$ ) Let  $\tilde{N} = \mathbb{H} \setminus \{0\}$  with complex coordinates  $(z, w)$ , and let  $(I, J, K)$  be the standard hypercomplex structure of  $\tilde{N}$  as in (2.2). Let  $g$  be the hyperhermitian metric on  $\tilde{N}$  defined by

$$g = \frac{2(|dz|^2 + |dw|^2)}{|z|^2 + |w|^2}.$$

Then the fundamental forms  $F_1, F_2, F_3$  are given by

$$F_1 = \frac{\sqrt{-1}(dz \wedge d\bar{z} + dw \wedge d\bar{w})}{|z|^2 + |w|^2}, \quad F_2 = \frac{dz \wedge dw + d\bar{z} \wedge d\bar{w}}{|z|^2 + |w|^2},$$

$$F_3 = \frac{\sqrt{-1}(d\bar{z} \wedge d\bar{w} - dz \wedge dw)}{|z|^2 + |w|^2},$$

respectively.

Let  $\mu = -\log(|z|^2 + |w|^2)$ , a smooth function on  $\tilde{N}$ , and define three two-forms  $\Omega_a$  on  $\tilde{N}$  by  $\Omega_a = d(I_a d\mu)$ . They are given explicitly by

$$\Omega_1 = \frac{2\sqrt{-1}(|w|^2 dz \wedge d\bar{z} + |z|^2 dw \wedge d\bar{w})}{(|z|^2 + |w|^2)^2} - \frac{2\sqrt{-1}(\bar{z}w dz \wedge d\bar{w} - z\bar{w} d\bar{z} \wedge dw)}{(|z|^2 + |w|^2)^2},$$

$$\Omega_2 = \frac{dz \wedge dw + d\bar{z} \wedge d\bar{w}}{|z|^2 + |w|^2} - \frac{(zw - \bar{z}\bar{w})(dz \wedge d\bar{z} - dw \wedge d\bar{w})}{(|z|^2 + |w|^2)^2}$$

$$- \frac{(\bar{z}^2 + w^2)dz \wedge d\bar{w} + (z^2 + \bar{w}^2)d\bar{z} \wedge dw}{(|z|^2 + |w|^2)^2},$$

$$\begin{aligned} \Omega_3 = & \frac{\sqrt{-1}(d\bar{z} \wedge d\bar{w} - dz \wedge dw)}{|z|^2 + |w|^2} + \frac{\sqrt{-1}(zw + \bar{z}\bar{w})(dz \wedge d\bar{z} - dw \wedge d\bar{w})}{(|z|^2 + |w|^2)^2} \\ & - \sqrt{-1} \frac{(z^2 - w^2)dz \wedge d\bar{w} - (z^2 - \bar{w}^2)d\bar{z} \wedge dw}{(|z|^2 + |w|^2)^2}. \end{aligned}$$

Let  $N := \tilde{N}/\langle\alpha\rangle$ , where  $\alpha$  is a complex constant with  $|\alpha| > 1$ , acting on  $\tilde{N}$  by

$$\alpha \cdot (z, w) := (\alpha z, \bar{\alpha} w), \quad (z, w) \in \tilde{N},$$

and  $\langle\alpha\rangle$  is the infinite cyclic group generated by  $\alpha$ . Then  $N$  is a smooth manifold diffeomorphic to  $S^1 \times S^3$ , and  $(g, I, J, K)$  descends to a hyperhermitian structure on  $N$ . Thus we obtain a hyperhermitian Hopf surface  $(N, g, I, J, K)$ . Note that though the function  $\mu$  does not descend to a function on  $N$ , the differential  $d\mu$  descends to a one-form on  $N$ , and therefore  $\Omega_a$  descend to two-forms on  $N$ .

Let  $\pi : M = N \times T^3 \rightarrow N$  be the trivial  $T^3$ -bundle with fibre-coordinates  $(t_1, t_2, t_3)$ . (Therefore,  $M$  is diffeomorphic to  $S^3 \times T^4$ .) Define an  $\mathbb{R}^3$ -valued one-form  $\theta = (\theta_1, \theta_2, \theta_3)$  on  $M$  by  $\theta_a = dt_a + \pi^* I_a d\mu$ . Then  $\theta$  is a connection one-form in the bundle  $\pi : M \rightarrow N$  with curvature form  $\Omega = (\pi^* \Omega_1, \pi^* \Omega_2, \pi^* \Omega_3)$ . It is straightforward to verify that the hyperhermitian Hopf surface  $N$  and the forms  $\Omega_a$  satisfy the conditions (i) and (ii) before Proposition 2.1. Therefore,  $M$  comes equipped with a hyper pseudohermitian structure. We will revisit this example in Section 3, Section 5.

The above construction of the hyper pseudohermitian structure on  $S^3 \times T^4$  is a special case of the following more general one. An *HKT manifold* is a hyperhermitian manifold  $(N, g, I, J, K)$ , characterized by the property that the fundamental forms  $F_1, F_2, F_3$  satisfy  $IdF_1 = JdF_2 = KdF_3$ , where  $I\omega = \omega(I \cdot, \dots, I \cdot)$  for a  $k$ -form  $\omega$ . By a result of Banos-Swann [3], there exists an *HKT-potential*  $\mu$ , that is, a locally defined function  $\mu$  on  $N$  such that

$$F_a = \frac{1}{2}(d(I_a d\mu) - I_b d(I_a d\mu)),$$

where the indices  $a, b$  are as before. If  $d(I_a d\mu)$  are globally defined on  $N$  and

determine integral cohomology classes of  $N$ , as in the preceding example, then there exists a principal  $T^3$ -bundle  $\pi: M \rightarrow N$  with connection form  $\theta$  whose curvature form  $d\theta$  coincides with  $(\pi^*d(I_a d\mu))$ . Now Proposition 2.1 applies, and we obtain a hyper pseudohermitian structure on the total space  $M$ . This construction is a generalization of that due to Hernandez [7] for hyperkähler manifolds.

### 3. Canonical connection

In this section we shall construct a quaternionic analogue of the Tanaka-Webster connection [17], [19] in CR geometry. Throughout this section, we shall assume that the hyper and quaternionic CR structures are strongly pseudoconvex. Since our construction is modelled on that in the CR case, we first review it briefly.

Let  $(M, \theta)$  be a pseudohermitian manifold. (Recall that the underlying almost CR structure is assumed to be partially integrable.) As in [15], let  $T$  be an *arbitrary* transverse vector field such that  $\theta(T) = 1$ ; except for this point, we follow the explanation of the Tanaka-Webster connection due to Rumin [14], where  $T$  is the Reeb field from the beginning. For each  $k > 0$ , define a Riemannian metric  $g_{M,k}$  on  $M$  by

$$g_{M,k} = g + k\theta^2,$$

where  $g$  is extended to a positive semidefinite form on  $TM$  by defining  $g(T, \cdot) = 0$ . There is a unique connection  $\nabla$  which satisfies  $\nabla g_{M,k} = 0$  for all  $k$  and among such connections, has as small torsion as possible. It is characterized by the following conditions:

- (i) the subbundle  $Q$  is preserved by  $\nabla$ ;
- (ii)  $g$  and  $T$  are  $\nabla$ -parallel;
- (iii) the torsion tensor  $\text{Tor}$  of  $\nabla$  satisfies
  - (a)  $\text{Tor}(X, Y)_Q = 0$ ,  $X, Y \in Q$ ;
  - (b)  $X \in Q \mapsto \text{Tor}(T, X)_Q \in Q$  is  $g$ -symmetric,

where  $E_Q$  denotes the  $Q$ -component of a tangent vector  $E$  with respect to the splitting  $TM = Q \oplus \mathbb{R}T$ . It follows from  $\nabla g = 0$  and (iii-a) that for  $X, Y \in \Gamma(Q)$ ,  $\nabla_X Y$  is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y]_Q, Z) - g([X, Z]_Q, Y) - g([Y, Z]_Q, X) \quad \text{for all } Z \in \Gamma(Q). \quad (3.1)$$

We now determine  $T$  so that the corresponding connection  $\nabla$  be as close to being a unitary connection as possible. Since any orthogonal connection on a hermitian line bundle is unitary, this step does not work when  $n = 1$ . Hence we assume  $n \geq 2$  hereafter. Fix an arbitrary  $T$ , and write  $\widehat{T} = T + 2JV$  for  $V \in \Gamma(Q)$ . Then by (3.1), the corresponding connections  $\nabla$  and  $\widehat{\nabla}$  are related by

$$\widehat{\nabla}_X Y = \nabla_X Y + g(JX, Y)JV - g(JV, Y)JX - g(JV, X)JY. \quad (3.2)$$

Let  $\{e_1, \dots, e_n\}$  be a local unitary basis for  $Q^{1,0}$ , and write

$$\nabla e_i = \sum_{j=1}^n (\omega_{i\bar{j}} e_j + \omega_{ij} \bar{e}_j), \quad \widehat{\nabla} e_i = \sum_{j=1}^n (\widehat{\omega}_{i\bar{j}} e_j + \widehat{\omega}_{ij} \bar{e}_j).$$

Note that  $\widehat{\nabla}$  is a unitary connection if and only if  $\widehat{\omega}_{ij} = 0$  for all  $i, j$ . Using (3.2) we obtain

$$\widehat{\omega}_{ij}(e_k) = \omega_{ij}(e_k), \quad \widehat{\omega}_{ij}(\bar{e}_k) = \omega_{ij}(\bar{e}_k) + \delta_{kj} \bar{V}_i - \delta_{ki} \bar{V}_j,$$

where we write  $V = \sum_{i=1}^n (V_i e_i + \bar{V}_i \bar{e}_i)$ .  $\widehat{\omega}_{ij}(e_k)$  are independent of  $V$ , and they all vanish if and only if the underlying almost CR structure is integrable, while  $\widehat{\omega}_{ij}(\bar{e}_k)$  can always be made zero by an appropriate choice of  $V$ . Indeed, using (3.1) and  $d(d\theta)(e_i, e_j, \bar{e}_k) = 0$ , we obtain

$$\omega_{ij}(\bar{e}_k) = \frac{1}{2} (\delta_{ki} d\theta(T, e_j) - \delta_{kj} d\theta(T, e_i)). \quad (3.3)$$

Hence, by choosing

$$\bar{V}_i = \frac{1}{2} d\theta(T, e_i), \quad i = 1, \dots, n, \quad (3.4)$$

we can achieve  $\widehat{\omega}_{ij}(\bar{e}_k) = 0$ . Note that (3.4) is equivalent to  $\widehat{T}$  being the Reeb field associated with  $\theta$ . In particular, the resulting connection  $\widehat{\nabla}$  is the

Tanaka-Webster connection, as generalized to the partially integrable case by Tanno [18].

We now turn to the quaternionic case, and first treat the hyper CR case. So let  $(M, \theta)$  be a strongly pseudoconvex hyper pseudohermitian manifold. As above, our construction proceeds in two steps: first, for each choice of an admissible three-plane field  $Q^\perp$ , we construct a certain uniquely determined connection  $\nabla$  on  $TM$ . Next we determine  $Q^\perp$  so that, when restricted to a connection on  $Q$ ,  $\nabla$  be “as close to an  $Sp(n) \cdot Sp(1)$ -connection as possible.”

Let  $g$  denote the Levi form of  $\theta$ ; it is a metric on  $Q$ . Let  $Q^\perp$  be an admissible three-plane field, so that we have the splitting

$$TM = Q \oplus Q^\perp. \quad (3.5)$$

Set  $g^\perp := \theta_1^2 + \theta_2^2 + \theta_3^2$ , and denote its restriction to  $Q^\perp$  by the same symbol. We define a family of Riemannian metrics  $g_{M,k}$  on  $M$  by  $g_{M,k} = g + kg^\perp$ , where  $k > 0$  and  $g$  is extended to a positive semidefinite form on  $TM$  by defining  $g(U, \cdot) = 0$  for all  $U \in Q^\perp$ . The splitting (3.5) is orthogonal with respect to all  $g_{M,k}$ . As in the CR case, there is a unique connection  $\nabla$  which satisfies  $\nabla g_{M,k} = 0$  for all  $k$  and among such connections, has as small torsion as possible. We state a characterization of this connection as

**Proposition 3.1** *Let  $(M, \theta)$  be a hyper pseudohermitian manifold, and let  $Q^\perp$  be an admissible three-plane field. Then there exists a unique connection  $\nabla$  on  $TM$  satisfying the following conditions:*

- (i) *the subbundles  $Q$  and  $Q^\perp$  are preserved by  $\nabla$ ;*
- (ii)  *$g$  and  $g^\perp$  are  $\nabla$ -parallel;*
- (iii) *for  $X, Y \in Q$  and  $U, V \in Q^\perp$ ,*
  - (a)  $\text{Tor}(X, Y)_Q = 0$ ;
  - (b)  $\text{Tor}(U, V)_{Q^\perp} = 0$ ;
  - (c)  $X \in Q \mapsto \text{Tor}(U, X)_Q \in Q$  *is  $g$ -symmetric;*
  - (d)  $U \in Q^\perp \mapsto \text{Tor}(U, X)_{Q^\perp} \in Q^\perp$  *is  $g^\perp$ -symmetric,*

where  $E_Q$  and  $E_{Q^\perp}$  respectively denote the  $Q$ - and  $Q^\perp$ -components of a tangent vector  $E$  with respect to the splitting (3.5).

*Proof.* Throughout the proof, let  $X, Y, Z \in \Gamma(Q)$  and  $U, V, W \in \Gamma(Q^\perp)$ . Suppose that  $\nabla$  is a connection on  $TM$  satisfying the conditions stated in the proposition. Then the conditions (i), (ii), (iii-a), (iii-b) force  $\nabla$  to satisfy

$$\begin{aligned}
2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y]_Q, Z) \\
&\quad - g([X, Z]_Q, Y) - g([Y, Z]_Q, X)
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
2g^\perp(\nabla_U V, W) &= Ug^\perp(V, W) + Vg^\perp(U, W) - Wg^\perp(U, V) + g^\perp([U, V]_{Q^\perp}, W) \\
&\quad - g^\perp([U, W]_{Q^\perp}, V) - g^\perp([V, W]_{Q^\perp}, U).
\end{aligned} \tag{3.7}$$

Since  $\text{Tor}(U, X)_Q = \nabla_U X - [U, X]_Q$ , the condition  $\nabla g = 0$  implies that

$$\begin{aligned}
g(\text{Tor}(U, X)_Q, Y) + g(X, \text{Tor}(U, Y)_Q) \\
= Ug(X, Y) - g([U, X]_Q, Y) - g(X, [U, Y]_Q).
\end{aligned}$$

Therefore, the condition (iii-c) determines  $\text{Tor}(U, X)_Q$  by

$$g(\text{Tor}(U, X)_Q, Y) = \frac{1}{2}(Ug(X, Y) - g([U, X]_Q, Y) - g(X, [U, Y]_Q)),$$

and this gives

$$g(\nabla_U X, Y) = \frac{1}{2}(Ug(X, Y) + g([U, X]_Q, Y) - g(X, [U, Y]_Q)). \tag{3.8}$$

Similarly, the condition (iii-d) determines  $\text{Tor}(U, X)_{Q^\perp}$  by

$$\begin{aligned}
g^\perp(\text{Tor}(U, X)_{Q^\perp}, V) \\
= -\frac{1}{2}(Xg^\perp(U, V) + g^\perp([U, X]_{Q^\perp}, V) + g^\perp(U, [V, X]_{Q^\perp})),
\end{aligned}$$

which gives

$$g^\perp(\nabla_X U, V) = \frac{1}{2}(Xg^\perp(U, V) - g^\perp([U, X]_{Q^\perp}, V) + g^\perp(U, [V, X]_{Q^\perp})). \tag{3.9}$$

Conversely, (3.6), (3.7), (3.8), (3.9) determine a connection  $\nabla$  on  $TM$  uniquely, and  $\nabla$  satisfies the conditions stated in the proposition.  $\square$

**Remark 3** One can generalize Proposition 3.1 to a quaternionic pseudo-hermitian structure in an obvious manner.

Our next task is to determine  $Q^\perp$ . As in the CR case, we shall work with complex frames. Let  $Q^{1,0} = \{X \in Q \otimes \mathbb{C} \mid IX = \sqrt{-1}X\}$ . Then we have the decomposition  $Q \otimes \mathbb{C} = Q^{1,0} \oplus \overline{Q^{1,0}}$ , orthogonal with respect to the Levi form  $g$  regarded as a hermitian form on  $Q \otimes \mathbb{C}$ . Take a local orthonormal frame  $\{\varepsilon_1, \dots, \varepsilon_{4n}\}$  for  $Q$  satisfying

$$\begin{aligned} \varepsilon_{4k-2} = I\varepsilon_{4k-3}, \quad \varepsilon_{4k-1} = J\varepsilon_{4k-3}, \quad \varepsilon_{4k} = K\varepsilon_{4k-3} (= I\varepsilon_{4k-1}), \\ k = 1, \dots, n. \end{aligned} \quad (3.10)$$

(Such a local orthonormal frame for  $Q$  is said to be *adapted*.) Then

$$\{e_{2k-1} = (\varepsilon_{4k-3} - \sqrt{-1}\varepsilon_{4k-2})/\sqrt{2}, \quad e_{2k} = (\varepsilon_{4k-1} - \sqrt{-1}\varepsilon_{4k})/\sqrt{2}\}_{1 \leq k \leq n} \quad (3.11)$$

is a local unitary frame for  $Q^{1,0}$ , and  $J, K : Q^{1,0} \rightarrow \overline{Q^{1,0}}$  are given by

$$Je_{2k-1} = \overline{e_{2k}}, \quad Je_{2k} = -\overline{e_{2k-1}}; \quad Ke_{2k-1} = -\sqrt{-1}\overline{e_{2k}}, \quad Ke_{2k} = \sqrt{-1}\overline{e_{2k-1}}.$$

Choose an admissible three-plane field  $Q^\perp$ , and let  $\nabla$  be the corresponding connection as in Proposition 3.1. Regarding  $\nabla$  as a connection on  $Q$ , let  $\omega$  be the matrix of connection forms with respect to the above local frame; its components are given by

$$\nabla e_i = \sum_{j=1}^{2n} (\omega_{i\bar{j}} e_j + \omega_{ij} \overline{e_j}).$$

Since  $\nabla g = 0$ , we have  $\omega_{i\bar{j}} = -\overline{\omega_{\bar{j}i}}$  and  $\omega_{ij} = -\omega_{ji}$ , namely,  $(\omega_{i\bar{j}})$  is skew-hermitian and  $(\omega_{ij})$  is skew-symmetric. Note that  $\nabla \overline{e_i} = \overline{\nabla e_i}$  since  $\nabla$  is a real connection.

We now further restrict the domain of  $\nabla$  by considering the  $Q$ -partial connection

$$\nabla^Q : (X, Y) \in \Gamma(Q) \times \Gamma(Q) \mapsto \nabla_X Y \in \Gamma(Q).$$

In other words, we regard  $\omega$  as being defined on  $Q$ . Let  $(sp(n) + sp(1))^\perp$  denote the orthogonal complement of  $sp(n) + sp(1)$  in  $so(4n)$  with respect to the Killing inner product. Then  $\omega$ 's  $(sp(n) + sp(1))^\perp$ -component  $\omega^{\text{obs}}$  gives an obstruction for the  $Q$ -partial connection  $\nabla^Q$  to preserve the  $Sp(n) \cdot Sp(1)$ -structure of  $Q$ , and  $\omega^{\text{obs}}$  is small if and only if  $\nabla^Q$  is close to being an  $Sp(n) \cdot Sp(1)$ -partial connection.  $\omega^{\text{obs}}$  is tensorial, and  $(\omega^{\text{obs}})_q$  is an element of  $Q_q^* \otimes (sp(n) + sp(1))^\perp$  for each point  $q$ .

Note that when  $n = 1$ , since  $Sp(1) \cdot Sp(1) = SO(4)$ , the  $SO(4)$ -connection  $\nabla$  necessarily preserves the  $Sp(1) \cdot Sp(1)$ -structure of  $Q$ , and therefore the obstruction tensor  $\omega^{\text{obs}}$  vanishes irrespective of the choice of  $Q^\perp$ . Hence we assume  $n \geq 2$  hereafter, and use representation theory to make the requirement that  $\omega^{\text{obs}}$  be small more explicit.  $Q_q^* \otimes (sp(n) + sp(1))^\perp$  is an  $Sp(n) \times Sp(1)$ -module whose model is  $\mathbb{H}^n \otimes (sp(n) + sp(1))^\perp$ . Swann [16] wrote down the irreducible decomposition of this module explicitly, which we shall review. Let  $E$  (resp.  $H$ ) be the standard complex  $Sp(n)$ (resp.  $Sp(1)$ )-module, with the left action of  $Sp(n)$  (resp.  $Sp(1)$ ) through the inclusion  $Sp(n) \subset SU(2n)$  (resp.  $Sp(1) = SU(2)$ ). Then we have

$$\mathbb{H}^n \otimes \mathbb{C} \cong E \otimes H \tag{3.12}$$

$$(sp(n) + sp(1))^\perp \otimes \mathbb{C} \cong \Lambda_0^2 E \otimes S^2 H \tag{3.13}$$

as complex  $Sp(n) \times Sp(1)$ -modules. In fact,

$$\begin{aligned} so(4n) \otimes \mathbb{C} &\cong \Lambda^2 \mathbb{H}^n \otimes \mathbb{C} \quad (\text{as } SO(4n)\text{-modules}) \\ &\cong \Lambda^2(E \otimes H) \\ &\cong S^2 E \oplus S^2 H \oplus (\Lambda_0^2 E \otimes S^2 H). \end{aligned} \tag{3.14}$$

Since  $S^2 E \cong sp(n)$  and  $S^2 H \cong sp(1)$ , we conclude (3.13). It follows from (3.12) and (3.13) that

$$\begin{aligned} (\mathbb{H}^n \otimes (sp(n) + sp(1))^\perp) \otimes \mathbb{C} &\cong (E \otimes H) \otimes (\Lambda_0^2 E \otimes S^2 H) \\ &\cong (E \otimes \Lambda_0^2 E) \otimes (H \otimes S^2 H) \\ &\cong (K \oplus \Lambda_0^3 E \oplus E) \otimes (S^3 H \oplus H), \end{aligned} \tag{3.15}$$

where  $K$  is the irreducible complex  $Sp(n)$ -module with highest weight  $(2, 1, 0, \dots, 0)$ . Therefore, we have the irreducible decomposition

$$\begin{aligned} & (\mathbb{H}^n \otimes (sp(n) + sp(1))^\perp) \otimes \mathbb{C} \\ & \cong (K \otimes S^3 H) \oplus (\Lambda_0^3 E \otimes S^3 H) \oplus (E \otimes S^3 H) \\ & \oplus (K \otimes H) \oplus (\Lambda_0^3 E \otimes H) \oplus (E \otimes H). \end{aligned} \quad (3.16)$$

It can be shown that all the components of  $\omega^{\text{obs}}$  other than the one corresponding to  $E \otimes H$  are stable under a change of  $Q^\perp$ . It is also not possible in general to remove the component  $\omega^{E \otimes H}$  of  $\omega^{\text{obs}}$  in  $E \otimes H$ . As we shall prove in Theorem 3.4 below,  $\omega^{E \otimes H}$  can be removed by a suitable choice of  $Q^\perp$  if one assumes a stronger condition than strong pseudoconvexity which is called ultra-pseudoconvexity and will be defined below.

To proceed, we shall first express the condition

$$\omega^{E \otimes H} = 0 \quad (3.17)$$

in a more explicit form.

**Lemma 3.2** *The condition (3.17) is rewritten as follows: for  $l = 1, \dots, n$ ,*

$$\begin{aligned} & \sum_{k=1}^n \left\{ \frac{1}{2} (\overline{\omega_{2k-1, 2l-1}} - \omega_{2k, 2l})(e_{2k-1}) + \omega_{2k-1, 2l-1}(\overline{e_{2k-1}}) \right. \\ & \quad + \frac{1}{2} (\omega_{2k-1, 2l} + \overline{\omega_{2k, 2l-1}})(e_{2k}) + \omega_{2k, 2l-1}(\overline{e_{2k}}) \\ & \quad \left. + \frac{1}{2n} (\omega_{2k-1, 2k-1} + \omega_{2k, 2k})(e_{2l-1}) + \frac{1}{n} \omega_{2k-1, 2k}(\overline{e_{2l}}) \right\} = 0, \end{aligned} \quad (3.18)$$

and

$$\begin{aligned} & \sum_{k=1}^n \left\{ \frac{1}{2} (\overline{\omega_{2k-1, 2l}} + \omega_{2k, 2l-1})(e_{2k-1}) + \omega_{2k-1, 2l}(\overline{e_{2k-1}}) \right. \\ & \quad + \frac{1}{2} (-\omega_{2k-1, 2l-1} + \overline{\omega_{2k, 2l}})(e_{2k}) + \omega_{2k, 2l}(\overline{e_{2k}}) \\ & \quad \left. + \frac{1}{2n} (\omega_{2k-1, 2k-1} + \omega_{2k, 2k})(e_{2l}) - \frac{1}{n} \omega_{2k-1, 2k}(\overline{e_{2l-1}}) \right\} = 0. \end{aligned} \quad (3.19)$$

We shall postpone the proof of this lemma to the next section.

**Definition 3.3** We say that a hyper CR structure is *ultra-pseudoconvex* if the symmetric bilinear form  $h$  on the subbundle  $Q$  defined by

$$h(X, Y) = (2n + 4) \text{Levi}_\theta(X, Y) - \sum_{a=1}^3 d\theta_a(X, I_a Y), \quad X, Y \in Q,$$

is positive or negative definite for some (hence any) compatible,  $\mathbb{R}^3$ -valued one-form  $\theta$ .

Since the component of  $h$  invariant under  $I, J, K$  is  $(2n + 1)g$ , a hyper CR structure is strongly pseudoconvex if it is ultra-pseudoconvex. The sphere  $S^{4n+3}$  and the quaternionic Heisenberg group  $\mathcal{H}^{4n+3}$  are ultra-pseudoconvex, since on these hyper CR manifolds, the three complex Levi forms  $d\theta_a(\cdot, I_a \cdot)$  coincide on  $Q$  and therefore they are equal to  $\text{Levi}_\theta$ . It is easy to see that strictly convex real hypersurfaces in  $\mathbb{H}^{n+1}$  are ultra-pseudoconvex. In particular, the ellipsoids as in Section 2 are ultra-pseudoconvex. We give a less obvious example.

**Example 6** Let  $(M, \theta)$  be the hyper pseudohermitian manifold as in Example 5. Recall that  $M$  is the total space of the (trivial)  $T^3$ -bundle  $\pi: M \rightarrow N$  over the hyperhermitian Hopf surface  $(N, g, I, J, K)$ . We have

$$\text{Levi}_\theta = \pi^* g = \frac{2(|dz|^2 + |dw|^2)}{|z|^2 + |w|^2}$$

and

$$\begin{aligned} h &= 6 \text{Levi}_\theta - \sum_{a=1}^3 d\theta_a(\cdot, I_a \cdot) \\ &= \frac{4(|dz|^2 + |dw|^2)}{|z|^2 + |w|^2} + \frac{2(\bar{z}dz + zd\bar{z} + \bar{w}dw + wd\bar{w})^2}{(|z|^2 + |w|^2)^2} \\ &= 2 \text{Levi}_\theta + 2(d\mu)^2. \end{aligned} \tag{3.20}$$

Therefore, the hyper CR structure of  $M$  is ultra-pseudoconvex.

**Example 7** For the hyper pseudohermitian structure as in Example 4,

we have

$$\begin{aligned}
 h = \frac{1}{2} \sum_{\alpha} \left\{ \right. & \left( (n+2)\Lambda_{\alpha} - B_{\alpha}^1 - C_{\alpha}^2 - D_{\alpha}^3 - \sum_{a=1}^3 A_{\alpha}^a \right) (dx_{\alpha}^0)^2 \\
 & + \left( (n+2)\Lambda_{\alpha} - A_{\alpha}^1 - D_{\alpha}^2 - C_{\alpha}^3 - \sum_{a=1}^3 B_{\alpha}^a \right) (dx_{\alpha}^1)^2 \\
 & + \left( (n+2)\Lambda_{\alpha} - D_{\alpha}^1 - A_{\alpha}^2 - B_{\alpha}^3 - \sum_{a=1}^3 C_{\alpha}^a \right) (dx_{\alpha}^2)^2 \\
 & \left. + \left( (n+2)\Lambda_{\alpha} - C_{\alpha}^1 - B_{\alpha}^2 - A_{\alpha}^3 - \sum_{a=1}^3 D_{\alpha}^a \right) (dx_{\alpha}^3)^2 \right\}.
 \end{aligned}$$

Note that  $h$  can be degenerate or even indefinite according to various choices of  $A_{\alpha}^a, \dots, D_{\alpha}^a$  (e.g. if  $A_{\alpha}^a = n+1$  and  $B_{\alpha}^a = C_{\alpha}^a = D_{\alpha}^a = -n/3$ , then  $h$  is indefinite). Thus the hyper CR structure may not be ultra-pseudoconvex, even though it is strongly pseudoconvex.

Note that  $h$  remains unchanged under the deformation of hyper CR structure and  $\theta$  as in (1.10), (1.12). Thus the definition of ultra-pseudoconvexity extends to the quaternionic CR structure.

**Theorem 3.4** *Let  $(M, \theta)$  be an ultra-pseudoconvex hyper pseudohermitian manifold of dimension  $> 7$ . Then there exists a unique admissible three-plane field  $Q^{\perp}$  such that the corresponding connection  $\nabla$  as in Proposition 3.1 satisfies (3.17).*

We call  $Q^{\perp}$  of the theorem the *canonical three-plane field*, and the corresponding admissible triple  $(T_1, T_2, T_3)$  the *canonical triple*. The corresponding connection, denoted by  $D$ , is a quaternionic analogue of the Tanaka-Webster connection in complex CR geometry. We call it the *canonical connection* associated with  $(M, \theta)$ .

*Proof of Theorem 3.4.* Fix an arbitrary admissible triple  $(T_1, T_2, T_3)$  of reference, and let  $\nabla$  be the corresponding connection given by Proposition 3.1. Let  $\widehat{T}_a = T_a + 2I_a V$  for  $V \in \Gamma(Q)$ . We will show that  $V$  can be chosen uniquely so that  $\widehat{\nabla}$ , the connection corresponding to  $(\widehat{T}_1, \widehat{T}_2, \widehat{T}_3)$ , satisfies (3.17).

First, we have for  $X, Y \in Q$ ,

$$[X, Y]_{\widehat{Q}} = [X, Y]_Q + 2 \sum_{a=1}^3 d\theta_a(X, Y)I_a V, \quad (3.21)$$

where  $[X, Y]_{\widehat{Q}}$  is the  $Q$ -component of  $[X, Y]$  with respect to  $(\widehat{T}_1, \widehat{T}_2, \widehat{T}_3)$ . Using (3.21) and (3.6) with  $(Y, Z) = (e_I, e_J)$ , we obtain

$$\begin{aligned} \widehat{\omega}_{IJ}(X) &= \omega_{IJ}(X) + \sum_{a=1}^3 d\theta_a(X, e_I)g(I_a V, e_J) \\ &\quad - \sum_{a=1}^3 d\theta_a(X, e_J)g(I_a V, e_I) - \sum_{a=1}^3 d\theta_a(e_I, e_J)g(I_a V, X). \end{aligned} \quad (3.22)$$

Here,  $\omega_{IJ}$  (resp.  $\widehat{\omega}_{IJ}$ ) are connection forms of  $\nabla$  (resp.  $\widehat{\nabla}$ ), and the indices  $I, J$  range over  $1, 2, \dots, 2n, \bar{1}, \bar{2}, \dots, \bar{2n}$ .

Let  $\omega_{2l-1}$  (resp.  $\widehat{\omega}_{2l-1}$ ) denote the left-hand side of (3.18), computed for  $\nabla$  (resp.  $\widehat{\nabla}$ ). Then by using (3.22), we obtain

$$\begin{aligned} &\widehat{\omega}_{2l-1} - \omega_{2l-1} \\ &= \sum_{a=1}^3 \left\{ - \left( \frac{3}{2} + \frac{1}{2n} \right) d\theta_a(I_a V, e_{2l-1}) \right. \\ &\quad - \left( \frac{1}{2} + \frac{1}{2n} \right) \sum_{k=1}^n (d\theta_a(e_{2k-1}, \overline{e_{2k-1}}) + d\theta_a(e_{2k}, \overline{e_{2k}}))g(I_a V, e_{2l-1}) \\ &\quad - \left( 1 + \frac{1}{n} \right) \sum_{k=1}^n d\theta_a(e_{2k-1}, e_{2k})g(I_a V, \overline{e_{2l}}) \\ &\quad + \frac{1}{n} \sum_{k=1}^n [(d\theta_a(\overline{e_{2k}}, e_{2l-1}) - d\theta_a(e_{2k-1}, \overline{e_{2l}}))g(I_a V, e_{2k}) \\ &\quad \quad \left. + (d\theta_a(\overline{e_{2k-1}}, e_{2l-1}) + d\theta_a(e_{2k}, \overline{e_{2l}}))g(I_a V, e_{2k-1})] \right\}. \end{aligned} \quad (3.23)$$

The right-hand side is simplified as follows. Since  $d\theta_a$  is  $I_a$ -invariant,

the sum on the second line vanishes if  $a = 2, 3$ . We compute the sum for  $a = 1$ :

$$\begin{aligned}
& \sum_{k=1}^n (d\theta_1(e_{2k-1}, \overline{e_{2k-1}}) + d\theta_1(e_{2k}, \overline{e_{2k}}))g(I_1V, e_{2l-1}) \\
&= - \sum_{k=1}^n (d\theta_1(e_{2k-1}, \overline{e_{2k-1}}) + d\theta_1(e_{2k}, \overline{e_{2k}}))g(V, I_1e_{2l-1}) \\
&= \sum_{k=1}^n (d\theta_1(e_{2k-1}, I_1\overline{e_{2k-1}}) + d\theta_1(\overline{e_{2k}}, I_1e_{2k}))g(V, e_{2l-1}) \\
&= \sum_{k=1}^n 2g(e_{2k-1}, \overline{e_{2k-1}})g(V, e_{2l-1}) \\
&= 2n g(V, e_{2l-1}). \tag{3.24}
\end{aligned}$$

The sum on the third line vanishes if  $a = 1$ ; for  $a = 2$ , we compute:

$$\begin{aligned}
\sum_{k=1}^n d\theta_2(e_{2k-1}, e_{2k})g(I_2V, \overline{e_{2l}}) &= \sum_{k=1}^n d\theta_2(e_{2k-1}, I_2\overline{e_{2k-1}})g(V, e_{2l-1}) \\
&= \sum_{k=1}^n g(e_{2k-1}, \overline{e_{2k-1}})g(V, e_{2l-1}) \\
&= n g(V, e_{2l-1}). \tag{3.25}
\end{aligned}$$

Likewise, for  $a = 3$ , we obtain

$$\sum_{k=1}^n d\theta_3(e_{2k-1}, e_{2k})g(I_3V, \overline{e_{2l}}) = n g(V, e_{2l-1}). \tag{3.26}$$

We compute the sum on the fourth and fifth lines:

$$\begin{aligned}
& \sum_{k=1}^n [(d\theta_a(\overline{e_{2k}}, e_{2l-1}) - d\theta_a(e_{2k-1}, \overline{e_{2l}}))g(I_aV, e_{2k}) \\
& \quad + (d\theta_a(\overline{e_{2k-1}}, e_{2l-1}) + d\theta_a(e_{2k}, \overline{e_{2l}}))g(I_aV, e_{2k-1})]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{2n} [(d\theta_a(\bar{e}_i, e_{2l-1}) + d\theta_a(I_2\bar{e}_i, \bar{e}_{2l}))g(I_a V, e_i) \\
&\quad + (d\theta_a(e_i, e_{2l-1}) + d\theta_a(I_2 e_i, \bar{e}_{2l}))g(I_a V, \bar{e}_i)] \\
&\quad - \sum_{i=1}^{2n} (d\theta_a(e_i, e_{2l-1}) + d\theta_a(I_2 e_i, \bar{e}_{2l}))g(I_a V, \bar{e}_i) \\
&= d\theta_a(I_a V, e_{2l-1}) + d\theta_a(I_2 I_a V, \bar{e}_{2l}) \\
&\quad - \sum_{i=1}^{2n} (d\theta_a(e_i, e_{2l-1}) + d\theta_a(I_2 e_i, \bar{e}_{2l}))g(I_a V, \bar{e}_i).
\end{aligned}$$

We find that the right-hand side is equal to  $2[g(V, e_{2l-1}) - d\theta_a(V, I_a e_{2l-1})]$  for all  $a$ . Indeed, if  $a = 1$ , the sum on the second line vanishes, and

$$\begin{aligned}
d\theta_1(I_2 I_1 V, \bar{e}_{2l}) &= d\theta_1(I_2 V, I_1 I_2 e_{2l-1}) \\
&= (d\theta_1(I_2 V, I_1 I_2 e_{2l-1}) + d\theta_1(V, I_1 e_{2l-1})) - d\theta_1(V, I_1 e_{2l-1}) \\
&= 2g(V, e_{2l-1}) - d\theta_1(V, I_1 e_{2l-1}).
\end{aligned}$$

If  $a = 2$ , the sum on the first line is equal to  $2d\theta_2(I_2 V, e_{2l-1})$ , and the sum on the second line becomes

$$-2 \sum_{i=1}^{2n} d\theta_2(e_i, I_2 \bar{e}_{2l})g(I_2 V, \bar{e}_i) = -2g(I_2 V, \bar{e}_{2l}) = -2g(V, e_{2l-1}),$$

since  $d\theta_2(e_i, I_2 \bar{e}_{2l}) = g(e_i, \bar{e}_{2l}) = \delta_{i, 2l}$ . The  $a = 3$  case is similar.

We thus conclude that

$$\begin{aligned}
&\widehat{\omega}_{2l-1} - \omega_{2l-1} \\
&= \left(-3n - 3 + \frac{6}{n}\right)g(V, e_{2l-1}) + \left(\frac{3}{2} - \frac{3}{2n}\right) \sum_{a=1}^3 d\theta_a(V, I_a e_{2l-1}) \\
&= -\frac{3n-3}{2n}h(V, e_{2l-1}).
\end{aligned}$$

We obtain a similar result for the left-hand side  $\omega_{2l}$  of (3.19), and conclude that  $\widehat{\omega}^{E \otimes H}$  vanishes if and only if

$$\frac{3n-3}{2n}h(V, e_{2l-1}) = \omega_{2l-1} \quad \text{and} \quad \frac{3n-3}{2n}h(V, e_{2l}) = \omega_{2l}$$

for  $l = 1, \dots, n$ . By ultra-pseudoconvexity, there exists a unique  $V$  which satisfies this last system of linear equations.  $\square$

We now extend the above construction of the canonical connection for a hyper pseudohermitian structure to that for a quaternionic pseudohermitian structure. To do this, it suffices to verify that given a quaternionic pseudohermitian structure, the condition (3.17) over each local hyper pseudohermitian structure is actually global. Let  $(M, \Theta)$  be an ultra-pseudoconvex quaternionic pseudohermitian manifold of dimension  $> 7$ . There is a global  $Sp(n) \cdot Sp(1)$ -bundle  $Q$  over  $M$ , and let  $\mathcal{P}$  be the bundle of frames for  $Q$  which are adapted (cf. (3.10)) with respect to some triple  $(I_a)$  of compatible complex structures; this is a principal  $Sp(n) \cdot Sp(1)$ -bundle over  $M$ . Let  $\nabla$  be any  $SO(4n)$ -connection on  $Q$ , and for any local section  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{4n})$  of  $\mathcal{P}$ , let  $\omega$  be the corresponding matrix of (real-valued) connection forms, given by  $\nabla\varepsilon = \varepsilon \otimes \omega$ . Note that this  $\omega$  is essentially the same as the previous one (when  $\nabla$  is the connection given by Proposition 3.1); as a collection of local matrix-valued forms, the present  $\omega$  is the expression of the previous one in terms of real frames (3.10) rather than complex ones (3.11). As before, we regard  $\omega$  as being defined on  $Q$ . If  $\varepsilon$  changes as  $\varepsilon \mapsto \varepsilon a$ , where  $a$  is a local  $Sp(n) \cdot Sp(1)$ -valued function, then  $\omega$  transforms as  $\omega \mapsto a^{-1}\omega a + a^{-1}da$ .

Now let  $\iota$  be the standard representation of  $Sp(n) \cdot Sp(1)$  on  $\mathbb{H}^n$ ,  $\iota^*$  its dual, and let  $\text{Ad}$  be the adjoint representation of  $SO(4n)$  on its Lie algebra  $so(4n)$ . By restriction, the last representation induces one of  $Sp(n) \cdot Sp(1)$  on  $(sp(n) + sp(1))^\perp$ , which we denote by the same symbol. We then consider the representation  $\iota^* \otimes \text{Ad}$  of  $Sp(n) \cdot Sp(1)$  on  $\mathbb{H}^{n*} \otimes (sp(n) + sp(1))^\perp$ , and construct the vector bundle

$$\mathcal{E} = \mathcal{P} \times_{\iota^* \otimes \text{Ad}} (\mathbb{H}^{n*} \otimes (sp(n) + sp(1))^\perp) = Q^* \otimes \mathcal{P} \times_{\text{Ad}} (sp(n) + sp(1))^\perp.$$

Let  $\omega^{\text{obs}}$  be the  $(sp(n) + sp(1))^\perp$ -component of  $\omega$ . Then the above transformation law for  $\omega$  ensures that the local forms  $\omega^{\text{obs}}$  give a global section of  $\mathcal{E}$ . According to the irreducible decomposition (3.16), the bundle  $\mathcal{E} \otimes \mathbb{C}$  splits and  $\omega^{\text{obs}}$  thereby decomposes, both globally. Therefore,  $\omega^{E \otimes H}$ , the  $E \otimes H$ -component of  $\omega^{\text{obs}}$ , is also global.

By the observation we just made, we obtain the following conclusion.

**Theorem 3.5** *Let  $(M, \Theta = \{\theta_U\})$  be an ultra-pseudoconvex quaternionic pseudohermitian manifold of dimension  $> 7$ . Then the local canonical three-plane fields  $\{(Q^\perp)_U\}$  and the local canonical connections  $\{D_U\}$  for local hyper pseudohermitian structures patch together to give a global admissible three-plane field  $Q^\perp$  and a global connection  $D$ , respectively.*

**Definition 3.6** *Let  $(M, \Theta)$  be an ultra-pseudoconvex quaternionic pseudohermitian manifold of dimension  $> 7$ . We call  $Q^\perp$  and  $D$  of Theorem 3.5 the *canonical three-plane field* and the *canonical connection*, respectively, associated with  $\Theta$ .*

We now derive, for future use, the transformation law for the canonical triple under a conformal change of (hyper) pseudohermitian structure.

**Proposition 3.7** *Let  $(M, \theta)$  be an ultra-pseudoconvex hyper pseudohermitian manifold of dimension  $> 7$ . Let  $\theta' = e^{2f}\theta$ , and  $(T_1, T_2, T_3)$  (resp.  $(T'_1, T'_2, T'_3)$ ) the canonical triple corresponding to  $\theta$  (resp.  $\theta'$ ). Then we have*

$$T'_a = e^{-2f}(T_a + 2I_a W),$$

where  $W \in \Gamma(Q)$  is uniquely determined by

$$h(W, X) = -(2n + 1)d_b f(X), \quad X \in Q. \tag{3.27}$$

*Proof.* We regard  $W$  as the unknown and verify that it must satisfy (3.27). Set  $\alpha = d_b f$ ,  $g = \text{Levi}_\theta$  and  $g' = \text{Levi}_{\theta'} = e^{2f}g$ . Let  $D$  (resp.  $D'$ ) be the canonical connection for  $\theta$  (resp.  $\theta'$ ). As in the proof of Theorem 3.4, we obtain for  $X \in Q$ ,

$$\begin{aligned} \omega'_{IJ}(X) &= \omega_{IJ}(X) + \alpha(X)g(e_I, e_J) + \alpha(e_I)g(X, e_J) - \alpha(e_J)g(X, e_I) \\ &\quad + \sum_{a=1}^3 d\theta_a(X, e_I)g(I_a W, e_J) - \sum_{a=1}^3 d\theta_a(X, e_J)g(I_a W, e_I) \\ &\quad - \sum_{a=1}^3 d\theta_a(e_I, e_J)g(I_a W, X), \end{aligned} \tag{3.28}$$

where  $\omega_{IJ}$  (resp.  $\omega'_{IJ}$ ) are connection forms of  $D$  (resp.  $D'$ ),  $\{e_1, \dots, e_{2n}\}$

is a  $g$ -unitary frame as in (3.11) and the indices  $I, J$  range over  $1, 2, \dots, 2n, \bar{1}, \bar{2}, \dots, \bar{2n}$ . It should be also noted that  $\omega'_{I, J}$  are computed with respect to the  $g'$ -unitary frame  $\{e^{-f}e_i\}$ . Since the last three sums on the right-hand side appear in (3.22), we can use the computation in the proof of Theorem 3.4. Denoting the left-hand sides of (3.18), (3.19) by  $\omega_{2l-1}$ ,  $\omega_{2l}$ , respectively, we obtain

$$e^f \omega'_i - \omega_i = -\frac{3n-3}{2n} \left\{ h(e_i, W) + (2n+1)\alpha(e_i) \right\}$$

for  $i = 1, \dots, 2n$ . Again, note that  $\omega'_i$  are computed with respect to  $\{e^{-f}e_i\}$ . Since the left-hand sides of these identities vanish, we must have  $h(e_i, W) = -(2n+1)\alpha(e_i)$  for  $i = 1, \dots, 2n$ . This completes the proof of Proposition 3.7.  $\square$

**Remark 4** For the sphere  $S^{4n+3}$ , we have  $h = (2n+1)\text{Levi}_\theta$ . Therefore, (3.27) gives  $W = -d_b f^\#$ , which is consistent with the transformation law (2.5).

We conclude this section with some comments on the curvature of the canonical connection. Let  $(M, \Theta)$  be an ultra-pseudoconvex quaternionic pseudohermitian manifold of dimension  $> 7$ , and  $D$  the associated canonical connection. Let  $R$  and  $\text{Ric}$  denote the curvature and Ricci tensors of  $D$ , respectively. For  $X, Y \in Q$ , we have  $\text{Ric}(X, Y) = \sum_{i=1}^{4n} g(R(\varepsilon_i, X)Y, \varepsilon_i)$ , where  $\{\varepsilon_1, \dots, \varepsilon_{4n}\}$  is an orthonormal basis for  $Q$  with respect to the Levi form  $g = \text{Levi}_\theta$ . The *pseudohermitian Ricci tensor*  $r$  is the component of  $\text{Ric}|_Q$  (restriction to  $Q$ ) which is symmetric and invariant under  $I, J, K$ . The *pseudohermitian scalar curvature* is  $s = \text{tr}_g(\text{Ric}|_Q) = \sum_{i=1}^{4n} \text{Ric}(\varepsilon_i, \varepsilon_i)$ .

Let  $\theta_S$  and  $\theta_H$  be the standard pseudohermitian structures of the sphere  $S^{4n+3}$  and the quaternionic Heisenberg group  $\mathcal{H}^{4n+3}$ , respectively. Recall from Section 2 that they are related by  $\theta_S = e^{2f}\sigma\theta_H\sigma^{-1}$  for some real-valued function  $f$  and  $Sp(1)$ -valued function  $\sigma$ .

The curvature of  $\theta_H$  vanishes identically, and the curvature of  $\theta_S$  coincides with that of  $e^{2f}\theta_H$ . There are formulas computing the curvature of the pseudohermitian structure of the form  $e^{2f}\theta_H$ , and by using them, we obtain  $r_{\theta_S} = 2(n+2)\text{Levi}_{\theta_S}$  and  $s_{\theta_S} = 8n(n+2)$ .

In a future work, we shall study the curvature of quaternionic pseudohermitian manifold in detail.

#### 4. Proof of Lemma 3.2

Let  $\omega$ ,  $\omega^{\text{obs}}$  and  $\omega^{E \otimes H}$  be the forms as in the previous section. Recall that we regard them as being defined on  $Q$ . Then we have

**Lemma 4.1** *The coefficients of  $\omega^{E \otimes H}$  corresponding to a standard basis of  $E \otimes H$  are given by the left-hand sides of (3.18) and (3.19) in Lemma 3.2 with  $l = 1, \dots, n$ , and their complex conjugates.*

The rest of this section is devoted to the proof of Lemma 4.1.

To prove Lemma 4.1, we start by making the correspondences (3.12) and (3.13) more explicit. For (3.12), let  $I : \mathbb{H}^n \rightarrow \mathbb{H}^n$  be the complex structure given by the right multiplication of  $\mathbf{i}^{-1}$ , and set  $V = \{X \in \mathbb{H}^n \otimes \mathbb{C} \mid IX = \sqrt{-1}X\}$ , so that we have  $\mathbb{H}^n \otimes \mathbb{C} = V \oplus \bar{V}$ . Let  $(\varepsilon_1, \dots, \varepsilon_{4n})$  be the standard basis for  $\mathbb{H}^n = \mathbb{R}^{4n}$ , and define a complex basis for  $V$  by  $e_{2k-1} = (\varepsilon_{4k-3} - \sqrt{-1}\varepsilon_{4k-2})/\sqrt{2}$ ,  $e_{2k} = (\varepsilon_{4k-1} - \sqrt{-1}\varepsilon_{4k})/\sqrt{2}$  ( $k = 1, \dots, n$ ). Also, let  $(\mathbf{e}_1, \dots, \mathbf{e}_{2n})$  and  $(\mathbf{f}_1, \mathbf{f}_2)$  respectively denote the standard basis for  $E = \mathbb{C}^{2n}$  and  $H = \mathbb{C}^2$ . Then the correspondence

$$e_{2k-1} \leftrightarrow \mathbf{e}_{2k-1} \otimes \mathbf{f}_2, \quad e_{2k} \leftrightarrow \mathbf{e}_{2k} \otimes \mathbf{f}_2, \quad \overline{e_{2k-1}} \leftrightarrow -\mathbf{e}_{2k} \otimes \mathbf{f}_1, \quad \overline{e_{2k}} \leftrightarrow \mathbf{e}_{2k-1} \otimes \mathbf{f}_1$$

( $k = 1, \dots, n$ ) gives an isomorphism  $\mathbb{H}^n \otimes \mathbb{C} \cong E \otimes H$ . For (3.13), we can find the elements of  $(sp(n) + sp(1))^\perp \otimes \mathbb{C}$  corresponding to generators of  $\Lambda_0^2 E \otimes S^2 H$ , by tracing the isomorphisms in (3.14) backwards:

$$\begin{aligned} (\mathbf{e}_{2k-1} \wedge \mathbf{e}_{2l-1}) \otimes (\mathbf{f}_1 \cdot \mathbf{f}_1) &\leftrightarrow \overline{e_{2k}} \wedge \overline{e_{2l}}, \\ (\mathbf{e}_{2k-1} \wedge \mathbf{e}_{2l-1}) \otimes (\mathbf{f}_2 \cdot \mathbf{f}_2) &\leftrightarrow e_{2k-1} \wedge e_{2l-1}, \\ (\mathbf{e}_{2k-1} \wedge \mathbf{e}_{2l-1}) \otimes (\mathbf{f}_1 \cdot \mathbf{f}_2) &\leftrightarrow \frac{1}{2}(e_{2k-1} \wedge \overline{e_{2l}} - e_{2l-1} \wedge \overline{e_{2k}}), \\ (\mathbf{e}_{2k} \wedge \mathbf{e}_{2l}) \otimes (\mathbf{f}_1 \cdot \mathbf{f}_1) &\leftrightarrow \overline{e_{2k-1}} \wedge \overline{e_{2l-1}}, \\ (\mathbf{e}_{2k} \wedge \mathbf{e}_{2l}) \otimes (\mathbf{f}_2 \cdot \mathbf{f}_2) &\leftrightarrow e_{2k} \wedge e_{2l}, \\ (\mathbf{e}_{2k} \wedge \mathbf{e}_{2l}) \otimes (\mathbf{f}_1 \cdot \mathbf{f}_2) &\leftrightarrow \frac{1}{2}(-\overline{e_{2k-1}} \wedge e_{2l} + \overline{e_{2l-1}} \wedge e_{2k}), \\ (\mathbf{e}_{2k} \wedge \mathbf{e}_{2l-1})_0 \otimes (\mathbf{f}_1 \cdot \mathbf{f}_1) &\leftrightarrow -\overline{e_{2k-1}} \wedge \overline{e_{2l}} + \frac{1}{n} \delta_{kl} \sum_{m=1}^n \overline{e_{2m-1}} \wedge \overline{e_{2m}}, \end{aligned}$$

$$\begin{aligned}
(\mathbf{e}_{2k} \wedge \mathbf{e}_{2l-1})_0 \otimes (\mathbf{f}_2 \cdot \mathbf{f}_2) &\leftrightarrow e_{2k} \wedge e_{2l-1} + \frac{1}{n} \delta_{kl} \sum_{m=1}^n e_{2m-1} \wedge e_{2m}, \\
(\mathbf{e}_{2k} \wedge \mathbf{e}_{2l-1})_0 \otimes (\mathbf{f}_1 \cdot \mathbf{f}_2) &\leftrightarrow \frac{1}{2} (-\overline{e_{2k-1}} \wedge e_{2l-1} - \overline{e_{2l}} \wedge e_{2k}) \\
&\quad + \frac{1}{2n} \delta_{kl} \sum_{m=1}^n (\overline{e_{2m-1}} \wedge e_{2m-1} + \overline{e_{2m}} \wedge e_{2m}).
\end{aligned}$$

The isomorphism  $S^3 H \oplus H \cong H \otimes S^2 H$  embeds  $H$  into  $H \otimes S^2 H$  by

$$s_H : w \in H \mapsto \mathbf{f}_1 \otimes (\mathbf{f}_2 \cdot w) - \mathbf{f}_2 \otimes (\mathbf{f}_1 \cdot w) \in H \otimes S^2 H.$$

Likewise, the isomorphism  $K \oplus \Lambda_0^3 E \oplus E \cong E \otimes \Lambda_0^2 E$  embeds  $E$  into  $E \otimes \Lambda_0^2 E$  by

$$s_E : w \in E \mapsto \sum_{k=1}^n [\mathbf{e}_{2k-1} \otimes (\mathbf{e}_{2k} \wedge w)_0 - \mathbf{e}_{2k} \otimes (\mathbf{e}_{2k-1} \wedge w)_0] \in E \otimes \Lambda_0^2 E.$$

We are now ready to give

*Proof of Lemma 4.1.* We shall identify the coefficient of  $\omega^{E \otimes H}$  corresponding to  $\mathbf{e}_{2l-1} \otimes \mathbf{f}_1 \in E \otimes H$ , to which corresponds the following element of  $(\mathbb{H}^n \otimes \Lambda^2 \mathbb{H}^n) \otimes \mathbb{C}$ :

$$\begin{aligned}
&\sum_{k=1}^n \left\{ \overline{e_{2k}} \otimes \left[ \frac{1}{2} (-\overline{e_{2k-1}} \wedge e_{2l-1} - \overline{e_{2l}} \wedge e_{2k}) \right. \right. \\
&\quad \left. \left. + \frac{1}{2n} \delta_{kl} \sum_{m=1}^n (\overline{e_{2m-1}} \wedge e_{2m-1} + \overline{e_{2m}} \wedge e_{2m}) \right] \right. \\
&\quad \left. + \overline{e_{2k-1}} \otimes \frac{1}{2} (e_{2k-1} \wedge \overline{e_{2l}} - e_{2l-1} \wedge \overline{e_{2k}}) \right. \\
&\quad \left. - e_{2k-1} \otimes \left[ -\overline{e_{2k-1}} \wedge \overline{e_{2l}} + \frac{1}{n} \delta_{kl} \sum_{m=1}^n \overline{e_{2m-1}} \wedge \overline{e_{2m}} \right] + e_{2k} \otimes (\overline{e_{2k}} \wedge \overline{e_{2l}}) \right\}.
\end{aligned}$$

Note that at each point  $q \in M$ , we can express  $\omega$  as

$$\omega = \sum_{1 \leq i < j \leq 2n} \omega_{ij} \otimes \varphi_i \wedge \varphi_j + \sum_{1 \leq i < j \leq 2n} \omega_{i\bar{j}} \otimes \overline{\varphi_i} \wedge \overline{\varphi_j} + \sum_{i,j=1}^{2n} \omega_{i\bar{j}} \otimes \varphi_i \wedge \overline{\varphi_j},$$

where  $(\varphi_1, \dots, \varphi_{2n})$  is the dual of the unitary basis  $(e_1, \dots, e_{2n})$  for  $(Q_q)^{1,0} \cong V$ . Then the coefficient of  $\omega^{E \otimes H}$  corresponding to  $\mathbf{e}_{2l-1} \otimes \mathbf{f}_1$  is given by

$$\begin{aligned} & \sum_{k=1}^n \left\{ \frac{1}{2} (-\omega_{\overline{2k-1}, 2l-1} - \omega_{\overline{2l}, 2k})(\overline{e_{2k}}) + \frac{1}{2n} (\omega_{\overline{2k-1}, 2k-1} + \omega_{\overline{2k}, 2k})(\overline{e_{2l}}) \right. \\ & \quad + \frac{1}{2} (\omega_{2k-1, \overline{2l}} - \omega_{2l-1, \overline{2k}})(\overline{e_{2k-1}}) + \omega_{\overline{2k-1}, \overline{2l}}(e_{2k-1}) \\ & \quad \left. - \frac{1}{n} \omega_{\overline{2k-1}, \overline{2k}}(e_{2l-1}) + \omega_{\overline{2k}, \overline{2l}}(e_{2k}) \right\}, \end{aligned}$$

which is the complex conjugate of the left-hand side of (3.19). Likewise, computing the coefficients corresponding to the other basis elements of  $E \otimes H$ , we obtain the left-hand sides of (3.18), their complex conjugates and those of (3.19).  $\square$

### 5. Comparison to quaternionic contact structure

As mentioned in the introduction, some quaternionic analogues of CR structures other than those in this paper have been studied by several authors (cf. [1], [2], [4], [7]).

In this section, we first review the definition of quaternionic contact structure, introduced by Biquard [4], and the canonical connection, called the Biquard connection, associated with a choice of metric. We then compare the quaternionic CR structure to the quaternionic contact structure. We observe that while a quaternionic contact structure can always be “extended” to a quaternionic CR structure, the quaternionic contact structure is more restrictive than the quaternionic CR structure.

**Definition 5.1** A *quaternionic contact structure* on a  $(4n+3)$ -dimensional manifold  $M$  is a corank three bundle  $Q$  equipped with a  $CSp(n) \cdot Sp(1)$ -structure satisfying a compatibility condition. That is, we have a conformal class  $[\gamma]$  of metrics on  $Q$  and a two-sphere bundle  $\mathbb{I}$  over  $M$  of complex structures  $I: Q \rightarrow Q$ ,  $I^2 = -\text{Id}$ , and these satisfy the following conditions:

- (i)  $\gamma(IX, IY) = \gamma(X, Y)$  for all  $I \in \mathbb{I}$  and  $X, Y \in Q$ .
- (ii)  $\mathbb{I}$  locally admits sections  $I_a$ ,  $a = 1, 2, 3$ , satisfying the quaternion relations  $I_1 I_2 = -I_2 I_1 = I_3$  and  $\mathbb{I} = \{v_1 I_1 + v_2 I_2 + v_3 I_3 \mid v_1^2 + v_2^2 + v_3^2 = 1\}$ .
- (iii)  $Q$  is locally the kernel of an  $\mathbb{R}^3$ -valued one-form  $\eta = (\eta_1, \eta_2, \eta_3)$  satisfying the compatibility relations

$$\gamma(I_a X, Y) = d\eta_a(X, Y), \quad a = 1, 2, 3, \quad (5.1)$$

where  $X, Y \in Q$ .

Note that (5.1) is equivalent to  $\gamma(X, Y) = d\eta_a(X, I_a Y)$ ; in particular, (5.1) implies the condition (i). Note also that if (5.1) holds, then for any other triple  $(I'_1, I'_2, I'_3)$  of complex structures as in the condition (ii), there exists an  $\mathbb{R}^3$ -valued one-form  $\eta' = (\eta'_1, \eta'_2, \eta'_3)$  so that the compatibility relations  $\gamma(I'_a X, Y) = d\eta'_a(X, Y)$  hold. Indeed, if  $I'_a = \sum_{p=1}^3 s_{ap} I_p$ , where  $(s_{ap})$  is an  $SO(3)$ -valued function, then it suffices to choose  $\eta'_a = \sum_{p=1}^3 s_{ap} \eta_p$ . (Actually, this is a unique choice of  $\eta'_a$ , as verified by argument similar to that in the proof of Proposition 5.3 below.)

On a quaternionic contact manifold of dimension  $> 7$  with a choice of metric  $\gamma$  on  $Q$  in the conformal class, Biquard constructed a canonical connection  $D^B$ , called the *Biquard connection* (cf. [5] for the seven-dimensional case). He also gave a distinguished rank three subbundle  $Q^\perp$  of  $TM$  complementary to  $Q$ . The connection  $D^B$  and our canonical connection  $D$  on a quaternionic pseudohermitian manifold are similar but differ in some respects: first,  $D^B$  preserves the  $Sp(n) \cdot Sp(1)$ -structure of  $Q$ , while  $D$  does not in general, because of the generality of our structure; second, the torsion tensor  $\text{Tor}$  of  $D^B$ , restricted to  $Q \times Q^\perp$ , has no  $Q^\perp$ -component and is more sensitive to the  $GL(n, \mathbb{H}) \cdot Sp(1)$ -structure of  $Q$ , because of the “quaternionic extension” used in the construction of  $D^B$ .

The bundle  $Q^\perp$  can be explicitly described. Choose a local  $\mathbb{R}^3$ -valued one-form  $(\eta_1, \eta_2, \eta_3)$  as in Definition 5.1. Then  $Q^\perp$  is locally generated by vector fields  $\{R_a\}_{a=1,2,3}$  characterized by

$$\eta_a(R_b) = \delta_{ab}, \quad d\eta_a(R_a, X) = 0, \quad X \in Q, \quad (5.2)$$

and they further satisfy

$$d\eta_b(R_a, X) = -d\eta_a(R_b, X), \quad X \in Q.$$

Let  $M$  be a quaternionic contact manifold, with the associated corank three subbundle  $Q$  of  $TM$  and two-sphere bundle  $\mathbb{I}$  of complex structures of  $Q$ . Then there are quaternionic CR structures having  $(Q, \mathbb{I})$  as the underlying structure. To show this, fix a metric  $\gamma$  on  $Q$  and choose an arbitrary rank three bundle  $Q^\perp$  transverse to  $Q$ . First, we construct a local hyper CR structure. So choose  $(I_1, I_2, I_3)$  locally and then choose a local  $\mathbb{R}^3$ -valued one-form  $(\eta_1, \eta_2, \eta_3)$  so that (5.1) holds. Since  $\eta_a|_{Q^\perp}$  form a local coframe for  $Q^\perp$ , there is a unique triple  $(T_1, T_2, T_3)$  of local sections of  $Q^\perp$  such that  $\eta_a(T_b) = \delta_{ab}$ . Then set  $Q_a = Q \oplus \mathbb{R}T_b \oplus \mathbb{R}T_c$ , and extend  $I_a: Q \rightarrow Q$  to  $I_a: Q_a \rightarrow Q_a$  by defining  $I_aT_b = T_c$  and  $I_aT_c = -T_b$ . Note that we have  $\ker \eta_a = Q_a$  and  $\eta_a \circ I_b = \eta_c$ , and  $\{(Q_a, I_a)\}_{a=1,2,3}$  satisfies the conditions for an almost hyper CR structure. Moreover, it is integrable. Indeed, for  $X, Y \in \Gamma(Q)$ , we compute using (5.1):

$$\begin{aligned} \eta_a([X, Y] - [I_aX, I_aY]) &= -d\eta_a(X, Y) + d\eta_a(I_aX, I_aY) \\ &= -\gamma(I_aX, Y) - \gamma(X, I_aY) \\ &= 0, \end{aligned}$$

$$\begin{aligned} &\eta_b(I_a([X, Y] - [I_aX, I_aY]) - ([X, I_aY] + [I_aX, Y])) \\ &= -\eta_c([X, Y] - [I_aX, I_aY]) - \eta_b([X, I_aY] + [I_aX, Y]) \\ &= d\eta_c(X, Y) - d\eta_c(I_aX, I_aY) + d\eta_b(X, I_aY) + d\eta_b(I_aX, Y) \\ &= \gamma(I_cX, Y) - \gamma(I_cI_aX, I_aY) + \gamma(I_bX, I_aY) + \gamma(I_bI_aX, Y) \\ &= 0, \end{aligned}$$

and likewise,

$$\eta_c(I_a([X, Y] - [I_aX, I_aY]) - ([X, I_aY] + [I_aX, Y])) = 0.$$

In this way, for each local choice of  $(I_1, I_2, I_3)$ , we have the corresponding local hyper CR structure  $\{(Q_a, I_a)\}$ . We now verify that two such local hyper CR structures  $\{(Q_a, I_a)\}$  and  $\{(Q'_a, I'_a)\}$  satisfy the gluing condition (1.10) if they overlap. Suppose that  $I'_a = \sum_p s_{ap}I_p$  as endomorphisms of  $Q$ , where  $(s_{ap})$  is an  $SO(3)$ -valued function. Choose  $\eta'_a = \sum_p s_{ap}\eta_p$  and

$T'_1, T'_2, T'_3$  be the corresponding local sections of  $Q^\perp$ . We must show that

$$T'_a = \sum_p s_{ap} T_p \quad \text{and} \quad I'_a = \left( \sum_p s_{ap} \tilde{I}_p \right) \Big|_{Q'_a}, \quad (5.3)$$

where the notation  $\tilde{I}_p$  is as in Section 1. Since the former relations are clear, it suffices to verify the latter relations of (5.3). Set  $I''_a = \sum_p s_{ap} \tilde{I}_p$ . We compute

$$I''_a T'_b = \sum_{p,q} s_{ap} s_{bq} \tilde{I}_p T_q,$$

and restricting the indices  $p, q$  to those which extends to a cyclic permutation  $(p, q, r)$  of  $(1, 2, 3)$ , we further rewrite the right-hand side as

$$\sum (s_{ap} s_{bq} - s_{aq} s_{bp}) \tilde{I}_p T_q = \sum_r s_{cr} T_r = T'_c.$$

Thus  $I''_a = I'_a$ , which gives the latter relations of (5.3).

The above construction actually gives a quaternionic pseudohermitian structure such that the associated Levi form is the metric  $\gamma$ , and therefore, we have the canonical three-plane field  $(Q^\perp)'$ . While  $(Q^\perp)'$  differs from  $Q^\perp$  in general,  $(Q^\perp)' = Q^\perp$  holds when  $Q^\perp$  is Biquard's one, locally generated by the vector fields  $\{R_a\}_{a=1,2,3}$  satisfying the Reeb condition (5.2). We record this fact as the following

**Proposition 5.2** *Let  $M$  be a quaternionic contact manifold of dimension  $> 7$  with a choice of metric  $\gamma$  on the corank three bundle  $Q$ . Let  $Q^\perp$  be the rank three bundle locally generated by the vector fields  $\{R_a\}_{a=1,2,3}$  satisfying (5.2), and equip  $M$  with a quaternionic pseudohermitian structure in the way as above. Then  $Q^\perp$  gives the canonical three-plane field associated with the quaternionic pseudohermitian structure.*

*Proof.* Let  $D^B$  be the Biquard connection associated with the metric  $\gamma$ , and  $\nabla$  the affine connection, given by Proposition 3.1, associated with the quaternionic pseudohermitian structure and the admissible three-plane field  $Q^\perp$ . When regarded as  $Q$ -partial connections, they coincide with each other, since they are characterized by the same condition (3.6). We know that  $D^B$

restricts to an  $Sp(n) \cdot Sp(1)$ -connection on  $Q$ , and therefore  $\nabla$  restricts to a  $Q$ -partial connection preserving the  $Sp(n) \cdot Sp(1)$  structure of  $Q$ . So the obstruction  $\omega^{\text{obs}}$  vanishes, and in particular,  $\omega^{E \otimes H} = 0$ . This means that  $Q^\perp$  is the canonical three-plane field (and  $\nabla$  is the canonical connection) associated with the quaternionic pseudohermitian structure. We are done.  $\square$

The following proposition generalizes [6, Proposition 2.1], which characterizes a quaternionic contact real hypersurface in a quaternionic manifold, to an arbitrary quaternionic CR manifold. It shows that the above mentioned coincidence of the Levi form with the metric  $\gamma$  is actually the case for any quaternionic CR structure which has  $(Q, \mathbb{I})$  as the underlying structure.

**Proposition 5.3** *Let  $M$  be a quaternionic contact manifold, with the associated corank three subbundle  $Q$  of  $TM$ , two-sphere bundle  $\mathbb{I}$  of complex structures of  $Q$  and conformal class  $[\gamma]$  of metrics on  $Q$ . Then for any quaternionic CR structure on  $M$  having  $(Q, \mathbb{I})$  as the underlying structure, the conformal class consisting of all Levi forms coincides with  $[\gamma]$ . Furthermore, if  $\theta = (\theta_1, \theta_2, \theta_3)$  is a local  $\mathbb{R}^3$ -valued one-form compatible with a local hyper CR structure  $\{(Q_a, I_a)\}_{a=1,2,3}$  (constituting the quaternionic CR structure), then  $\text{Levi}_\theta$ ,  $(I_1, I_2, I_3)$  and  $\theta$  satisfy the compatibility relations*

$$\text{Levi}_\theta(X, Y) = d\theta_a(X, I_a Y), \quad a = 1, 2, 3, \tag{5.4}$$

where  $X, Y \in Q$ . In particular, the quaternionic CR structure under consideration must be ultra-pseudoconvex.

*Proof.* Let  $(I_1, I_2, I_3)$  be as in the statement of the proposition. The condition (iii) says that there exists a local  $\mathbb{R}^3$ -valued one-form  $\eta = (\eta_1, \eta_2, \eta_3)$  such that the kernel of  $\eta$  coincides with  $Q$  and (5.1) holds:  $\gamma(I_a X, Y) = d\eta_a(X, Y)$ . First observe that  $\theta_a$  may be expressed as  $\theta_a = \sum_{p=1}^3 s_{ap} \eta_p$ , where  $(s_{ap})$  is a  $GL(3, \mathbb{R})$ -valued function. Then for  $X, Y \in Q$ ,

$$d\theta_a(X, Y) = \sum_p s_{ap} d\eta_p(X, Y) = \sum_p s_{ap} \gamma(I_p X, Y) = \gamma(J_a X, Y),$$

where we set  $J_a = \sum_p s_{ap} I_p$ . We compute  $d\theta_a(X, I_a Y)$  in two ways:

$$d\theta_a(X, I_a Y) = \gamma(J_a X, I_a Y) = -\gamma(I_a J_a X, Y)$$

and

$$d\theta_a(X, I_a Y) = -d\theta_a(I_a X, Y) = -\gamma(J_a I_a X, Y).$$

This implies  $I_a J_a = J_a I_a$  as endomorphisms of  $Q$ , and therefore  $J_a$  is a multiple of  $I_a$  by a scalar-valued function:  $J_a = \lambda_a I_a$ ,  $\lambda_a \neq 0$ . Now  $d\theta_a(X, I_a Y) = \lambda_a \gamma(X, Y)$ , and so  $\text{Levi}_\theta(X, Y) = \lambda_a \gamma(X, Y)$ . Therefore,  $\lambda_1 = \lambda_2 = \lambda_3$ , and denoting this function by  $\lambda$ , we have  $d\theta_a(X, I_a Y) = \text{Levi}_\theta(X, Y) = \lambda \gamma(X, Y)$ .  $\square$

We now look at a real hypersurface  $M$  in a quaternionic manifold, and compare the quaternionic CR structure to the quaternionic contact structure in this case. As explained in Section 2,  $M$  has a canonical quaternionic CR structure, and therefore there exists a canonical corank three subbundle  $Q$  of  $TM$ , together with a canonical two-sphere bundle  $\mathbb{I}$  of complex structures of  $Q$  as in (ii) of the above definition. In contrast, a real hypersurface in a quaternionic manifold does not admit in general a quaternionic contact structure which the canonical  $(Q, \mathbb{I})$  underlies. Ellipsoids as in Section 2 supply concrete examples; an ellipsoid in  $\mathbb{H}^{n+1}$  does not admit a quaternionic contact structure having the canonical  $(Q, \mathbb{I})$  as the underlying structure, unless the ellipsoid is a quaternionic one. This follows from Proposition 5.3. Indeed, for an ellipsoid which is not quaternionic, we observed in Section 2 that the complex Levi forms  $\text{Levi}_{\theta_a} = d\theta_a(\cdot, I_a \cdot)$  do not coincide on  $Q$  for the standard choice of  $\theta$ . In particular, (5.4) cannot hold. Note that, since any ellipsoid in  $\mathbb{H}^{n+1}$  is diffeomorphic to the sphere  $S^{4n+3}$ , it does admit a quaternionic contact structure by pulling back that of the sphere. However, the underlying structure  $(Q, \mathbb{I})$  is different from the canonical one of the ellipsoid.

The hyper CR manifolds of Example 4 and Example 5 give concrete examples of intrinsic quaternionic CR manifold which does not admit a quaternionic contact structure with the same underlying structure  $(Q, \mathbb{I})$ . Indeed, for the pseudohermitian structure  $\theta$  of Example 4, (5.4) holds if and only if

$$A_\alpha^1 + B_\alpha^1 = C_\alpha^1 + D_\alpha^1 = A_\alpha^2 + C_\alpha^2 = B_\alpha^2 + D_\alpha^2 = A_\alpha^3 + D_\alpha^3 = B_\alpha^3 + C_\alpha^3 = \frac{\Lambda_\alpha}{2}$$

for all  $\alpha$ . In other words, unless this last condition is satisfied, (5.4) cannot

hold. For  $\theta$  of Example 5, (3.20) shows that  $h$  and  $\text{Levi}_\theta$  are not proportional, and therefore (5.4) cannot hold.

## 6. Appendix

### 6.1. All $(Q_{\mathbf{v}}, I_{\mathbf{v}})$ satisfy (1.6) and (1.7)

Let  $M$  be a hyper CR manifold, and fix a function  $\mathbf{v} = (v_1, v_2, v_3)$  with values in  $S^2 \subset \mathbb{R}^3$ . In this subsection, we will show that  $(Q_{\mathbf{v}}, I_{\mathbf{v}})$  satisfies the conditions (1.6) and (1.7) for all  $X, Y \in \Gamma(Q)$ .

Let  $\theta = (\theta_1, \theta_2, \theta_3)$  be an  $\mathbb{R}^3$ -valued one-form on  $M$  compatible with the hyper CR structure. Set  $\theta_{\mathbf{v}} := v_1\theta_1 + v_2\theta_2 + v_3\theta_3$ , so that  $Q_{\mathbf{v}} = \ker \theta_{\mathbf{v}}$ . Fix an admissible triple  $(T_1, T_2, T_3)$  such that  $\theta_a(T_a) = 1$ . Recall from Section 1 that endomorphisms  $\tilde{I}_a$  of  $TM$  are defined by setting  $\tilde{I}_a X = I_a X$  for  $X \in Q_a$  and  $\tilde{I}_a T_a = 0$ , and that  $I_{\mathbf{v}} = (v_1\tilde{I}_1 + v_2\tilde{I}_2 + v_3\tilde{I}_3)|_{Q_{\mathbf{v}}}$ . We will use the following relations:

$$\theta_c = \theta_a \circ \tilde{I}_b = -\theta_b \circ \tilde{I}_a, \quad \theta_a \circ \tilde{I}_a = 0, \tag{6.1}$$

$$\tilde{I}_c = \begin{cases} \tilde{I}_a I_b & \text{on } Q_b, \\ -\tilde{I}_b I_a & \text{on } Q_a. \end{cases} \tag{6.2}$$

**Proposition 6.1** *For any  $S^2$ -valued function  $\mathbf{v}$ ,  $(Q_{\mathbf{v}}, I_{\mathbf{v}})$  satisfies*

$$[X, Y] - [I_{\mathbf{v}}X, I_{\mathbf{v}}Y] \in \Gamma(Q_{\mathbf{v}}), \tag{6.3}$$

$$I_{\mathbf{v}}([X, Y] - [I_{\mathbf{v}}X, I_{\mathbf{v}}Y]) - [X, I_{\mathbf{v}}Y] - [I_{\mathbf{v}}X, Y] \in \Gamma(Q) \tag{6.4}$$

for all  $X, Y \in \Gamma(Q)$ .

*Proof.* We first prove (6.3) by verifying  $\theta_{\mathbf{v}}([X, I_{\mathbf{v}}Y] + [I_{\mathbf{v}}X, Y]) = 0$  for  $X, Y \in \Gamma(Q)$ . Plug (1.7) into  $\theta_c$ , use (1.5) and replace  $Y$  by  $I_c Y$ . We then obtain

$$\theta_b([X, I_c Y] + [I_a X, I_b Y]) + \theta_c([X, I_b Y] - [I_a X, I_c Y]) = 0.$$

Rewriting this as

$$\theta_b([X, I_c Y] + [I_b I_c X, I_b Y]) = -\theta_c([X, I_b Y] + [I_c I_b X, I_c Y])$$

and using (1.6), we conclude

$$\theta_b([X, I_c Y] + [I_c X, Y]) = -\theta_c([X, I_b Y] + [I_b X, Y]), \quad (6.5)$$

which also holds when  $b = c$ . Therefore,

$$\theta_{\mathbf{v}}([X, I_{\mathbf{v}} Y] + [I_{\mathbf{v}} X, Y]) = \sum_{b,c} v_b v_c \theta_b([X, I_c Y] + [I_c X, Y]) = 0.$$

Note that the terms involving the derivatives of  $v_c$  disappear, since  $\theta_b$  vanishes on  $Q$ . This proves (6.3).

Next we prove (6.4) by showing that

$$\theta_a(I_{\mathbf{v}}([X, I_{\mathbf{v}} Y] + [I_{\mathbf{v}} X, Y]) + [X, Y] - [I_{\mathbf{v}} X, I_{\mathbf{v}} Y]) = 0$$

for each  $a$ . The left-hand side is computed as

$$\begin{aligned} & \sum_{b,c} v_b v_c \theta_a(\tilde{I}_b([X, I_c Y] + [I_c X, Y]) - [I_b X, I_c Y]) + \theta_a([X, Y]) \\ &= \sum_b v_b^2 \theta_a(I_b([X, I_b Y] + [I_b X, Y]) + [X, Y] - [I_b X, I_b Y]) \\ &+ \sum_{b \neq a} v_b v_a \theta_a(\tilde{I}_b([X, I_a Y] + [I_a X, Y]) + \tilde{I}_a([X, I_b Y] + [I_b X, Y]) \\ &\quad - [I_b X, I_a Y] - [I_a X, I_b Y]) \\ &+ v_b v_c \theta_a(\tilde{I}_b([X, I_c Y] + [I_c X, Y]) + \tilde{I}_c([X, I_b Y] + [I_b X, Y]) \\ &\quad - [I_b X, I_c Y] - [I_c X, I_b Y]). \end{aligned}$$

Note that the sum over  $b, c$  in the left-hand side is divided into four parts: the sums over  $b = c$ ,  $b \neq c = a$ ,  $c \neq b = a$ , and  $b \neq c \neq a \neq b$ . The second and third sums are grouped into the second sum of the right-hand side, and the fourth sum into the last term, in which  $b, c$  are so that  $(a, b, c)$  is a cyclic permutation of  $(1, 2, 3)$ . The first sum of the right-hand side vanishes by (1.7); the last term by (6.1), (1.6). The second sum also vanishes, since

$$\begin{aligned} \theta_a(\tilde{I}_b([X, I_a Y] + [I_a X, Y])) &= \theta_c([X, I_a Y] + [I_a X, Y]) \\ &= -\theta_a([X, I_c Y] + [I_c X, Y]) = \theta_a([I_a X, I_b Y] + [I_b X, I_a Y]) \end{aligned}$$

by (6.1), (6.5) and (1.6). This completes the proof of Proposition 6.1.  $\square$

**6.2. Levi form**

In this subsection, by applying Proposition 6.1, we prove Proposition 1.7, asserting that the Levi form on a quaternionic CR manifold is well-defined.

**Lemma 6.2** *Let  $\mathbf{u}, \mathbf{v}$  be mutually orthogonal unit vectors in  $\mathbb{R}^3$  and  $X, Y \in Q$ . Then  $d\theta_{\mathbf{u}}(I_{\mathbf{v}}X, I_{\mathbf{v}}Y)$  is independent of the choice of  $\mathbf{v}$  orthogonal to  $\mathbf{u}$ .*

*Proof.* Let  $\mathbf{v}'$  be another unit vector orthogonal to  $\mathbf{u}$ . Then  $\mathbf{v}'$  can be expressed as  $\mathbf{v}' = \lambda\mathbf{v} + \mu\mathbf{u} \times \mathbf{v}$  with  $\lambda^2 + \mu^2 = 1$ , and we have

$$\begin{aligned} d\theta_{\mathbf{u}}(I_{\mathbf{v}'}X, I_{\mathbf{v}'}Y) &= \lambda^2 d\theta_{\mathbf{u}}(I_{\mathbf{v}}X, I_{\mathbf{v}}Y) + \mu^2 d\theta_{\mathbf{u}}(I_{\mathbf{u}}I_{\mathbf{v}}X, I_{\mathbf{u}}I_{\mathbf{v}}Y) \\ &\quad + \lambda\mu[d\theta_{\mathbf{u}}(I_{\mathbf{v}}X, I_{\mathbf{u}}I_{\mathbf{v}}Y) + d\theta_{\mathbf{u}}(I_{\mathbf{u}}I_{\mathbf{v}}X, I_{\mathbf{v}}Y)]. \end{aligned}$$

Since  $d\theta_{\mathbf{u}}$ , restricted to  $Q$ , is  $I_{\mathbf{u}}$ -invariant, the right-hand side is equal to  $d\theta_{\mathbf{u}}(I_{\mathbf{v}}X, I_{\mathbf{v}}Y)$ .  $\square$

*Proof of Proposition 1.7.* Let  $\theta_U = (\theta_a)$  and  $\theta_{U'} = (\theta'_a)$ . Then  $\theta'_a = \sum_{p=1}^3 s_{ap}\theta_p$  for an  $SO(3)$ -valued function  $(s_{ap})$ . We must show that

$$d\theta'_1(X, I'_1Y) + d\theta'_1(I'_2X, I'_3Y) = d\theta_1(X, I_1Y) + d\theta_1(I_2X, I_3Y)$$

for all  $X, Y \in Q$ . Since  $d\theta'_1 = \sum_{p=1}^3 (s_{1p}d\theta_p + ds_{1p} \wedge \theta_p)$  and  $\theta_p$ 's vanish on  $Q$ , we may assume that  $s_{ap}$ 's are constants. Therefore, it suffices to verify that

$$d\theta_{\mathbf{u}}(X, I_{\mathbf{u}}Y) + d\theta_{\mathbf{u}}(I_{\mathbf{v}}X, I_{\mathbf{u} \times \mathbf{v}}Y) = d\theta_1(X, I_1Y) + d\theta_1(I_2X, I_3Y) \quad (6.6)$$

for all  $X, Y \in Q$ , where  $\mathbf{u}, \mathbf{v}$  are any mutually orthogonal unit vectors in  $\mathbb{R}^3$ .

Note that by Lemma 6.2, the left-hand side of (6.6) is independent of  $\mathbf{v}$  (orthogonal to  $\mathbf{u}$ ). It is easy to verify that one can choose  $\mathbf{v}$  orthogonal to  $\mathbf{u}$ , and  $\mathbf{v}'$  orthogonal to  $\mathbf{u}' = \mathbf{u} \times \mathbf{v}$ , so that  $\mathbf{u}' \times \mathbf{v}' = \mathbf{e}_1 = {}^t(1, 0, 0)$ .

Using (6.4) and  $\theta_{\mathbf{u}} \circ I_{\mathbf{v}} = \theta_{\mathbf{u} \times \mathbf{v}}$  (cf. (6.1)), we can rewrite the left-hand side of (6.6) as

$$\begin{aligned}
& d\theta_{\mathbf{u}}(X, I_{\mathbf{u}}Y) - d\theta_{\mathbf{u}}(I_{\mathbf{v}}X, I_{\mathbf{v}}(I_{\mathbf{u}}Y)) \\
&= -d\theta_{\mathbf{u} \times \mathbf{v}}(X, I_{\mathbf{v}}(I_{\mathbf{u}}Y)) - d\theta_{\mathbf{u} \times \mathbf{v}}(I_{\mathbf{v}}X, I_{\mathbf{u}}Y) \\
&= d\theta_{\mathbf{u}'}(X, I_{\mathbf{u}'}Y) - d\theta_{\mathbf{u}'}(I_{\mathbf{v}}X, I_{\mathbf{v}}(I_{\mathbf{u}'}Y)) \\
&= d\theta_{\mathbf{u}'}(X, I_{\mathbf{u}'}Y) - d\theta_{\mathbf{u}'}(I_{\mathbf{v}'}X, I_{\mathbf{v}'}(I_{\mathbf{u}'}Y)). \tag{6.7}
\end{aligned}$$

For the last equality, we have used Lemma 6.2 again to replace  $\mathbf{v}$  by  $\mathbf{v}'$ . Computing similarly as in (6.7) while noting that  $\mathbf{v}'$  and  $\mathbf{e}_2 = {}^t(0, 1, 0)$  are both orthogonal to  $\mathbf{e}_1$ , we can verify that the last expression is equal to the right-hand side of (6.6).  $\square$

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