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On h-vectors of Buchsbaum Stanley-Reisner rings

Naoki TERAI

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Abstract. We give a necessary condition for a sequence of integers to be the h-vector of a Buchsbaum complex (or equivalently a Buchsbaum Stanley-Reisner ring). We construct 3-dimensional Buchsbaum Stanley-Reisner rings with depth 2 which give lower bounds of the h-vectors among those of the Buchsbaum Stanley-Reisner rings with the above conditions.

Key words: Stanley-Reisner ring Buchsbaum complex, f-vector, h-vector, Hilbert function, O-sequence.

Introduction

It is one of important problems to characterize the h-vectors (or equivalently f-vectors) of good classes of Stanley-Reisner rings (or equivalently simplicial complexes) in combinatorial commutative ring theory. See Björner-Kalai [Bj-Ka] to survey this topic.

Let f and i be positive integers. Then f can be uniquely written in the form

$$f=\binom{n_i}{i}+\binom{n_{i-1}}{i-1}+\cdots+\binom{n_j}{j},$$

where $n_i > n_{i-1} > \cdots > n_j \geq j \geq 1$. Define

$$\begin{aligned} f^{(i)} &= \binom{n_i}{i+1}+\binom{n_{i-1}}{i}+\cdots+\binom{n_j}{j+1}, \\ f^{<i>} &= \binom{n_i+1}{i+1}+\binom{n_{i-1}+1}{i}+\cdots+\binom{n_j+1}{j+1}, \\ 0^{<i>} &= 0. \end{aligned}$$

Then the following two results are classical.

Theorem 0.1 (Kruskal [Kr], Katona [Ka]). *Let $f=(f_0, f_1, \cdots, f_{d-1})$ be a sequence of integers. Then the following conditions are equivalent :*

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(1) *The vector f is the f-vector of some $(d-1)$-dimensional simplicial complex.*
(2) $0 < f_{i+1} \leq f_i^{(i+1)}$ *for* $0 \leq i \leq d-2$.

Theorem 0.2 (Macaulay, Stanley [St$_2$, Theorem 6]). *Let $h = (h_0, h_1, \cdots, h_d)$ be a sequence of integers. Then the following conditions are equivalent :*
(1) *The vector h is the h-vector of some $(d-1)$-dimensional Cohen-Macaulay simplicial complex over a field.*
(2) $h_0 = 1$ *and* $0 \leq h_{i+1} \leq h_i^{<i>}$ *for* $1 \leq i \leq d-1$.

We say that a sequence $h = (h_0, h_1, \cdots, h_d)$ of integers is an *O-sequence* if it satisfies the conditions in Theorem 0.2.

Then the following problem is very natural.

Problem 0.3 (Hibi [Hi$_1$, Open Problem]). Find a combinatorial characterization of the h-vectors of Buchsbaum simplicial complexes.

This paper gives partial results on the above problem. In fact, we give a necessary condition to be the h-vectors of Buchsbaum simplicial complexes as follows:

Theorem 0.4 *Let Δ be a $(d-1)$-dimensional Buchsbaum complex over a field, where $d \geq 2$, and $h(\Delta) = (h_0, h_1, \cdots, h_d)$ its h-vector. Then we have following inequalities :*

$$
\begin{aligned}
&dh_d + h_{d-1} \geq 0, \\
&\binom{d}{2} h_d + (d-1)h_{d-1} + h_{d-2} \geq 0, \\
&\binom{d}{3} h_d + \binom{d-1}{2} h_{d-1} + (d-2)h_{d-2} + h_{d-3} \geq 0, \\
&\binom{d+1}{4} h_d + \binom{d}{3} h_{d-1} + \binom{d-1}{2} h_{d-2} + (d-2)h_{d-3} + h_{d-4} \geq 0, \\
&\qquad\vdots \\
&\binom{2d-3}{d} h_d + \binom{2d-4}{d-1} h_{d-1} + \cdots + (d-2)h_1 + h_0 \geq 0,
\end{aligned}
$$

$$\binom{2d-2}{d+1}h_d + \binom{2d-3}{d}h_{d-1} + \cdots + \binom{d-1}{2}h_1 + (d-2)h_0 \geq 0,$$

$$\vdots$$

$$.$$

In the last section we consider sufficiency of the above condition in 2-dimensional case. We construct some Buchsbaum complexes which give lower bounds among the h-vectors of 3-dimensional Buchsbaum Stanley-Reisner rings with depth 2.

1. Preliminaries

We first fix notation. Let $\mathbf{N}$ (resp. $\mathbf{Z}$) denote the set of nonnegative integers (resp. integers). For a real number a, we define

$$\lceil a \rceil = \min\{n \in \mathbf{Z} |\ n \geq a\},$$
$$\lfloor a \rfloor = \max\{n \in \mathbf{Z} |\ n \leq a\}.$$

Let $\sharp(S)$ denote the cardinality of a set S.

We recall some notation on simplicial complexes and Stanley-Reisner rings according to [Hi$_2$] and [St$_1$]. We refer the reader to, e.g., [Br–He], [Hi$_1$], [Ho] and [St$_1$] for the detailed information about combinatorial and algebraic background.

(1.1) A *simplicial complex* Δ on the *vertex set* $V = \{x_1, x_2, \ldots, x_v\}$ is a collection of subsets of V such that (i) $\{x_i\} \in \Delta$ for every $1 \leq i \leq v$ and (ii) $\sigma \in \Delta$, $\tau \subset \sigma \Rightarrow \tau \in \Delta$. Each element σ of Δ is called a *face* of Δ. We call $\sigma \in \Delta$ *i-face* if $\sharp(\sigma) = i+1$ We set $d = \max\{\sharp(\sigma) \mid \sigma \in \Delta\}$ and define the *dimension* of Δ to be $\dim \Delta = d-1$. We say that Δ is *pure* if every maximal face has the same cardinality.

We say that a simplicial complex Δ is *spanned* by $\{\sigma_1, \cdots, \sigma_s\}$ if $\Delta = \mathbf{2}^{\sigma_1} \cup \cdots \cup \mathbf{2}^{\sigma_s}$, where $\mathbf{2}^{\sigma}$ is the family of all subsets of σ.

Let $f_i = f_i(\Delta)$, $0 \leq i \leq d-1$, denote the number of i-faces in Δ. We define $f_{-1} = 1$. We call $f(\Delta) = (f_0, f_1, \cdots, f_{d-1})$ the *f-vector* of Δ. Define the *h-vector* $h(\Delta) = (h_0, h_1, \cdots, h_d)$ of Δ by

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} t^{d-i}.$$

If σ is a face of Δ, then we define a subcomplex $\mathrm{link}_\Delta(\sigma)$ as follows:

$$\mathrm{link}_\Delta(\sigma) = \{\tau \in \Delta \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta\}.$$

Let $\tilde{H}_i(\Delta; k)$ denote the i-th *reduced simplicial homology group* of Δ with the coefficient field k. Note that $\tilde{H}_{-1}(\Delta; k) = 0$ if $\Delta \neq \{\emptyset\}$ and

$$\tilde{H}_i(\{\emptyset\}; k) = \begin{cases} 0 & (i \geq 0) \\ k & (i = -1). \end{cases}$$

(1.2) Let $R = k[x_1, x_2, \ldots, x_v]$ be the polynomial ring in v-variables over a field k. Here, we identify each $x_i \in V$ with the indeterminate x_i of R. Define I_Δ to be the ideal of R which is generated by square-free monomials $x_{i_1}x_{i_2}\cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq v$, with $\{x_{i_1}, x_{i_2}, \cdots, x_{i_r}\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := R/I_\Delta$ is the *Stanley-Reisner ring* of Δ over k. We consider $k[\Delta]$ as the graded algebra $k[\Delta] = \bigoplus_{n\geq 0} k[\Delta]_n$ with the standard grading, i.e., each $\deg x_i = 1$.

Let k be a field and A a neotherian graded k-algebra with $A_0 = k$. The *Hilbert series* of A is defined by

$$F(A, t) = \sum_{n\geq 0} (\dim_k A_n) t^n,$$

where $\dim_k A_n$ denotes the dimension of A_n as a k-vector space. When A is generated by A_1 as a k-algebra, it is well known that the Hilbert series $F(A, t)$ of A can be written in the form

$$F(A, t) = \frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^{\dim A}},$$

where $h_0(= 1)$, $h_1, \cdots, h_s$ are integers with $h_s \neq 0$. The vector $h(A) = (h_0, h_1, \cdots, h_s)$ is called the *h-vector* of A.

The Hilbert series $F(k[\Delta], t)$ of a Stanley-Reisner ring $k[\Delta]$ can be written as follows:

$$\begin{aligned} F(k[\Delta], t) &= 1 + \sum_{i=1}^{d-1} \frac{f_i t^{i+1}}{(1-t)^{i+1}} \\ &= \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d}, \end{aligned}$$

where $\dim \Delta = d-1$, $(f_0, f_1, \cdots, f_{d-1})$ is the f-vector of Δ, and $(h_0, h_1, \cdots, h_d)$ is the h-vector of Δ.

(1.3) A simplicial complex Δ is called *Buchsbaum* over a field k if it satisfies one of the following equivalent conditions:

(1) The Stanley-Reisner ring $k[\Delta]$ of Δ is Buchsbaum.

(2) (a) For every σ $(\neq \emptyset) \in \Delta$, and for every $i < \dim \operatorname{link}_\Delta(\sigma)$, $\tilde{H}_i(\operatorname{link}_\Delta(\sigma);\, k) = 0$

(b) Δ is pure.

See Stücklad-Vogel [St-Vo] for detailed information on Buchsbaum complexes.

2. H-vectors of Buchsbaum complexes

We give a necessary condition for h-vectors of Buchsbaum complexes.

Theorem 2.1 *Let Δ be a $(d-1)$-dimensional Buchsbaum complex over a field, where $d \geq 2$, and $h(\Delta) = (h_0, h_1, \cdots, h_d)$ its h-vector. Then we have the following inequalities :*

$$
\begin{aligned}
& d h_d + h_{d-1} \geq 0, \\
& \binom{d}{2} h_d + (d-1) h_{d-1} + h_{d-2} \geq 0, \\
& \binom{d}{3} h_d + \binom{d-1}{2} h_{d-1} + (d-2) h_{d-2} + h_{d-3} \geq 0, \\
& \binom{d+1}{4} h_d + \binom{d}{3} h_{d-1} + \binom{d-1}{2} h_{d-2} + (d-2) h_{d-3} + h_{d-4} \geq 0, \\
& \qquad \vdots \\
& \binom{2d-3}{d} h_d + \binom{2d-4}{d-1} h_{d-1} + \cdots + (d-2) h_1 + h_0 \geq 0, \\
& \binom{2d-2}{d+1} h_d + \binom{2d-3}{d} h_{d-1} + \cdots + \binom{d-1}{2} h_1 + (d-2) h_0 \geq 0, \\
& \qquad \vdots
\end{aligned}
$$

Proof. We may assume $\sharp(k) = \infty$. Let $e = \operatorname{depth} k[\Delta]$ and let $H^i_{\boldsymbol{m}}(k[\Delta])$ be the i-th local cohomology of $k[\Delta]$ with respect to the graded maximal ideal $\boldsymbol{m}$. By [St$_1$, Theorem 6.4] we have

$$\begin{aligned}\sum_{i=e}^{d}(-1)^i F(H^i_{\boldsymbol{m}}(k[\Delta]),\ t) &= F(k[\Delta],t)_\infty \\ &= \left(\frac{h_0+h_1t+\cdots+h_dt^d}{(1-t)^d}\right)_\infty \\ &= \frac{h_d+h_{d-1}t^{-1}+\cdots+h_0t^{-d}}{(-1)^d(1-t^{-1})^d} \\ &= (-1)^d(h_d+h_{d-1}t^{-1}+\cdots+h_0t^{-d}) \\ &\qquad \left(1+dt^{-1}+\binom{d+1}{2}t^{-2}+\cdots\right),\end{aligned}$$

where $F(k[\Delta],t)_\infty$ signifies that $F(k[\Delta],t)$ is to be expanded as a Laurent series around ∞. Since Δ is Buchsbaum, we have

$$F(H^i_{\boldsymbol{m}}(k[\Delta]),\ t) = \dim_k(H^i_{\boldsymbol{m}}(k[\Delta]))_0.$$

for $i < d$. Hence we have

$$\begin{aligned}F(H^d_{\boldsymbol{m}}(k[\Delta]),\ t) &= a+(dh_d+h_{d-1})t^{-1} \\ &\quad +\left(\binom{d+1}{2}h_d+dh_{d-1}+h_{d-2}\right)t^{-2}+\cdots,\end{aligned}$$

for some $a \in \mathbf{Z}$. Therefore we have

$$dh_d+h_{d-1} = \dim_k(H^d_{\boldsymbol{m}}(k[\Delta]))_{-1} \geq 0,$$

which is the first inequality.

Let $K_{k[\Delta]}$ be the canonical module of $k[\Delta]$. Then

$$\begin{aligned}F(K_{k[\Delta]},\ t) &= a+(dh_d+h_{d-1})t^1 \\ &\quad +\left(\binom{d+1}{2}h_d+dh_{d-1}+h_{d-2}\right)t^2+\cdots.\end{aligned}$$

By [Sch, Lemma 3.1.1] we have depth $K_{k[\Delta]} \geq 2$. Hence there exist $x,\ y \in (k[\Delta])_1$ such that $x,\ y$ is a $K_{k[\Delta]}$-sequence. Hence we can write

$$\begin{aligned}F(K_{k[\Delta]}/xK_{k[\Delta]},\ t) &= a+bt \\ &\quad +\left(\binom{d}{2}h_d+(d-1)h_{d-1}+h_{d-2}\right)t^2 \\ &\quad +\cdots.\end{aligned}$$

for some $b \in \mathbf{Z}$. Hence we have

$$\binom{d}{2}h_d + (d-1)h_{d-1} + h_{d-2} \geq 0,$$

which is the second inequality.

Similarly as above, we have

$$\begin{aligned} F(K_{k[\Delta]}/(x,\ y)K_{k[\Delta]},\ t) &= a + (b-a)t + ct^2 \\ &+ \left(\binom{d}{3}h_d + \binom{d-1}{2}h_{d-1} + (d-2)h_{d-2} + h_{d-3}\right)t^3 \\ &+ \cdots . \end{aligned}$$

for some $c \in \mathbf{Z}$. Then we have the remaining inequalities. □

The next proposition is essentially due to Schenzel.

Proposition 2.2 *Let Δ be a $(d-1)$-dimensional Buchsbaum complex, where $d \geq 2$, and $h(\Delta) = (h_0, h_1, \cdots, h_d)$ its h-vector. We put depth $k[\Delta] = e$. Then $(h_0,\ h_1, \cdots,\ h_e)$ is an O-sequence. In particular, we have $h_i \geq 0$ for $0 \leq i \leq e$.*

Proof. We may assume $\sharp(k) = \infty$. Let $y_1,\ y_2, \cdots y_d$ be a homogeneous system of parameters in $k[\Delta]_1$. By [Sch$_2$,Theorem 4.3], we have

$$F(k[\Delta]/(y_1,\ \cdots, y_d),\ t) = g_0 + g_1 t + \cdots + g_d t^d,$$

where

$$g_j = h_j + \binom{d}{j}\sum_{i=0}^{j-1}(-1)^{j-i-1}\dim_k(H^i_{\mathfrak{m}}(k[\Delta]))_0,$$

for $0 \leq j \leq d$.

Note that depth$k[\Delta] = e$ implies $H^i_{\mathfrak{m}}(k[\Delta]) = 0$ for $i < e$. Then we have $h_j = g_j \geq 0$ for $j \leq e$. □

Then we conjecture the following:

Conjecture 2.3 Let $h = (h_0,\ h_1, \cdots, h_d)$ be an integer sequence, where $d \geq 2$, and let k be a field. Then the following conditions are equivalent:

(1) There exists $(d-1)$-dimensional Buchsbaum complex with $\dim k[\Delta] -$ depth $k[\Delta] \leq 1$ such that $h = h(\Delta)$.

(2) $(h_0,\ h_1, \cdots, h_{d-1})$ is an O-sequence and $-\frac{1}{d}h_{d-1} \leq h_d \leq h_{d-1}^{<d-1>}$ holds.

Remark 2.4.
(1) Conjecture 2.3 holds in the case of $d=2$. In fact, a 1-dimensional complex Δ is Buchsbaum if and only if Δ is pure. And the 1-dimensional simplicial complexes Δ always satisfy the condition $\dim k[\Delta] - \operatorname{depth} k[\Delta] \leq 1$. The f-vectors of 1-dimensional pure complexes can be characterized by the conditions $f_0 \geq 0$ and $\frac{f_0}{2} \leq f_1 \leq \binom{f_0}{2}$, which is equivalent to the condition (2) in Conjecture 2.3.
(2) In Conjecture 2.3, (1) $\Rightarrow$ (2) always holds by Theorem 2.1.

3. 2-dimensional Buchsbaum complexes

In this section, we consider the case of $d=3$. Let $h=(h_0,\ h_1,\ h_2,\ h_3)$ be the h-vector of a 2-dimensional Buchsbaum complex. Suppose h_3 is negative and we put $h_3=-n$, where $n>0$. Then we have $h_2>0$ by Theorem 2.1. Since $h_1=\binom{h_1}{1}$, we have $\binom{h_1+1}{2}=h_1^{<1>}\geq h_2\geq 3n$. Therefore $h_1^2+h_1-6n\geq 0$. We have $h_1\geq \frac{-1+\sqrt{24n+1}}{2}$.

Definition 3.1 Let n be a natural number. We call the sequence

$$(1,\ \left\lceil \frac{-1+\sqrt{24n+1}}{2} \right\rceil,\ 3n,\ -n)$$

a *lower bound sequence.*

The following question is a special case of Conjecture 2.3.

Question 3.2 Are all lower bound sequences the h-vectors of Buchsbaum complexes?

We construct some 2-dimensional Buchsbaum complexes whose h-vectors are lower bound sequences. For simplicity we fix the vertex set $V=\{1,2,\cdots,v\}$, where $v>3$.

Theorem 3.3 *Let Δ be the simplicial complex which is spanned by*

$$\begin{aligned} S=\{\{a,b,a+b\}\ &|\ 1\leq a<b,\ a+b\leq v\}\\ \cup\{\{a,b,c\}\ &|\ 1\leq a<b<c\leq v,\ a+b+c=2v+1\ \}. \end{aligned}$$

If $2v+1$ is a prime number, then Δ is Buchsbaum and

$$h(\Delta)=(1,\ v-3,\ \frac{(v-2)(v-3)}{2},\ -\frac{(v-2)(v-3)}{6}).$$

Corollary 3.4 *Let $v > 3$ be an integer such that $2v+1$ is a prime number. Then lower bound sequences*

$$(1,\ v-3,\ \frac{(v-2)(v-3)}{2},\ -\frac{(v-2)(v-3)}{6})$$

are the h-vectors of Buchsbaum complexes.

Corollary 3.5 *There exist infinite number of lower bound sequences which are the h-vectors of Buchsbaum complexes.*

To prove Theorem 3.3 we prepare the following lemma.

Lemma 3.6

$$\sharp(S) = \frac{v(v-2)}{3}.$$

Proof. For a fixed i we define

$$S_i = \{\{i,j,l\} \in S |\ i < j < l\}$$

For $i < \frac{v}{2}$ we have

$$\begin{aligned} S_i &= \{\{i,i+1,2i+1\},\ \{i,i+2,2i+2\},\cdots,\{i,v-i,v\}\} \\ &\cup\ \{\{i,v-i+1,v\},\ \{i,v-i+2,v-1\}, \\ &\qquad \cdots,\{i,v-\left\lceil\frac{i}{2}\right\rceil,v-\left\lfloor\frac{i}{2}\right\rfloor+1\}\}. \end{aligned}$$

Therefore we have

$$\sharp(S_i) = v - 2i + \left\lfloor\frac{i}{2}\right\rfloor.$$

For $i \geq \frac{v}{2}$ with $S_i \neq \emptyset$ we have

$$\begin{aligned} S_i &= \{\{i,i+1,2v-2i\},\ \{i,i+2,2v-2i-1\}, \\ &\qquad \cdots,\{i,v-\left\lceil\frac{i}{2}\right\rceil,v-\left\lfloor\frac{i}{2}\right\rfloor+1\}\}. \end{aligned}$$

Therefore we have

$$\sharp(S_i) = 2v-2i-(v-\left\lfloor\frac{i}{2}\right\rfloor) = v-2i+\left\lfloor\frac{i}{2}\right\rfloor.$$

Then

$$\sharp(S)=\sum_{\substack{i\geq 1\\ v-2i+\lfloor\frac{i}{2}\rfloor>0}}\sharp(S_i)=\sum_{\substack{i\geq 1\\ v-2i+\lfloor\frac{i}{2}\rfloor>0}}(v-2i+\left\lfloor\frac{i}{2}\right\rfloor).$$

Since $2v+1$ is prime, $v\equiv 0,2 \pmod 3$. First suppose $3|v$.

$$\begin{aligned}\sharp(S)&=(v-2)+(v-3)+(v-5)+\ldots+4+3+1\\&=\sum_{i=0}^{\frac{v}{3}-1}\{(3i+1)+3i\}\\&=\frac{v(v-2)}{3}.\end{aligned}$$

Next suppose $3|(v-2)$.

$$\begin{aligned}\sharp(S)&=(v-2)+(v-3)+(v-5)+\ldots+5+3+2\\&=\sum_{i=0}^{\frac{v-2}{3}}\{3i+(3i-1)\}\\&=\frac{v(v-2)}{3}.\end{aligned}$$

□

Proof of Theorem 3.3. Note that $\{a,b\}\in\Delta$ for $1\leq a<b\leq v$. In fact, if $a+b\leq v$, then $\{a,b,a+b\}\in\Delta$. If $a+b\geq v+1$ and $b\neq 2a$, then $\{b-a,a,b\}\in\Delta$. If $a+b\geq v+1$ and $b=2a$, then $\{a,b,(2v+1)-3a\}\in\Delta$.

By Lemma 3.6 we have

$$f(\Delta)=(v,\ \frac{v(v-1)}{2},\ \frac{v(v-2)}{3}),$$

and

$$h(\Delta)=(1,\ v-3,\ \frac{(v-2)(v-3)}{2},\ -\frac{(v-2)(v-3)}{6}).$$

We must prove that Δ is Buchsbaum. We have only to show that $\mathrm{link}_\Delta(\{a\})$ is connected for $1\leq a\leq v$. First we assume that $a\leq\frac{v}{2}$ and that a is even. Then there exist paths in $\mathrm{link}_\Delta(\{a\})$ as below:

$$\begin{array}{l}
1 \text{ —— } a+1 \text{ —— } 2a+1 \text{ —— } \cdots\cdots\cdots \text{ —— } na+1,\\
2 \text{ —— } a+2 \text{ —— } 2a+2 \text{ —— } \cdots\cdots\cdots \text{ —— } na+2,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{ —— } v,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
a-1 \text{ —— } 2a-1 \text{ —— } 3a-1 \text{ —— } \cdots \text{ —— } na-1,\\
\qquad 2a \text{ —— } 3a \text{ —— } \cdots \text{ —— } na.
\end{array}$$

Next we join two points by arcs as follows: For left end-points, we connect couples of numbers whose sums are a. For right end-points, we connect couples of numbers whose sums are $(2v+1)-a$. We claim that it becomes a segment. Hence it is connected. Put $p = 2v+1$. Since p and a are coprime, we have

$$\mathbf{Z}/a\mathbf{Z} = \{0,\ \pm p,\ \pm 2p, \cdots, \pm(\frac{a}{2}-1)p,\ \frac{a}{2}p\}.$$

Hence the above link is as follows:

$$\overline{0} \text{ —— } \overline{p} \text{ —— } \overline{(-p)} \text{ —— } \overline{2p} \text{ —— } \cdots \text{ —— } \overline{(-(\frac{a}{2}+1)p)} \text{ —— } \overline{\frac{a}{2}p},$$

where $\overline{lp}$ stands for

$$(m\text{—})(a+m)\text{—}(2a+m)\text{—}\cdots\text{—}\{(n-1)a+m\}(\text{—}(na+m))$$

with $m \equiv lp (\mathrm{mod}\ a)$. Then it is a segment. In the case that $a > \frac{v}{2}$ or a is odd, we can prove it by a similar fashion. $\square$

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Department of Mathematics
Faculty of Education
Saga University
Saga 840, Japan
E-mail: terai@cc.saga-u.ac.jp