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Symmetry problems for elliptic systems

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Abstract. We consider some overdetermined boundary value problems for elliptic systems. Using the maximum principle and the technique of moving up planes perpendicular to a fixed direction we show that if a solution exists, then the domain must be a ball and the solution radially symmetric.

Key words: Elliptic systems, maximum principle.

1. Introduction

Recently Payne and Schaefer [7] considered several overdetermined boundary value problems for the biharmonic operator. Among other things they proved the following theorem.

Theorem A [7]. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $C^{2+\varepsilon}$ boundary $\partial\Omega$. Let u be a classical solution of the boundary value problem*

$$\Delta^2 u = 1 \qquad in\ \Omega, \tag{1.1}$$

$$u = \Delta u = 0 \qquad on\ \partial\Omega. \tag{1.2}$$

If

$$\frac{\partial u}{\partial \nu} = c\,(const.) \qquad on\ \partial\Omega \tag{1.3}$$

(where ν denotes the unit outer normal to $\partial\Omega$) and Ω is star-shaped with respect to the origin, then Ω is a disk.

Payne and Schaefer conjectured that theorem A holds in $\mathbb{R}^n$ with $n > 2$ for more general domains. Our first purpose here is to prove this conjecture. In fact we shall consider a more general situation than (1.1)–(1.2). We shall prove the following theorem.

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with C^2 boundary $\partial\Omega$. Let $f : \mathbb{R}^2 \to (0, \infty)$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be two functions satisfying the*

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following conditions :

(H_1) *For each* $v \in \mathbb{R}$, $u \to f(u,v)$ (*resp.* $u \to g(u,v)$) *is nondecreasing* (*resp. is nonincreasing*);

(H_2) *For each* $u \in \mathbb{R}$, $v \to f(u,v)$ (*resp.* $v \to g(u,v)$) *is nonincreasing* (*resp. is strictly increasing*);

(H_3) $g(u,0) = 0$ *for* $u \in \mathbb{R}$.

If $(u,v) \in C^2(\overline{\Omega}) \times (C^2(\Omega) \cap C^1(\overline{\Omega}))$ *satisfies the system of differential equations*

$$\begin{cases} \Delta u = g(u,v) & \text{in } \Omega, \\ \Delta v = f(u,v) & \text{in } \Omega \end{cases} \tag{1.4}$$

and the boundary conditions (1.3) *and*

$$u = 0 \qquad \text{on } \partial\Omega, \tag{1.5}$$

$$v = d\,(const.) \leq 0 \qquad \text{on } \partial\Omega, \tag{1.6}$$

then Ω *is a ball. If* $\Omega = \{x \in \mathbb{R}^n;\, |x - x_0| < R\}$ *for some* $x_0 \in \mathbb{R}^n$, *then* $u(x) = y(|x-x_0|)$, $v(x) = z(|x-x_0|)$, $y' < 0$ *in* $(0,R]$ *and* $z' > 0$ *in* $(0,R]$.

Our method of proof is based on the maximum principle and the technique of moving parallel planes used by Serrin [8] and Gidas, Ni and Nirenberg [4] for second order equations and by the author [2], [3] for fourth order equations.

We shall use repeatedly the maximum principle and the Hopf boundary lemma which we recall. Let $D \subset \mathbb{R}^n$ be a domain and let $v \in C^2(D)$ satisfy the differential inequality $\Delta v \geq 0$ in D.

Maximum Principle (Gilbarg and Trudinger [5] p. 15). *If* $v \leq M$ *in* D *and* $v = M$ *at some point in* D, *then* $v \equiv M$ *in* D.

Hopf Lemma ([5] p. 33). *Let* $P \in \partial D$ *be such that :*

(i) v *is continuous at* P;

(ii) $v(x) < v(P)$ *for all* $x \in D$;

(iii) *There is a ball* B *in* D *with* $P \in \partial B$.

Then the outer normal derivative of v *at* P, *if it exists, satisfies the strict inequality* $\partial v(P)/\partial \nu > 0$.

Finally we also recall a version of the Hopf lemma which applies to domains with corners.

Lemma S (Serrin [8] p. 308). *Let $D^\star \subset \mathbb{R}^n$ be a domain with C^2 boundary and let T be a plane containing the normal to $\partial D^\star$ at some point Q. Let D be the portion of $D^\star$ lying on some particular side of T.*

Suppose that w is of class C^2 in $\overline{D}$ and satisfies

$$\Delta w \le 0 \qquad \text{in } D,$$

while also $w \ge 0$ in D and $w = 0$ at Q. Let $\vec{s}$ denote any direction at Q which enters D non tangentially. Then either

$$\frac{\partial w}{\partial \vec{s}}(Q) > 0 \qquad \text{or} \qquad \frac{\partial^2 w}{\partial \vec{s}^2}(Q) > 0$$

unless $w \equiv 0$ in D.

Our paper is organized as follows. In Section 2 we prove theorem 1. In Section 3 we show that our result can be applied to a somewhat more general boundary condition than (1.3) as in Serrin's paper. Finally in Section 4 we conclude with some remarks and we give a characterization of open balls in $\mathbb{R}^n$ by means of an integral identity.

2. Proof of Theorem 1

As in [8], we use the procedure of moving up planes perpendicular to a fixed direction and we briefly describe it.

Let γ be a unit vector in $\mathbb{R}^n$ and let T_λ denote the hyperplane $\gamma.x = \lambda$. For $\tilde{\lambda} > 0$ large the plane $T_{\tilde{\lambda}}$ does not intersect $\overline{\Omega}$ since Ω is bounded. We decrease λ until T_λ begins to intersect $\overline{\Omega}$. From that moment on, the plane T_λ cuts off from Ω an open cap, $\Sigma(\lambda)$, the part of Ω on the same side of T_λ as $T_{\tilde{\lambda}}$. Let $\Sigma'(\lambda)$ denote the reflection of $\Sigma(\lambda)$ in the plane T_λ. At the beginning $\Sigma'(\lambda) \subset \Omega$ and as λ decreases $\Sigma'(\lambda) \subset \Omega$ at least until one of the following occurs :

(i) $\Sigma'(\lambda)$ becomes internally tangent to $\partial\Omega$ at some point P not on T_λ;

(ii) T_λ reaches a position at which it is orthogonal to $\partial\Omega$ at some point $Q \in T_\lambda \cap \partial\Omega$.

We denote by $T_{\lambda_1} : \gamma.x = \lambda_1$ the plane T_λ when it first reaches a position such that (i) or (ii) holds. Clearly $\Sigma'(\lambda_1) \subset \Omega$. Also we define λ_0 to be the first value of λ for which T_λ intersects $\overline{\Omega}$, that is

$$\lambda_0 = \inf\{\hat{\lambda} < \tilde{\lambda};\ T_\lambda \cap \overline{\Omega} = \emptyset \quad \text{for } \hat{\lambda} < \lambda < \tilde{\lambda}\}.$$

Finally, for $\lambda \in [\lambda_1, \lambda_0)$ and $x \in \Sigma'(\lambda)$ we define x^λ to be the reflection of x in the plane T_λ.

We first show that Ω is symmetric about the plane T_{λ_1}. Since this is true for an arbitrary direction, Ω must be simply connected. Then Ω must be a ball.

Lemma 1 *With the above notations, for all $\lambda \in (\lambda_1, \lambda_0)$ and for all $x \in \overline{\partial\Sigma(\lambda)\backslash T_\lambda}$ we have*

$$\gamma.\nabla u(x) < 0 \qquad \text{and} \qquad \gamma.\nabla v(x) > 0.$$

Proof. Since $\Delta v > 0$ in Ω and $v = d$ on $\partial\Omega$ the maximum principle implies that $v < d$ in Ω and then the Hopf lemma implies that $\frac{\partial v}{\partial \nu} > 0$ on $\partial\Omega$, hence $\gamma.\nabla v(x) > 0$ for $x \in \overline{\partial\Sigma(\lambda)\backslash T_\lambda}$ with $\lambda \in (\lambda_1, \lambda_0)$. We have $v < d \leq 0$ in Ω. Then (H_2) and (H_3) imply that $\Delta u < 0$ in Ω. Since $u = 0$ on $\partial\Omega$, in the same way we have $u > 0$ in Ω and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$, hence $\gamma.\nabla u(x) < 0$ for $x \in \overline{\partial\Sigma(\lambda)\backslash T_\lambda}$ with $\lambda \in (\lambda_1, \lambda_0)$. □

Let $\lambda \in [\lambda_1, \lambda_0)$ and define the functions

$$u_\lambda(x) = u(x^\lambda) \quad \text{and} \quad v_\lambda(x) = v(x^\lambda) \qquad \text{for } x \in \Sigma'(\lambda).$$

We have

$$\begin{cases} \Delta u_\lambda = g(u_\lambda, v_\lambda) & \text{in } \Sigma'(\lambda), \\ \Delta v_\lambda = f(u_\lambda, v_\lambda) & \text{in } \Sigma'(\lambda), \end{cases}$$

with the boundary conditions

$$\begin{aligned} u_\lambda = u,\ v_\lambda = v \qquad & \text{on } \partial\Sigma'(\lambda) \cap T_\lambda, \\ u_\lambda = 0,\ v_\lambda = d \qquad & \text{on } \partial\Sigma'(\lambda)\backslash T_\lambda, \\ \tfrac{\partial u_\lambda}{\partial \nu} = c \qquad & \text{on } \partial\Sigma'(\lambda)\backslash T_\lambda \end{aligned}$$

(here ν denotes the unit outer normal to $\partial\Sigma'(\lambda)\backslash T_\lambda$). By virtue of lemma 1, there exists $\eta > 0$ such that for $\lambda \in (\max(\lambda_1, \lambda_0 - \eta), \lambda_0)$, we have

$$\begin{cases} u_\lambda - u < 0 & \text{in } \Sigma'(\lambda) \quad \text{and} \quad \gamma.\nabla u < 0 \quad \text{in } \Sigma(\lambda), \\ v_\lambda - v > 0 & \text{in } \Sigma'(\lambda) \quad \text{and} \quad \gamma.\nabla v > 0 \quad \text{in } \Sigma(\lambda). \end{cases} \tag{2.1}$$

Decrease λ until a critical value $\mu \geq \lambda_1$ is reached, beyond which (2.1) is no longer true. Then (2.1) holds for $\lambda \in (\mu, \lambda_0)$ while for $\lambda = \mu$ we have by continuity

$$\begin{cases} u_\mu - u \leq 0 & \text{in } \Sigma'(\mu) \quad \text{and} \quad \gamma.\nabla u < 0 \quad \text{in } \Sigma(\mu), \\ v_\mu - v \geq 0 & \text{in } \Sigma'(\mu) \quad \text{and} \quad \gamma.\nabla v > 0 \quad \text{in } \Sigma(\mu). \end{cases} \tag{2.2}$$

Suppose $\mu > \lambda_1$. (H_1), (H_2) and (2.2) imply that $\Delta(v_\mu - v) \leq 0$ in $\Sigma'(\mu)$. Since $v < d$ in Ω, we have $v_\mu - v \not\equiv 0$ in $\Sigma'(\mu)$. The maximum principle and the Hopf lemma imply that

$$v_\mu - v > 0 \quad \text{in } \Sigma'(\mu) \quad \text{and} \quad \gamma.\nabla v > 0 \quad \text{on } T_\mu \cap \Omega \tag{2.3}$$

where the second inequality follows from the fact that $\gamma.\nabla(v_\mu - v) = -2\gamma.\nabla v$ on $T_\mu \cap \Omega$. Now, using (H_1), (H_2), (2.2) and (2.3) we get $\Delta(u_\mu - u) > 0$ in $\Sigma'(\mu)$. Then the maximum principle and the Hopf lemma imply that

$$u_\mu - u < 0 \quad \text{in } \Sigma'(\mu) \quad \text{and} \quad \gamma.\nabla u < 0 \quad \text{on } T_\mu \cap \Omega \tag{2.4}$$

where the second inequality follows from the fact that $\gamma.\nabla(u_\mu - u) = -2\gamma.\nabla u$ on $T_\mu \cap \Omega$. (2.2), (2.3) and (2.4) show that (2.1) holds for $\lambda = \mu$.

Using lemma 1, (2.1) with $\lambda = \mu$, (2.3) and (2.4) we see that for some $\varepsilon > 0$ such that $\mu - \varepsilon > \lambda_1$ we have

$$\gamma.\nabla u < 0 \quad \text{in } \Sigma(\mu - \varepsilon). \tag{2.5}$$

and

$$\gamma.\nabla v > 0 \quad \text{in } \Sigma(\mu - \varepsilon). \tag{2.6}$$

Thus our definition of μ implies that either there is a strictly increasing sequence (λ_j) with $\lim_{j\to\infty} \lambda_j = \mu$ $(\lambda_j \in (\mu - \varepsilon, \mu)\, \forall j)$ such that for each j there is a point $x_j \in \Sigma'(\lambda_j)$ for which

$$u_{\lambda_j}(x_j) - u(x_j) \geq 0 \qquad \forall\, j \tag{2.7}$$

or that there is a strictly increasing sequence (μ_j) with $\lim_{j\to\infty} \mu_j = \mu$ $(\mu_j \in (\mu - \varepsilon, \mu)\, \forall j)$ such that for each j there is a point $z_j \in \Sigma'(\mu_j)$ for which

$$v_{\mu_j}(z_j) - v(z_j) \leq 0 \qquad \forall\, j. \tag{2.8}$$

In the situation (2.7), a subsequence which we still call x_j will converge to some point $x \in \overline{\Sigma'(\mu)}$; then $u_\mu(x) - u(x) \geq 0$. Since (2.1) holds for $\lambda = \mu$ we must have $x \in \partial\Sigma'(\mu)$; If $x \in \partial\Sigma'(\mu)\backslash T_\mu$ then $0 = u_\mu(x) < u(x)$, a contradiction. Therefore $x \in T_\mu$. The straight segment joining x_j to its symmetric about T_{λ_j} belongs to Ω and by the theorem of the mean it contains a point y_j such that

$$\gamma.\nabla u(y_j) \geq 0.$$

Since $\lim_{j\to\infty} y_j = x$, we obtain a contradiction to (2.5).

In the situation (2.8), a subsequence which we still call z_j will converge to some point $z \in \overline{\Sigma'(\mu)}$; then $v_\mu(z) - v(z) \leq 0$. Since (2.1) holds for $\lambda = \mu$ we must have $z \in \partial\Sigma'(\mu)$; If $z \in \partial\Sigma'(\mu)\backslash T_\mu$ then $d = v_\mu(z) > v(z)$, a contradiction. Therefore $z \in T_\mu$. The straight segment joining z_j to its symmetric about T_{μ_j} belongs to Ω and by the theorem of the mean it contains a point t_j such that

$$\gamma.\nabla v(t_j) \leq 0.$$

Since $\lim_{j\to\infty} t_j = z$, we obtain a contradiction to (2.6).

Thus we have proved that $\mu = \lambda_1$ and that (2.1) holds for $\lambda \in (\lambda_1, \lambda_0)$. By continuity we have

$$\begin{cases} u_{\lambda_1} - u \leq 0 & \text{in } \Sigma'(\lambda_1) \quad \text{and} \quad \gamma.\nabla u < 0 \quad \text{in } \Sigma(\lambda_1), \\ v_{\lambda_1} - v \geq 0 & \text{in } \Sigma'(\lambda_1) \quad \text{and} \quad \gamma.\nabla v > 0 \quad \text{in } \Sigma(\lambda_1). \end{cases} \tag{2.9}$$

Using (H_1), (H_2) and (2.9) we obtain

$$\Delta(u_{\lambda_1} - u) \geq 0 \qquad \text{in } \Sigma'(\lambda_1). \tag{2.10}$$

The maximum principle implies that

$$u_{\lambda_1} \equiv u \qquad \text{in } \Sigma'(\lambda_1) \tag{2.11}$$

or

$$u_{\lambda_1} - u < 0 \qquad \text{in } \Sigma'(\lambda_1). \tag{2.12}$$

If (2.11) holds then $u = 0$ on $\partial\Sigma'(\lambda_1)\backslash T_{\lambda_1}$ and, since $u > 0$ in Ω, this implies that $\Sigma'(\lambda_1)$ coincides with that part of Ω on the same side of T_{λ_1} as $\Sigma'(\lambda_1)$; that is Ω is symmetric about T_{λ_1}. Now we show that (2.12) cannot hold. Indeed suppose first that we are in case (i), that is $\Sigma'(\lambda_1)$ is internally tangent to $\partial\Omega$ at some point P not on T_{λ_1}. Since $(u_{\lambda_1} - u)(P) = 0$, (2.10), (2.12) and the Hopf lemma imply that

$$\frac{\partial}{\partial\nu}(u_{\lambda_1} - u)(P) > 0, \tag{2.13}$$

and this contradicts the fact that $\frac{\partial u_{\lambda_1}}{\partial \nu} = \frac{\partial u}{\partial \nu} = c$ at P. In case (ii) T_{λ_1} is orthogonal to $\partial\Omega$ at some point Q. It now follows as in the proof given by Serrin ([8] p. 307–308) that $u_{\lambda_1} - u$ has a zero of order two at Q and lemma S gives a contradiction.

We have thus proved that Ω is symmetric about T_{λ_1}. Therefore, as we have already seen, we can conclude that Ω is a ball. Now (2.11) shows that u is symmetric about T_{λ_1}. Using the first equation in (1.4), (H_2) and (2.11) we find that v is also symmetric about T_{λ_1}. Since this is true for an arbitrary direction we conclude that u and v are radially symmetric. The other assertions of the theorem follow easily from (2.9) and lemma 1.

3. A different boundary condition

In this section we extend theorem 1 to a more general boundary condition than (1.3). Let H denote the mean curvature of the boundary $\partial\Omega$, chosen so that H is positive when $\partial\Omega$ is convex. We have

Theorem 2 *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with C^3 boundary $\partial\Omega$. Let f and g be as in theorem* 1. *If $(u, v) \in C^2(\overline{\Omega}) \times (C^2(\Omega) \cap C^1(\overline{\Omega}))$ satisfies the system of differential equations* (1.4) *and the boundary conditions* (1.5), (1.6) *and*

$$\frac{\partial u}{\partial \nu} = c(H)$$

where c is a continuously differentiable nonincreasing function of H, then the conclusions of theorem 1 *remain valid.*

Proof. Since we use the same arguments as in the proof of theorem 1 we only mention the modifications in the above discussion. □

Lemma 1 still holds. Thus in the same way we arrive at the situation (2.9)–(2.13). In case (i), as in Serrin's paper ([8] p. 317) we show that

$$\frac{\partial}{\partial \nu}(u_{\lambda_1} - u)(P) = c(H'(P)) - c(H(P)) \leq 0$$

where $H'(P)$ is the mean curvature of $\partial\Sigma'(\lambda_1)$ at P and this contradicts (2.13). Now, in case (ii) the arguments given by Serrin ([8] p. 317–318) imply that $u_{\lambda_1} - u$ has a zero of order two at Q and lemma S gives a contradiction.

Remark 1. Note that the assumption $\partial\Omega \in C^3$ can be weakened (see [8]).

4. Concluding remarks

In this section we first examine the case where condition (1.6) is replaced by $v = d > 0$ on $\partial\Omega$. We begin with a theorem obtained in [3] (théorème 3.1).

Theorem 3 *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with C^2 boundary $\partial\Omega$. Let f be as in theorem 1. Let $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ satisfy the differential equation*

$$\Delta^2 u = f(u, \Delta u) \qquad \text{in } \Omega,$$

and the boundary conditions

$$u = \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega,$$
$$\Delta u = d\,(\text{const.}) \qquad \text{on } \partial\Omega.$$

If $u \geq 0$ in Ω, then Ω is a ball. If $\Omega = \{x \in \mathbb{R}^n;\, |x - x_0| < R\}$ for some $x_0 \in \mathbb{R}^n$, then $u(x) = y(|x - x_0|)$, $y' < 0$ in $(0, R)$ and $(\Delta y)' > 0$ in $(0, R]$.

Remark 2. Notice that $u \in C^4(\overline{\Omega})$ in [3], but it is enough to assume that $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$.

Remark 3. We easily show that $d > 0$ in theorem 3 (see lemma 2.1 in [3]).

Remark 4. Assume that $f \equiv 1$, $\partial\Omega \in C^{4+\varepsilon}$ and $u \in C^4(\overline{\Omega})$. Then the assumption $u \geq 0$ in Ω can be removed. Indeed this is just Bennett's result [1].

Remark 5. Clearly the above result can be extended to overdetermined elliptic systems.

Now we shall examine the case where $c \neq 0$ in (1.3).

Theorem 4 *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with C^2 boundary $\partial\Omega$. Let f and g be as in theorem* 1. *Let $(u, v) \in C^2(\overline{\Omega}) \times (C^2(\Omega) \cap C^1(\overline{\Omega}))$ satisfy the system of differential equations* (1.4) *and the boundary conditions* (1.5),

$$\frac{\partial u}{\partial \nu} = c\,(\text{const.}) < 0 \qquad \text{on } \partial\Omega \tag{4.1}$$

and

$$v = d\,(const.) > 0 \qquad on\ \partial\Omega.$$

If $u \geq 0$ in Ω, then Ω is a ball. If $\Omega = \{x \in \mathbb{R}^n;\ |x - x_0| < R\}$ for some $x_0 \in \mathbb{R}^n$, then $u(x) = y(|x - x_0|)$, $v(x) = z(|x - x_0|)$, $y' < 0$ in $(0, R]$ and $z' > 0$ in $(0, R]$.

Proof. Since we make use of the same arguments as in the proofs of theorem 1 and theorem 3 (see théorème 3.1 in [3]), we shall be sketchy. We first note that lemma 1 holds. Now using the notations of section 2 we arrive at the situation (2.7)–(2.8). In the situation (2.7), in the same way we have $u_\mu(x) - u(x) \geq 0$ and $x \in \partial\Sigma'(\mu)$. If $x \in \partial\Sigma'(\mu)\backslash T_\mu$ we get $u(x) = -(u_\mu - u)(x) \leq 0$. Since $u \geq 0$ in Ω, we deduce that $u(x) = 0$. Using (2.1) with $\lambda = \mu$, (H_1), (H_2) and the Hopf lemma we obtain

$$\frac{\partial}{\partial\nu}(u_\mu - u)(x) > 0$$

(here ν denotes the unit outer normal to $\partial\Sigma'(\mu)\backslash T_\mu$). Since $\frac{\partial u_\mu}{\partial\nu}(x) = c$ we deduce that $\frac{\partial u}{\partial\nu}(x) < c < 0$ and we get a contradiction with the fact that $u \geq 0$ in Ω. Therefore $x \in T_\mu$ and the proof is the same in this case. Also, in the situation (2.8) the proof is the same and we arrive at (2.10). (H_1), (H_2) and (2.9) imply that $\Delta(v_{\lambda_1} - v) \leq 0$ in $\Sigma'(\lambda_1)$. Then, using the maximum principle we get

$$v_{\lambda_1} - v \equiv 0 \qquad \text{in } \Sigma'(\lambda_1) \tag{4.2}$$

or

$$v_{\lambda_1} - v > 0 \qquad \text{in } \Sigma'(\lambda_1). \tag{4.3}$$

If (4.2) holds then $v = d$ on $\partial\Sigma'(\lambda_1)\backslash T_{\lambda_1}$ and, since $v < d$ in Ω, this implies that $\Sigma'(\lambda_1)$ coincides with that part of Ω on the same side of T_{λ_1} as $\Sigma'(\lambda_1)$; that is Ω is symmetric about T_{λ_1}. Now assume that (4.3) holds. Then, using (H_1)–(H_3), (2.9) and (4.3) we obtain $\Delta(u_{\lambda_1} - u) > 0$ in $\Sigma'(\lambda_1)$, from which we deduce (2.12). We show that (2.12) cannot hold and we get the conclusion as in the proof of theorem 1. $\square$

Remark 6. Clearly our method of proof cannot be used to treat the case where the condition $c < 0$ in (4.1) is replaced by $c > 0$. On the other hand theorem 4 can be extended to a somewhat more general condition than (4.1). With the notations of section 3, theorem 4 remains valid if we replace (4.1) by

$$\frac{\partial u}{\partial \nu} = c(H) \qquad \text{on } \partial\Omega$$

where now c is a continuously differentiable nonincreasing function of H such that $c < 0$.

Finally, just as in [1], [6] and [7] we obtain a characterization of open balls in $\mathbb{R}^n$ by means of an integral identity.

Theorem 5 *If $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with $C^{4+\varepsilon}$ boundary $\partial\Omega$ and*

$$\int_\Omega B\,dx = c\int_{\partial\Omega} \Delta B\,ds \tag{4.4}$$

for some constant c and for every $B \in C^4(\overline{\Omega})$ such that $\Delta^2 B = 0$ and $B = 0$ on $\partial\Omega$, then Ω is a ball.

Proof. We shall show that (4.4) is equivalent to the following statement :

$$\begin{cases} u \in C^{4+\varepsilon}(\overline{\Omega}) \text{ satisfies the differential equation } \Delta^2 u = 1 \text{ in } \Omega \\ \text{and the boundary conditions (1.3), (1.5) and } \Delta u = 0 \text{ on } \partial\Omega. \end{cases}$$

Then the theorem follows from theorem 1. □

Suppose that $u \in C^{4+\varepsilon}(\overline{\Omega})$ satisfies the above statement. Let $B \in C^4(\overline{\Omega})$ be a biharmonic function such that $B = 0$ on $\partial\Omega$. Then, using Green's formula we get

$$\int_\Omega B\,dx = \int_\Omega B\Delta^2 u\,dx = \int_{\partial\Omega} \Delta B \frac{\partial u}{\partial \nu}\,ds\,. \tag{4.5}$$

Thus (1.3) implies (4.4).

Now suppose that (4.4) holds. Let $u \in C^{4+\varepsilon}(\overline{\Omega})$ be the solution of $\Delta^2 u = 1$ in Ω satisfying (1.5) and $\Delta u = 0$ on $\partial\Omega$. Choose $B \in C^4(\overline{\Omega})$ such that $\Delta^2 B = 0$ in Ω, $B = 0$ on $\partial\Omega$ and $\Delta B = \frac{\partial u}{\partial \nu} - c$ on $\partial\Omega$. Then (4.5) implies that (1.3) is satisfied.

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