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Oscillations of delay difference equations

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Abstract. We obtain some new sufficient conditions for oscillations of all solutions of the delay difference equation

$$y_{n+1}-y_n+p_n y_{n-k}=0, \quad n=0,1,2,\dots$$

where $\{p_n\}$ is a sequence of nonnegative numbers and k is a positive integer. Our theorems improve several previous well-known results. Some examples are given to demonstrate the advantage of our results.

Key words: oscillation, eventually positive solution, difference equation.

1. Introduction

In the recent papers [1–12], the oscillation of all solutions of the delay difference equation

$$y_{n+1}-y_n+p_n y_{n-k}=0, \quad n=0,1,2,\dots \tag{1}$$

has been investigated, where $\{p_n\}$ is a sequence of nonnegative numbers and k is a positive integer.

A solution $\{y_n\}$ of Eq.(1) is said to be oscillatory if the terms y_n of the sequence are not eventually positive or eventually negative. Otherwise, the solution is called nonoscillatory.

In [1], Erbe and Zhang first proved that all solutions of (1) oscillate if

$$\liminf_{n\to\infty} p_n > \frac{k^k}{(k+1)^{k+1}}, \tag{2}$$

or

$$\Lambda = \limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i > 1. \tag{3}$$

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Later, condition (2) was improved, by Ladas, Philos, Sficas [2], to

$$\alpha = \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > \left(\frac{k}{k+1}\right)^{k+1}. \tag{4}$$

We remark that conditions (3) and (4) are two well-known oscillation criterion for (1) which have been extensively employed in the study of oscillation of various delay differences. For example, see the monographs [3, 4, 5]. However, there is an obvious gap between the conditions (3) and (4). It would be interesting to fill the gap, i.e. to obtain sufficient conditions for the oscillation of (1) when $\alpha \le k^{k+1}/(k+1)^{k+1}$ and $\Lambda \le 1$.

Recently, there are many papers which devoted oneself to filling the gap between conditions (3) and (4). For instance, Tang [6] proved that all solutions of (1) oscillate if

$$\sum_{i=n-k}^{n-1} p_i \ge \left(\frac{k}{k+1}\right)^{k+1} \quad \text{for large } n \tag{5}$$

and

$$\sum_{n=k}^{\infty} p_n \left[\sum_{i=n-k}^{n-1} p_i - \left(\frac{k}{k+1}\right)^{k+1}\right] = \infty. \tag{6}$$

Clearly, conditions (5) and (6) improve (4). Afterwards, Tang and Yu [7] further improved the above conditions, proved that

$$\sum_{n=0}^{\infty} p_n \left[(k+1)\left(\sum_{i=n+1}^{n+k} p_i\right)^{\frac{1}{k+1}} - k\right] = \infty \tag{7}$$

also implies that all solutions of (1) oscillate.

In a different direction, Yu, Zhang and Qian [8] proved that all solutions of (1) oscillate if

$$\alpha \le \left(\frac{k}{k+1}\right)^{k+1} \quad \text{and} \quad \Lambda > 1 - \frac{\alpha^2}{4}, \tag{8}$$

or

$$\alpha \le \left(\frac{k}{k+1}\right)^{k+1} \quad \text{and} \quad \Lambda > \frac{2}{\sqrt{f(\alpha)}}, \tag{9}$$

where $f(\alpha) \in [1, k/(k+1)\alpha]$ satisfies the following equation

$$f(\alpha)\left[1 - \frac{\alpha}{k} f(\alpha)\right]^k = 1. \tag{10}$$

In [9], Chen and Yu proved that (8) can be replaced by the weaker condition

$$\alpha \leq \left(\frac{k}{k+1}\right)^{k+1} \quad \text{and} \quad \Lambda > 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2}. \tag{11}$$

Conditions (8), (9) and (11) all improve (3), but (8) and (11) are independent of (9).

The aim in this note is to further improve conditions (9) and (11). As a consequent of our main results, we prove that

$$\alpha \leq \left(\frac{k}{k+1}\right)^{k+1} \quad \text{and} \quad \Lambda > \frac{1 + \ln f(\alpha)}{f(\alpha)} - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2} \tag{12}$$

guarantee that all solutions of (1) oscillate, where $f(\alpha)$ is the value determined by α from (10). It is not difficult to verify that (12) improves (9) and (11).

2. Preliminaries

For $0 < \alpha \leq k^{k+1}/(k+1)^{k+1}$, since the function $x(1 - \alpha x/k)^k$ is strictly increasing in $[1, k/(k+1)\alpha]$ from $(1 - \alpha/k)^k$ to $(1/\alpha)k^{k+1}/(k+1)^{k+1}$, it follows there exists a unique function $f(\alpha) \in [1, k/(k+1)\alpha]$ such that (10) holds. It is easy to see that $f(0) = 1$, $f(k^{k+1}/(k+1)^{k+1}) = (k+1)^k/k^k$, and $1 < f(\alpha) < k/(k+1)\alpha$ for $0 < \alpha < k^{k+1}/(k+1)^{k+1}$. From this and (10) we obtain

$$\left[1 - \frac{k+1}{k}\alpha f(\alpha)\right] f'(\alpha) = f^2(\alpha),$$

which leads to that $f'(\alpha) > 0$ for $0 < \alpha < k^{k+1}/(k+1)^{k+1}$. This shows that function $f(\alpha)$ is strictly increasing in $[0, k^{k+1}/(k+1)^{k+1}]$.

Lemma 1 [9] *Assume that* $0 \leq \alpha \leq k^{k+1}/(k+1)^{k+1}$, *and let* $\{y_n\}$ *be an* *eventually positive solution of* (1). *Then*

$$\liminf_{n\to\infty} \frac{y_{n+1}}{y_{n-k}} \geq A(\alpha) : \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}. \tag{13}$$

Lemma 2 [8] *Assume that* $0 \leq \alpha \leq k^{k+1}/(k+1)^{k+1}$, *and let* $\{y_n\}$ *be an eventually positive solution of* (1). *Then*

$$\liminf_{n\to\infty} \frac{y_{n-k}}{y_n} \geq f(\alpha). \tag{14}$$

Lemma 3 *Assume that* $0 \leq \alpha \leq k^{k+1}/(k+1)^{k+1}$, *and let* $\{y_n\}$ *be an eventually positive solution of* (1). *Then*

$$\limsup_{n\to\infty} p_n \leq \frac{1}{f(\alpha)} - A(\alpha). \tag{15}$$

Proof. From (1), we have eventually

$$p_n = \frac{y_n}{y_{n-k}} - \frac{y_{n+1}}{y_{n-k}}. \tag{16}$$

By Lemmas 1 and 2, it follows from (16) that

$$\limsup_{n\to\infty} p_n \leq 1 \Big/ \liminf_{n\to\infty} \frac{y_{n-k}}{y_n} - \liminf_{n\to\infty} \frac{y_{n+1}}{y_{n-k}} \leq \frac{1}{f(\alpha)} - A(\alpha).$$

The proof is complete. □

Lemma 4 *Assume that* $0 \leq \alpha \leq k^{k+1}/(k+1)^{k+1}$, *and let* $\{y_n\}$ *be an eventually positive solution of* (1). *Then*

$$\liminf_{n\to\infty} \left[\frac{y_{n-k}}{y_n} \prod_{i=n-k}^{n-1} (1 - p_i f(\alpha)) \right] \geq 1. \tag{17}$$

Proof. Let $n_0 > 0$ be an integer such that $y_n > 0$ and $y_{n+1} - y_n \leq 0$ for $n \geq n_0 - 2k$. From (1), we have

$$\frac{y_{n-k}}{y_n} = \prod_{i=n-k}^{n-1} \left(1 - p_i \frac{y_{i-k}}{y_i}\right)^{-1}, \quad n \geq n_0. \tag{18}$$

If $\alpha = 0$, then $f(\alpha) = 1$. It follows from (18) that

$$\frac{y_{n-k}}{y_n} \geq \prod_{i=n-k}^{n-1} (1-p_i)^{-1}, \quad n \geq n_0,$$

or

$$\frac{y_{n-k}}{y_n} \prod_{i=n-k}^{n-1} (1-p_i) \geq 1, \quad n \geq n_0$$

which implies that (17) holds for the case $\alpha = 0$. If $0 < \alpha \leq k^{k+1}/(k+1)^{k+1}$, then $f(\alpha) > 1$ and $A(\alpha) > 0$. By Lemma 3, there exists an integer $n_1 > n_0 + k$ such that

$$p_n \leq \frac{1}{f(\alpha)} - \frac{1}{2}A(\alpha) \quad \text{for} \quad n \geq n_1. \tag{19}$$

Set $\omega_n = \min\{y_{n-k}/y_n, f(\alpha)\}$ for $n \geq n_0$. Then by Lemma 2, $\liminf_{n\to\infty} \omega_n = f(\alpha)$. Hence, from (18), we obtain

$$\frac{y_{n-k}}{y_n} \geq \prod_{i=n-k}^{n-1} (1-p_i\omega_i)^{-1}, \quad n \geq n_1.$$

It follows that for $n \geq n_1 + k$

$$\begin{aligned}
\frac{y_{n-k}}{y_n} \prod_{i=n-k}^{n-1} (1-p_i f(\alpha)) &\geq \prod_{i=n-k}^{n-1} \frac{1-p_i f(\alpha)}{1-p_i\omega_i} \\
&\geq \prod_{i=n-k}^{n-1} \frac{1-\left[\frac{1}{f(\alpha)} - \frac{1}{2}A(\alpha)\right] f(\alpha)}{1-\left[\frac{1}{f(\alpha)} - \frac{1}{2}A(\alpha)\right] \omega_i} \\
&= \prod_{i=n-k}^{n-1} \frac{\frac{1}{2}A(\alpha)f(\alpha)}{1-\left[\frac{1}{f(\alpha)} - \frac{1}{2}A(\alpha)\right] \omega_i},
\end{aligned}$$

and so

$$\begin{aligned}
\liminf_{n\to\infty} \left[\frac{y_{n-k}}{y_n} \prod_{i=n-k}^{n-1} (1-p_i f(\alpha))\right] &\geq \left[\frac{\frac{1}{2}A(\alpha)f(\alpha)}{1-\left[\frac{1}{f(\alpha)} - \frac{1}{2}A(\alpha)\right] \liminf_{n\to\infty}\omega_n}\right]^k \\
&= 1.
\end{aligned}$$

The proof is complete. □

3. Main Results

The first Theorem is a direct corollary of Lemma 3.

Theorem 1 *Assume that* $0 \le \alpha \le k^{k+1}/(k+1)^{k+1}$. *If*

$$\limsup_{n\to\infty} p_n > \frac{1}{f(\alpha)} - A(\alpha), \tag{20}$$

then all solutions of (1) *oscillate.*

Next we are going to deal with the case when the inequality in condition (20) is reversed. Without loss of generality, we assume that

$$p_n \le \frac{1}{f(\alpha)} - \frac{1}{2}A(\alpha) \quad \text{for} \quad n = 0, 1, 2, \dots .$$

Theorem 2 *Assume that* $0 \le \alpha \le k^{k+1}/(k+1)^{k+1}$. *If*

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i \prod_{j=i-k}^{n-k-1} (1 - p_j f(\alpha))^{-1} > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{21}$$

then all solutions of (1) *oscillate.*

Proof. Suppose the contrary,and let $\{y_n\}$ be an eventually positive solution of (1). Then there exists an integer $n_0 > k$ such that

$$y_n > 0 \quad \text{and} \quad y_{n+1} - y_n \le 0, \quad n \ge n_0 - k.$$

For the case $\alpha = 0$, since $f(\alpha) = 1$, (21) reduces to

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i \prod_{j=i-k}^{n-k-1} (1 - p_j)^{-1} > 1. \tag{22}$$

From (1), we have

$$\frac{y_{n+1}}{y_n} = 1 - p_n \frac{y_{n-k}}{y_n}, \quad n \ge n_0.$$

It follows that for $n - k \le i \le n$ and $n \ge n_0 + k$

$$\frac{y_{i-k}}{y_{n-k}} = \prod_{j=i-k}^{n-k-1} \left(1 - p_j \frac{y_{j-k}}{y_j}\right)^{-1}. \tag{23}$$

Note that $y_{n-k}/y_n \geq 1$ for $n \geq n_0$, from (23) we get

$$\frac{y_{i-k}}{y_{n-k}} \geq \prod_{j=i-k}^{n-k-1} (1-p_j)^{-1}, \quad n \geq n_0 + k \quad \text{and} \quad n-k \leq i \leq n.$$

Summing (1) from $n-k$ to n and using the above inequalities, we obtain

$$\begin{aligned} y_{n-k} - y_{n+1} &= \sum_{i=n-k}^{n} p_i y_{i-k} \\ &\geq y_{n-k} \sum_{i=n-k}^{n} p_i \prod_{j=i-k}^{n-k-1} (1-p_j)^{-1}, \quad n \geq n_0 + k, \end{aligned}$$

or

$$1 - \frac{y_{n+1}}{y_{n-k}} \geq \sum_{i=n-k}^{n} p_i \prod_{j=i-k}^{n-k-1} (1-p_j)^{-1}, \quad n \geq n_0 + k.$$

Hence

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i \prod_{j=i-k}^{n-k-1} (1-p_j)^{-1} \leq 1$$

which contradicts with (22).

For the other case $0 < \alpha \leq k^{k+1}/(k+1)^{k+1}$, we have $f(\alpha) > 1$ and $A(\alpha) > 0$. Rewrite (21) as

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i \prod_{j=i-k}^{n-k-1} (1-p_j f(\alpha))^{-1} + A(\alpha) > 1. \tag{24}$$

This implies that there exists $\eta \in (1/f(\alpha), 1)$ such that

$$\limsup_{n\to\infty} \lambda^k \sum_{i=n-k}^{n} p_i \prod_{j=i-k}^{n-k-1} (1-p_j f(\alpha))^{-1} + \eta A(\alpha) > 1. \tag{25}$$

where

$$\lambda = \frac{f(\alpha)A(\alpha)}{2(1-\eta) + \eta f(\alpha) A(\alpha)}. \tag{26}$$

By Lemmas 1 and 2, there exists an integer $n_1 > n_0$ such that

$$\frac{y_{n-k}}{y_n} \geq \eta f(\alpha) \quad \text{and} \quad \frac{y_{n+1}}{y_{n-k}} \geq \eta A(\alpha), \quad n \geq n_1. \tag{27}$$

From (25), we may choose an integer $N > n_1 + 2k$ so large that

$$\lambda^k \sum_{i=N-k}^{N} p_i \prod_{j=i-k}^{N-k-1} (1 - p_j f(\alpha))^{-1} + \eta A(\alpha) > 1. \tag{28}$$

On the other hand, from (23), (26) and (27), we have for $N - k \leq i \leq N$

$$\begin{aligned}
\frac{y_{i-k}}{y_{N-k}} &= \prod_{j=i-k}^{N-k-1} \left(1 - p_j \frac{y_{j-k}}{y_j}\right)^{-1} \geq \prod_{j=i-k}^{N-k-1} (1 - p_j \eta f(\alpha))^{-1} \\
&= \prod_{j=i-k}^{N-k-1} \frac{\lambda}{\lambda - 1 + p_j f(\alpha)(1 - \lambda\eta) + 1 - p_j f(\alpha)} \\
&\geq \prod_{j=i-k}^{N-k-1} \frac{\lambda}{\lambda - 1 + [1 - f(\alpha)A(\alpha)/2](1 - \lambda\eta) + 1 - p_j f(\alpha)} \\
&= \prod_{j=i-k}^{N-k-1} \lambda(1 - p_j f(\alpha))^{-1} \geq \lambda^k \prod_{j=i-k}^{N-k-1} (1 - p_j f(\alpha))^{-1}.
\end{aligned}$$

Summing (1) from $N - k$ to N and using the above inequalities, we obtain

$$\begin{aligned}
y_{N-k} - y_{N+1} &= \sum_{i=N-k}^{N} p_i y_{i-k} \\
&\geq \lambda^k y_{N-k} \sum_{i=N-k}^{N} p_i \prod_{j=i-k}^{N-k-1} (1 - p_j f(\alpha))^{-1}
\end{aligned}$$

or

$$1 - \frac{y_{N+1}}{y_{N-k}} \geq \lambda^k \sum_{i=N-k}^{N} p_i \prod_{j=i-k}^{N-k-1} (1 - p_j f(\alpha))^{-1}. \tag{29}$$

Substituting $y_{N+1}/y_{N-k} \geq \eta A(\alpha)$ into (29), we have

$$1 \geq \eta A(\alpha) + \lambda^k \sum_{i=N-k}^{N} p_i \prod_{j=i-k}^{N-k-1} (1 - p_j f(\alpha))^{-1}$$

which contradicts with (28), and so the proof is complete. □

Theorem 3 *Assume that* $0 \le \alpha \le k^{k+1}/(k+1)^{k+1}$. *If*

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i \left[\min\left\{\prod_{j=i-k}^{n-k-1}(1-p_j f(\alpha))^{-1}, \frac{1}{f(\alpha)}\prod_{j=i-k}^{i-1}(1-p_j f(\alpha))^{-1}\right\}\right] > \frac{1+\ln f(\alpha)}{f(\alpha)} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \quad (30)$$

then all solutions of (1) *oscillate.*

Proof. For the case $\alpha = 0$, since $f(\alpha) = 1$, it is easy to see that (30) is the same to (21). By Theorem 2, the conclusion of Theorem 3 is true. In the sequel, we only consider the other case $0 < \alpha \le k^{k+1}/(k+1)^{k+1}$. Suppose that the conclusion of the theorem is false, and that (1) has an eventually positive solution $\{y_n\}$. Choose a positive integer $n_0 > k$ such that $y_n > 0$ and $y_{n+1} - y_n \le 0$ for $n \ge n_0 - k$. Rewrite (30) as

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i \left[\min\left\{\prod_{j=i-k}^{n-k-1}(1-p_j f(\alpha))^{-1}, \frac{1}{f(\alpha)}\prod_{j=i-k}^{i-1}(1-p_j f(\alpha))^{-1}\right\}\right] + A(\alpha) - \frac{1+\ln f(\alpha)}{f(\alpha)} > 0. \quad (31)$$

Since $f(\alpha) > 1$ and $A(\alpha) > 0$, (31) implies that there exists $\eta \in (1/f(\alpha), 1)$ such that

$$\limsup_{n\to\infty} \lambda^k \sum_{i=n-k}^{n} p_i \left[\min\left\{\prod_{j=i-k}^{n-k-1}(1-p_j f(\alpha))^{-1}, \frac{1}{f(\alpha)}\prod_{j=i-k}^{i-1}(1-p_j f(\alpha))^{-1}\right\}\right] + \eta A(\alpha) - \frac{1+\ln \eta f(\alpha)}{\eta f(\alpha)} > 0, \quad (32)$$

where λ is defined by (26). For given η, by Lemmas 1, 2 and 4, there exists an integer $n_1 > n_0 + k$ such that for $n \ge n_1$

$$\frac{y_{n-k}}{y_n} \ge \eta f(\alpha) \quad \text{and} \quad \frac{y_{n+1}}{y_{n-k}} \ge \eta A(\alpha), \quad (33)$$

and

$$\frac{y_{n-k}}{y_n}\prod_{j=n-k}^{n-1}(1-p_jf(\alpha))\geq\eta. \tag{34}$$

It follows from (32) that there exists an integer $N>n_1+2k$ such that

$$\lambda^k\sum_{i=N-k}^{N}p_i\left[\min\left\{\prod_{j=i-k}^{N-k-1}(1-p_jf(\alpha))^{-1},\frac{1}{f(\alpha)}\prod_{j=i-k}^{i-1}(1-p_jf(\alpha))^{-1}\right\}\right]$$
$$>\frac{1+\ln\eta f(\alpha)}{\eta f(\alpha)}-\eta A(\alpha). \tag{35}$$

Since

$$y_{N-k}/y_{N-k}=1\quad\text{and}\quad y_{N-k}/y_N>\eta f(\alpha)>1.$$

Then there exists an integer l with $0\leq l\leq k$ such that

$$y_{N-k}/y_{N-l}\leq\eta f(\alpha)\quad\text{and}\quad y_{N-k}/y_{N-l+1}>\eta f(\alpha).$$

Let $\xi\in[0,1)$ such that

$$y_{N-k}/\left[y_{N-l}+\xi(y_{N-l+1}-y_{N-l})\right]=\eta f(\alpha). \tag{36}$$

From (1) and (34), we have for $t\in[0,1]$ and $n\geq n_0+k$

$$-\frac{y_{n+1}-y_n}{y_n+t(y_{n+1}-y_n)}=p_n\frac{y_{n-k}}{y_n+t(y_{n+1}-y_n)}\geq p_n\frac{y_{n-k}}{y_n}$$
$$\geq\eta p_n\prod_{j=n-k}^{n-1}(1-p_jf(\alpha))^{-1}. \tag{37}$$

For $n\in\{N-k,N-k+1,\ldots,N-l-1\}$, integrating (37) over $[0,1]$, we get

$$\ln\frac{y_n}{y_{n+1}}\geq\eta p_n\prod_{j=n-k}^{n-1}(1-p_jf(\alpha))^{-1},$$
$$n=N-k,N-k+1,\ldots,N-l-1.$$

For $n=N-l$, integrating again (37) over $[0,\xi]$, we have

$$\ln\frac{y_{N-l}}{y_{N-l}+\xi(y_{N-l+1}-y_{N-l})}\geq\xi\eta p_{N-l}\prod_{j=N-k-l}^{N-l-1}(1-p_jf(\alpha))^{-1}.$$

Summing the above inequalities, we obtain

$$\ln \frac{y_{N-k}}{y_{N-l} + \xi(y_{N-l+1} - y_{N-l})} \geq \eta \sum_{i=N-k}^{N-l-1} p_i \prod_{j=i-k}^{i-1} (1 - p_j f(\alpha))^{-1} + \xi \eta p_{N-l} \prod_{j=N-k-l}^{N-l-1} (1 - p_j f(\alpha))^{-1}.$$

In view of (36),

$$\frac{\ln \eta f(\alpha)}{\eta f(\alpha)} \geq \frac{1}{f(\alpha)} \left[\sum_{i=N-k}^{N-l-1} p_i \prod_{j=i-k}^{i-1} (1 - p_j f(\alpha))^{-1} + \xi p_{N-l} \prod_{j=N-k-l}^{N-l-1} (1 - p_j f(\alpha))^{-1} \right]. \tag{38}$$

Similar to proof Theorem 2, from (1), (26) and (33) we may obtain

$$\begin{aligned} \frac{y_{i-k}}{y_{N-k}} &= \prod_{j=i-k}^{N-k-1} \left(1 - p_j \frac{y_{j-k}}{y_j}\right)^{-1} \\ &\geq \lambda^k \prod_{j=i-k}^{N-k-1} (1 - p_j f(\alpha))^{-1}, \quad N - k \leq i \leq N. \end{aligned}$$

Hence

$$\begin{aligned} & y_{N-l} + \xi(y_{N-l+1} - y_{N-l}) - y_{N+1} \\ &\quad = - \sum_{i=N-l}^{N} (y_{i+1} - y_i) + \xi(y_{N-l+1} - y_{N-l}) \\ &\quad = \sum_{i=N-l+1}^{N} p_i y_{i-k} + (1 - \xi) p_{N-l} y_{N-l-k} \\ &\quad \geq \lambda^k y_{N-k} \left[\sum_{i=N-l+1}^{N} p_i \prod_{j=i-k}^{N-k-1} (1 - p_j f(\alpha))^{-1} + (1 - \xi) p_{N-l} \prod_{j=N-k-l}^{N-k-1} (1 - p_j f(\alpha))^{-1} \right] \end{aligned}$$

It follows that

$$\frac{y_{N-l}+\xi(y_{N-l+1}-y_{N-l})}{y_{N-k}}-\frac{y_{N+1}}{y_{N-k}}$$
$$\geq \lambda^k\left[\sum_{i=N-l+1}^{N} p_i \prod_{j=i-k}^{N-k-1}(1-p_jf(\alpha))^{-1} + (1-\xi)p_{N-l}\prod_{j=N-k-l}^{N-k-1}(1-p_jf(\alpha))^{-1}\right]$$

Substituting (33) and (36) into this,

$$\frac{1}{\eta f(\alpha)}-\eta A(\alpha)$$
$$\geq \lambda^k\left[\sum_{i=N-l+1}^{N} p_i \prod_{j=i-k}^{N-k-1}(1-p_jf(\alpha))^{-1} + (1-\xi)p_{N-l}\prod_{j=N-k-l}^{N-k-1}(1-p_jf(\alpha))^{-1}\right] \quad (39)$$

Adding (38) and (39) leads to

$$\frac{1+\ln\eta f(\alpha)}{\eta f(\alpha)}-\eta A(\alpha)$$
$$\geq \lambda^k \sum_{i=N-k}^{N} p_i\left[\min\left\{\prod_{j=i-k}^{N-k-1}(1-p_jf(\alpha))^{-1}, \frac{1}{f(\alpha)}\prod_{j=i-k}^{i-1}(1-p_jf(\alpha))^{-1}\right\}\right]$$

which contradicts with (35), and so the proof is complete. □

From Theorems 2 and 3, we have immediately

Corollary 1 *Assume that* $0 \leq \alpha \leq k^{k+1}/(k+1)^{k+1}$. *If*

$$\limsup_{n\to\infty}\sum_{i=n-k}^{n} p_i\prod_{j=i-k}^{n-k-1}(1-p_j)^{-1} > 1-\frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \quad (40)$$

then all solutions of (1) *oscillate.*

Corollary 2 *Assume that* $0 \le \alpha \le k^{k+1}/(k+1)^{k+1}$. *If*

$$\Lambda = \limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i > \frac{1+\ln f(\alpha)}{f(\alpha)} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \quad (41)$$

then all solutions of (1) *oscillate.*

Remark 1. Obviously, Condition (41) improves (9) and (11) when $0 \le \alpha \le k^{k+1}/(k+1)^{k+1}$. However, as $\alpha \to 0$, (41), together with (8), (9) and (11), reduces to (3). Nevertheless, the following Example 2 illustrate that Corollary 1 still possible improve (3) for the case $\alpha = 0$.

4. Several Examples

In this section, we give some examples to show the effect of our results.

Example 1. Consider the difference equation

$$y_{n+1} - y_n + p_n y_{n-2} = 0, \quad n = 1, 2, \ldots, \quad (42)$$

where $p_{10n} = p_{10n+1} = \cdots = p_{10n+8} = 0.1$, $p_{10n+9} = 0.73$, $n = 0, 1, 2, \ldots$. It is easy to observe that

$$\alpha = \liminf_{n\to\infty} \sum_{i=n-2}^{n-1} p_i = 0.2 < \left(\frac{2}{3}\right)^3,$$

$$\Lambda = \limsup_{n\to\infty} \sum_{i=n-2}^{n} p_i = 0.93 < 1.$$

In addition, we find

$$f(\alpha) = 1.336 \text{ and } A(\alpha) = \left(1 - \alpha - \sqrt{1-2\alpha-\alpha^2}\right)/2 = 0.026.$$

By these, one can easy verify that

$$\Lambda < \frac{1+\ln f(\alpha)}{f(\alpha)} - A(\alpha).$$

These show conditions (3), (4), (8), (9), (11) and (12) are not satisfied. But

$$\limsup_{n\to\infty} p_n = 0.73 > \frac{1}{f(\alpha)} - A(\alpha) = 0.7225.$$

Hence, the conditions of Theorem 1 are satisfied and therefore every solution of (42) is oscillatory.

Example 2. Consider the difference equation

$$y_{n+1} - y_n + p_n y_{n-3} = 0, \quad n = 0, 1, 2, \dots \tag{43}$$

where $k = 3$ and $p_{15n} = p_{15n+1} = \cdots = p_{15n+7} = 0$, $p_{15n+8} = p_{15n+9} = \cdots = p_{15n+14} = 0.2$, $n = 0, 1, 2, \dots$. It is easy to observe that

$$\alpha = \liminf_{n\to\infty} \sum_{i=n-3}^{n-1} p_i = 0,$$

$$\Lambda = \limsup_{n\to\infty} \sum_{i=n-3}^{n} p_i = 0.8 < 1,$$

which show that conditions (3), (4), (5), (9) and (11) are not satisfied. In addition, it is easy to verify that (7) is not satisfied either. But we

$$\sum_{i=15n+11}^{15n+14} p_i \prod_{j=i-3}^{15n+10} (1-p_j)^{-1} = \frac{369}{320},$$

and so

$$\limsup_{n\to\infty} \sum_{i=n-3}^{n} p_i \prod_{j=i-3}^{n-3-1} (1-p_j)^{-1} > 1.$$

Hence, the conditions of Corollary 1 are satisfied and therefore all solution of (43) oscillate.

Example 3. Consider the difference equation

$$y_{n+1} - y_n + p_n y_{n-3} = 0, \quad n = 1, 2, \dots, \tag{44}$$

where $p_{15n} = p_{15n+1} = \cdots = p_{15n+7} = 0.1$, $p_{15n+8} = p_{15n+9} = \cdots = p_{15n+14} = 0.16$, $n = 0, 1, 2, \dots$. It is easy to observe that

$$\alpha = \liminf_{n\to\infty} \sum_{i=n-3}^{n-1} p_i = 0.3 < \left(\frac{3}{4}\right)^4,$$

$$\Lambda = \limsup_{n\to\infty} \sum_{i=n-3}^{n-1} p_i = 0.64 < 1,$$

$$f(\alpha) = f(0.3) = 1.842,$$

$$A(\alpha) = \left(1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}\right)/2 = 0.0716.$$

Hence

$$\frac{1 + \ln f(\alpha)}{f(\alpha)} - A(\alpha) = 0.802 > \Lambda,$$

which shows that condition (41) is not satisfied. But

$$\begin{aligned}
&\sum_{i=15n+11}^{15n+14} p_i \left[\min\left\{\prod_{j=i-3}^{15n+10} (1 - p_j f(\alpha))^{-1}, \frac{1}{f(\alpha)} \prod_{j=i-3}^{i-1} (1 - p_j f(\alpha))^{-1}\right\}\right] \\
&= 0.16 \sum_{i=15n+11}^{15n+14} \left[\min\left\{(1 - 0.16 \times 1.842)^{-(15n+14-i)}, \right.\right. \\
&\qquad\qquad \left.\left. \frac{1}{1.842}(1 - 0.16 \times 1.842)^{-3}\right\}\right] \\
&= 0.16 \left(\frac{1.418^3}{1.842} + \frac{1.418^3}{1.842} + 1.418 + 1\right) = 0.882 \\
&> 0.802 = \frac{1 + \ln f(\alpha)}{f(\alpha)} - A(\alpha).
\end{aligned}$$

These show that the conditions of Theorem 3 are satisfied and therefore every solution of (44) is oscillatory.

Remark 2. Example 3 shows that Theorem 3 can ameliorate Corollary 2 in general case.

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