Hokkaido Mathematical Journal Vol. 40 (2011) p. 313–335

Equivalence problem of second order PDE for scale transformations

(Dedicated to Professor Hajime Sato on his first retirement)

Takahiro Noda

(Received September 17, 2009; Revised July 14, 2010)

Abstract. The purpose of the paper is to consider an equivalence problem of second order partial differential equations for one unknown function of two independent variables under scale transformations. For this equivalence problem, explicit forms of invariant functions are given. In particular, if all of these invariant functions vanish, then PDEs are equivalent to the flat equation.

Key words: second order partial differential equations, equivalence problem, scale transformations, G-structure

1. Introduction

Sophus Lie initiated the study of geometric structures associated with differential equations by considering a certain equivalence problem of second order ordinary differential equations. To explain his work, we introduce a notion of the (local) equivalence problem of differential equations in general. We remark that every notions (e.g. coordinate transformations, functions) appearing in this paper are assumed to be in the local category. We need to fix classes of differential equations and a group of coordinate transformations to consider this problem. Then, the local equivalence problem of differential equations is a problem how differential equations change under local coordinate transformations. We can also express this problem in terms of group actions. Let X be a set of certain differential equations and $\mathcal{G}$ be a local coordinate transformation group which acts on X. Then the equivalence problem for differential equations in X is interpreted as the problem of determining the orbit decomposition under the action of $\mathcal{G}$ on X. Lie studied the equivalence problem in the case of

2000 Mathematics Subject Classification : Primary 58A15; Secondary 58A17.

The author is partially supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

$$\mathcal{G} = \mathrm{Cont}(J^1(\mathbb{R},\mathbb{R})), \quad X = \{y'' = f(x,y,y') \mid f \in C^\infty(J^1(\mathbb{R},\mathbb{R}))\},$$

where $\mathrm{Cont}(J^1(\mathbb{R},\mathbb{R}))$ is the contact diffeomorphism group preserving the canonical contact structure on $J^1(\mathbb{R},\mathbb{R})$. For this problem, he obtained the fact that this action is transitive. Namely, the orbit decomposition of X for the action of $\mathcal{G}$ has just one orbit. After the work of Lie, A. Tresse studied the following case. Let $\mathcal{G}$ be the subgroup $\mathrm{Diff}(\mathbb{R}^2)^{\mathrm{cont}}$ consisting of contact prolongations of diffeomorphisms on $\mathbb{R}^2$ to the jet space $J^1(\mathbb{R},\mathbb{R})$, and X be the same set of differential equations. Under this set up, Tresse considered an orbit decomposition of the action of $\mathcal{G}$ on X. In contrast to the above problem considered by Lie, Tresse proved that this action is not transitive.

At the same time, Élie Cartan also considered the same problem with a different method which is now called the equivalence method ([Gar], [O2], [St]). Tresse and Cartan proved independently the following result by using their original methods [GTW].

Theorem 1.1 (Tresse, Cartan) *Let $G = \mathrm{Diff}(\mathbb{R}^2)$ be the diffeomorphism group of $\mathbb{R}^2$. Two second order ordinary differential equations $y'' = f(x,y,y')$ and $y'' = g(x,y,y')$ are transformed for one to another by contact prolongations of elements of G if and only if $A(f) = A(g)$ and $B(f) = B(g)$, where A and B are functions expressed by*:

$$A = A(f) = \frac{d^2 f_{y'y'}}{dx^2} - 4\frac{df_{y'y}}{dx} - 3f_y f_{y'y'} + 6f_{yy} + f_{y'}\left(4f_{y'y} - \frac{df_{y'y'}}{dx}\right),$$

$$B = B(f) = f_{y'y'y'y'} \qquad \left(\frac{d}{dx} = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + f\frac{\partial}{\partial y'}\right).$$

It is well-known that this result is also obtained by using the theory of construction of Cartan-Tanaka connections by N. Tanaka ([Tan2], [Ya3]). After their works, some researchers studied the equivalence problem in the case of more restricted diffeomorphism groups ([Gar], [O2]). For example, Kamran, Lamb and Shadwick considered the equivalence problem with respect to the fiber preserving diffeomorphisms ϕ on $\mathbb{R}^2$ [KLS]:

$$\phi : (x,y) \mapsto (X(x), Y(x,y)).$$

Along this historical background, it is natural to extend the above-mentioned theory to the case of two independent variables. We consider an equivalence problem for the following second order PDE for one unknown function of two variables $y = y(x_1, x_2)$:

$$\frac{\partial^2 y}{\partial x_i \partial x_j} = f_{ij}(x_1, x_2, y, z_1, z_2), \tag{1}$$

where f_{ij} $(1 \leq i,\ j \leq 2)$ satisfying $f_{ij} = f_{ji}$ are C^∞ functions on the 1-jet space $J^1(\mathbb{R}^2, \mathbb{R}) := \{(x_1, x_2, y, z_1, z_2)\}$ with the canonical contact structure $C^1 = \{\theta := dy - z_1 dx_1 - z_2 dx_2 = 0\}$. By this contact structure C^1, we have the identification $z_1 = y_{x_1}$, $z_2 = y_{x_2}$ with respect to the dependent variable $z = z(x, y)$, hence we have second order PDEs (1) of normalized type. If f_{ij} all vanish, (1) is called the flat equation. We set $\mathcal{M} = \{$second order PDEs (1)$\}$. For these PDEs, we can choose many coordinate transformation groups as well as the second order ODEs. As a typical example, there is the following pseudo Lie group:

$$\mathrm{Diff}(\mathbb{R}^3)^{\mathrm{cont}} = \text{The contact prolongation of } \mathrm{Diff}(\mathbb{R}^3) \text{ to } J^1(\mathbb{R}^2, \mathbb{R}).$$

In this case, we can apply the Tanaka theory to the equivalence problem as well as the case of Tresse-Cartan for second order ODEs ([Tan2], [Ya3]). This problem is also studied precisely by Ozawa, Sato, Suzuki [SOS]. However they did not use the Tanaka theory and the Cartan's equivalence method. They characterized the orbit of the flat equation under contact prolongations. On the other hand, there are no results of equivalence problems associated with more restricted diffeomorphism groups for PDEs (1). Thus, it is natural to research an equivalence problem in the case of a restricted transformation group as well as second order ODEs. So, we take the group

$$\begin{aligned}&\mathrm{ScaleDiff}(\mathbb{R}^3)^{\mathrm{cont}}\\ &\quad = \text{The contact prolongation of } \mathrm{ScaleDiff}(\mathbb{R}^3) \text{ to } J^1(\mathbb{R}^2, \mathbb{R}),\end{aligned}$$

where $\mathrm{ScaleDiff}(\mathbb{R}^3)$ is the diffeomorphism group consisting of scale transformations defined by,

$$\phi(x_1, x_2, y) = \big(X_1(x_1), X_2(x_2), Y(x_1, x_2, y)\big). \tag{2}$$

Since ϕ is a transformation on $J^0(\mathbb{R}^2, \mathbb{R}) \cong \mathbb{R}^3$, we can also characterize this transformation geometrically as follows. Scale transformations preserve not only fibers on $J^0(\mathbb{R}^2, \mathbb{R})$ but also the web-structure on the base space $\mathbb{R}^2$ consisting of by parallel translation of x_1-axis and x_2-axis. Now we state the main problem treated in the present paper as follows.

Problem 1.2 *Examine the orbit decomposition under the action of* $\mathrm{ScaleDiff}(\mathbb{R}^3)^{\mathrm{cont}}$ *on* $\mathcal{M}$.

We can not apply the Tanaka theory to this equivalence problem, because a symmetry group under $\mathrm{ScaleDiff}(\mathbb{R}^3)^{\mathrm{cont}}$ is not semi-simple. Thus, it is necessary to use Cartan's classical method. We will calculate explicitly invariant functions for this equivalence problem by using Cartan's equivalence method ([Gar], [O2], [St]). To apply the theory of G-structure, we assume the integrability condition (6) with respect to the equation (1). Then, our main result can be stated as follows.

Main Theorem. *For* Problem 1.2 *of equations* (1) *satisfying the integrability condition, we obtain the* $\{e\}$*-structure* $\mathcal{F}^{(1)}_{G_2}$. *The structure equation of this* $\{e\}$*-structure* $\mathcal{F}^{(1)}_{G_2}$ *is given by*

$$
d\begin{bmatrix}\hat{\theta}_0\\ \hat{\theta}_1\\ \hat{\theta}_2\\ \hat{\omega}_1\\ \hat{\omega}_2\\ \hat{\alpha}\\ \hat{\gamma}\\ \hat{\psi}\end{bmatrix}
=\begin{bmatrix}
(\hat{\alpha}+\hat{\gamma})\wedge\hat{\theta}_0+\hat{\omega}_1\wedge\hat{\theta}_1+\hat{\omega}_2\wedge\hat{\theta}_2\\
\hat{\alpha}\wedge\hat{\theta}_1+M_1\hat{\theta}_2\wedge\hat{\omega}_1+M_3\hat{\theta}_2\wedge\hat{\theta}_0+M_4\hat{\omega}_1\wedge\hat{\theta}_0+M_5\hat{\omega}_2\wedge\hat{\theta}_0\\
(\hat{\alpha}+\hat{\gamma}-\hat{\psi})\wedge\hat{\theta}_2+M_6\hat{\theta}_1\wedge\hat{\omega}_2+M_7\hat{\theta}_1\wedge\hat{\theta}_0\\
\qquad +M_8\hat{\omega}_1\wedge\hat{\theta}_0+M_9\hat{\omega}_2\wedge\hat{\theta}_0\\
\hat{\gamma}\wedge\hat{\omega}_1\\
\hat{\psi}\wedge\hat{\omega}_2\\
S_1\hat{\omega}_1\wedge\hat{\theta}_0+S_2\hat{\omega}_2\wedge\hat{\theta}_0+S_3\hat{\theta}_1\wedge\hat{\theta}_0+S_4\hat{\theta}_2\wedge\hat{\theta}_0+S_5\hat{\omega}_1\wedge\hat{\theta}_1\\
\qquad +S_6\hat{\omega}_1\wedge\hat{\omega}_2+S_7\hat{\theta}_2\wedge\hat{\omega}_1-M_7\hat{\theta}_1\wedge\hat{\omega}_2\\
S_8\hat{\omega}_1\wedge\hat{\omega}_2+S_9\hat{\omega}_1\wedge\hat{\theta}_0+S_5\hat{\theta}_1\wedge\hat{\omega}_1+S_{10}\hat{\theta}_2\wedge\hat{\omega}_1\\
S_{11}\hat{\omega}_1\wedge\hat{\omega}_2+S_{12}\hat{\omega}_2\wedge\hat{\theta}_0+S_{13}\hat{\theta}_1\wedge\hat{\omega}_2+S_{14}\hat{\theta}_2\wedge\hat{\omega}_2
\end{bmatrix},
$$

where torsions M_i, S_j *are found on* (24). *Moreover, torsions* $M_4, M_9, S_3, S_4, S_7, S_{10}, S_{13}$ *are described by other torsions. Thus, we have* 15 *invariant functions* M_i, S_j $(i = 1, 3, 5, 6, 7, 8,\ j = 1, 2, 5, 6, 8, 9, 11, 12, 14)$.

This theorem is obtained by Theorem 3.8 and Proposition 3.9. Moreover, we obtain the following necessary and sufficient condition with respect to this equivalence problem. For the second order PDE (1) satisfying the integrability condition, if these invariant functions (24) vanish, then this equation is locally equivalent to the flat equation via the theory of G-structure [St].

Corollary 1.3 *Suppose that the second order PDE* (1) *satisfies the integrability condition. Then, the equation* (1) *is locally equivalent to the flat equation under contact prolongations of scale transformations if and only if invariant functions M_i, S_j vanish. In particular, we assume that defining functions f_{ij} in the equation* (1) *are given by the following form*:

$$f_{11} = P(x_1, x_2, y), \qquad f_{12} = Q(x_1, x_2, y), \qquad f_{22} = R(x_1, x_2, y).$$

Then, the equation (1) *is locally equivalent to the flat equation under contact prolongations of scale transformations if and only if the integrability condition $P_y = Q_y = R_y = 0$, $P_{x_2} = Q_{x_1}$, $Q_{x_2} = R_{x_1}$ is satisfied.*

This corollary is given by Corollary 3.12 and Corollary 3.13.

2. Equivalence problem and G-structure

In this section, we introduce the G-structure associated with Problem 1.2.

First, we consider contact prolongations $\phi^{(1)}$ on $J^1(\mathbb{R}^2, \mathbb{R})$ of scale transformations ϕ in (2) as follows:

$$\phi^{(1)}(x_1, x_2, y, z_1, z_2) = (X_1, X_2, Y, Z_1, Z_2), \tag{3}$$

where $Z_1 = \frac{Y_{x_1} + Y_y z_1}{(X_1)_{x_1}}$, $Z_2 = \frac{Y_{x_2} + Y_y z_2}{(X_2)_{x_2}}$. Indeed, we can see that $\phi^{(1)}$ are contact diffeomorphisms by:

$$\phi^{(1)^*}\theta = Y_y \theta,$$

where $\theta = dy - z_1 dx_1 - z_2 dx_2$ is the contact 1-form. Next, we introduce exterior differential systems $\mathcal{I}$ corresponding to PDEs (1) as follows. We choose the following adapted coframe of $J^1(\mathbb{R}^2, \mathbb{R})$ corresponding to the equation (1),

$$\begin{aligned}
\underline{\theta}_0 &= dy - z_1 dx_1 - z_2 dx_2,\\
\underline{\theta}_1 &= dz_1 - f_{11} dx_1 - f_{12} dx_2,\\
\underline{\theta}_2 &= dz_2 - f_{21} dx_1 - f_{22} dx_2,\\
\underline{\omega}_1 &= dx_1,\\
\underline{\omega}_2 &= dx_2.
\end{aligned} \tag{4}$$

We consider the completely integrable system (Frobenius system)

$$\mathcal{I} := \left\{\underline{\theta}_0, \underline{\theta}_1, \underline{\theta}_2\right\}_{\mathrm{diff}} \quad \text{with} \quad \underline{\omega}_1 \wedge \underline{\omega}_2 \neq 0 \tag{5}$$

consisting of this coframe. This system $\mathcal{I}$ is a differential ideal of the algebra $\Omega(J^1) := \bigoplus \Gamma(\Lambda^k T^* J^1)$ consisting of differential forms defined on $J^1(\mathbb{R}^2, \mathbb{R})$. The correspondence between the second order PDE (1) and the integrable system $\mathcal{I}$ is described as follows. We consider vector fields X satisfying the following property. 1-forms $\underline{\theta}_i$ annihilate X, while $\underline{\omega}_i$ do not annihilate X. At any point on $J^1(\mathbb{R}^2, \mathbb{R})$, such vector fields are generated by two vector fields v_1, v_2. The integral surfaces which are tangent to the 2-plane $\mathrm{span}\{v_1, v_2\}$ at any point are the graphs of solutions of the second order PDE (1). Then, the parameters (x_1, x_2) are regarded as a local coordinate system of this integral surface.

The integrability condition (Frobenius condition) of the integrable system $\mathcal{I}$ is:

$$d\underline{\theta}_i \equiv 0 \quad \left(\mathrm{mod}\ \underline{\theta}_0, \underline{\theta}_1, \underline{\theta}_2\right) \quad (i = 0, 1, 2). \tag{6}$$

Note that this condition is equivalent to the integrability condition of the PDE (1). Then, the above integrability condition is equivalent to $A = B = 0$, where A and B are given by

$$\begin{aligned}
A = {} & (f_{11})_{x_2} - (f_{12})_{x_1} + (f_{11})_y z_2 + (f_{11})_{z_1} f_{12} + (f_{11})_{z_2} f_{22}\\
& - (f_{12})_y z_1 - (f_{12})_{z_1} f_{11} - (f_{12})_{z_2} f_{12},\\
B = {} & (f_{12})_{x_2} - (f_{22})_{x_1} + (f_{12})_y z_2 + (f_{12})_{z_1} f_{12} + (f_{12})_{z_2} f_{22}\\
& - (f_{22})_y z_1 - (f_{22})_{z_1} f_{11} - (f_{22})_{z_2} f_{12}.
\end{aligned}$$

Remark 2.1 From now on, we discuss only overdetermined systems (1) satisfying this integrability condition.

A family of integral surfaces of $\mathcal{I}$ gives a 2-dimensional foliation on $J^1(\mathbb{R}^2,\mathbb{R})$. We describe the infinitesimal automorphism group of the foliation, and consider the principal bundle over $J^1(\mathbb{R}^2,\mathbb{R})$ with this group as a structure group. To define this structure group, we take another PDE of the same form:

$$\frac{\partial^2 Y}{\partial X_i \partial X_j} = F_{ij}(X_1, X_2, Y, Z_1, Z_2), \tag{7}$$

where this equation is defined on the jet space $J^1(\mathbb{R}^2,\mathbb{R}) = \{(X_1, X_2, Y, Z_1, Z_2)\}$ with the canonical contact form $\theta := dY - Z_1 dX_1 - Z_2 dX_2$. For this PDE, we also have the following adapted coframe:

$$\begin{aligned}
\theta_0 &= dY - Z_1 dX_1 - Z_2 dX_2,\\
\theta_1 &= dZ_1 - F_{11} dX_1 - F_{12} dX_2,\\
\theta_2 &= dZ_2 - F_{21} dX_1 - F_{22} dX_2,\\
\omega_1 &= dX_1,\\
\omega_2 &= dX_2.
\end{aligned} \tag{8}$$

If the contact prolongation $\phi^{(1)}$ of the scale transformation ϕ transforms a solution of the PDE (1) to a solution of the PDE (7), then $\phi^{(1)}$ induces a linear transformation between the adapted coframe (4) and the another coframe (8). We express an explicit form of these linear transformations. First, we have the following relation by the form of $\phi^{(1)}$:

$$\begin{aligned}
{\phi^{(1)}}^*\theta_0 &= a_0\underline{\theta}_0 && (a_0 \neq 0),\\
{\phi^{(1)}}^*\theta_1 &= b_0\underline{\theta}_0 + b_1\underline{\theta}_1 + b_2\underline{\omega}_1 + b_3\underline{\omega}_2, &&\\
{\phi^{(1)}}^*\theta_2 &= c_0\underline{\theta}_0 + c_1\underline{\theta}_2 + c_2\underline{\omega}_1 + c_3\underline{\omega}_2, &&\\
{\phi^{(1)}}^*\omega_1 &= e\underline{\omega}_1 && (e \neq 0),\\
{\phi^{(1)}}^*\omega_2 &= f\underline{\omega}_2 && (f \neq 0).
\end{aligned} \tag{9}$$

Moreover, we have $b_2 = b_3 = c_2 = c_3 = 0$, because $\phi^{(1)}$ transforms a solution of the PDE (1) to a solution of the PDE (7). More precisely, if we restrict coefficient functions b_2, b_3, c_2, c_3 to a solution surface, then b_2, b_3, c_2, c_3 vanish. Now, by the integrability condition, the integral surfaces of $\mathcal{I}$ form a foliation in $J^1(\mathbb{R}^2, \mathbb{R})$. Hence, if we take any point v in $J^1(\mathbb{R}^2, \mathbb{R})$, then there exists a solution surface of (1) which contains v, and we have $b_2 = b_3 = c_2 = c_3 = 0$ in the above transformation (9). Consequently, we have the following linear transformation of adapted coframes:

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 & 0 & 0 \\ b & c & 0 & 0 & 0 \\ e & 0 & g & 0 & 0 \\ 0 & 0 & 0 & h & 0 \\ 0 & 0 & 0 & 0 & k \end{bmatrix} \begin{bmatrix} \underline{\theta}_0 \\ \underline{\theta}_1 \\ \underline{\theta}_2 \\ \underline{\omega}_1 \\ \underline{\omega}_2 \end{bmatrix}, \tag{10}$$

where a, b, c, e, g, h, k are functions defined on $J^1(\mathbb{R}^2, \mathbb{R})$. Thus we have linear transformations of coframes determined by $\phi^{(1)}$. Moreover, we have the condition that contact prolongations $\phi^{(1)}$ must satisfy the following structure equation of the exterior differential system $\mathcal{I}$:

$$\begin{aligned} d\theta_0 &\equiv -\theta_1 \wedge \omega_1 - \theta_2 \wedge \omega_2 && (\mathrm{mod}\ \theta_0), \\ d\theta_1 &\equiv 0 && (\mathrm{mod}\ \theta_0, \theta_1, \theta_2), \\ d\theta_2 &\equiv 0 && (\mathrm{mod}\ \theta_0, \theta_1, \theta_2). \end{aligned} \tag{11}$$

In this equation, the first equation means that contact prolongations preserve a linear symplectic structure on the contact distribution. Moreover, the second and third conditions mean that contact prolongations $\phi^{(1)}$ preserve the integrability condition of $\mathcal{I}$. The first relation gives the condition $a = ch = gk$. Summarizing, we get the linear transformations of coframes of the following form:

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} ch & 0 & 0 & 0 & 0 \\ b & c & 0 & 0 & 0 \\ e & 0 & g & 0 & 0 \\ 0 & 0 & 0 & h & 0 \\ 0 & 0 & 0 & 0 & k \end{bmatrix} \begin{bmatrix} \underline{\theta}_0 \\ \underline{\theta}_1 \\ \underline{\theta}_2 \\ \underline{\omega}_1 \\ \underline{\omega}_2 \end{bmatrix}. \tag{12}$$

Therefore, we obtain the following 5-dimensional Lie group as the infinitesimal automorphism group:

$$G = \left\{ \begin{bmatrix} ch & 0 & 0 & 0 & 0 \\ b & c & 0 & 0 & 0 \\ e & 0 & g & 0 & 0 \\ 0 & 0 & 0 & h & 0 \\ 0 & 0 & 0 & 0 & k \end{bmatrix} \in GL(5, \mathbb{R}) \;\middle|\; ch = gk \right\}. \tag{13}$$

Then, we have the reduced G-bundle $\mathcal{F}_G$ of the coframe bundle $\mathcal{F}_{GL}(\mathbb{R}^5)$ over $J^1(\mathbb{R}^2, \mathbb{R})$. This bundle $\mathcal{F}_G$ is G-structure associated with the equivalence problem of second order PDE (1) for scale transformations.

3. Cartan's equivalence method

In the previous section, we introduced the G-structure $\mathcal{F}_G$ associated with the second order PDE (1). In this section we compute local invariant functions for the equivalence problem. For this purpose, we adopt the Cartan's equivalence method ([Gar], [O2], [St]).

To compute the structure equation on $\mathcal{F}_G$, we take the tautological 1-form of $\mathcal{F}_G$ defined as follows.

Definition 3.1 The tautological 1-form ω on $\mathcal{F}_G$ is a $\mathbb{R}^5$-valued 1-form on $\mathcal{F}_G$ defined by

$$\omega|_{(x,g_x)}(X) = g_x^{-1} \pi_*(X) \quad \text{for } X \in T_{(x,g_x)}\mathcal{F}_G, \tag{14}$$

where π is the bundle projection of $\mathcal{F}_G \to J^1(\mathbb{R}^2, \mathbb{R})$.

From this definition, we have the tautological 1-form $(\theta_0, \theta_1, \theta_2, \omega_1, \omega_2)$ in (12) on $\mathcal{F}_G$. To obtain the structure equation, we compute the exterior derivative of this tautological 1-forms $(\theta_0, \theta_1, \theta_2, \omega_1, \omega_2)$.

$$d \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} \frac{dc}{c} + \frac{dh}{h} & 0 & 0 & 0 & 0 \\ \frac{db}{ch} - \frac{bdc}{c^2h} & \frac{dc}{c} & 0 & 0 & 0 \\ \frac{de}{ch} - \frac{edg}{cgh} & 0 & \frac{dg}{g} & 0 & 0 \\ 0 & 0 & 0 & \frac{dh}{h} & 0 \\ 0 & 0 & 0 & 0 & \frac{dk}{k} \end{bmatrix} \wedge \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{bmatrix}$$

$$+\begin{bmatrix} T_1\omega_1\wedge\theta_0+T_2\omega_2\wedge\theta_0-\theta_1\wedge\omega_1-\theta_2\wedge\omega_2 \\ \theta_0\wedge(T_3\omega_1+T_4\omega_2)+\theta_1\wedge(T_5\omega_1+T_6\omega_2) \\ \qquad +\theta_2\wedge(T_7\omega_1+T_8\omega_2) \\ \theta_0\wedge(T_9\omega_1+T_{10}\omega_2)+\theta_1\wedge(T_{11}\omega_1+T_{12}\omega_2) \\ \qquad +\theta_2\wedge(T_{13}\omega_1+T_{14}\omega_2) \\ 0 \\ 0 \end{bmatrix}, \tag{15}$$

where

$$T_1=-\frac{b}{ch},\qquad T_2=-\frac{e}{ch},\qquad T_3=\frac{b^2}{(ch)^2}-\frac{(f_{11})_y}{h^2}+\frac{b(f_{11})_{z_1}}{ch^2}+\frac{e(f_{11})_{z_2}}{gh^2},$$

$$T_4=\frac{be}{(ch)^2}-\frac{(f_{12})_y}{hk}+\frac{b(f_{12})_{z_1}}{chk}+\frac{e(f_{12})_{z_2}}{ch^2},\qquad T_5=-\frac{b}{ch}-\frac{(f_{11})_{z_1}}{h},$$

$$T_6=-\frac{(f_{12})_{z_1}}{k},\qquad T_7=-\frac{c(f_{11})_{z_2}}{gh},\qquad T_8=-\frac{b}{ch}-\frac{(f_{12})_{z_2}}{h},$$

$$T_9=\frac{be}{(ch)^2}-\frac{g(f_{12})_y}{ch^2}+\frac{bg(f_{12})_{z_1}}{(ch)^2}+\frac{e(f_{12})_{z_2}}{ch^2},$$

$$T_{10}=\frac{e^2}{(ch)^2}-\frac{g(f_{22})_y}{chk}+\frac{bg(f_{22})_{z_1}}{c^2hk}+\frac{e(f_{22})_{z_2}}{chk},$$

$$T_{11}=-\frac{e}{ch}-\frac{g(f_{12})_{z_1}}{ch},\qquad T_{12}=-\frac{g(f_{22})_{z_1}}{ck},\qquad T_{13}=-\frac{(f_{12})_{z_2}}{h},$$

$$T_{14}=-\frac{e}{ch}-\frac{(f_{22})_{z_2}}{k}.$$

Remark 3.2 We put ω=$(\theta_0,\theta_1,\theta_2,\omega_1,\omega_2)$ and write the structure equation as follows:

$$d\omega=-\theta\wedge\omega+T\omega\wedge\omega.$$

In the above, we note that θ is a $\mathfrak{g}$-valued 1-form and $T\omega\wedge\omega$ is a $\mathbb{R}^5$-valued 2-form, where $\mathfrak{g}$ is the Lie algebra of G. In fact,

$$d\omega=d(g\underline{\omega})=dg\cdot g^{-1}\wedge\omega+T\omega\wedge\omega,$$

where $g \in G$ and $\underline{\omega} = (\underline{\theta_0}, \underline{\theta_1}, \underline{\theta_2}, \underline{\omega_1}, \underline{\omega_2})$. In the structure equation, each component of θ is called the pseudo-connection form and $T\omega \wedge \omega$ is called the torsion 2-form, and coefficient functions of 2-forms in each component of $T\omega \wedge \omega$ are called torsions [IL].

To simplify the structure equation, we set:

$$
\begin{aligned}
\alpha &:= \frac{dc}{c} - \frac{b}{ch}\omega_1 - \frac{e}{ch}\omega_2, \\
\beta &:= \frac{db}{ch} - \frac{bdc}{c^2h} - \left\{ \frac{b^2}{(ch)^2} - \frac{(f_{11})_y}{h^2} + \frac{b(f_{11})_{z_1}}{ch^2} + \frac{e(f_{11})_{z_2}}{gh^2} \right\}\omega_1 \\
&\quad - \left\{ \frac{be}{(ch)^2} - \frac{(f_{12})_y}{hk} + \frac{b(f_{12})_{z_1}}{chk} + \frac{e(f_{12})_{z_2}}{ch^2} \right\}\omega_2, \\
\varepsilon &:= \frac{de}{ch} - \frac{edg}{cgh} - \left\{ \frac{be}{(ch)^2} - \frac{g(f_{12})_y}{ch^2} + \frac{bg(f_{12})_{z_1}}{(ch)^2} + \frac{e(f_{12})_{z_2}}{ch^2} \right\}\omega_1 \\
&\quad - \left\{ \frac{e^2}{(ch)^2} - \frac{g(f_{22})_y}{chk} + \frac{bg(f_{22})_{z_1}}{c^2hk} + \frac{e(f_{22})_{z_2}}{chk} \right\}\omega_2, \\
\delta &:= \frac{dg}{g} - \frac{b}{ch}\omega_1 - \frac{e}{ch}\omega_2, \quad \gamma := \frac{dh}{h}, \quad \psi := \frac{dk}{k}.
\end{aligned}
$$

By substituting the above terms into the structure equation (15), we get the following proposition.

Proposition 3.3 *The structure equation on $\mathcal{F}_G$ is written as*:

$$
d\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} \alpha+\gamma & 0 & 0 & 0 & 0 \\ \beta & \alpha & 0 & 0 & 0 \\ \varepsilon & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & \psi \end{bmatrix} \wedge \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \omega_1 \\ \omega_2 \end{bmatrix}
+ \begin{bmatrix} -\theta_1 \wedge \omega_1 - \theta_2 \wedge \omega_2 \\ L_1\theta_1 \wedge \omega_1 + L_2\theta_1 \wedge \omega_2 + L_3\theta_2 \wedge \omega_1 + L_4\theta_2 \wedge \omega_2 \\ L_2\theta_1 \wedge \omega_1 + L_5\theta_1 \wedge \omega_2 + L_4\theta_2 \wedge \omega_1 + L_6\theta_2 \wedge \omega_2 \\ 0 \\ 0 \end{bmatrix}, \tag{16}
$$

where

$$L_1 := -\frac{2b}{ch} - \frac{(f_{11})_{z_1}}{h}, \quad L_2 := -\frac{e}{ch} - \frac{(f_{12})_{z_1}}{k}, \quad L_3 := -\frac{c(f_{11})_{z_2}}{gh},$$

$$L_4 := -\frac{b}{ch} - \frac{(f_{12})_{z_2}}{h}, \quad L_5 := -\frac{g(f_{22})_{z_1}}{ck}, \qquad L_6 := -\frac{2e}{ch} - \frac{(f_{22})_{z_2}}{k},$$

$$\alpha + \gamma = \delta + \psi.$$

Remark 3.4 In the structure equation (16), some torsions in (15) are absorbed in pseudo-connection forms in the $\mathfrak{g}$-valued 1-form. This procedure is called absorption of torsions, the above expression of pseudo-connection forms $\alpha, \beta, \varepsilon, \delta, \gamma, \psi$ are obtained by solving the absorption equation (precisely, see Chapter 10 in [O2]).

There exists ambiguity for the pseudo-connection forms of $\mathcal{F}_G$. Hence, we consider a reduction of G-structure $\mathcal{F}_G$. Precisely, refer to Lecture 4 in [Gar] or Chapter 10 in [O2] (normalization of torsions and group reduction). To eliminate the group parameter b of G, we choose an element $(x, g_x) \in \mathcal{F}_G$ which satisfies $L_4(x, g_x) = 0$, for example,

$$(x, g_x) = \left(x, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -(f_{12})_{z_2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right). \tag{17}$$

The isotropy subgroup G_1 for (x, g_x) above is

$$\begin{aligned} G_1 &= \{g \in G \mid L_4(x, gg_x) = 0\} \\ &= \left\{ \begin{bmatrix} ch & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 \\ e & 0 & g & 0 & 0 \\ 0 & 0 & 0 & h & 0 \\ 0 & 0 & 0 & 0 & k \end{bmatrix} \in GL(5, \mathbb{R}) \mid ch = gk \right\}. \end{aligned} \tag{18}$$

We consider the reduced G_1-structure $\mathcal{F}_{G_1}$ which has the structure group G_1. For two G-structures $\mathcal{F}_G(U)$ on an open set U and $\mathcal{F}_G(V)$ on an open set V,

$\mathcal{F}_G(U)$ and $\mathcal{F}_G(V)$ are locally isomorphic if and only if $\mathcal{F}_{G_1}(U)$ and $\mathcal{F}_{G_1}(V)$ are locally isomorphic ([Gar, Lecture 4, Theorem]). Hence, it is sufficient to apply the equivalence method to the G_1-structure $\mathcal{F}_{G_1}$. To compute the structure equation of $\mathcal{F}_{G_1}$, we need to take the tautological form of $\mathcal{F}_{G_1}$. In this case, this tautological 1-form is obtained by substituting the condition $L_4 = 0$ into the tautological form (12) of $\mathcal{F}_G$:

$$\begin{bmatrix} \hat{\theta}_0 \\ \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\omega}_1 \\ \hat{\omega}_2 \end{bmatrix} = \begin{bmatrix} ch\underline{\theta_0} \\ -c(f_{12})_{z_2}\underline{\theta_0} + c\underline{\theta_1} \\ e\underline{\theta_0} + g\underline{\theta_2} \\ h\underline{\omega_1} \\ k\underline{\omega_2} \end{bmatrix}. \tag{19}$$

Then, the structure equation on $\mathcal{F}_{G_1}$ is given by

$$d\begin{bmatrix} \hat{\theta}_0 \\ \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\omega}_1 \\ \hat{\omega}_2 \end{bmatrix} = \begin{bmatrix} \alpha+\gamma & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ \varepsilon & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & \psi \end{bmatrix} \wedge \begin{bmatrix} \hat{\theta}_0 \\ \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\omega}_1 \\ \hat{\omega}_2 \end{bmatrix}$$
$$+ \begin{bmatrix} N_1\hat{\omega}_1\wedge\hat{\theta}_0 + N_2\hat{\omega}_2\wedge\hat{\theta}_0 - \hat{\theta}_1\wedge\hat{\omega}_1 - \hat{\theta}_2\wedge\hat{\omega}_2 \\ N_3\hat{\theta}_1\wedge\hat{\omega}_1 + N_4\hat{\theta}_1\wedge\hat{\omega}_2 + N_5\hat{\theta}_1\wedge\hat{\theta}_0 + N_6\hat{\theta}_2\wedge\hat{\omega}_1 \\ \qquad +N_7\hat{\theta}_2\wedge\hat{\theta}_0 + N_8\hat{\omega}_1\wedge\hat{\theta}_0 + N_9\hat{\omega}_2\wedge\hat{\theta}_0 \\ N_{10}\hat{\theta}_1\wedge\hat{\omega}_1 + N_{11}\hat{\theta}_1\wedge\hat{\omega}_2 - N_1\hat{\theta}_2\wedge\hat{\omega}_1 + N_{12}\hat{\theta}_2\wedge\hat{\omega}_2 \\ \qquad +N_{13}\hat{\omega}_1\wedge\hat{\theta}_0 + N_{14}\hat{\omega}_2\wedge\hat{\theta}_0 \\ 0 \\ 0 \end{bmatrix},$$

where

$$\alpha = \frac{dc}{c}, \quad \varepsilon = \frac{de}{ch} - \frac{edg}{cgh}, \quad \delta = \frac{dg}{g}, \quad \gamma = \frac{dh}{h}, \quad \psi = \frac{dk}{k},$$

$$N_1 = \frac{(f_{12})_{z_2}}{h}, \quad N_2 = -\frac{e}{ch}, \quad N_3 = \frac{1}{h}\{-(f_{11})_{z_1} + (f_{12})_{z_2}\},$$

$$N_4 = -\frac{(f_{12})_{z_1}}{k}, \quad N_5 = -\frac{(f_{12})_{z_2z_1}}{ch}, \quad N_6 = -\frac{c}{gh}(f_{11})_{z_2}, \quad N_7 = -\frac{(f_{12})_{z_2z_2}}{gh},$$

$$N_8 = -\frac{1}{h^2}\Big\{(f_{12})^2_{z_2} - (f_{11})_y - (f_{12})_{z_2}(f_{11})_{z_1} + \frac{e}{g}(f_{11})_{z_2} + (f_{12})_{z_2x_1} + (f_{12})_{z_2y}z_1 + (f_{12})_{z_2z_1}f_{11} + (f_{12})_{z_2z_2}f_{21}\Big\},$$

$$N_9 = \frac{1}{hk}\{(f_{12})_y + (f_{12})_{z_2}(f_{12})_{z_1} - (f_{12})_{z_2x_2} - (f_{12})_{z_2y}z_2 - (f_{12})_{z_2z_1}f_{12} - (f_{12})_{z_2z_2}f_{22}\},$$

$$N_{10} = -\frac{e + g(f_{12})_{z_1}}{ch}, \quad N_{11} = -\frac{g(f_{22})_{z_1}}{ck}, \quad N_{12} = -\frac{e}{ch} - \frac{(f_{22})_{z_2}}{k},$$

$$N_{13} = \frac{g}{ch^2}\{(f_{12})_y + (f_{12})_{z_1}(f_{12})_{z_2}\},$$

$$N_{14} = -\frac{e^2}{(ch)^2} + \frac{g(f_{22})_y}{chk} + \frac{g(f_{12})_{z_2}(f_{22})_{z_1}}{chk} - \frac{e(f_{22})_{z_2}}{chk}.$$

To simplify this structure equation, we set:

$$\hat{\alpha} = \alpha - N_5\hat{\theta}_0 - N_3\hat{\omega}_1 + N_2\hat{\omega}_2,$$

$$\hat{\varepsilon} = \varepsilon - N_5\hat{\theta}_2 + N_{13}\hat{\omega}_1 + N_{14}\hat{\omega}_2,$$

$$\hat{\delta} = \delta - N_5\hat{\theta}_0 + N_1\hat{\omega}_1 - N_{12}\hat{\omega}_2,$$

$$\hat{\gamma} = \gamma + (N_1 + N_3)\hat{\omega}_1,$$

$$\hat{\psi} = \psi + (N_2 + N_{12})\hat{\omega}_2.$$

By substituting these terms into the above structure equation, we get the following.

Proposition 3.5 *The structure equation on $\mathcal{F}_{G_1}$ is written as*:

$$d\begin{bmatrix}\hat{\theta}_0\\ \hat{\theta}_1\\ \hat{\theta}_2\\ \hat{\omega}_1\\ \hat{\omega}_2\end{bmatrix} = \begin{bmatrix}\hat{\alpha}+\hat{\gamma} & 0 & 0 & 0 & 0\\ 0 & \hat{\alpha} & 0 & 0 & 0\\ \hat{\varepsilon} & 0 & \hat{\delta} & 0 & 0\\ 0 & 0 & 0 & \hat{\gamma} & 0\\ 0 & 0 & 0 & 0 & \hat{\psi}\end{bmatrix} \wedge \begin{bmatrix}\hat{\theta}_0\\ \hat{\theta}_1\\ \hat{\theta}_2\\ \hat{\omega}_1\\ \hat{\omega}_2\end{bmatrix}$$

$$+\begin{bmatrix} -\hat{\theta}_1 \wedge \hat{\omega}_1 - \hat{\theta}_2 \wedge \hat{\omega}_2 \\ N_{10}\hat{\theta}_1 \wedge \hat{\omega}_2 + N_6\hat{\theta}_2 \wedge \hat{\omega}_1 + N_7\hat{\theta}_2 \wedge \hat{\theta}_0 \\ \qquad +N_8\hat{\omega}_1 \wedge \hat{\theta}_0 + N_9\hat{\omega}_2 \wedge \hat{\theta}_0 \\ N_{10}\hat{\theta}_1 \wedge \hat{\omega}_1 + N_{11}\hat{\theta}_1 \wedge \hat{\omega}_2 \\ 0 \\ 0 \end{bmatrix}. \tag{20}$$

In the structure equation (20), there still remains ambiguity of pseudo-connection forms. Hence, we shall take the next step of reduction. To eliminate the group parameter e of G_1, we choose an element $(x, g_x) \in \mathcal{F}_{G_1}$ which satisfies $N_{10}(x, g_x) = 0$, for example,

$$(x, g_x) = \left(x, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -(f_{12})_{z_1} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right). \tag{21}$$

The isotropy subgroup G_2 for (x, g_x) above is

$$\begin{aligned} G_2 &= \{ g \in G \mid N_{10}(x, g g_x) = 0 \} \\ &= \left\{ \begin{bmatrix} ch & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 \\ 0 & 0 & g & 0 & 0 \\ 0 & 0 & 0 & h & 0 \\ 0 & 0 & 0 & 0 & k \end{bmatrix} \in GL(5, \mathbb{R}) \mid ch = gk \right\}. \end{aligned} \tag{22}$$

We take the reduced G_2-structure $\mathcal{F}_{G_2}$ which has the structure group G_2. Similarly to the case of G_1-structure, we apply the equivalence method to the G_2-structure $\mathcal{F}_{G_2}$. We have the tautological 1-form on $\mathcal{F}_{G_2}$ by substituting the condition $N_{10} = 0$ into the equation (12):

$$\begin{bmatrix} \hat{\theta}_0 \\ \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\omega}_1 \\ \hat{\omega}_2 \end{bmatrix} = \begin{bmatrix} ch\underline{\theta_0} \\ -c(f_{12})_{z_2}\underline{\theta_0} + c\underline{\theta_1} \\ -g(f_{12})_{z_1}\underline{\theta_0} + g\underline{\theta_2} \\ h\underline{\omega_1} \\ k\underline{\omega_2} \end{bmatrix}.$$

Then, the structure equation on $\mathcal{F}_{G_2}$ is given by

$$d\begin{bmatrix} \hat{\theta}_0 \\ \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\omega}_1 \\ \hat{\omega}_2 \end{bmatrix} = \begin{bmatrix} \alpha+\gamma & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & \delta & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & \psi \end{bmatrix} \wedge \begin{bmatrix} \hat{\theta}_0 \\ \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\omega}_1 \\ \hat{\omega}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} M_{12}\hat{\omega}_1 \wedge \hat{\theta}_0 + M_{11}\hat{\omega}_2 \wedge \hat{\theta}_0 - \hat{\theta}_1 \wedge \hat{\omega}_1 - \hat{\theta}_2 \wedge \hat{\omega}_2 \\ M_1\hat{\theta}_2 \wedge \hat{\omega}_1 + M_2\hat{\theta}_1 \wedge \hat{\theta}_0 + M_3\hat{\theta}_2 \wedge \hat{\theta}_0 + M_4\hat{\omega}_1 \wedge \hat{\theta}_0 \\ \qquad + M_5\hat{\omega}_2 \wedge \hat{\theta}_0 + M_{10}\hat{\omega}_1 \wedge \hat{\theta}_1 + M_{11}\hat{\omega}_2 \wedge \hat{\theta}_1 \\ M_6\hat{\theta}_1 \wedge \hat{\omega}_2 + M_7\hat{\theta}_1 \wedge \hat{\theta}_0 + M_2\hat{\theta}_2 \wedge \hat{\theta}_0 + M_8\hat{\omega}_1 \wedge \hat{\theta}_0 \\ \qquad + M_9\hat{\omega}_2 \wedge \hat{\theta}_0 + M_{12}\hat{\omega}_1 \wedge \hat{\theta}_2 + M_{13}\hat{\omega}_2 \wedge \hat{\theta}_2 \\ 0 \\ 0 \end{bmatrix},$$

where

$$\alpha = \frac{dc}{c}, \qquad \delta = \frac{dg}{g}, \qquad \gamma = \frac{dh}{h}, \qquad \psi = \frac{dk}{k},$$

$$M_1 = -\frac{c(f_{11})_{z_2}}{gh}, \qquad M_2 = -\frac{(f_{12})_{z_2 z_1}}{ch}, \qquad M_3 = -\frac{(f_{12})_{z_2 z_2}}{gh},$$

$$M_4 = -\frac{1}{h^2}\big\{(f_{12})_{z_2}^2 - (f_{11})_y - (f_{12})_{z_2}(f_{11})_{z_1} - (f_{11})_{z_2}(f_{12})_{z_1} \\ + (f_{12})_{z_2 x_1} + (f_{12})_{z_2 y} z_1 + (f_{12})_{z_2 z_1} f_{11} + (f_{12})_{z_2 z_2} f_{21}\big\},$$

$$M_5 = \frac{1}{hk}\big\{(f_{12})_y + (f_{12})_{z_2}(f_{12})_{z_1} - (f_{12})_{z_2 x_2} \\ - (f_{12})_{z_2 y} z_2 - (f_{12})_{z_2 z_1} f_{12} - (f_{12})_{z_2 z_2} f_{22}\big\},$$

$$M_6 = -\frac{g(f_{22})_{z_1}}{ck}, \qquad M_7 = -\frac{(f_{12})_{z_1z_1}}{ck},$$

$$M_8 = \frac{1}{hk}\big\{(f_{12})_y + (f_{12})_{z_1}(f_{12})_{z_2} - (f_{12})_{z_1x_1} - (f_{12})_{z_1y}z_1 \\ - (f_{12})_{z_1z_1}f_{11} - (f_{12})_{z_1z_2}f_{21}\big\},$$

$$M_9 = -\frac{1}{k^2}\big\{(f_{12})_{z_1}^2 - (f_{22})_y - (f_{12})_{z_2}(f_{22})_{z_1} - (f_{12})_{z_1}(f_{22})_{z_2} \\ + (f_{12})_{z_1x_2} + (f_{12})_{z_1y}z_2 + (f_{12})_{z_1z_1}f_{12} + (f_{12})_{z_1z_2}f_{22}\big\},$$

$$M_{10} = \frac{1}{h}\big\{(f_{11})_{z_1} - (f_{12})_{z_2}\big\}, \qquad M_{11} = \frac{(f_{12})_{z_1}}{k},$$

$$M_{12} = \frac{(f_{12})_{z_2}}{h}, \qquad M_{13} = \frac{1}{k}\big\{(f_{22})_{z_2} - (f_{12})_{z_1}\big\}.$$

To simplify the structure equation, we set

$$\begin{aligned}
\hat{\alpha} &= \alpha - M_2\hat{\theta}_0 + M_{10}\hat{\omega}_1 + M_{11}\hat{\omega}_2, \\
\hat{\gamma} &= \gamma + (M_{12} - M_{10})\hat{\omega}_1, \\
\hat{\delta} &= \delta - M_2\hat{\theta}_0 + M_{12}\hat{\omega}_1 + M_{13}\hat{\omega}_2, \\
\hat{\psi} &= \psi + (M_{11} - M_{13})\hat{\omega}_2.
\end{aligned}$$

Then, we obtain the following:

Proposition 3.6 *We have the following structure equation on $\mathcal{F}_{G_2}$.*

$$d\begin{bmatrix}\hat{\theta}_0\\ \hat{\theta}_1\\ \hat{\theta}_2\\ \hat{\omega}_1\\ \hat{\omega}_2\end{bmatrix} = \begin{bmatrix}\hat{\alpha}+\hat{\gamma} & 0 & 0 & 0 & 0\\ 0 & \hat{\alpha} & 0 & 0 & 0\\ 0 & 0 & \hat{\delta} & 0 & 0\\ 0 & 0 & 0 & \hat{\gamma} & 0\\ 0 & 0 & 0 & 0 & \hat{\psi}\end{bmatrix} \wedge \begin{bmatrix}\hat{\theta}_0\\ \hat{\theta}_1\\ \hat{\theta}_2\\ \hat{\omega}_1\\ \hat{\omega}_2\end{bmatrix} \\ + \begin{bmatrix}\hat{\omega}_1\wedge\hat{\theta}_1 + \hat{\omega}_2\wedge\hat{\theta}_2\\ M_1\hat{\theta}_2\wedge\hat{\omega}_1 + M_3\hat{\theta}_2\wedge\hat{\theta}_0 + M_4\hat{\omega}_1\wedge\hat{\theta}_0 + M_5\hat{\omega}_2\wedge\hat{\theta}_0\\ M_6\hat{\theta}_1\wedge\hat{\omega}_2 + M_7\hat{\theta}_1\wedge\hat{\theta}_0 + M_8\hat{\omega}_1\wedge\hat{\theta}_0 + M_9\hat{\omega}_2\wedge\hat{\theta}_0\\ 0\\ 0\end{bmatrix} \tag{23}$$

We note that the structure equation (23) defines uniquely the pseudo-connection forms $\hat{\alpha}$, $\hat{\gamma}$, $\hat{\delta}$, $\hat{\psi}$. Hence, we can obtain the G_2-invariant 1-forms $(\hat{\theta}_0, \hat{\theta}_1, \hat{\theta}_2, \hat{\omega}_1, \hat{\omega}_2, \hat{\alpha}, \hat{\gamma}, \hat{\psi})$ on $\mathcal{F}_{G_2}$. To consider invariant functions for Problem 1.2, we need to take the prolongation $\mathcal{F}^{(1)}_{G_2}$ of $\mathcal{F}_{G_2}$ defined as follows.

Definition 3.7 Let $\mathcal{F}_G$ be a G-structure and $\mathfrak{g} \subset Hom(V, V)$ be the Lie algebra of the structure group G. Then, the prolongation $\mathcal{F}_{G^1}$ of $\mathcal{F}_G$ is a principal bundle over $\mathcal{F}_G$ with the structure group G^1, where G^1 is the group which has the corresponding Lie algebra $\mathfrak{g}^1 := (\mathfrak{g} \otimes V^*) \cup (V \otimes S^2(V^*))$.

For the G_2-structure, we see that $\mathfrak{g}_2^{(1)} = 0$ and the group $G_2^{(1)} = \{e\}$. Hence, $\mathcal{F}^{(1)}_{G_2}$ is the $\{e\}$-structure over $\mathcal{F}_{G_2}$. That is, $\mathcal{F}^{(1)}_{G_2}$ is absolute parallelism on $\mathcal{F}_{G_2}$. Now we choose the tautological 1-form $(\hat{\theta}_0, \hat{\theta}_1, \hat{\theta}_2, \hat{\omega}_1, \hat{\omega}_2, \hat{\alpha}, \hat{\gamma}, \hat{\psi})$ on $\mathcal{F}^{(1)}_{G_2}$. By taking the exterior derivation of this tautological 1-form, we obtain the following structure equation on the $\{e\}$-structure.

Theorem 3.8 *The structure equation of the $\{e\}$-structure $\mathcal{F}^{(1)}_{G_2}$ with the tautological 1-form $(\hat{\theta}_0, \hat{\theta}_1, \hat{\theta}_2, \hat{\omega}_1, \hat{\omega}_2, \hat{\alpha}, \hat{\gamma}, \hat{\psi})$ is given by*

$$
d \begin{bmatrix} \hat{\theta}_0 \\ \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\omega}_1 \\ \hat{\omega}_2 \\ \hat{\alpha} \\ \hat{\gamma} \\ \hat{\psi} \end{bmatrix} = \begin{bmatrix} (\hat{\alpha} + \hat{\gamma}) \wedge \hat{\theta}_0 + \hat{\omega}_1 \wedge \hat{\theta}_1 + \hat{\omega}_2 \wedge \hat{\theta}_2 \\ \hat{\alpha} \wedge \hat{\theta}_1 + M_1 \hat{\theta}_2 \wedge \hat{\omega}_1 + M_3 \hat{\theta}_2 \wedge \hat{\theta}_0 + M_4 \hat{\omega}_1 \wedge \hat{\theta}_0 + M_5 \hat{\omega}_2 \wedge \hat{\theta}_0 \\ (\hat{\alpha} + \hat{\gamma} - \hat{\psi}) \wedge \hat{\theta}_2 + M_6 \hat{\theta}_1 \wedge \hat{\omega}_2 + M_7 \hat{\theta}_1 \wedge \hat{\theta}_0 \\ \qquad + M_8 \hat{\omega}_1 \wedge \hat{\theta}_0 + M_9 \hat{\omega}_2 \wedge \hat{\theta}_0 \\ \hat{\gamma} \wedge \hat{\omega}_1 \\ \hat{\psi} \wedge \hat{\omega}_2 \\ S_1 \hat{\omega}_1 \wedge \hat{\theta}_0 + S_2 \hat{\omega}_2 \wedge \hat{\theta}_0 + S_3 \hat{\theta}_1 \wedge \hat{\theta}_0 + S_4 \hat{\theta}_2 \wedge \hat{\theta}_0 + S_5 \hat{\omega}_1 \wedge \hat{\theta}_1 \\ \qquad + S_6 \hat{\omega}_1 \wedge \hat{\omega}_2 + S_7 \hat{\theta}_2 \wedge \hat{\omega}_1 - M_7 \hat{\theta}_1 \wedge \hat{\omega}_2 \\ S_8 \hat{\omega}_1 \wedge \hat{\omega}_2 + S_9 \hat{\omega}_1 \wedge \hat{\theta}_0 + S_5 \hat{\theta}_1 \wedge \hat{\omega}_1 + S_{10} \hat{\theta}_2 \wedge \hat{\omega}_1 \\ S_{11} \hat{\omega}_1 \wedge \hat{\omega}_2 + S_{12} \hat{\omega}_2 \wedge \hat{\theta}_0 + S_{13} \hat{\theta}_1 \wedge \hat{\omega}_2 + S_{14} \hat{\theta}_2 \wedge \hat{\omega}_2 \end{bmatrix},
$$

where torsions M_i, S_j are given by

$$
\begin{aligned}
M_1 &= -\frac{c}{gh}(f_{11})_{\underline{\theta}_2}, \qquad M_3 = -\frac{1}{gh}(f_{12})_{\underline{\theta}_2\underline{\theta}_2},\\
M_4 &= -\frac{1}{h^2}\big\{(f_{11})_{\underline{\theta}_2\underline{\omega}_2} - 2(f_{11})_{\underline{\theta}_2}(f_{12})_{\underline{\theta}_1} + (f_{11})_{\underline{\theta}_2}(f_{22})_{\underline{\theta}_2}\big\},\\
M_5 &= \frac{1}{hk}\big\{(f_{12})_{\underline{\theta}_0} + (f_{12})_{\underline{\theta}_2}(f_{12})_{\underline{\theta}_1} - (f_{12})_{\underline{\theta}_2\underline{\omega}_2}\big\},\\
M_6 &= -\frac{g}{ck}(f_{22})_{\underline{\theta}_1}, \qquad M_7 = -\frac{1}{ck}(f_{12})_{\underline{\theta}_1\underline{\theta}_1},\\
M_8 &= \frac{1}{hk}\left\{(f_{12})_{\underline{\theta}_0} + (f_{12})_{\underline{\theta}_1}(f_{12})_{\underline{\theta}_2} - (f_{12})_{\underline{\theta}_1\underline{\omega}_1}\right\},\\
M_9 &= -\frac{1}{k^2}\big\{-2(f_{12})_{\underline{\theta}_2}(f_{22})_{\underline{\theta}_1} + (f_{22})_{\underline{\theta}_1\underline{\omega}_1} + (f_{11})_{\underline{\theta}_1}(f_{22})_{\underline{\theta}_1}\big\},\\
S_1 &= \frac{1}{ch^2}\big\{(f_{11})_{\underline{\theta}_2\underline{\theta}_1\underline{\omega}_2} + (f_{11})_{\underline{\theta}_2\underline{\theta}_2}(f_{22})_{\underline{\theta}_1} + (f_{11})_{\underline{\theta}_2\underline{\theta}_1}(f_{22})_{\underline{\theta}_2} - (f_{12})_{\underline{\theta}_2\underline{\theta}_1}(f_{11})_{\underline{\theta}_2}\\
&\qquad - (f_{12})_{\underline{\theta}_2\underline{\theta}_2}(f_{12})_{\underline{\theta}_2} - (f_{12})_{\underline{\theta}_2\underline{\theta}_1}(f_{11})_{\underline{\theta}_1} + 2(f_{12})_{\underline{\theta}_2\underline{\theta}_1}(f_{12})_{\underline{\theta}_2}\big\},\\
S_2 &= \frac{1}{chk}\big\{(f_{12})_{\underline{\theta}_2\underline{\theta}_1\underline{\omega}_2} - (f_{12})_{\underline{\theta}_1\underline{\theta}_0} - (f_{12})_{\underline{\theta}_1\underline{\theta}_1}(f_{12})_{\underline{\theta}_2}\big\}, \qquad (24)\\
S_3 &= \frac{(f_{12})_{\underline{\theta}_2\underline{\theta}_1\underline{\theta}_1}}{c^2h}, \quad S_4 = \frac{(f_{12})_{\underline{\theta}_2\underline{\theta}_1\underline{\theta}_2}}{cgh}, \quad S_5 = \frac{2(f_{12})_{\underline{\theta}_2\underline{\theta}_1} - (f_{11})_{\underline{\theta}_1\underline{\theta}_1}}{ch},\\
S_6 &= \frac{1}{hk}\big\{-(f_{12})_{\underline{\theta}_0} - (f_{12})_{\underline{\theta}_1}(f_{12})_{\underline{\theta}_2} + (f_{11})_{\underline{\theta}_2}(f_{22})_{\underline{\theta}_1} + (f_{12})_{\underline{\theta}_2\underline{\omega}_2}\big\},\\
S_7 &= \frac{(f_{11})_{\underline{\theta}_1\underline{\theta}_2} - (f_{12})_{\underline{\theta}_2\underline{\theta}_2}}{gh}, \qquad S_8 = \frac{1}{hk}\big\{(f_{11})_{\underline{\theta}_1\underline{\omega}_2} - 2(f_{12})_{\underline{\theta}_2\underline{\omega}_2}\big\},\\
S_9 &= \frac{1}{ch^2}\big\{(f_{11})_{\underline{\theta}_1\underline{\theta}_0} - 2(f_{12})_{\underline{\theta}_2\underline{\theta}_0} + (f_{11})_{\underline{\theta}_1\underline{\theta}_1}(f_{12})_{\underline{\theta}_2}\\
&\qquad + (f_{11})_{\underline{\theta}_1\underline{\theta}_2}(f_{12})_{\underline{\theta}_1} - 2(f_{12})_{\underline{\theta}_1\underline{\theta}_2}(f_{12})_{\underline{\theta}_2} - 2(f_{12})_{\underline{\theta}_2\underline{\theta}_2}(f_{12})_{\underline{\theta}_1}\big\},\\
S_{10} &= \frac{-(f_{11})_{\underline{\theta}_1\underline{\theta}_2} + 2(f_{12})_{\underline{\theta}_2\underline{\theta}_2}}{gh}, \qquad S_{11} = \frac{1}{hk}\big\{2(f_{12})_{\underline{\theta}_1\underline{\omega}_1} - (f_{22})_{\underline{\theta}_2\underline{\omega}_1}\big\},\\
S_{12} &= \frac{1}{chk}\big\{-2(f_{12})_{\underline{\theta}_1\underline{\theta}_0} - 2(f_{12})_{\underline{\theta}_1\underline{\theta}_1}(f_{12})_{\underline{\theta}_2} - 2(f_{12})_{\underline{\theta}_1\underline{\theta}_2}(f_{12})_{\underline{\theta}_1} + (f_{22})_{\underline{\theta}_2\underline{\theta}_0}\\
&\qquad + (f_{22})_{\underline{\theta}_1\underline{\theta}_2}(f_{12})_{\underline{\theta}_2} + (f_{22})_{\underline{\theta}_2\underline{\theta}_2}(f_{12})_{\underline{\theta}_1}\big\},\\
S_{13} &= \frac{2(f_{12})_{\underline{\theta}_1\underline{\theta}_1} - (f_{22})_{\underline{\theta}_1\underline{\theta}_2}}{ck}, \qquad S_{14} = \frac{2(f_{12})_{\underline{\theta}_1\underline{\theta}_2} - (f_{22})_{\underline{\theta}_2\underline{\theta}_2}}{gk},
\end{aligned}
$$

and we used the dual frame of the coframe $(\underline{\theta}_0, \underline{\theta}_1, \underline{\theta}_2, \underline{\omega}_1, \underline{\omega}_2)$:

$$\partial_{\underline{\theta}_0} = \frac{\partial}{\partial y}, \qquad \partial_{\underline{\theta}_1} = \frac{\partial}{\partial z_1}, \qquad \partial_{\underline{\theta}_2} = \frac{\partial}{\partial z_2},$$
$$\partial_{\underline{\omega}_1} = \frac{\partial}{\partial x_1} + z_1 \frac{\partial}{\partial y} + f_{11} \frac{\partial}{\partial z_1} + f_{12} \frac{\partial}{\partial z_2},$$
$$\partial_{\underline{\omega}_2} = \frac{\partial}{\partial x_2} + z_2 \frac{\partial}{\partial y} + f_{21} \frac{\partial}{\partial z_1} + f_{22} \frac{\partial}{\partial z_2}.$$

In the above torsions, there are the following relations.

Proposition 3.9 *Torsions* $M_4, M_9, S_3, S_4, S_7, S_{10}, S_{13}$ *are given by*:

$$M_4 = -\frac{1}{h^2}\left\{ \frac{-gh}{c}(M_1)_{\underline{\omega}_2} + \frac{2gh}{c} M_1 (f_{12})_{\underline{\theta}_1} - \frac{gh}{c} M_1 (f_{22})_{\underline{\theta}_2} \right\},$$

$$M_9 = -\frac{1}{k^2}\left\{ -\frac{ck}{g}(M_6)_{\underline{\omega}_1} - \frac{ck}{g} M_6 (f_{11})_{\underline{\theta}_1} + \frac{2ck}{g} M_6 (f_{12})_{\underline{\theta}_2} \right\},$$

$$S_3 = -\frac{k}{ch}(M_7)_{\underline{\theta}_2}, \qquad S_4 = -\frac{1}{c}(M_3)_{\underline{\theta}_1}, \qquad S_7 = -\frac{1}{c}(M_1)_{\underline{\theta}_1} + M_3,$$

$$S_{10} = -\frac{1}{c}(M_1)_{\underline{\theta}_1} + 2M_3, \qquad S_{13} = -2M_7 + \frac{1}{g}(M_6)_{\underline{\theta}_2}.$$

Hence, the vanishing of $M_4, M_9, S_3, S_4, S_7, S_{10}, S_{13}$ is given by the vanishing of other torsions. Consequently, we obtain fifteen invariant functions. By the theory of G-structure [St], we have the following results.

Theorem 3.10 ([St]) *If a G-structure is locally flat then its structure function vanishes identically.*

Theorem 3.11 ([St]) *Let G be a group of finite type. A necessary and sufficient condition for a G-structure to be locally flat is that the structure function of all prolongations of G be constant and equal to the corresponding structure constants of the flat G-structure.*

From these theorems, the vanishing condition of invariant functions M_i, S_j $(i = 1,3,5,6,7,8,\ j = 1,2,5,6,8,9,11,12,14)$ gives the following corollary.

Corollary 3.12 *Suppose that the second order PDE* (1) *satisfies the* *integrability condition* $A = B = 0$. *Then, the equation* (1) *is locally equivalent to the flat equation under contact prolongations of scale transformations if and only if invariant functions* M_i, S_j $(i = 1,3,5,6,7,8,$ $j = 1,2,5,6,8,9,11,12,14)$ *vanish.*

First, it is easy to check that the functions f_{ij} satisfying $A = B = M_i = S_j = 0$ are written as quadratic polynomials in z_1, z_2. Hence, if there is a polynomial z_1, z_2 of degree three among f_{ij}, then the corresponding equation (1) is not equivalent to the flat equation under contact prolongations of scale transformations.

Next, we give some examples of equations which are equivalent to the flat equation. To show the vanishing condition of invariant functions more explicitly, we consider the functions f_{ij} given by:

$$f_{11} = P(x_1, x_2, y), \qquad f_{12} = Q(x_1, x_2, y), \qquad f_{22} = R(x_1, x_2, y). \tag{25}$$

Then, Corollary 3.12 gives the following corollary.

Corollary 3.13 *Suppose that the functions* f_{ij} *in* (1) *are given by* (25). *Then the equation* (1) *is locally equivalent to the flat equation under contact prolongations of scale transformations if and only if* $P_y = Q_y = R_y = 0$, $P_{x_2} = Q_{x_1}$, $Q_{x_2} = R_{x_1}$.

The condition $P_y = Q_y = R_y = 0$, $P_{x_2} = Q_{x_1}$, $Q_{x_2} = R_{x_1}$ in this corollary are obtained by the integrability condition $A = B = 0$. Namely, the vanishing condition of invariant functions (i.e. $M_i = S_j = 0$) is absorbed into the integrability condition. Therefore, we can see that the second order PDE (1) for the functions f_{ij} given by (25) is locally equivalent to the flat equation if and only if it is integrable.

Acknowledgements. The author would like to thank Professors. Hajime Sato and Tatsuya Tate for their lectures and supports through this work. He also would like to thank the referee for his many helpful suggestions.

References

[BCG3] Bryant R., Chern S. S., Gardner R., Goldscmidt H. and Griffiths P., Exterior Differential Systems. MSRI Publ. vol. **18**, Springer Verlag, Berlin, (1991).

[Bry] Bryant R., *Élie Cartan and geometric duality*, Journees Élie Cartan 1998 et 1999, Intitut Élie Cartan. **16** (2000), 5–20.

[Car] Cartan E., *Les systèmes de Pfaff à cing variables et les équations aux dérivées partielles du second ordre*. Ann. École Normale. **27** (1910), 109–192.

[Gar] Gardner R., "The method of equivalence and its applications", CBMS-NSF Regional Conf. Ser. in Appl. math, 58, SIAM, Philadelphia (1989).

[GTW] Grissom C., Thompson G. and Wilkens G., *Linearization of second order ordinary differential equations via Cartan's equivalence method*. J. Differential Equations. **77**, (1989), 1–15.

[IL] Ivey, Landsberg, *Cartan for Beginners*. AMS 2003.

[KLS] Kamran N., Lamb G. and Shadwick F., *The local equivalence problem for* $\frac{d^2y}{dx^2} = F\big(x, y, \frac{dy}{dx}\big)$ *and the Painlevé transcendents*. J. Differential Geom. **22** (1985), 139–150.

[Mo] Morimoto T., *Two great books by Élie Cartan*. Pleasure of mathematics. **no.29**, 98–104, Nippon-Hyouronsha, (2002), (Japanese).

[O1] Olver P., "Applications of Lie Groups to Differential Equations (Second Edition)", Graduate Texts in Mathematics, Springer 2000.

[O2] Olver P., "Equivalence, Invariants, and Symmetry", Cambridge University Press, 1995.

[OS] Ozawa T. and Sato H., *Linearization of ordinary differential equations by area preserving maps*. Nagoya Math. J. **156** (1999), 109–122.

[SOS] Sato H., Ozawa T. and Suzuki H., *Differential Equations and Schwarzian Derivatives*. noncommutative geometry and physics 2005, Proceedings of the International Sendai-Beijing Joint Workshop, (2007), 129–149.

[SY] Sato H. and Yoshikawa A. Y., *Third order ordinary differential equations and Legendre manifolds*. J. Math. Soc. Japan. **50** (1998), 993–1013.

[Sa] Sato H., *Orbit decomposition of space of differential equations*. mathematics of 21 century, $\sim$ untrodden peak of geometry $\sim$, Nippon-Hyouronsha, (2004), 267–280, (Japanese).

[St] Sternberg S., "Lectures on differential geometry", Chelsea, (1983).

[Tan1] Tanaka N., *On generalized graded Lie algebras and geometric structures I*. J. Math. Soc. Japan. **19** (1967), 215–254.

[Tan2] Tanaka N., *On the equivalence problems associated with simple graded Lie algebras*. Hokkaido Math. J. **8** (1979), no. 1, 23–84.

[Wan] Wang S. H., *Legendrian Submanifold Path Geometry*. Math. Ann. **325** (2003), no. 2, 249–277.

[Ya1] Yamaguchi K., *Contact geometry of higher order*. Japan. J. Math. **8** (1982), 109–176.

[Ya2] Yamaguchi K., *Geometrization of jet bundles*. Hokkaido Math. J. **12** (1983), 27–40.

[Ya3] Yamaguchi K., *Differential systems associated with simple graded Lie algebras*. Advanced Studies in Pure Math. **22** (1993), 413–449.

[YY] Yamaguchi K. and Yatsui T., *Geometry of higher order differential equations of finite type associated with symmetric spaces*. Advanced Studies in Pure Math. **37** (2002), 397–458.

[Yo] Yoshikawa A. Y., *Equivalence problem of third-order ordinary differential equations*. Internat. J. Math. **17** (2006), no. 9, 1103–1125.

Graduate School of Mathematics
Nagoya University
Chikusa-ku, Nagoya 464-8602, Japan

Osaka City University Advanced Mathematical Insititute
Osaka City University
Sugimoto, Sumiyoshi-Ku, Osaka, 558-8585, Japan
E-mail: m04031x@math.nagoya-u.ac.jp