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On the boundedness of a class of rough maximal operators on product spaces

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Abstract. In this paper, we study the L^p boundedness of a class of maximal operators $T^{(\gamma)}_{\{\Omega_j\}}$ and a related class of rough singular integrals on product spaces. We obtain appropriate L^p estimates for such maximal operators and singular integrals. These estimates are used in an extrapolation argument and allow us to obtain some new and improved results for certain maximal integral operators and singular integrals on product spaces under certain sharp conditions on the kernel functions. Also, one of our main results in this paper is a corrigendum of a result obtained by Ding-Lin.

Key words: maximal operator, rough kernel, L log L spaces, block spaces, singular integral, L^p boundedness, product spaces.

1. Introduction and statement of results

Throughout this paper, let $\mathbf{R}^n$, $n \geq 2$, be the n-dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue surface measure $d\sigma$. Also, we let ξ' denote $\xi/|\xi|$ for $\xi \in \mathbf{R}^n\backslash\{0\}$ and p' denote the exponent conjugate to p, that is, $1/p + 1/p' = 1$.

A problem that has attracted the attention of many authors in recent years is finding a class of kernels M so that the maximal operator $\sup_{K\in M} |T_K f|$ is bounded on L^p for some p, where T_K is the singular integral operator defined by

$$T_K f(x) = \text{p.v.} \int_{\mathbf{R}^n} f(x-y)K(y)dy.$$

Such maximal operators were studied initially in [11] and subsequently by many other authors. See for example, [1], [2], [4], [5], [12], [13], [26].

L. K. Chen and H. Lin [11] studied the L^p boundedness of the maximal operator $\sup_{K\in M} |T_K f|$ whenever the class of kernels M is given for a fixed function $\Omega \in L^1(\mathbf{S}^{n-1})$ and a fixed number $\gamma \geq 1$ by

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$$M =: M^{(\gamma)}(\Omega) = \Big\{ K(x) = \frac{\Omega(x')}{|x|^n} h(|x|) :$$
$$h \text{ is a radial function such that } \|h\|_{L^\gamma(\mathbf{R}_+, dr/r)} \le 1 \Big\}.$$

They proved the following:

Theorem A *Let Ω be an arbitrary but fixed function defined on $\mathbf{S}^{n-1}$ with $\Omega \in C(\mathbf{S}^{n-1})$ and satisfies $\int_{\mathbf{S}^{n-1}} \Omega(y) d\sigma(y) = 0$. Then $\mathcal{A}_\Omega^{(\gamma)}(f) = \sup_{K \in M^{(\gamma)}(\Omega)} |T_K f|$ is bounded on $L^p(\mathbf{R}^n)$ for $(n\gamma)/(n\gamma - 1) < p < \infty$ and $1 \le \gamma \le 2$. Moreover, the range of p is the best possible.*

In the case $\gamma = 2$, L. K. Chen and X. Wang in [12] investigated the L^p boundedness of the more general class of maximal operators $\sup_{K \in M} |T_K f|$, when $M =: M^{(2)}(\{\Omega_j\})$ where $\{\Omega_j\}$ is an arbitrary but fixed countable subset of $L^1(\mathbf{S}^{n-1})$ and

$$M^{(\gamma)}(\{\Omega_j\}) = \Big\{ K(x) = \sum_j \frac{\Omega_j(x')}{|x|^n} h_j(|x|) :$$
$$\Big(\int_0^\infty \sum_j |h_j(t)|^\gamma \frac{dt}{t} \Big)^{1/\gamma} \le 1 \Big\} \text{ for a given } \gamma \ge 1.$$

L. K. Chen and X. Wang [12] proved the following:

Theorem B *Assume that $2n/(2n-1) < p < \infty$. If $\{\Omega_j\}$ is a fixed countable subset of $L^2(\mathbf{S}^{n-1})$ with $\int_{\mathbf{S}^{n-1}} \Omega_j(y) d\sigma(y) = 0$ and $\sum_j \|\Omega_j\|^2_{L^2(\mathbf{S}^{n-1})} < \infty$, then $H^{(2)}_{\{\Omega_j\}}(f)$ is bounded on $L^p(\mathbf{R}^n)$, where $H^{(\gamma)}_{\{\Omega_j\}}(f) = \sup_{K \in M^{(\gamma)}(\{\Omega_j\})} |T_K f|$. That is, $\big\| H^{(2)}_{\{\Omega_j\}}(f) \big\|_{L^p(\mathbf{R}^n)} \le C_p \|f\|_{L^p(\mathbf{R}^n)}$ for all f in the Schwartz class. Moreover, the range of p is the best possible.*

We notice that if we take in the definition of $M^{(\gamma)}(\{\Omega_j\})$ our countable set $\{\Omega_j\}$ to be the singleton Ω, where Ω is a fixed function defined on $\mathbf{S}^{n-1}$ with $\Omega \in L^2(\mathbf{S}^{n-1})$ and satisfies $\int_{\mathbf{S}^{n-1}} \Omega(y) d\sigma(y) = 0$ and if we take the countable set $\{h_j\}$ to be the singleton h and letting h vary with h belongs to the class $L^2(\mathbf{R}_+, dr/r)$, the maximal function $H^{(\gamma)}_{\{\Omega_j\}}(f)$ will reduce to the maximal function $\mathcal{A}_\Omega^{(\gamma)}(f)$. Thus, the maximal operator $H^{(\gamma)}_{\{\Omega_j\}}(f)$ is a natural extension of the maximal operator $\mathcal{A}_\Omega^{(\gamma)}(f)$.

On the other hand, Ding and Lin [13] considered the analogue of Theorem B in the product space setting. Let $1 \le \gamma < \infty$ and $\{\Omega_j\}$ be an arbitrary but fixed countable subset of $L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ satisfying the following cancellation conditions for all j:

$$\int_{\mathbf{S}^{n-1}} \Omega_j(u, \cdot) d\sigma(u) = \int_{\mathbf{S}^{m-1}} \Omega_j(\cdot, v) d\sigma(v) = 0. \tag{1.1}$$

Let $E^{(\gamma)}(\{\Omega_j\})$, $1 \le \gamma < \infty$, denote the class of all kernels of the form

$$K(u, v) = \sum_j h_j(|u|, |v|) \frac{\Omega_j(u, v)}{|u|^n |v|^m},$$

where

$$\left(\int_0^\infty \int_0^\infty \sum_j |h_j(r, t)|^\gamma \frac{drdt}{rt} \right)^{1/\gamma} \le 1.$$

Now define the singular integral T_K by

$$T_K f(x, y) = \text{p.v.} \int_{\mathbf{R}^n \times \mathbf{R}^m} f(x - u, y - v) K(u, v) du dv$$

and $T_{\{\Omega_j\}}^{(\gamma)}(f) = \sup_{K \in E^{(\gamma)}(\{\Omega_j\})} |T_K f|$.

The following can be found in Ding and Lin in [13]:

Theorem C *Suppose* $\gamma = 2$ *and* $\{\Omega_j\} \subseteq L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ *for some* $1 < q \le \infty$ *and satisfies* $\sum_j \|\Omega_j\|^2_{L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} < \infty$. *Suppose that* p *and* q *satisfy one of the following conditions*:

(a) $1 < q < 2$ *and* $\max\{2nq'/(2n + nq' - 2), 2mq'/(2m + mq' - 2)\} < p < 2q'/(q' - 2)$,
(b) $2 \le q < \max\{2(n-1)/(n-2), 2(m-1)/(m-2)\}$ *and* $\max\{2nq'/(2n + nq' - 2), 2mq'/(2m + mq' - 2)\} < p < \infty$,
(c) $q > \max\{2(n-1)/(n-2), 2(m-1)/(m-2)\}$ *and* $1 < p < \infty$.

Then the maximal operator $T_{\{\Omega_j\}}^{(2)}(f)$ *can be extended to a bounded op-*

erator on $L^p(\mathbf{R}^n \times \mathbf{R}^m)$*. That is,*

$$\left\|T^{(2)}_{\{\Omega_j\}}(f)\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \le C_p\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}$$

for all $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$*.*

We should point out the range of p given in Theorem C is not true. In fact, if we take for example, $q = \infty$, then we notice from Theorem C that the range of p is $1 < p < \infty$ which is impossible because the best range of p for the maximal operator $T^{(2)}_{\{\Omega_j\}}(f)$ to be bounded on $L^p(\mathbf{R}^n \times \mathbf{R}^m)$ is $2n/(2n-1) < p < \infty$. So it is natural to ask what is the right range of p so that $T^{(2)}_{\{\Omega_j\}}$ is bounded on $L^p(\mathbf{R}^n \times \mathbf{R}^m)$. One of the main purposes of this paper is to determine the right range of p so that $T^{(2)}_{\{\Omega_j\}}(f)$ is bounded on $L^p(\mathbf{R}^n \times \mathbf{R}^m)$. In fact, we shall deal with the more general class of operators $T^{(\gamma)}_{\Phi,\Psi,\{\Omega_j\}}$ (for $\gamma \ge 1$) as described in the following theorems, where Φ and Ψ are suitable functions defined on $[0,\infty)$,

$$T^{(\gamma)}_{\Phi,\Psi,\{\Omega_j\}}(f) = \sup_{K\in E^{(\gamma)}(\{\Omega_j\})} |T_{K,\Phi,\Psi}f| \tag{1.2}$$

and

$$T_{K,\Phi,\Psi}f(x,y) = \text{p.v.}\int_{\mathbf{R}^n\times\mathbf{R}^m} f(x-\Phi(|u|)u', y-\Psi(|v|)v')K(u,v)dudv.$$

Theorem 1.1 *Let* $T^{(\gamma)}_{\Phi,\Psi,\{\Omega_j\}}$ *be given as in* (1.2) *with* $1 \le \gamma \le 2$*. Assume that* Φ *and* Ψ *are* $C^2([0,\infty))$*, convex, and increasing functions with* $\Phi(0) = \Psi(0) = 0$*. Suppose that* $\{\Omega_j\}$ *is a fixed countable subset of* $L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ *for some* $1 < q \le \infty$ *with* $\left\|\|\Omega_j\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\right\|_{l^{\gamma'}} < \infty$*. Then the inequality*

$$\begin{aligned}&\left\|T^{(\gamma)}_{\Phi,\Psi,\{\Omega_j\}}(f)\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}\\ &\qquad \le C_p\left(\frac{q}{q-1}\right)^{2/\gamma'}\left\|\|\Omega_j\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\right\|_{l^{\gamma'}}\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}\end{aligned} \tag{1.3}$$

holds for $(\alpha\beta\gamma')/(\gamma'\alpha+\alpha\beta-\gamma') < p < \infty$ *and* $1 \le \gamma \le 2$*, where* $\alpha = \min(m,n)$ *and* $\beta = \max\{2, q'\}$*.*

We notice that if we take in the definition of the maximal function $T^{(\gamma)}_{\Phi,\Psi,\{\Omega_j\}}(f)$ our countable set $\{\Omega_j\}$ to be the the singleton Ω, where Ω is a fixed function defined on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ with $\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and satisfies (1.1) with Ω_j replaced by Ω and if we take the countable set $\{h_j\}$ to be the singleton h and letting h vary with h belongs to the class $L^\gamma(\mathbf{R}_+ \times \mathbf{R}_+, \frac{drdt}{rt})$, the maximal function $T^{(\gamma)}_{\Phi,\Psi,\{\Omega_j\}}(f)$ will reduce to the maximal function $\mathcal{M}^{(\gamma)}_{\Phi,\Psi,\Omega}f(x,y)$, where $\mathcal{M}^{(\gamma)}_{\Phi,\Psi,\Omega}$ is the maximal operator defined by

$$\mathcal{M}^{(\gamma)}_{\Phi,\Psi,\Omega}f(x,y) = \sup_{h \in L^\gamma(\mathbf{R}_+ \times \mathbf{R}_+, \frac{drdt}{rt})} \left| S_{\Phi,\Psi,\Omega,h}f(x,y) \right| \tag{1.4}$$

and $S_{\Phi,\Psi,\Omega,h}$ is the singular integral operator defined on the product space $\mathbf{R}^n \times \mathbf{R}^m$ by

$$\begin{aligned} &S_{\Phi,\Psi,\Omega,h}f(x,y) \\ &\quad = \text{p.v.}\int_{\mathbf{R}^n \times \mathbf{R}^m} \frac{\Omega(u,v)}{|u|^n|v|^m} h(|u|,|v|) f\big(x-\Phi(|u|)u', y-\Psi(|v|)v'\big)dudv. \end{aligned} \tag{1.5}$$

Therefore, by Theorem 1.1 we immediately get the following:

Theorem 1.2 *Let $\mathcal{M}^{(\gamma)}_{\Phi,\Psi,\Omega}$ be given as in (1.4) with $1 \le \gamma \le 2$. Assume that Φ and Ψ are $C^2([0,\infty))$, convex, and increasing functions with $\Phi(0) = \Psi(0) = 0$. Suppose that $\Omega \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for some $1 < q \le \infty$. Then*

$$\begin{aligned} &\left\| \mathcal{M}^{(\gamma)}_{\Phi,\Psi,\Omega}(f) \right\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \\ &\qquad \le C_p \left(\frac{q}{q-1} \right)^{2/\gamma'} \|\Omega\|_{L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})} \|f\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \end{aligned} \tag{1.6}$$

holds for $(\alpha\beta\gamma')/(\gamma'\alpha + \alpha\beta - \gamma') < p < \infty$ and $1 \le \gamma \le 2$, where $\alpha = \min(m,n)$ and $\beta = \max\{2, q'\}$.

Here we point out that if $\alpha = \min(m,n)$, then we have

$$\begin{aligned} &\max\left\{(\gamma' n\beta)/(\gamma' n + n\beta - \gamma'), (\gamma' m\beta)/(\gamma' m + m\beta - \gamma')\right\} \\ &\quad = (\alpha\beta\gamma')/(\gamma'\alpha + \alpha\beta - \gamma'). \end{aligned}$$

Theorem 1.3 *Let $S_{\Phi,\Psi,\Omega,h}$ be given as in* (1.5). *Assume that Φ and Ψ are $C^2([0,\infty))$, convex, and increasing functions with $\Phi(0)=\Psi(0)=0$. Suppose that $\Omega \in L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ for some $1<q\le 2$ and $h\in L^\gamma(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt})$ for some $1<\gamma\le\infty$. Then*

$$\begin{aligned}\left\|S_{\Phi,\Psi,\Omega,h}(f)\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}\\ \le C_p(q-1)^{-2/\gamma'}\|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}\end{aligned} \tag{1.7}$$

holds for $1<p<\infty$.

By the conclusions in Theorems 1.2–1.3 and applying an extrapolation method, we get the following results:

Theorem 1.4 *Let $1\le\gamma\le 2$. Assume that Φ and Ψ are $C^2([0,\infty))$, convex, and increasing functions with $\Phi(0)=\Psi(0)=0$.*

(a) *If $\Omega\in L(\log L)^{2/\gamma'}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ and $1<\gamma\le 2$, the operator $\mathcal{M}^{(\gamma)}_{\Phi,\Psi,\Omega}$ is bounded on $L^p(\mathbf{R}^n\times\mathbf{R}^m)$ for $2\le p<\infty$;*
(b) *If $\Omega\in B_q^{(0,2/\gamma'-1)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ and $1<\gamma\le 2$, the operator $\mathcal{M}^{(\gamma)}_{\Phi,\Psi,\Omega}$ is bounded on $L^p(\mathbf{R}^n\times\mathbf{R}^m)$ for $2\le p<\infty$;*
(c) *If $\Omega\in L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ and $\gamma=1$, the operator $\mathcal{M}^{(\gamma)}_{\Phi,\Psi,\Omega}$ is bounded on $L^\infty(\mathbf{R}^n\times\mathbf{R}^m)$.*

Here, $L(\log L)^\alpha(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ (for $\alpha>0$) denotes the class of all measurable functions Ω on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$ which satisfy

$$\begin{aligned}&\|\Omega\|_{L(\log L)^\alpha(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\\ &\quad=\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}|\Omega(x,y)|\log^\alpha(2+|\Omega(x,y)|)d\sigma(x)d\sigma(y)<\infty\end{aligned}$$

and $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ denotes a special class of block spaces whose definition will be recalled in Section 2.

Theorem 1.5 *Suppose that Φ and Ψ are $C^2([0,\infty))$, convex, and increasing functions with $\Phi(0)=\Psi(0)=0$. Suppose that $h\in L^\gamma(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt})$ for some $1\le\gamma\le\infty$.*

(a) *If $\Omega\in L(\log L)^{2/\gamma'}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ and $1<\gamma\le\infty$, the operator $S_{\Phi,\Psi,\Omega,h}$ is bounded on $L^p(\mathbf{R}^n\times\mathbf{R}^m)$ for $1<p<\infty$;*

(b) *If* $B_q^{(0,2/\gamma'-1)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ *for some* $q>1$ *and* $1<\gamma\leq\infty$, *the operator* $S_{\Phi,\Psi,\Omega,h}$ *is bounded on* $L^p(\mathbf{R}^n\times\mathbf{R}^m)$ *for* $1<p<\infty$;

(c) *If* $\Omega\in L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ *and* $\gamma=1$, *the operator* $S_{\Phi,\Psi,\Omega,h}$ *is bounded on* $L^p(\mathbf{R}^n\times\mathbf{R}^m)$ *for* $1\leq p\leq\infty$.

Remarks

(1) Theorem 1.1 is a corrigendum of Theorem C in the sense that it provides us with the right range of p.

(2) The L^p boundedness of $\mathcal{M}^{(\gamma)}_{\Phi,\Psi,\Omega}$ under the same conditions on p,γ and Ω as in Theorem 1.2 was proved in [1], but the main thrust of Theorem 1.2 is that it provides us with estimates which will be useful in employing an extrapolation argument and in turn allow us to obtain the L^p boundedness of $\mathcal{M}^{(\gamma)}_{\Phi,\Psi,\Omega}$ under optimal size conditions on Ω.

(3) For any $q>1$, $0<\alpha<\beta$ and $-1<\upsilon$, the following inclusions hold and are proper:

$$L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})\subset L(\log L)^{\beta}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$$
$$\subset L(\log L)^{\alpha}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}),$$

$$\bigcup_{r>1} L^r(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})\subset B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}) \text{ for any } -1<\upsilon,$$

$$B_q^{(0,\upsilon_2)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})\subset B_q^{(0,\upsilon_1)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}) \text{ for any } -1<\upsilon_1<\upsilon_2.$$

The question with regard to the relationship between $B_q^{(0,\upsilon-1)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ and $L(\log L)^{\upsilon}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ (for $\upsilon>0$) remains open.

(4) Theorem 1.4 (a) was obtained in [5] only in the case $\gamma=2$ and in [4] in the case $1<\gamma\leq 2$, but with p in the smaller range $\gamma'\leq p<\infty$. Thus, Theorem 1.4 (a) improves the corresponding results in [26], [5] and [4]. Also, it is worth mentioning that the condition $\Omega\in L(\log L)^{2/\gamma'}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ was shown by Al-Qassem and Pan in [4] to be optimal in the case $\gamma=2$ in the sense that the exponent $2/\gamma'$ in $L(\log L)^{2/\gamma'}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ cannot be replaced by any smaller number.

(5) Theorem 1.4 (b) was obtained in [1] only in the case $\gamma=2$. Thus, Theorem 1.4 (b) improves the corresponding result in [1]. Also, we point out that the condition $\Omega\in B_q^{(0,2/\gamma'-1)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ for some $q>1$ was shown by Al-Qassem in [1] to be optimal in the case $\gamma=2$ in

the sense that the $2/\gamma'$ in $B_q^{(0,2/\gamma'-1)}$ cannot be replaced by any smaller number.

(6) If $\Phi(t) \equiv t$ and $\Psi(t) \equiv t$, we denote $S_{\Phi,\Psi,\Omega,h}$ by $S_{\Omega,h}$. In [6], Al-Salman, Al-Qassem and Pan were able to show that the $L^p(\mathbf{R}^n \times \mathbf{R}^m)$ $(1 < p < \infty)$ boundedness of $S_{\Omega,h}$ holds if $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and $h \in L^\infty(\mathbf{R}_+ \times \mathbf{R}_+)$. Also, the condition that $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ is the most desirable size condition for the L^p boundedness of $S_{\Omega,1}$ in the sense that the operator $S_{\Omega,1}$ mail fail to be bounded on L^p for any p if the condition is replaced by the condition $\Omega \in L(\log L)^{2-\varepsilon}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for any $\varepsilon > 0$. On the other hand, Theorem 1.5 (a) implies that if $h \in L^\gamma(\mathbf{R}_+ \times \mathbf{R}_+, \frac{drdt}{rt})$ for some $\gamma > 1$, the singular operator $S_{\Omega,h}$ is bounded on L^p under the much weaker condition $\Omega \in L(\log L)^{2/\gamma'}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. The reason for this new phenomena on singular integrals is that the singular operators $S_{\Omega,h}$ (with $h \in L^\gamma(\mathbf{R}_+ \times \mathbf{R}_+, \frac{drdt}{rt})$ for some $1 < \gamma < \infty$) have weaker singularities than the singular operators $S_{\Omega,1}$ due to the presence of the strong condition on h.

(7) In [3], Al-Qassem and Pan proved that $S_{\Omega,h}$ is bounded on $L^p(\mathbf{R}^n \times \mathbf{R}^m)$ for $1 < p < \infty$ if $\Omega \in B_q^{(0,1)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for some $q > 1$ and $h \in L^\infty(\mathbf{R}_+ \times \mathbf{R}_+)$. Again, as in Remark 6 above, under the condition $h \in L^\gamma(\mathbf{R}_+ \times \mathbf{R}_+, \frac{drdt}{rt})$ for some $\gamma > 1$ we obtain from Theorem 1.5 (b) that $S_{\Omega,h}$ is bounded on on $L^p(\mathbf{R}^n \times \mathbf{R}^m)$ for $1 < p < \infty$ if Ω satisfies the weaker condition $\Omega \in B_q^{(0,2/\gamma'-1)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for some $q > 1$.

(8) Theorem 1.5 (b) implies that the operator $S_{\Omega,h}$ when $h \in L^1(\mathbf{R}_+ \times \mathbf{R}_+, \frac{drdt}{rt})$ is bounded on $L^1(\mathbf{R}^n \times \mathbf{R}^m)$ and $L^\infty(\mathbf{R}^n \times \mathbf{R}^m)$, while $S_{\Omega,1}$ is not. Also, when $h \in L^1(\mathbf{R}_+ \times \mathbf{R}_+, \frac{drdt}{rt})$ the operator $S_{\Omega,h}$ is bounded on L^p if $\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$, while $S_{\Omega,1}$ is not bounded on L^p for any p if $\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ unless $\Omega(u,v)$ is an odd function in each one of the variables u and v, i.e., $\Omega(u,v) = -\Omega(-u,v) = -\Omega(u,-v) = \Omega(-u,-v)$.

Throughout the rest of the paper the letter C will stand for a constant but not necessarily the same one in each occurrence.

2. Some definitions and lemmas

The block spaces originated in the work of M. H. Taibleson and G. Weiss on the convergence of the Fourier series in connection with the developments of the real Hardy spaces. Below we shall recall the definition of block spaces on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$. For further background information about the theory of spaces generated by blocks and its applications to harmonic analysis one can consult the book [20]. The special class of block spaces $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ (for $\upsilon > -1$ and $q > 1$) was introduced by Jiang and Lu with respect to the study of singular integral operators on product domains [18].

Definition 2.1 A q-block on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ is an L^q $(1 < q \leq \infty)$ function $b(x, y)$ that satisfies (i) $\operatorname{supp}(b) \subset I$; (ii) $\|b\|_{L^q} \leq |I|^{-1/q'}$, where $|\cdot|$ denotes the product measure on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$, and I is an interval on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$, i.e., $I = \{x \in \mathbf{S}^{n-1} : |x - x_0| < \alpha\} \times \{y \in \mathbf{S}^{m-1} : |y - y_0| < \beta\}$ for some α, $\beta > 0$ and $(x_0, y_0) \in \mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$. The block space $B_q^{(0,\upsilon)} = B_q^{(0,\upsilon)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ is defined by $B_q^{(0,\upsilon)} = \{\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) : \Omega = \sum_{\mu=1}^{\infty} \lambda_\mu b_\mu,\ M_q^{(0,\upsilon)}(\{\lambda_\mu\}) < \infty\}$, where each λ_μ is a complex number; each b_μ is a q-block supported on an interval I_μ on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$, $\upsilon > -1$ and $M_q^{(0,\upsilon)}(\{\lambda_\mu\}) = \sum_{\mu=1}^{\infty} |\lambda_\mu| \{1 + \log^{(\upsilon+1)}(|I_\mu|^{-1})\}$. Let $\|\Omega\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} = N_q^{(0,\upsilon)}(\Omega) = \inf\{M_q^{(0,\upsilon)}(\{\lambda_\mu\})\}$, where the infimum is taken over all q-block decompositions of Ω.

Definition 2.2 For arbitrary functions $\Phi(\cdot)$ and $\Psi(\cdot)$ on $\mathbf{R}_+$, $\theta \geq 2$ and $\Omega : \mathbf{S}^{n-1} \times \mathbf{S}^{m-1} \to \mathbf{R}$, we define the sequence of measures $\{\sigma_{\Omega,h,\theta,k,d} : k, d \in \mathbf{Z}\}$ and the corresponding maximal operator $\sigma^*_{\Omega,h,\theta}$ on $\mathbf{R}^n \times \mathbf{R}^m$ by

$$\int_{\mathbf{R}^n \times \mathbf{R}^m} f \, d\sigma_{\Omega,h,\theta,k,d} = \int_{\theta^{d+1} \leq |v| < \theta^{d+1}} \int_{\theta^k \leq |u| < \theta^{k+1}} h(|u|, |v|) \times \frac{\Omega(u', v')}{|u|^n |v|^m} f\big(\Phi(|u|), \Psi(|v|)\big) du dv;$$

$$\sigma^*_{\Omega,h,\theta}(f) = \sup_{k,j \in \mathbf{Z}} ||\sigma_{\Omega,h,\theta,k,d}| * f|.$$

Lemma 2.3 *Let $\theta \geq 2$ and $\Omega \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for some $1 < q \leq 2$ and satisfies the cancellation conditions in* (1.1) *with Ω_j replaced by Ω. Assume that Φ, Ψ are $C^2([0, \infty))$, convex, and increasing functions with $\Phi(0) = \Psi(0) = 0$. Let*

$$J_{k,d}(\xi,\eta)=\Big(\int_{\theta^d}^{\theta^{d+1}}\int_{\theta^k}^{\theta^{k+1}}\Big|\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\Omega(x,y)$$
$$\times\, e^{-i(\Phi(t)\xi\cdot x+\Psi(s)\eta\cdot y)}d\sigma(x)d\sigma(y)\Big|^2\frac{dtds}{ts}\Big)^{1/2}.$$

Then there exist positive constants C and λ such that

$$|J_{k,d}(\xi,\eta)|\le C\log(\theta)\|\Omega\|_q\begin{cases}(\Phi(\theta^k)|\xi|)^{-\frac{\lambda}{q'}}(\Psi(\theta^d)|\eta|)^{-\frac{\lambda}{q'}};\\(\Phi(\theta^{k+1})|\xi|)^{\frac{\lambda}{q'}}(\Psi(\theta^{d+1})|\eta|)^{\frac{\lambda}{q'}};\\(\Phi(\theta^{k+1})|\xi|)^{\frac{\lambda}{q'}}(\Psi(\theta^{d})|\eta|)^{-\frac{\lambda}{q'}};\\(\Phi(\theta^{k})|\xi|)^{-\frac{\lambda}{q'}}(\Psi(\theta^{d+1})|\eta|)^{\frac{\lambda}{q'}},\end{cases}\tag{2.1}$$

where C is a constant independent of k,d,ξ,η,q and θ.

Proof. By Schwarz's inequality we have

$$\Big|\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\Omega(x,y)e^{-i(\Phi(\theta^kt)\xi\cdot x+\Psi(\theta^ds)\eta\cdot y)}d\sigma(x)d\sigma(y)\Big|^2$$
$$\le\int_{\mathbf{S}^{m-1}}\Big|\int_{\mathbf{S}^{n-1}}\Omega(x,y)e^{-i\Phi(\theta^kt)\xi\cdot x}d\sigma(x)\Big|^2d\sigma(y)$$
$$=\int_{\mathbf{S}^{m-1}}\Big(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}}\Omega(x,y)\overline{\Omega(u,y)}e^{-i\Phi(\theta^kt)\xi\cdot(x-u)}d\sigma(x)d\sigma(u)\Big)d\sigma(y)$$

and hence we have

$$|J_{k,d}(\xi,\eta)|^2\le(\log\theta)\int_{\mathbf{S}^{m-1}}\Big(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}}\Omega(x,y)\overline{\Omega(u,y)}$$
$$\times\Big(\int_1^\theta e^{-i\Phi(\theta^kt)\xi\cdot(x-u)}\frac{dt}{t}\Big)d\sigma(x)d\sigma(u)\Big)d\sigma(y).\tag{2.2}$$

Write

$$\int_1^\theta e^{-i\Phi(\theta^kt)\xi\cdot(x-u)}\frac{dt}{t}=\int_1^\theta H'(t)\frac{dt}{t},$$

$$\text{where } H(t) = \int_1^t e^{-i\Phi(\theta^k w)\xi\cdot(x-u)}dw,\ 1 \le t \le \theta.$$

By the assumptions on Φ and the mean value theorem we have

$$\frac{d}{dw}\big(\Phi(\theta^k w)\big) = \theta^k\Phi'(\theta^k w) \ge \frac{\Phi(\theta^k w)}{w} \ge \frac{\Phi(\theta^k)}{t} \text{ for } 1 \le w \le t \le \theta.$$

By van der Corput's lemma we get $|H(t)| \le |\Phi(\theta^k)\xi|^{-1}|\xi'\cdot(x-u)|^{-1}t$, for $1 \le t \le \theta$. Hence by integration by parts,

$$\Big|\int_1^\theta e^{-i\Phi(\theta^k t)\xi\cdot(x-u)}\frac{dt}{t}\Big| \le C(\log\theta)|\Phi(\theta^k)\xi|^{-1}|\xi'\cdot(x-u)|^{-1}.$$

By combining this estimate with the trivial estimate $\big|\int_1^\theta e^{-i\Phi(\theta^k t)\xi\cdot(x-u)}\frac{dt}{t}\big| \le (\log\theta)$ we get

$$\Big|\int_1^\theta e^{-i\Phi(\theta^k t)\xi\cdot(x-u)}\frac{dt}{t}\Big| \le C(\log\theta)|\Phi(\theta^k)\xi|^{-\alpha}|\xi'\cdot(x-u)|^{-\alpha} \quad \text{for any } 0 < \alpha \le 1. \tag{2.3}$$

By Hölder's inequality and (2.2)–(2.3) we get

$$|J_{k,d}(\xi,\eta)| \le C(\log\theta)\|\Omega\|_q|\Phi(\theta^k)\xi|^{-\frac{\alpha}{2q'}} \times\Big(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}}|\xi'\cdot(x-u)|^{-\alpha q'}d\sigma(x)d\sigma(u)\Big)^{\frac{1}{2q'}}.$$

By choosing α so that $\alpha q' < 1$ we obtain that the last integral is finite and hence

$$|J_{k,d}(\xi,\eta)| \le C(\log\theta)\|\Omega\|_q|\Phi(\theta^k)\xi|^{-\frac{\alpha}{2q'}}. \tag{2.4}$$

Similarly,

$$|J_{k,d}(\xi,\eta)| \le C(\log\theta)\|\Omega\|_q|\Psi(\theta^d)\eta|^{-\frac{\alpha}{2q'}}. \tag{2.5}$$

Also, by the cancellation conditions on Ω and by a change of variables we obtain

$$|J_{k,d}(\xi,\eta)|^2 \le \int_1^\theta \int_1^\theta \Big(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega(x,y)| \times \left|e^{-i\Phi(\theta^k t)\xi\cdot x} - 1\right| d\sigma(x)d\sigma(y)\Big)^2 \frac{dtds}{ts}.$$

Since Φ is an increasing function we get

$$|J_{k,d}(\xi,\eta)| \le C\log(\theta)\|\Omega\|_1 |\Phi(\theta^{k+1})\xi|.$$

By combining the last estimate with the trivial estimate $|J_{k,d}(\xi,\eta)| \le C\log(\theta)\|\Omega\|_1$ we get

$$|J_{k,d}(\xi,\eta)| \le C\log(\theta)\|\Omega\|_1 |\Phi(\theta^{k+1})\xi|^{\frac{\alpha}{2q'}}. \tag{2.6}$$

Similarly,

$$|J_{k,d}(\xi,\eta)| \le C\log(\theta)\|\Omega\|_1 |\Psi(\theta^{d+1})\eta|^{\frac{\alpha}{2q'}}. \tag{2.7}$$

By combining the estimates (2.4)–(2.7) we obtain the estimates in (2.1). Lemma 2.3 is proved.

Lemma 2.4 *Let θ, Φ, Ψ and Ω be as in Lemma 2.3 and let $h \in L^\gamma(\mathbf{R}_+ \times \mathbf{R}_+, \frac{drdt}{rt})$ for some $1 < \gamma < \infty$. Then there exist positive constants C and λ such that*

$$\begin{aligned} &\left|\hat{\sigma}_{\Omega,h,\theta,k,d}(\xi,\eta)\right| \\ &\quad \le C(\log\theta)^{2/\gamma'}\|\Omega\|_q \begin{cases} (\Phi(\theta^k)|\xi|)^{-\frac{\lambda}{\gamma' q'}} (\Psi(\theta^d)|\eta|)^{-\frac{\lambda}{\gamma' q'}}; \\ (\Phi(\theta^{k+1})|\xi|)^{\frac{\lambda}{\gamma' q'}} (\Psi(\theta^{d+1})|\eta|)^{\frac{\lambda}{\gamma' q'}}; \\ (\Phi(\theta^{k+1})|\xi|)^{\frac{\lambda}{\gamma' q'}} (\Psi(\theta^d)|\eta|)^{-\frac{\lambda}{\gamma' q'}}; \\ (\Phi(\theta^k)|\xi|)^{-\frac{\lambda}{\gamma' q'}} (\Psi(\theta^{d+1})|\eta|)^{\frac{\lambda}{\gamma' q'}}, \end{cases} \end{aligned} \tag{2.8}$$

where C is a constant independent of k, d, ξ, η, q and θ.

Proof. By Hölder's inequality we get

$$\begin{aligned}\left|\hat{\sigma}_{\Omega,h,\theta,k,d}(\xi,\eta)\right| \leq \|h\|_{L^{\gamma}(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt})} \\ \times\left(\int_{\theta^d}^{\theta^{d+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\Omega(x,y)e^{-i(\Phi(t)\xi\cdot x+\Psi(s)\eta\cdot y)}\right.\right. \\ \left.\left.\times\, d\sigma(x)d\sigma(y)\right|^{\gamma'}\frac{dtds}{ts}\right)^{1/\gamma'}.\end{aligned}$$

Now, if $2 \leq \gamma' < \infty$, by noticing that $\left|\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\Omega(x,y)e^{-i(\Phi(t)\xi\cdot x+\Psi(s)\eta\cdot y)}\right.$ $\left.d\sigma(x)d\sigma(y)\right| \leq \|\Omega\|_1$ we get

$$\begin{aligned}\left|\hat{\sigma}_{\Omega,h,\theta,k,d}(\xi,\eta)\right| \leq \|\Omega\|_1^{(1-2/\gamma')}\|h\|_{L^{\gamma}(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt})} \\ \times\left(\int_{\theta^d}^{\theta^{d+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\Omega(x,y)e^{-i(\Phi(t)\xi\cdot x+\Psi(s)\eta\cdot y)}\right.\right. \\ \left.\left.\times\, d\sigma(x)d\sigma(y)\right|^{2}\frac{dtds}{ts}\right)^{1/\gamma'}\end{aligned}$$

By the last estimate and Lemma 2.3 we easily get the estimates in (2.8). On the other hand, if $1 < \gamma' < 2$, the estimates in (2.8) follow by Lemma 2.3 and Hölder's inequality. This completes the proof of Lemma 2.4.

We shall need the following lemma which has its roots in [14] and [3]. A proof of this lemma can be obtained by the same proof (with only minor modifications) as that of Theorem 15 in [3]. We omit the details.

Lemma 2.5 *Let $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_j\}_{j\in\mathbb{Z}}$ be any two arbitrary lacunary sequences of positive numbers with $\inf_{k\in\mathbf{Z}}(a_{k+1}/a_k) \geq a > 1$ and $\inf_{j\in\mathbf{Z}}(b_{j+1}/b_j) \geq b > 1$. Let $\{\sigma_{k,j} : k,j \in \mathbf{Z}\}$ be a sequence of Borel measures on $\mathbf{R}^n \times \mathbf{R}^m$. Suppose that for some α,β, $C > 0$, $B > 1$ and $p_o \in (2,\infty)$ the following hold for k, $j \in \mathbf{Z}$, $(\xi,\eta) \in \mathbf{R}^n \times \mathbf{R}^m$ and for arbitrary functions $\{g_{k,j}\}$ on $\mathbf{R}^n \times \mathbf{R}^m$:*

$$\text{(i)}\quad \left|\hat{\sigma}_{k,j}(\xi,\eta)\right| \leq CB\begin{cases}\left|a_{k+1}\xi\right|^{\frac{\alpha}{\log(a)}}\left|b_{j+1}\eta\right|^{\frac{\beta}{\log(b)}}; \\ \left|a_{k+1}\xi\right|^{\frac{\alpha}{\log(a)}}\left|b_{j}\eta\right|^{-\frac{\beta}{\log(b)}}; \\ \left|a_{k}\xi\right|^{-\frac{\alpha}{\log(a)}}\left|b_{j+1}\eta\right|^{\frac{\beta}{\log(b)}}; \\ \left|a_{k}\xi\right|^{-\frac{\alpha}{\log(a)}}\left|b_{j}\eta\right|^{-\frac{\beta}{\log(b)}},\end{cases}$$

(ii) $\left\|\left(\sum_{k,j\in\mathbf{Z}}|\sigma_{k,j}*g_{k,j}|^2\right)^{1/2}\right\|_{p_0}\le CB\left\|\left(\sum_{k,j\in\mathbf{Z}}|g_{k,j}|^2\right)^{1/2}\right\|_{p_0}.$

Then for $p_0'<p<p_0$, there exists a positive constant C_p independent of B such that

$$\left\|\sum_{k,j\in\mathbf{Z}}\sigma_{k,j}*f\right\|_p\le C_pB\|f\|_p$$

holds for all f in $L^p(\mathbf{R}^n\times\mathbf{R}^m)$.

By the same argument as in ([24, p. 57]) we get

Lemma 2.6 *Let φ be a nonnegative, decreasing function on $[0,\infty)$ with $\int_0^\infty\varphi(t)dt=1$. Then*

$$\left|\int_0^\infty f(x-ty)\varphi(t)dt\right|\le M_yf(x),$$

where $M_yf(x)=\sup_{R\in\mathbf{R}}\frac{1}{R}\int_0^R|f(x-sy)|ds$ is the Hardy-Littlewood maximal function of f in the direction of y.

For $\theta\ge 2$ and $y\in\mathbf{S}^{n-1}$, let $\mathcal{M}_{\Phi,\theta,y}(f)$ denote the maximal function defined by

$$\mathcal{M}_{\Phi,\theta,y}f(x)=\sup_{k\in\mathbf{Z}}\left|\int_{\theta^k}^{\theta^{k+1}}f(x-\Phi(t)y)\frac{dt}{t}\right|.$$

Lemma 2.7 *Assume that Φ is as in Lemma 2.3. Then*

$$\left\|\mathcal{M}_{\Phi,\theta,y}(f)\right\|_p\le C_p(\log\theta)\|f\|_p \tag{2.9}$$

for some constant $C_p>0$ independent of y, all $1<p\le\infty$ and $f\in L^p$.

Proof. By a change of variable we have

$$\mathcal{M}_{\Phi,\theta,y}f(x)\le\sup_{k\in\mathbf{Z}}\left(\int_{\Phi(\theta^k)}^{\Phi(\theta^{k+1})}|f(x-ty)|\frac{dt}{\Phi^{-1}(t)\Phi'(\Phi^{-1}(t))}\right).$$

Without loss of generality, we may assume that $\Phi(t) > 0$ for all $t > 0$. By Lemma 2.6 and since the function $\frac{1}{\Phi^{-1}(t)\Phi'(\Phi^{-1}(t))}$ is non-negative, decreasing and its integral over $[\Phi(\theta^k),\, \Phi(\theta^{k+1})]$ is equal to $\log\theta$, we obtain

$$\mathcal{M}_{\Phi,\theta,y}f(x) \le C(\log\theta)M_y f(x). \tag{2.10}$$

Since $M_y(f)$ is bounded $L^p(\mathbf{R}^n)$ with bound independent of y, we immediately get (2.9). This completes the proof of Lemma 2.7.

Lemma 2.8 *Let θ, Φ and Ψ be as in Lemma 2.3 and $\sigma^*_{\Omega,h,\theta}$ be given as in Definition 2.2. Assume that $h \in L^\gamma(\mathbf{R}_+\times\mathbf{R}_+, \frac{drdt}{rt})$ for some $1<\gamma<\infty$ and $\Omega \in L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$. Then*

$$\left\|\sigma^*_{\Omega,h,\theta}(f)\right\|_p \le C_p(\log\theta)^{2/\gamma'}\|h\|_{L^\gamma(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt})}\|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\|f\|_p \tag{2.11}$$

for $1 < p \le \infty$ and $f \in L^p(\mathbf{R}^n\times\mathbf{R}^m)$, where C_p is independent of Ω, θ and f.

Proof. By Hölder's inequality we have

$$\begin{aligned}\left|\sigma_{\Omega,h,\theta,k,d} * f(x,y)\right| &\le \|h\|_{L^\gamma(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt})}\|\Omega\|^{1/\gamma}_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\\ &\quad\times\Bigg(\int_{\theta^d}^{\theta^{d+1}}\int_{\theta^k}^{\theta^{k+1}}\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\\ &\qquad\times|\Omega(u,v)|\left|f(x-\Phi(t)u, y-\Psi(s)v)\right|^{\gamma'}d\sigma(u)d\sigma(v)\frac{dtds}{ts}\Bigg)^{1/\gamma'}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sigma^*_{\Omega,h,\theta}f(x,y) &\le \|h\|_{L^\gamma(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt})}\|\Omega\|^{1/\gamma}_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\\ &\quad\times\Bigg(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}|\Omega(u,v)|\\ &\qquad\times(\mathcal{M}_{\Psi,\theta,v}\circ\mathcal{M}_{\Phi,\theta,u})(|f|^{\gamma'})(x,y)d\sigma(u)d\sigma(v)\Bigg)^{1/\gamma'}.\end{aligned}$$

and hence by Minkowski's inequality for integrals we get

$$\begin{aligned}\left\|\sigma^*_{\Omega,h,\theta}(f)\right\|_p &\le \|h\|_{L^\gamma(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt})}\|\Omega\|^{1/\gamma}_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\\ &\quad\times\bigg(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}|\Omega(u,v)|\\ &\qquad\times\left\|(\mathcal{M}_{\Psi,\theta,v}\circ\mathcal{M}_{\Phi,\theta,u})(|f|^{\gamma'})\right\|_{p/\gamma'}d\sigma(u)d\sigma(v)\bigg)^{1/\gamma'}.\end{aligned}$$

Thus, the last inequality and Lemma 2.7 imply (2.11) which completes the proof of the Lemma 2.8.

By following a similar argument as in [16] and [2] we obtain the following:

Lemma 2.9 *Let θ, Φ, Ψ, h and Ω be as in Lemma 2.3. Then for $\gamma' < p < \infty$, there exists a positive constant C_p which is independent of θ such that*

$$\begin{aligned}&\left\|\bigg(\sum_{k,d\in\mathbf{Z}}|\sigma_{\Omega,h,\theta,k,d}*g_{k,d}|^2\bigg)^{1/2}\right\|_p\\ &\le C_p(\log\theta)^{2/\gamma'}\|h\|_{L^\gamma(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt})}\\ &\quad\times\|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\left\|\bigg(\sum_{k,d\in\mathbf{Z}}|g_{k,d}|^2\bigg)^{1/2}\right\|_p \qquad (2.12)\end{aligned}$$

holds for arbitrary measurable functions $\{g_{k,d}\}$ on $\mathbf{R}^n\times\mathbf{R}^m$.

Let $\mathcal{M}_S$ be the spherical maximal operator defined by

$$\mathcal{M}_S f(x) = \sup_{r>0}\int_{\mathbf{S}^{n-1}}|f(x-r\theta)|d\sigma(\theta).$$

By the results of E. M. Stein [22] and J. Bourgain [10] we have

Lemma 2.10 *Suppose that $n \ge 2$ and $p > n/(n-1)$. Then $\mathcal{M}_S(f)$ is bounded on $L^p(\mathbf{R}^n)$.*

We shall need the spherical maximal operator $\mathcal{M}_{SP}$ defined on functions $f(x,y)$ on $\mathbf{R}^n\times\mathbf{R}^m$ by

$$\mathcal{M}_{SP}f(x,y) = \sup_{r,s>0}\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}|f(x-r\theta,y-sv)|d\sigma(\theta)d\sigma(v). \qquad (2.13)$$

Let $\mathcal{M}_S^{(1)}$ and $\mathcal{M}_S^{(2)}$ denote the operators defined on functions f on $\mathbf{R}^n \times \mathbf{R}^m$ by $(\mathcal{M}_S^{(1)} f)(x, y) = (\mathcal{M}_S^{(1)}) f(\cdot, y))(x)$ and $(\mathcal{M}_S^{(2)} f)(x, y) = (\mathcal{M}_S^{(2)}) f(x, \cdot))(y)$. By using Lemma 2.10 and the inequality $\mathcal{M}_{SP} f(x, y) \leq (\mathcal{M}_S^{(2)} \circ \mathcal{M}_S^{(1)}) f(x, y)$ we get the following:

Lemma 2.11 *Suppose that $n, m \geq 2$ and $p > \max\{n/(n-1), m/(m-1)\}$. Then $\mathcal{M}_{SP}(f)$ is bounded on $L^p(\mathbf{R}^n \times \mathbf{R}^m)$.*

Now, we need the following simple lemma.

Lemma 2.12 *Let $q > 1$ and $\beta = \max\{2, q'\}$. Suppose that $\Omega \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. Then for some positive constant C, we have*

$$\begin{aligned} &\left| \int_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} \Omega(\xi, \eta) f(\xi, \eta) d\sigma(\xi) d\sigma(\eta) \right|^2 \\ &\quad \leq C \|\Omega\|_q^{\min\{2,q\}} \int_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} |\Omega(\xi, \eta)|^{\max\{0,2-q\}} |f(\xi, \eta)|^2 d\sigma(\xi) d\sigma(\eta) \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} &\int_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} |\Omega(\xi, \eta)|^{\max\{0,2-q\}} |f(x - t\xi, y - r\eta)| d\sigma(\xi) d\sigma(\eta) \\ &\quad \leq C \|\Omega\|_q^{\max\{0,2-q\}} \big(\mathcal{M}_{SP}(|f|^{\beta/2})(x, y)\big)^{2/\beta} \end{aligned} \tag{2.15}$$

for all positive real numbers t and r, $(x, y) \in \mathbf{R}^n \times \mathbf{R}^m$ and arbitrary functions f.

Proof. First, we prove (2.14). If $q \geq 2$, by Hölder's inequality we have

$$\begin{aligned} &\left| \int_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} \Omega(\xi, \eta) f(\xi, \eta) d\sigma(\xi) d\sigma(\eta) \right|^2 \\ &\quad \leq \|\Omega\|_q^2 \left(\int_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} |f(\xi, \eta)|^{q'} d\sigma(\xi) d\sigma(\eta) \right)^{2/q'} \\ &\quad \leq \|\Omega\|_q^2 \int_{\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}} |f(\xi, \eta)|^2 d\sigma(\xi) d\sigma(\eta), \end{aligned}$$

which is the inequality (2.14) in the case $q \geq 2$. Next, if $1 < q < 2$, (2.14) follows from Schwarz's inequality.

Now, we prove (2.15). If $q \geq 2$, the inequality (2.15) is obvious. However, if $1 < q < 2$, (2.15) follows easily from Hölder's inequality and noticing that $(\frac{q}{2-q})' = q'/2$. The lemma is proved.

3. Proof of main results

Proof of Theorem 1.1. Since $L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \subseteq L^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for $q \geq 2$, Theorem 1.1 is proved once we establish that

$$\begin{aligned}\|T^{(\gamma)}_{\Phi,\Psi,\{\Omega_j\}}(f)\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}\\ \leq C_p(q-1)^{-2/\gamma'}\left\|\sum_j \|\Omega_j\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\right\|_{l^{\gamma'}}\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \qquad (3.1)\end{aligned}$$

holds for $1 < q \leq 2$, $(\alpha q'\gamma')/(\gamma'\alpha + \alpha q' - \gamma') < p < \infty$ and $1 \leq \gamma \leq 2$, where $\alpha = \min(m, n)$.

To prove (3.1), we consider three cases.

<u>**Case 1.**</u> $\gamma = 2$. By Hölder's inequality we obtain

$$\begin{aligned}&T^{(2)}_{\Phi,\Psi,\{\Omega_j\}}f(x,y)\\ &\leq \Big(\int_0^\infty\int_0^\infty \sum_j \Big|\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \Omega_j(u,v)\\ &\qquad \times f(x-\Phi(r)u, y-\Psi(t)v)d\sigma(u)d\sigma(v)\Big|^2 \frac{drdt}{rt}\Big)^{1/2}. \qquad (3.2)\end{aligned}$$

Let $\theta = 2^{q'}$. Since Φ is convex and increasing in $(0,\infty)$, we have $\Phi(t)/t$ is also increasing for $t > 0$. Therefore, the sequence $\{\Phi(\theta^k) : k \in \mathbf{Z}\}$ is a lacunary sequence with $\Phi(\theta^{k+1})/\Phi(\theta^k) \geq \theta > 1$. Let $\{\Gamma_{k,\Phi}\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0,\infty)$ adapted to the interval $I_{k,\Phi} = [(\Phi(\theta^{k+1}))^{-1}, (\Phi(\theta^{k-1}))^{-1}]$. To be precise, we require the following:

$$\Gamma_{k,\Phi} \in C^\infty, \quad 0 \leq \Gamma_{k,\Phi} \leq 1, \quad \sum_k \Gamma_{k,\Phi}(t) = 1,$$

$$\operatorname{supp}\Gamma_{k,\Phi} \subseteq I_{k,\Phi}, \quad \left|\frac{d^s\Gamma_{k,\Phi}(t)}{dt^s}\right| \le \frac{C_s}{t^s},$$

where C_s is independent of the lacunary sequence $\{\Phi(\theta^k) : k \in \mathbf{Z}\}$. Define the multiplier operators $M_{k,d}$ in $\mathbf{R}^n \times \mathbf{R}^m$ by $(\widehat{M_{k,d}f})(\xi,\eta) = \Gamma_{k,\Phi}(|\xi|)\Gamma_{d,\Psi}(|\eta|)\hat{f}(\xi,\eta)$. Then for any $f \in \mathcal{S}(\mathbf{R}^n \times \mathbf{R}^m)$ and $k,d \in \mathbf{Z}$ we have $f(x,y) = \sum_{l,s\in\mathbf{Z}}(M_{k+l,d+s}f)(x,y)$. Therefore, by Minkowski's inequality we have

$$\begin{aligned}
T^{(2)}_{\Phi,\Psi,\{\Omega_j\}}f(x,y) &\le \left(\sum_{k,d,j}\int_{\theta^d}^{\theta^{d+1}}\int_{\theta^k}^{\theta^{k+1}}\left|\sum_{l,s}A_{j,k,d,l,s,r,t}\right|^2\frac{drdt}{rt}\right)^{1/2}\\
&\le \sum_{l,s}H_{l,s}f(x,y),
\end{aligned}$$

where

$$H_{l,s}f(x,y) = \left(\sum_{k,d,j}\int_{\theta^d}^{\theta^{d+1}}\int_{\theta^k}^{\theta^{k+1}}\left|A_{j,k,d,l,s,r,t}f(x,y)\right|^2\frac{drdt}{rt}\right)^{1/2},$$

$$\begin{aligned}
A_{j,k,d,l,s,r,t}f(x,y) = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} &\Omega_j(u,v)(M_{k+l,d+s}f)\\
&\times(x-\Phi(r)u, y-\Psi(t)v)d\sigma(u)d\sigma(v).
\end{aligned}$$

Therefore, to prove (3.1) for the case $\gamma = 2$, it suffices to prove the inequality

$$\|H_{l,s}(f)\|_p \le C_p(q-1)^{-1}\left(\sum_j\|\Omega_j\|_q^2\right)^{1/2}2^{-\delta_p(|l|+|s|)}\|f\|_p \tag{3.3}$$

holds for $1 < q \le 2$, $(\alpha q'\gamma')/(\gamma'\alpha + \alpha q' - \gamma') < p < \infty$ and for some positive constants C_p and δ_p.

We start proving (3.3) for the case $p = 2$. By Plancherel's theorem and Lemma 2.3 we obtain

$$\|H_{l,s}(f)\|_2^2 = \int_{\mathbf{R}^n\times\mathbf{R}^m}\sum_{k,d,j}\int_{\theta^d}^{\theta^{d+1}}\int_{\theta^k}^{\theta^{k+1}}\left|A_{j,k,d,l,s,r,t}f(x,y)\right|^2\frac{drdt}{rt}dxdy$$

$$
\begin{aligned}
&= \int_{\Delta_{k+l,d+s}} \sum_{k,d,j} \int_{\theta^d}^{\theta^{d+1}} \int_{\theta^k}^{\theta^{k+1}} \bigg| \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \Omega_j(x,y) \\
&\qquad\qquad \times e^{-i(\Phi(r)\xi\cdot x+\Psi(t)\eta\cdot y)} d\sigma(x)d\sigma(y)\bigg|^2 \\
&\quad \times \left|\hat{f}(\xi,\eta)\right|^2 \frac{drdt}{rt} d\xi d\eta \\
&\le C2^{-2\lambda(|l|+|s|)}(q-1)^{-2} \\
&\quad \times \left(\sum_j \|\Omega_j\|_q^2\right)\left(\sum_{k,d}\int_{\Delta_{k+l,d+s}} \left|\hat{f}(\xi,\eta)\right|^2 d\xi d\eta\right) \\
&\le C2^{-2\lambda(|l|+|s|)}(q-1)^{-2}\left(\sum_j \|\Omega_j\|_q^2\right)\|f\|_2^2,
\end{aligned}
$$

where $\Delta_{k,d} = \{(\xi,\eta) \in \mathbf{R}^n \times \mathbf{R}^m : (|\xi|,|\eta|) \in I_{k,\Phi} \times I_{d,\Psi}\}$. Therefore we have

$$
\|H_{l,s}(f)\|_2 \le C2^{-\lambda(|l|+|s|)}(q-1)^{-1}\left(\sum_j \|\Omega_j\|_q^2\right)^{1/2}\|f\|_2. \tag{3.4}
$$

Next, we compute the L^p-norm of $H_{l,s}(f)$ for $p > 2$. By duality, there is a nonnegative function g in $L^{(p/2)'}(\mathbf{R}^n \times \mathbf{R}^m)$ with $\|g\|_{(p/2)'} \le 1$ such that

$$
\begin{aligned}
\|H_{l,s}(f)\|_p^2 &= \int_{\mathbf{R}^n\times\mathbf{R}^m} \sum_{k,d,j} \int_{\theta^d}^{\theta^{d+1}} \int_{\theta^k}^{\theta^{k+1}} \left|A_{j,k,d,l,s,r,t}f(x,y)\right|^2 \frac{drdt}{rt} g(x,y)dxdy \\
&\le \int_{\mathbf{R}^n\times\mathbf{R}^m} \sum_{k,d,j} \|\Omega_j\|_1 \int_{\theta^d}^{\theta^{d+1}} \int_{\theta^k}^{\theta^{k+1}} \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \\
&\quad \times |\Omega_j(u,v)|\left|M_{k+l,d+s}f(x,y)\right|^2 \\
&\quad \times g(x+\Phi(r)u, y+\Psi(t)v)d\sigma(u)d\sigma(v)\frac{drdt}{rt}dxdy \\
&\le C\left(\sum_j \|\Omega_j\|_1\right)\int_{\mathbf{R}^n\times\mathbf{R}^m}\left(\sum_{k,d}|M_{k+l,d+s}f(x,y)|^2\right) \\
&\quad \times \sigma^*_{\Omega_j,1,\theta}(\tilde{g})(-x,-y)dxdy
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\Big(\sum_j \|\Omega_j\|_1\Big)\big\|\sigma^*_{\Omega_j,1,\theta}(\tilde g)\big\|_{(p/2)'}\Big\|\sum_{k,d}\big|M_{k+l,d+s}f\big|^2\Big\|_{(p/2)}\\
&\leq C_p\Big(\sum_j \|\Omega_j\|_1^2\Big)(\log\theta)^2\|f\|_p^2,
\end{aligned}
$$

where $\tilde g(x,y) = g(-x,-y)$ and the last inequality follows from Lemma 2.8 and using Littlewood-Paley theory ([23, p. 96]). Thus we have

$$
\|H_{l,s}(f)\|_p \leq C_p(q-1)^{-1}\Big(\sum_{j\in\mathbf{Z}}\|\Omega_j\|_q^2\Big)^{1/2}\|f\|_p \text{ for } 2\leq p<\infty. \tag{3.5}
$$

By interpolation between (3.4) and (3.5) we get (3.3) for the case $p \geq 2$.

Finally, we compute the L^p-norm of $H_{l,s}(f)$ for $(2\alpha q')/(2\alpha+\alpha q'-2) < p < 2$. By changing variables and using the properties of Φ and Ψ we get

$$
H_{l,s}f(x,y) = \Big(\sum_{k,d,j}\int_{\Phi(1)}^{\Phi(\theta)}\int_{\Psi(1)}^{\Psi(\theta)}\big|X_{j,k,l,d,s}f(x,y)\big|\frac{drdt}{rt}\Big)^{1/2},
$$

where

$$
\begin{aligned}
&X_{j,k,l,d,s}f(x,y)\\
&\quad= \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\Omega_j(u,v)(M_{k+l,d+s}f)(x-\theta^k ru, y-\theta^d tv)d\sigma(u)d\sigma(v).
\end{aligned}
$$

By duality there is a function $h = h_{k,d,l,s,j}(x,y,r,t)$ satisfying $\|h\| \leq 1$ and

$$
\begin{aligned}
&h_{k,d,l,s,j}(x,y,r,t)\\
&\quad\in L^{p'}\Big(l^2\Big(l^2\Big[L^2\Big([\Phi(1),\Phi(\theta)]\times[\Psi(1),\Psi(\theta)],\frac{drdt}{rt}\Big),k,d\Big],j\Big),dxdy\Big)
\end{aligned}
$$

such that

$$
\begin{aligned}
\|H_{l,s}(f)\|_p = & \int_{\mathbf{R}^n\times\mathbf{R}^m} \sum_{k,d,j} \int_{\Phi(1)}^{\Phi(\theta)}\int_{\Psi(1)}^{\Psi(\theta)}\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \Omega_j(u,v) \\
& \times (M_{k+l,d+s}f)(x-\theta^k ru, y-\theta^d tv) h_{k,d,l,s,j}(x,y,r,t) \\
& \times d\sigma(u)d\sigma(v)\frac{drdt}{rt}dxdy \\
= & \int_{\mathbf{R}^n\times\mathbf{R}^m} \sum_{j}\sum_{k,d} \int_{\Phi(1)}^{\Phi(\theta)}\int_{\Psi(1)}^{\Psi(\theta)}\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \Omega_j(u,v) \\
& \times (M_{k+l,d+s}f)(x,y) h_{k,d,l,s,j}(x+\theta^k ru, y+\theta^d tv, r,t) \\
& \times d\sigma(u)d\sigma(v)\frac{drdt}{rt}dxdy \\
\le & \left\|(Y(h))^{1/2}\right\|_{p'} \left\|\Big(\sum_{k,d}|M_{k+l,d+s}f|^2\Big)^{1/2}\right\|_p,
\end{aligned}
$$

where

$$
\begin{aligned}
Y(h) = \sum_{k,d}\Bigg(\sum_j & \int_{\Phi(1)}^{\Phi(\theta)}\int_{\Psi(1)}^{\Psi(\theta)}\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \Omega_j(u,v) \\
& \times h_{k,d,l,s,j}\big(x+\theta^k ru, y+\theta^d tv, r, t\big) d\sigma(u)d\sigma(v)\frac{drdt}{rt}\Bigg)^2.
\end{aligned}
$$

By the Littlewood-Paley theory we get

$$
\|H_{l,s}(f)\|_p \le C_p\|f\|_p\left\|(Y(h))^{1/2}\right\|_{p'}. \tag{3.6}
$$

Since $p' > 2$ and $\|(Y(h))^{1/2}\|_{p'} = \|Y(h)\|_{p'/2}^{1/2}$, there is a function $b \in L^{(p'/2)'}$ $(\mathbf{R}^n\times\mathbf{R}^m)$ such that $\|b\|_{(p'/2)'} \le 1$ and

$$
\begin{aligned}
\|Y(h)\|_{p'/2} = \int_{\mathbf{R}^n\times\mathbf{R}^m} \sum_{k,d}\Bigg(\sum_j & \int_{\Phi(1)}^{\Phi(\theta)}\int_{\Psi(1)}^{\Psi(\theta)}\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \Omega_j(u,v) \\
& \times h_{k,d,l,s,j}\big(x+\theta^k ru, y+\theta^d tv, r, t\big) d\sigma(u)d\sigma(v)\frac{drdt}{rt}\Bigg)^2 \\
& \times b(x,y)dxdy.
\end{aligned}
$$

By Schwarz inequality and Lemma 2.12, we get

$$\begin{aligned}
&\Big(\sum_j \int_{\Phi(1)}^{\Phi(\theta)}\int_{\Psi(1)}^{\Psi(\theta)}\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \Omega_j(u,v)\\
&\quad\times h_{k,d,l,s,j}\big(x+\theta^k ru, y+\theta^d tv, r, t\big)d\sigma(u)d\sigma(v)\frac{drdt}{rt}\Big)^2\\
&\le C(\log\theta)^2\int_{\Phi(1)}^{\Phi(\theta)}\int_{\Psi(1)}^{\Psi(\theta)}\Big(\sum_j\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\Omega_j(u,v)\\
&\qquad\qquad\times h_{k,d,l,s,j}\big(x+\theta^k ru, y+\theta^d tv, r, t\big)d\sigma(u)d\sigma(v)\Big)^2\frac{drdt}{rt}\\
&\le C(\log\theta)^2\int_{\Phi(1)}^{\Phi(\theta)}\int_{\Psi(1)}^{\Psi(\theta)}\Big(\sum_j\|\Omega_j\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}^{\frac{1}{2}\min\{2,q\}}\\
&\quad\times\Big(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}|\Omega_j(u,v)|^{\max\{0,2-q\}}\\
&\qquad\qquad\times\big|h_{k,d,l,s,j}(x+\theta^k ru, y+\theta^d tv, r, t)\big|^2 d\sigma(u)d\sigma(v)\Big)^{1/2}\Big)^2\frac{drdt}{rt}\\
&\le C(\log\theta)^2\Big(\sum_j\|\Omega_j\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}^{\min\{2,q\}}\Big)\int_{\Phi(1)}^{\Phi(\theta)}\int_{\Psi(1)}^{\Psi(\theta)}\\
&\quad\times\sum_j\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}|\Omega_j(u,v)|^{\max\{0,2-q\}}\\
&\quad\times\big|h_{k,d,l,s,j}(x+\theta^k ru, y+\theta^d tv, r, t)\big|^2 d\sigma(u)d\sigma(v)\frac{drdt}{rt}.
\end{aligned}$$

Therefore, by a change of variable, Fubini's theorem, Hölder's inequality, and invoking Lemma 2.12 we get

$$\begin{aligned}
\|Y(h)\|_{p'/2} &\le C(\log\theta)^2\Big(\sum_j\|\Omega_j\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}^{\min\{2,q\}}\|\Omega_j\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}^{\max\{0,2-q\}}\Big)\\
&\qquad\times\int_{\mathbf{R}^n\times\mathbf{R}^m}\Big(\sum_j\sum_{k,d}\int_{\Phi(1)}^{\Phi(\theta)}\int_{\Psi(1)}^{\Psi(\theta)}\big|h_{k,d,l,s,j}(x,y,r,t)\big|^2\frac{drdt}{rt}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq \big(\mathcal{M}_{SP}(|\tilde{b}|^{q'/2})(-x,-y)\big)^{2/q'}dxdy \\
&\leq C(\log\theta)^2\bigg(\sum_j \|\Omega_j\|^2_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\bigg) \\
&\quad\times\bigg\|\sum_j\sum_{k,d}\int_{\Phi(1)}^{\Phi(\theta)}\int_{\Psi(1)}^{\Psi(\theta)}\big|h_{k,d,l,s,j}(x,y,r,t)\big|^2\frac{drdt}{rt}\bigg\|_{p'/2} \\
&\quad\times\Big\|\big(\mathcal{M}_{SP}\big(|\tilde{b}|^{q'/2}\big)\big)^{2/q'}\Big\|_{(p'/2)'}.
\end{aligned}
$$

By the condition on p we have $(2/q')(p'/2)' > \alpha'$. Thus by the choice of b and invoking Lemma 2.11 we get

$$
\|Y(h)\|_{p'/2} \leq C(q-1)^{-2}\bigg(\sum_j \|\Omega_j\|^2_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\bigg)
$$

which when combined with (3.6) and then interpolating with (3.4) we get (3.3). This completes the proof of (3.3).

<u>Case 2.</u> $\gamma = 1$. Write $T_{K,\Phi,\Psi}f(x,y) = \int_0^\infty\int_0^\infty \sum_j h_j(r,t)F_{r,t,j}(x,y)\frac{drdt}{rt}$, where

$$
F_{r,t,j}(x,y) = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} f(x-\Phi(r)u, y-\Psi(t)v)\Omega_j(u,v)d\sigma(u)d\sigma(v).
$$

By duality we have

$$
\begin{aligned}
&T^{(\gamma)}_{\Phi,\Psi,\{\Omega_j\}}f(x,y) \\
&\quad= \big\|F_{r,t,j}(\cdot,\cdot)\big\|_{l^\infty(L^\infty(\mathbf{R}_+\times\mathbf{R}_+,\frac{dtdr}{tr}),j)} \\
&\quad= \big\|F_{r,t,j}(\cdot,\cdot)\big\|_{l^\infty(L^\infty(\mathbf{R}_+\times\mathbf{R}_+,drdt),j)} \\
&\quad\leq \sup_{j,t,r>0}\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} f(x-\Phi(r)u, y-\Psi(t)v)\Omega_j(u,v)d\sigma(u)d\sigma(v).
\end{aligned}
$$

Now, by Hölder's inequality, we have

$$
F_{r,t,j}(x,y) \leq \sup_j \|\Omega_j\|_q\big(\mathcal{M}_{SP}\big(|f|^{q'}\big)(x,y)\big)^{1/q'}
$$

and hence we have

$$T^{(\gamma)}_{\Phi,\Psi,\{\Omega_j\}} f(x,y) \leq \big\| \|\Omega_j\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \big\|_{l^\infty} \big(\mathcal{M}_{SP}(|f|^{q'})(x,y)\big)^{1/q'}$$

By the last inequality and Lemma 2.11 we obtain (3.1) for the case $\gamma = 1$.

Case 3. $1 < \gamma < 2$. For $1 \leq p, q, \omega < \infty$, let $L^p(l^\omega(L^q(\mathbf{R}_+ \times \mathbf{R}_+, \frac{drdt}{rt}), j), \mathbf{R}^n \times \mathbf{R}^m)$ denote the space of all measurable functions $G_j(x,y,r,t)$ defined on $\mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}_+ \times \mathbf{R}_+$ with the mixed norm $\|G\|_{L^p(l^\omega(L^q(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt}),j),\mathbf{R}^n\times\mathbf{R}^m)}$, where

$$\begin{aligned}
&\|G\|_{L^p(l^\omega(L^q(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt}),j),\mathbf{R}^n\times\mathbf{R}^m)} \\
&\quad = \Big\| \big\| \|G_{(\cdot)}(\cdot,\cdot,\cdot)\|_{L^q(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt})} \big\|_{l^\omega} \Big\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \\
&\quad = \left(\int_{\mathbf{R}^n\times\mathbf{R}^m} \left(\sum_j \left(\int_{\mathbf{R}_+\times\mathbf{R}_+} |G_j(x,y,r,t)|^q \frac{drdt}{rt} \right)^{\omega/q} \right)^{p/\omega} dxdy \right)^{1/p}.
\end{aligned}$$

If $p = \infty$, $q = \infty$ or $\omega = \infty$, we can define $L^p(l^\omega(L^q(\mathbf{R}_+ \times \mathbf{R}_+, \frac{drdt}{rt}), j), \mathbf{R}^n \times \mathbf{R}^m)$ by the usual modification.

By duality we have

$$\big\| T^{(\gamma)}_{\Phi,\Psi,\{\Omega_j\}}(f) \big\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} = \|F(f)\|_{L^p(l^{\gamma'}(L^{\gamma'}(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt}),j),\mathbf{R}^n\times\mathbf{R}^m,dxdy)},$$

where $F : L^p(\mathbf{R}^n \times \mathbf{R}^m) \to L^p(l^{\gamma'}(L^{\gamma'}(\mathbf{R}_+ \times \mathbf{R}_+, \frac{drdt}{rt}), j), \mathbf{R}^n \times \mathbf{R}^m)$ is a linear operator defined by

$$F(f)(x,y;r,t;j) = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} f(x - \Phi(r)u, y - \Psi(t)v)\Omega_j(u,v)d\sigma(u)d\sigma(v).$$

From the inequalities (3.1) (for the case $\gamma = 2$) and (3.1) (for the case $\gamma = 1$), we interpret that

$$\begin{aligned}
&\|F(f)\|_{L^p(l^2(L^2(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt}),j),\mathbf{R}^n\times\mathbf{R}^m)} \\
&\quad \leq C(q-1)^{-1} \big\| \|\Omega_j\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \big\|_{l^2} \|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \qquad (3.7)
\end{aligned}$$

for $(2\alpha q')/(2\alpha + \alpha q' - 2) < p < \infty$ and

$$\|F(f)\|_{L^p(l^\infty(L^\infty(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt}),j),\mathbf{R}^n\times\mathbf{R}^m)} \leq C\Big\|\|\Omega_j\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\Big\|_{l^\infty}\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \tag{3.8}$$

for $q'\alpha' \leq p < \infty$. Applying the real interpolation theorem for Lebesgue mixed normed spaces to the above results (see [9]), we conclude that

$$\|F(f)\|_{L^p(l^{\gamma'}(L^{\gamma'}(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt}),j),\mathbf{R}^n\times\mathbf{R}^m)} \leq C_p(q-1)^{-2/\gamma'}\Big\|\|\Omega_j\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\Big\|_{l^{\gamma'}}\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \tag{3.9}$$

holds for $1 < q \leq 2$, $(\alpha q'\gamma')/(\gamma'\alpha + \alpha q' - \gamma') < p < \infty$ and $1 \leq \gamma \leq 2$, where $\alpha = \min(m, n)$. This completes the proof of Theorem 1.1.

Proof of Theorem 1.3 (a). To prove Theorem 1.3 we need to consider two cases.

Case 1. $1 < \gamma \leq 2$. First, we notice that $S_{\Phi,\Psi,\Omega,h}f(x,y) = \lim_{\varepsilon_1\to 0,\varepsilon_2\to 0} \cdot S^{(\varepsilon_1,\varepsilon_2)}_{\Phi,\Psi,\Omega,h}f(x,y)$ for $f \in \mathcal{S}(\mathbf{R}^n \times \mathbf{R}^m)$, where $S^{(\varepsilon_1,\varepsilon_2)}_{\Phi,\Psi,\Omega,h}$ is the truncated singular integral operator given by

$$S^{(\varepsilon_1,\varepsilon_2)}_{\Phi,\Psi,\Omega,h}f(x,y) = \int_{|v|>\varepsilon_2}\int_{|u|>\varepsilon_1}\frac{\Omega(u,v)}{|u|^n|v|^m} \times h(|u|,|v|)f(x-\Phi(|u|)u', y-\Psi(|v|)v')dudv. \tag{3.10}$$

We may assume without loss of generality that $\|h\|_{L^\gamma(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt})} = 1$. By Hölder's inequality and duality we have

$$\begin{aligned}\big|S^{(\varepsilon_1,\varepsilon_2)}_{\Phi,\Psi,\Omega,h}f(x,y)\big| &\leq \int_{\varepsilon_2}^\infty\int_{\varepsilon_1}^\infty |h(r,t)|\bigg|\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} f(x-\Phi(r)u, y-\Psi(t)v) \\ &\qquad\times \Omega(u,v)d\sigma(u)d\sigma(v)\bigg|\frac{drdt}{rt} \\ &\leq \bigg(\int_0^\infty\int_0^\infty\bigg|\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} f(x-\Phi(r)u, y-\Psi(t)v) \\ &\qquad\times \Omega(u,v)d\sigma(u)d\sigma(v)\bigg|^{\gamma'}\frac{drdt}{rt}\bigg)^{1/\gamma'} \\ &= \mathcal{M}^{(\gamma)}_{\Phi,\Psi,\Omega}f(x,y).\end{aligned}$$

Thus, by Theorem 1.2 we have

$$\left\|S^{(\varepsilon_1,\varepsilon_2)}_{\Phi,\Psi,\Omega,h}(f)\right\|_p \leq C_p(q-1)^{-2/\gamma'}\|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\|f\|_p \tag{3.11}$$

for $(\alpha q'\gamma')/(\gamma'\alpha+\alpha q'-\gamma') < p < \infty$ with $\alpha = \min(m,n)$ and $1 < \gamma \leq 2$ and for some positive constant C_p is independent of ε_1 and ε_2. In particular, (3.11) holds for $2 \leq p < \infty$ and $1 < \gamma \leq 2$. By a routine duality argument, (3.11) also holds for $1 < p \leq 2$ and $1 < \gamma \leq 2$. By Fatou's lemma and (3.11) we get (1.7) for $1 < p < \infty$ and $1 < \gamma \leq 2$.

Case 2. $2 < \gamma \leq \infty$. As above, we deal with $S^{(\varepsilon_1,\varepsilon_2)}_{\Phi,\Psi,\Omega,h}$. Write $S^{(\varepsilon_1,\varepsilon_2)}_{\Phi,\Psi,\Omega,h}(f) = \sum_{k,d\in\mathbf{Z}}\sigma_{\Omega,h,\theta,k,d} * f$. By invoking Lemmas 2.4, 2.5 and 2.9 with $\theta = 2^{q'}$, $a_k = \Phi(\theta^k)$ and $b_d = \Psi(\theta^d)$ we obtain the inequality (3.11) for $\gamma' < p < \infty$ with C_p independent of ε_1 and ε_2. Since $\gamma' < 2$, we get (3.11) for $2 \leq p < \infty$ and $2 < \gamma \leq \infty$. As above, by duality and Fatou's lemma we get (1.7) for $1 < p < \infty$ and $2 < \gamma \leq \infty$. This completes the proof of Theorem 1.3.

Proof of Theorem 1.4 (a). We employ the extrapolation method of Yano (see [27] or [28, Chap. XII, pp. 119–120]). Assume $1 < \gamma \leq 2$ and $\Omega \in L(\log L)^{2/\gamma'}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ are fixed and Ω satisfies (1.1). Fix p with $2 \leq p < \infty$ and a function f with $\|f\|_p \leq 1$. Let $R(\Omega) = \|\mathcal{M}^{(\gamma)}_{\Phi,\Psi,\Omega}(f)\|_p$. Decompose Ω as follows: For $\kappa \in \mathbf{N}$, let $\mathbf{J}_\kappa(\Omega) = \{(x,y) \in \mathbf{S}^{n-1}\times\mathbf{S}^{m-1}\colon 2^\kappa \leq |\Omega(x,y)| < 2^{\kappa+1}\}$. For $\kappa \in \mathbf{N}$, set $\tilde{a}_\kappa = \Omega\chi_{\mathbf{J}_\kappa(\Omega)}$, where $\chi_{\mathbf{A}}$ is the characteristic function of a set A. Set $I(\Omega) = \{\kappa \in \mathbf{N} : \|\tilde{a}_\kappa\|_1 \geq 2^{-4\kappa}\}$ and define the sequence of functions $\{\Omega^{(\kappa)}\}_{\kappa\in I(\Omega)\cup\{0\}}$ by

$$\begin{aligned}\Omega^{(\kappa)}(x,y) = \left\|\tilde{a}_\kappa\right\|_1^{-1}\bigg(\tilde{a}_\kappa(x,y) - \int_{\mathbf{S}^{n-1}_\kappa}\tilde{a}_\kappa(u,y)d\sigma(u) - \int_{\mathbf{S}^{m-1}}\tilde{a}_\kappa(x,v)d\sigma(v)\\ + \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\tilde{a}_\kappa(u,v)d\sigma(u)d\sigma(v)\bigg);\end{aligned} \tag{3.12}$$

$$\Omega^{(0)}(x,y) = \Omega(x,y) - \sum_{\kappa\in I(\Omega)}\left\|\tilde{a}_\kappa\right\|_1\Omega^{(\kappa)}(x,y). \tag{3.13}$$

It is easy to verify that the following hold for all $\kappa \in I(\Omega)\cup\{0\}$ and for some positive constant C:

$$\sum_{\kappa\in I(\Omega)} \kappa^{2/\gamma'}\|\tilde{a}_\kappa\|_1 \le \frac{1}{\sqrt{\log 2}}\|\Omega\|_{L(\log L)^{2/\gamma'}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}; \tag{3.14}$$

$$\int_{\mathbf{S}^{n-1}} \Omega^{(\kappa)}(u,\cdot)d\sigma(u) = \int_{\mathbf{S}^{m-1}} \Omega^{(\kappa)}(\cdot,v)d\sigma(v) = 0; \tag{3.15}$$

$$\Omega(x,y) = \Omega^{(0)}(x,y) + \sum_{\kappa\in I(\Omega)} \|\tilde{a}_\kappa\|_1\Omega^{(\kappa)}(x,y); \tag{3.16}$$

$$\|\Omega^{(\kappa)}\|_{1+\frac{1}{\kappa}} \le 2^7 \text{ for } \kappa\in I(\Omega) \text{ and } \|\Omega^{(0)}\|_2 \le 2^2. \tag{3.17}$$

Thus, by (3.14)–(3.17), Minkowski's inequality and applying Theorem 1.2 we get

$$\begin{aligned}
&\big\|\mathcal{M}^{(\gamma)}_{\Phi,\Psi,\Omega}(f)\big\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \\
&= \Bigg\|\mathcal{M}^{(\gamma)}_{\Phi,\Psi,\Omega^{(0)}}(f) + \sum_{\kappa\in I(\Omega)} \|\tilde{a}_\kappa\|_1\mathcal{M}^{(\gamma)}_{\Phi,\Psi,\Omega^{(\kappa)}}(f)\Bigg\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \\
&\le R(\Omega^{(0)}) + \sum_{\kappa\in I(\Omega)} \|\tilde{a}_\kappa\|_1 R(\Omega^{(\kappa)}) \\
&\le C\|\Omega^{(0)}\|_2 + \sum_{\kappa\in I(\Omega)} \kappa^{2/\gamma'}\|\tilde{a}_\kappa\|_1\|\Omega^{(\kappa)}\|_{1+\frac{1}{\kappa}} \\
&\le C\big(1+\|\Omega\|_{L(\log L)^{2/\gamma'}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\big).
\end{aligned}$$

Proof of Theorem 1.4 (b). Assume $1<\gamma\le 2$ and $\Omega\in B_q^{(0,2/\gamma'-1)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ for some $q>1$ are fixed and Ω satisfies (1.1). We may assume without loss of generality that $1<q\le 2$. Fix p with $2\le p<\infty$ and a function f with $\|f\|_p\le 1$ and let $A(\Omega)=\|\mathcal{M}^{(\gamma)}_{\Phi,\Psi,\Omega}(f)\|_p$. Since $\Omega\in B_q^{(0,2/\gamma'-1)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$, we can write Ω as $\Omega=\sum_{\mu=1}^{\infty}\lambda_\mu b_\mu$, where $\lambda_\mu\in\mathbf{C}$, b_μ is a q-block supported on an interval I_μ on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$ and $M_q^{(0,2/\gamma'-1)}(\{\lambda_\mu\})<\infty$. To each block function $b_\mu(\cdot,\cdot)$, let $\tilde{\Omega}_\mu(\cdot,\cdot)$ be a function defined by

$$\begin{aligned}
\tilde{\Omega}_\mu(x,y) = b_\mu(x,y) &- \int_{\mathbf{S}^{n-1}} b_\mu(u,y)d\sigma(u) - \int_{\mathbf{S}^{m-1}} b_\mu(x,v)d\sigma(v) \\
&+ \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} b_\mu(u,v)d\sigma(u)d\sigma(v).
\end{aligned}$$

Let $\mathbf{D} = \{\mu \in \mathbf{N} : |I_\mu| < e^{-\frac{q'}{q}}\}$, $\delta = (\sum_{\mu=1}^{\infty} |\lambda_\mu|)$ and let $\tilde{\Omega}_0 = \Omega - \sum_{\mu\in\mathbf{D}}^{\infty} \lambda_\mu \tilde{\Omega}_\mu$. Also, for $\mu \in \mathbf{D}$ we let $\omega_\mu = \log(|I_\mu|^{-1})$. Then one can easily verify the following:

$$\Omega = \tilde{\Omega}_0 + \sum_{\mu\in\mathbf{D}}^{\infty} \lambda_\mu \tilde{\Omega}_\mu; \tag{3.18}$$

$$\int_{\mathbf{S}^{n-1}} \tilde{\Omega}_\mu(u,\cdot)d\sigma(u) = \int_{\mathbf{S}^{m-1}} \tilde{\Omega}_\mu(\cdot,v)d\sigma(v) = 0 \text{ for all } \mu \in \mathbf{D} \cup \{0\}; \tag{3.19}$$

$$\left\|\tilde{\Omega}_0\right\|_q \leq \delta e^{\frac{1}{q}}. \tag{3.20}$$

Also, for $\mu \in \mathbf{D}$ we have $1 + \frac{1}{\omega_\mu} < q$ and hence by Hölder's inequality we have

$$\begin{aligned} \left\|\tilde{\Omega}_\mu\right\|_{1+\frac{1}{\omega_\mu}} &\leq 4\|b_\mu\|_q |I_\mu|^{\frac{q-1-\frac{1}{\omega_\mu}}{q(1+\frac{1}{\omega_\mu})}} \\ &\leq 4\big(|I_\mu|^{-\frac{1}{q'}}\big)|I_\mu|^{\frac{q-1-\frac{1}{\omega_\mu}}{q(1+\frac{1}{\omega_\mu})}} = 4|I_\mu|^{-\frac{1}{\omega_\mu+1}} \leq 8. \end{aligned} \tag{3.21}$$

By (3.18)–(3.21) and invoking Theorem 2.1 we get

$$\begin{aligned} A(\Omega) &\leq A\big(\tilde{\Omega}_0\big) + \sum_{\mu\in\mathbf{D}} |\lambda_\mu| A\big(\tilde{\Omega}_\mu\big) \\ &\leq C_p\bigg((q-1)^{-2/\gamma'}\left\|\tilde{\Omega}_0\right\|_q + \sum_{\mu\in\mathbf{D}} |\lambda_\mu|\big(\log|I_\mu|^{-1}\big)^{2/\gamma'}\left\|\tilde{\Omega}_\mu\right\|_{1+\frac{1}{\omega_\mu}}\bigg) \\ &\leq C_p\bigg(\delta e^{\frac{1}{q}}(q-1)^{-2/\gamma'} + 8\sum_{\mu\in\mathbf{D}}\big(|\lambda_\mu|\big(\log|I_\mu|^{-1}\big)^{2/\gamma'}\big)\bigg) \\ &\leq C_p\big(1 + \|\Omega\|_{B_q^{(0,2/\gamma'-1)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\big). \end{aligned}$$

Proof of Theorem 1.5 (a) and (b). A proof of Theorem 1.5 (a) can be constructed by Theorem 1.3, using an extrapolation argument and using a similar argument employed in the proof of Theorem 1.4 (a). Details will be omitted. Also, a proof of Theorem 1.5 (b) can be obtained by Theorem 1.3, extrapolation and following a similar argument employed in the proof of Theorem 1.4 (b). Again details will be omitted.

Theorem 1.5 (c). Assume $\gamma = 1$. It is easy to see that the inequality

$$\begin{aligned}&\left|S^{(\varepsilon_1,\varepsilon_2)}_{\Phi,\Psi,\Omega,h} f(x,y)\right| \\ &\quad\le \|h\|_{L^\gamma(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt})}\|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\|f\|_{L^\infty(\mathbf{R}^n\times\mathbf{R}^m)}\end{aligned}$$

holds for all $f \in L^\infty(\mathbf{R}^n \times \mathbf{R}^m)$ and for almost every $(x, y) \in \mathbf{R}^n \times \mathbf{R}^m$. By (3.22) we get

$$\begin{aligned}&\left\|S^{(\varepsilon_1,\varepsilon_2)}_{\Phi,\Psi,\Omega,h}(f)\right\|_{L^\infty(\mathbf{R}^n\times\mathbf{R}^m)} \\ &\quad\le \|h\|_{L^\gamma(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt})}\|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\|f\|_{L^\infty(\mathbf{R}^n\times\mathbf{R}^m)}\end{aligned}$$

for all $f \in \mathcal{S}(\mathbf{R}^n \times \mathbf{R}^m)$. By duality, we have

$$\begin{aligned}&\left\|S^{(\varepsilon_1,\varepsilon_2)}_{\Phi,\Psi,\Omega,h}(f)\right\|_{L^1(\mathbf{R}^n\times\mathbf{R}^m)} \\ &\quad\le \|h\|_{L^\gamma(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt})}\|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\|f\|_{L^1(\mathbf{R}^n\times\mathbf{R}^m)}\end{aligned}$$

for all $f \in \mathcal{S}(\mathbf{R}^n \times \mathbf{R}^m)$. Thus by interpolation between the last two estimates we get

$$\begin{aligned}&\left\|S^{(\varepsilon_1,\varepsilon_2)}_{\Phi,\Psi,\Omega,h}(f)\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \\ &\quad\le \|h\|_{L^\gamma(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt})}\|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}\end{aligned}$$

for $1 < p < \infty$ and all $f \in \mathcal{S}(\mathbf{R}^n \times \mathbf{R}^m)$. Finally, using density argument we get

$$\begin{aligned}&\left\|S^{(\varepsilon_1,\varepsilon_2)}_{\Phi,\Psi,\Omega,h}(f)\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \\ &\quad\le \|h\|_{L^\gamma(\mathbf{R}_+\times\mathbf{R}_+,\frac{drdt}{rt})}\|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}\end{aligned}$$

for $1 \le p \le \infty$ and for all $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$.

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