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The Borel-de Siebenthal Theorem, the classification of equi-rank groups, and related compact and semi-compact dual pairs

Hubert RUBENTHALER

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Abstract. Let G be a connected simple real Lie group with finite center and let K be a maximal compact subgroup of G. Such a group is called an equi-rank group if $\mathrm{rk}(G) = \mathrm{rk}(K)$. In this paper we give a new proof of the classification of the equi-rank groups (Borel-de Siebenthal Theorem). We also obtain some dual pairs inside $\mathfrak{g}_{\mathbb{R}}$ (the Lie algebra of G) such that at least one of the members of the dual pair is compact.

Key words: equi-rank group, dual pair.

1. Introduction

Let G be a connected simple real Lie group with finite center and let K be a maximal compact subgroup of G. By a well known result of Harish-Chandra ([HC]) the group G has a discrete series of representations if and only if the rank of K is equal to the rank of G.

The list of such G's, which we shall call *equi-rank groups*, has been known for a long time, mostly from case by case examination of the list of real simple Lie algebras (see for example [War]).

On the other hand there exists an intrinsic classification which is due essentially to Borel and de Siebenthal [BdS] (more precisely what is now called the Borel-de Siebenthal Theorem is Remarque 1, p. 218 of [BdS]). Their theorem was presented in terms of compact groups, then the transcription to equi-rank groups was given by Murakami, who called these equi-rank groups *groups of interior type* ([M], Theorème 1 p. 295). The paper by Wallach ([Wal]) is also relevant.

Recently A. Knapp [K1], gave a quick proof of the classification of real semi-simple Lie algebras which also relies on the Borel-de Siebenthal Theorem. In his book ([K2]), A. Knapp gave a proof of the theorem (Thm. 6.96. p. 350) that uses the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of G and its complexification

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$\mathfrak{g}$, together with the classification of roots into complex, imaginary, compact, non-compact roots.

In the beginning of this paper we propose a new proof of the Borel-de Siebenthal Theorem or to be more precise, a new classification of equi-rank groups, which is totally complex (i.e. we work only at the level of the complex Lie algebra $\mathfrak{g}$) and is based on Dynkin's notion of elementary operation ([D], see [T] for a nice résumé).

Our result avoids any case by case computation and allows an immediate classification (§ 4).

The proof illustrates what one could call the *parabolic* and *$\mathbb{Z}$-graded* nature of the structures involved and this seems to be a rather new aspect to us. In addition, our results give some new information about the different "parabolic realizations" of the complexified Lie algebra $\mathfrak{k}$ of K (§ 5) and lead naturally to the construction of some families of dual pairs $(\mathfrak{k}_1, \mathfrak{k}_2)$ inside $\mathfrak{g}_{\mathbb{R}}$, where at least one of these subalgebras is compact (§ 6).

2. Maximal compact subalgebras and complex involutions

Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$ be a Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$ relative to a Cartan involution θ. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition of $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$ obtained by complexification. We will denote by σ the complex linear extension of θ. Conversely, let us show the following (certainly well known) result:

Proposition 2.1 *If one has a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ (resp. $\mathfrak{p}$) is the $+1$ (resp. -1) eigenspace of a complex involution σ of $\mathfrak{g}$, then there exists a real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$ whose maximal compact subalgebra $\mathfrak{k}_{\mathbb{R}}$ is a real form of $\mathfrak{k}$. Moreover all such real forms are conjugate under the group* $\exp \mathfrak{k}$.

Proof. First we need the following lemma.

Lemma 2.2 *Let θ_1 and θ_2 be two Cartan involutions of the complex Lie algebra $\mathfrak{g}$ which commute with a given complex linear involution σ whose fixed point set is denoted by $\mathfrak{k}$. Then there exists an element $X \in \mathfrak{k}$ such that $\theta_2 = (\exp X)\theta_1(\exp -X)$.*

To prove the lemma, we define a linear operator N by $N = \theta_2\theta_1$. One shows easily that if B_{θ_1} is defined by $B_{\theta_1}(X, Y) = -B(X, \theta_1(Y))$, then the linear operator N is symmetric with respect to B_{θ_1}. As B_{θ_1} is hermitian positive definite, therefore N is diagonalizable with real eigenvalues λ_i. Hence, for any $t \in \mathbb{R}$, one can define an operator P^t whose eigenvalues are

$(\lambda_i^2)^t$. In particular $P = N^2$. Of course P^t is a one parameter subgroup inside $\mathrm{Aut}(\mathfrak{g})$, therefore there exists $X \in \mathfrak{g}$ such that $P^t = \exp(t \operatorname{ad} X)$. Moreover this one parameter subgroup commutes with σ (because it is so for N), i.e. $\sigma P^t \sigma^{-1} = \exp(t \operatorname{ad} \sigma(X)) = P^t = \exp(t \operatorname{ad}(X))$ for all $t \in \mathbb{R}$. Hence $\operatorname{ad} X = \operatorname{ad} \sigma(X)$, and one gets $X = \sigma(X)$. Therefore X is in $\mathfrak{k}$.

Set $\theta' = P^{\frac{1}{4}} \theta_1 P^{-\frac{1}{4}}$. This again is a Cartan involution of $\mathfrak{g}$. Let us show that θ' and θ_2 commute. Notice first that $\theta_1 N \theta_1^{-1} = \theta_1 \theta_2 \theta_1 \theta_1^{-1} = \theta_1 \theta_2 = N^{-1}$, hence $\theta_1 P^t \theta_1^{-1} = P^{-t}$. Then we have

$$\theta' \theta_2 = P^{\frac{1}{4}} \theta_1 P^{-\frac{1}{4}} \theta_1 \theta_1 \theta_2 = P^{\frac{1}{4}} P^{\frac{1}{4}} \theta_1 \theta_2 = P^{\frac{1}{2}} N^{-1} .$$

On the other hand we have

$$\theta_2 \theta' = \theta_2 \theta_1 \theta_1 P^{\frac{1}{4}} \theta_1 P^{-\frac{1}{4}} = N P^{-\frac{1}{2}} = P^{\frac{1}{2}} N^{-1} \quad (\text{because } P = N^2).$$

So we get $\theta' \theta_2 = \theta_2 \theta'$. But two Cartan involutions which commute are equal, therefore $\theta_2 = P^{\frac{1}{4}} \theta_1 P^{-\frac{1}{4}} = \exp\big(\operatorname{ad} \frac{1}{4} X\big) \theta_1 \exp\big(-\operatorname{ad} \frac{1}{4} X\big)$, with $X \in \mathfrak{k}$. □

We need a second lemma. We keep the same notation as in Lemma 2.2.

Lemma 2.3 *Let $\mathfrak{g} = \mathfrak{u}_1 \oplus i\mathfrak{u}_1 = \mathfrak{u}_2 \oplus i\mathfrak{u}_2$ be the Cartan decompositions of $\mathfrak{g}$ relatively to the Cartan involutions θ_1 and θ_2. Let $\mathfrak{g}_{\mathbb{R}}^1$ (resp. $\mathfrak{g}_{\mathbb{R}}^2$) be the fixed point algebra of $\sigma\theta_1$ (resp. $\sigma\theta_2$). Then $\mathfrak{g}_{\mathbb{R}}^1$ and $\mathfrak{g}_{\mathbb{R}}^2$ are real forms of $\mathfrak{g}$ and there exists $k \in \exp(\operatorname{ad} \mathfrak{k})$ such that $\mathfrak{g}_2 = k\mathfrak{g}_1$.*

Proof of the Lemma. As σ commutes with θ_1 and θ_2, one has the decompositions $\mathfrak{g}_{\mathbb{R}}^1 = \mathfrak{k} \cap \mathfrak{u}_1 \oplus \mathfrak{p} \cap i\mathfrak{u}_1$ and $\mathfrak{g}_{\mathbb{R}}^2 = \mathfrak{k} \cap \mathfrak{u}_2 \oplus \mathfrak{p} \cap i\mathfrak{u}_2$. The fact that $\mathfrak{g}_{\mathbb{R}}^1$ and $\mathfrak{g}_{\mathbb{R}}^2$ are real forms is now obvious. From the preceeding Lemma one has $\theta_2 = k\theta_1 k^{-1}$ where $k = \exp\big(\operatorname{ad} \frac{1}{4} X\big)$ $(X \in \mathfrak{k})$. This implies that $k\mathfrak{u}_1 = \mathfrak{u}_2$. As $k\mathfrak{k} = \mathfrak{k}$ and $k\mathfrak{p} = \mathfrak{p}$, one gets $k\mathfrak{g}_{\mathbb{R}}^1 = \mathfrak{g}_{\mathbb{R}}^2$. □

Let us now prove Proposition 2.1. There exists always a Cartan involution θ of $\mathfrak{g}$ which commutes with σ. Let $\mathfrak{g} = \mathfrak{u} \oplus i\mathfrak{u}$ be the corresponding Cartan decomposition and let $\mathfrak{g}_{\mathbb{R}}$ be the fixed point algebra of the involution $\sigma\theta$. One has $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \cap \mathfrak{u} \oplus \mathfrak{p} \cap i\mathfrak{u}$ and this is a Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$. The Lie algebra $\mathfrak{g}_{\mathbb{R}}$ is a real form of $\mathfrak{g}$ and $\mathfrak{k}_{\mathbb{R}} = \mathfrak{k} \cap \mathfrak{u}$ is a real form of $\mathfrak{k}$. This proves the first assertion of the proposition.

Let now $\mathfrak{g}'_{\mathbb{R}}$ be another real form of $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g}'_{\mathbb{R}} = \mathfrak{k}'_{\mathbb{R}} \oplus \mathfrak{p}'_{\mathbb{R}}$ such that $\mathfrak{k} = \mathfrak{k}'_{\mathbb{R}} \oplus i\mathfrak{k}'_{\mathbb{R}}$. Then automatically $\mathfrak{p} = \mathfrak{p}'_{\mathbb{R}} \oplus i\mathfrak{p}'_{\mathbb{R}}$. Set $u' = \mathfrak{k}'_{\mathbb{R}} \oplus i\mathfrak{p}'_{\mathbb{R}}$. Then $\mathfrak{g} = \mathfrak{u}' \oplus i\mathfrak{u}'$ is a Cartan decomposition and the corresponding

Cartan involution θ' is easily verified to commute with σ. By the preceeding lemma there exists $k \in \exp(\operatorname{ad}\mathfrak{k})$ such that $\mathfrak{g}'_{\mathbb{R}} = k\mathfrak{g}_{\mathbb{R}}$. □

3. The classification of equi-rank complex simple Lie algebras

Proposition 2.1, reduces the problem of classifying all equi-rank groups to the classification of all $\mathbb{Z}_2$-gradations $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ ($[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$) of the complex simple Lie algebra $\mathfrak{g}$ having the property $\operatorname{rank}(\mathfrak{g}) = \operatorname{rank}(\mathfrak{k})$, up to conjugacy by an element of the adjoint group of $\mathfrak{g}$. The required Lie algebras are simple because a simple equi-rank group is never a complex group.

Henceforth $\mathfrak{g}$ will denote a simple complex Lie algebra.

The data consisting of a simple complex Lie algebra together with a $\mathbb{Z}_2$-gradation having the above mentioned properties is called an *equi-rank algebra*.

Our main result is the following Theorem.

Theorem 3.1 *Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be an equi-rank algebra, with $\mathfrak{k} \neq \mathfrak{g}$. Then either*

1) we have the following $\mathbb{Z}$-gradation: $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Here $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a maximal parabolic subalgebra, and $\mathfrak{k} = \mathfrak{g}_0$, $\mathfrak{p} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ is the corresponding $\mathbb{Z}_2$-gradation. This corresponds exactly to the case where $\mathfrak{k}$ is reductive not semi-simple, i.e. has a non-trivial center (necessarily one dimensional). This is also exactly the case where G/K is hermitian.

or 2) we have the following $\mathbb{Z}$-gradation: $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. The parabolic subalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is again maximal, and $\mathfrak{k} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$ and $\mathfrak{p} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ is the corresponding $\mathbb{Z}_2$-gradation. This corresponds exactly to the case where $\mathfrak{k}$ is semi-simple.

Conversely given $\mathfrak{g}$ together with a gradation of one of the preceeding types, then the corresponding $\mathfrak{k}$ and $\mathfrak{p}$ (given as is 1) or 2)) determine an equi-rank algebra.

Let us now prove this theorem. First of all we need to recall the following definition due to Dynkin ([D]).

A subalgebra $\mathfrak{r}$ of $\mathfrak{g}$ is called a *regular* subalgebra if there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{r}] \subseteq \mathfrak{r}$.

Lemma 3.2 *Let $\mathfrak{r} = \mathfrak{z} \oplus \mathfrak{r}'$ be a reductive subalgebra of $\mathfrak{g}$ where $\mathfrak{z}$ is the center of $\mathfrak{r}$ and $\mathfrak{r}' = [\mathfrak{r}, \mathfrak{r}]$. Suppose moreover that the rank of $\mathfrak{r}$ is equal to* *the rank of* $\mathfrak{g}$. *Then* $\mathfrak{z}$ *is the center of a Levi subalgebra* $\mathfrak{l} = \mathfrak{z} \oplus \mathfrak{l}'$ *of* $\mathfrak{g}$, *and* $\mathfrak{r}'$ *is a regular subalgebra of* $\mathfrak{l}'$ *of maximal rank, i.e.* $\operatorname{rank}(\mathfrak{r}') = \operatorname{rank}(\mathfrak{l}')$.

Proof. As $\operatorname{rank}(\mathfrak{r}) = \operatorname{rank}(\mathfrak{g})$ one can choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ of the form

$$\mathfrak{h} = \mathfrak{z} \oplus \mathfrak{h}_{\mathfrak{r}'}$$

where $\mathfrak{h}_{\mathfrak{r}'}$ is a Cartan subalgebra of $\mathfrak{r}'$. Let $\mathfrak{l} = \mathcal{Z}_{\mathfrak{g}}(\mathfrak{z})$ be the centralizer of $\mathfrak{z}$. The Lie subalgebra $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and therefore of the form:

$$\mathfrak{l} = \mathcal{Z}_{\mathfrak{g}}(\mathfrak{z}) = \mathfrak{c} \oplus \mathfrak{l}',$$

where $\mathfrak{c}$ is the center of $\mathfrak{l}$ and $\mathfrak{l}' = [\mathfrak{l}, \mathfrak{l}]$. Moreover, $\mathfrak{r}' \subseteq \mathfrak{r} \subseteq \mathfrak{l}$.

From the fact that $\mathfrak{c}$ is the centralizer of $\mathfrak{l}$ in $\mathfrak{g}$, one deduces that $\mathfrak{z} \subseteq \mathfrak{c}$.

Let B be the Killing form of $\mathfrak{g}$. It is well known that $\mathfrak{c}^{\perp} \cap \mathfrak{l} = \mathfrak{l}'$ where $\mathfrak{c}^{\perp}$ is the orthogonal of $\mathfrak{c}$ with respect to B. Let $\mathfrak{h}_{\mathfrak{l}'} = \mathfrak{c}^{\perp} \cap \mathfrak{h}$. Then

$$\mathfrak{h} = \mathfrak{c} \oplus \mathfrak{h}_{\mathfrak{l}'}.$$

One has $\mathfrak{h}_{\mathfrak{l}'} \subseteq \mathfrak{h}_{\mathfrak{r}'}$, and therefore $\operatorname{rank}(\mathfrak{l}') \leq \operatorname{rank}(\mathfrak{r}')$.

Let $\Sigma(\mathfrak{g}, \mathfrak{h})$ be the root system of the pair $(\mathfrak{g}, \mathfrak{h})$. For $\alpha \in \Sigma(\mathfrak{g}, \mathfrak{h})$, let $\mathfrak{g}^{\alpha}$ be the corresponding root space. Then $\mathfrak{g}^{\alpha} \perp \mathfrak{h}$ and hence $\mathfrak{g}^{\alpha} \perp \mathfrak{c}$. Therefore, as $\mathfrak{r}' \subseteq \mathfrak{l}$, any root space in $\mathfrak{r}'$ is in fact in $\mathfrak{l}'$, this implies that $\mathfrak{r}' \subseteq \mathfrak{l}'$, and consequently $\operatorname{rank}(\mathfrak{r}') \leq \operatorname{rank}(\mathfrak{l}')$.

As the reverse inequality was proved earlier we get $\operatorname{rank}(\mathfrak{r}') = \operatorname{rank}(\mathfrak{l}')$.

Recall that $\mathfrak{z} \subseteq \mathfrak{c}$. From the equality

$$\dim \mathfrak{h} = \dim \mathfrak{z} + \dim \mathfrak{h}_{\mathfrak{r}'} = \dim \mathfrak{c} + \dim \mathfrak{h}_{\mathfrak{l}'}$$

one gets $\mathfrak{c} = \mathfrak{z}$. The lemma is proved. □

Let us return to the proof of Theorem 3.1.

First case : $\mathfrak{k}$ is reductive and not semi-simple.

From the previous lemma we have $\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{k}'$ where $\mathfrak{k}'$ is the derived algebra of $\mathfrak{k}$ and where $\mathfrak{z}$ is the center of a Levi subalgebra $\mathfrak{l} = \mathfrak{z} \oplus \mathfrak{l}'$ with $\mathfrak{k}' \subseteq \mathfrak{l}'$ and $\operatorname{rank}(\mathfrak{k}') = \operatorname{rank}(\mathfrak{l}')$.

Let $\mathfrak{P}$ be a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$. One has $\mathfrak{P} = \mathfrak{l} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{P}$. It is well known ([Bo2]) that one can choose a system of simple roots $\Psi \subseteq R = \sum(\mathfrak{g}, \mathfrak{h})$, and a subsystem $\Gamma \subseteq \Psi$, such that the set of roots $\langle \Gamma \rangle$ which are linear combinations of elements of

Γ is the root system of $(\mathfrak{l}, \mathfrak{h})$, and such that

$$\mathfrak{n} = \sum_{\alpha \in R^+ \setminus \langle \Gamma \rangle^+} \mathfrak{g}^\alpha$$

where the sets of positive roots R^+ and $\langle \Gamma \rangle^+$ are defined by Ψ and Γ respectively.

Let h_Γ be the unique element of $\mathfrak{z}$ defined by the equations

$$\begin{aligned} \alpha(h_\Gamma) &= 1 \quad \text{if } \alpha \in \Psi \setminus \Gamma \\ \alpha(h_\Gamma) &= 0 \quad \text{if } \alpha \in \Gamma. \end{aligned} \tag{3-1}$$

For $i \in \mathbb{Z}$, set $\mathfrak{g}_i = \{X \in \mathfrak{g} \mid [h_\Gamma, X] = iX\}$. Then one has the $\mathbb{Z}$-gradation $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, where $\mathfrak{l} = \mathfrak{g}_0$ and $\mathfrak{n} = \bigoplus_{i>0} \mathfrak{g}_i$.

Let n be the greatest integer such that $\mathfrak{g}_n \neq \{0\}$.

Lemma 3.3 *Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a simple equi-rank algebra with $\mathfrak{k}$ reductive and not semi-simple. Then $n = 1$ (i.e. the nilradical $\mathfrak{n}$ is commutative) and* $\dim \mathfrak{z} = 1$.

Proof of the Lemma. The assertion $\dim \mathfrak{z} = 1$ is an easy consequence of $n = 1$.

Suppose that $n \geq 2$. It is well known that then $[\mathfrak{g}_1, \mathfrak{g}_1] \neq \{0\}$ (in fact $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2$). As $\mathfrak{p}^\perp = \mathfrak{k}$ and $\mathfrak{l}^\perp = \mathfrak{g}_{-n} \oplus \cdots \mathfrak{g}_{-1} \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n \subseteq \mathfrak{p}$, one should have $[\mathfrak{g}_1, \mathfrak{g}_1] \subseteq [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k} \subseteq \mathfrak{l} = \mathfrak{g}_0$. If $\mathfrak{g}_2 \neq \{0\}$ this is not true. Therefore $n = 1$. $\square$

Note that the condition $n = 1$ just proved, implies (because $\mathfrak{g}$ is simple) that the parabolic subalgebra $\mathfrak{P} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is maximal.

In order to prove Theorem 3.1. in the case where $\mathfrak{k}$ is reductive and not semi-simple, it is now sufficient to prove that $\mathfrak{k} = \mathfrak{g}_0$.

Let $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}_1$ where $\mathfrak{p}_1 = \mathfrak{l} \cap \mathfrak{p}$. Recall that then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_1 \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$.

Put $\mathfrak{a} = \mathfrak{g}_{-1} \oplus [\mathfrak{g}_{-1}, \mathfrak{g}_1] \oplus \mathfrak{g}_1$. It is easily seen that $\mathfrak{a}$ is an ideal of $\mathfrak{g}$. As $\mathfrak{g}$ is simple, one has $\mathfrak{g} = \mathfrak{a}$.

Consider now the vector space $[\mathfrak{p}_1, \mathfrak{g}_1]$. As $\mathfrak{p}_1 \subseteq \mathfrak{p}$ and $\mathfrak{g}_1 \subseteq \mathfrak{p}$, one has $[\mathfrak{p}_1, \mathfrak{g}_1] \subseteq \mathfrak{k}$. But $\mathfrak{p}_1 \subseteq \mathfrak{l} = \mathfrak{g}_0$, therefore $[\mathfrak{p}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_1 \subseteq \mathfrak{p}$. Hence $[\mathfrak{p}_1, \mathfrak{g}_1] = \{0\}$. The same arguments shows that $[\mathfrak{p}_1, \mathfrak{g}_{-1}] = \{0\}$, and hence $[\mathfrak{p}_1, [\mathfrak{g}_{-1}, \mathfrak{g}_1]] = \{0\}$; and finally that $\mathfrak{p}_1$ is central in $\mathfrak{g}$, and therefore $\mathfrak{p}_1 = \{0\}$, $\mathfrak{k} = \mathfrak{g}_0$, and $\mathfrak{p} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$.

Conversely suppose that one has a $\mathbb{Z}$-gradation

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

Then the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = \mathfrak{g}_0$ and $\mathfrak{p} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ determines an equi-rank algebra.

Thus Theorem 3.1. is proved in the first case.

Second case: $\mathfrak{k}$ is semi-simple.

Let us recall what is an *elementary operation* in the sense of Dynkin.

Let D be the Dynkin diagram of a simple Lie algebra $\mathfrak{g}$. Once a Cartan subalgebra $\mathfrak{h}$ is chosen, as well as a set of simple roots Ψ, the vertices of D are associated to the elements of Ψ, and the different vertices are connected by edges according to well known rules. Let ω be the highest root with respect to Ψ. Then one gets the so-called *extended* Dynkin diagram $\widetilde{D}$ by adding a vertex associated to $-\omega$ and connecting this new vertex to the other vertices by using the same rules.

It turns out that $\widetilde{D}$ is *not* a Dynkin Diagram, but if one removes any vertex from it, as well as the edges connected to this vertex, then one gets a, possibly disconnected, new Dynkin diagram D_1 (with of course the same number of vertices as D).

This new Dynkin diagram D_1 is said to be obtained from D by an elementary operation.

Let now $(X_{-\alpha}, H_\alpha, X_\alpha)_{\alpha \in D_1}$ be the collection of the usual $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ associated to the roots corresponding to the vertices in D_1.

This collection of triples generates a semi-simple Lie subalgebra $\mathfrak{a}_1$ of $\mathfrak{g}$ whose Dynkin diagram is precisely D_1 ([D], [T]). The Lie algebra $\mathfrak{a}_1$ is said to be obtained from $\mathfrak{g}$ by an elementary operation.

Another way to understand this algebra $\mathfrak{a}_1$ is as follows. Let α_0 be the root which is removed by the elementary operation, and let $\Gamma = \Psi \setminus \{\alpha_0\}$. This subset Γ defines, as in the proof of the first case, a (maximal) parabolic subalgebra and therefore a $\mathbb{Z}$-gradation

$$\mathfrak{g} = \bigoplus_{i=-k}^{i=k} \mathfrak{g}_i. \tag{3-2}$$

One can prove that

$$\mathfrak{a}_1 = \mathfrak{g}_{-k} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_k$$

(see [Bo1] Exercice 4, Chap. VI, par. 4, p. 229 and [Bo2] Exercice 2, Chap. VIII par. 3, p. 223).

But now, one can make a new elementary operation on any of the connected components of D_1. Of course each time that a diagram of type A_n occurs, any elementary operation does not change the algebra.

Any semi-simple Lie algebra of maximal rank of a simple Lie algebra is obtained by a finite number of elementary operations (see [D], Th. 5.3. page 145 or [T]).

As $\mathfrak{k}$ is maximal rank, $\mathfrak{k}$ can be obtained this way; namely, one has a chain of subalgebras

$$\mathfrak{g} = \mathfrak{a}_0 \supseteq \mathfrak{a}_1 \supseteq \cdots \supseteq \mathfrak{a}_m = \mathfrak{k},$$

where each $\mathfrak{a}_i$ is obtained from $\mathfrak{a}_{i-1}$ by an elementary operation.

Let us focus on $\mathfrak{a}_1$. As we explained before there exists a $\mathbb{Z}$-gradation of $\mathfrak{g}$ as in (3-2) coming from a maximal parabolic subalgebra such that

$$\mathfrak{a}_1 = \mathfrak{g}_{-k} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_k.$$

As $\mathfrak{k} \subseteq \mathfrak{a}_1$ one has

$$\mathfrak{p} = \mathfrak{k}^{\perp} \supseteq \mathfrak{g}_{-k+1} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_1 \oplus \cdots \mathfrak{g}_{k-1}.$$

If $k \geq 3$, that is if $k-1 \geq 2$, one should have $[\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_2 \subseteq \mathfrak{p}$ and also $[\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{k} \subseteq \mathfrak{a}_1 = \mathfrak{g}_{-k} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_k$. This is impossible since $[\mathfrak{g}_1, \mathfrak{g}_1] \neq \{0\}$.

Hence $k \leq 2$. This means in fact that $k = 2$, because the case $k = 1$ would lead to $\mathfrak{a}_1 = \mathfrak{g}$ which is excluded since $\mathfrak{k} \neq \mathfrak{g}$.

Now we are in the following situation:

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

$$\mathfrak{k} \subseteq \mathfrak{a}_1 = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2, \qquad \mathfrak{g}_{-1} \oplus \mathfrak{g}_1 \subseteq \mathfrak{p}.$$

As $\mathfrak{g}$ is simple one has $[\mathfrak{g}_{-1}, \mathfrak{g}_1] = \mathfrak{g}_0$ and hence $\mathfrak{g}_0 \subseteq \mathfrak{k}$.

Moreover the gradation is defined by the eigenvalues of an element h_Γ as in (3-1) where $\Gamma = \Psi \setminus \{\alpha_0\}$ for some simple root α_0 (the parabolic subalgebra $\mathfrak{P} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is maximal).

This implies that

$$\mathfrak{k} = \mathfrak{g}_2 \cap \mathfrak{k} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2 \cap \mathfrak{k} \tag{3-3}$$

We shall need the following Lemma.

Lemma 3.4 *Let $\mathfrak{P}$ be a maximal parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{g}_{-n} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ be the associated $\mathbb{Z}$-gradation (as in (3-2)). Then for any $i = 1, \ldots, n$ the space $\mathfrak{g}_i$ is an irreducible $\mathfrak{g}_0$-module.*

Proof of the Lemma. It is well known that the result is true for $i = 1$ (see for example [Ru1]). As $\mathfrak{P}$ is maximal, the center of $\mathfrak{g}_0$ is one dimensional. Let us consider $\mathfrak{g}' = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_{pi}$. This algebra is certainly reductive. If it has a non trivial center, then this center is the one dimensional center of $\mathfrak{g}_0$. But it does not act trivially on the $\mathfrak{g}_i$'s, therefore $\mathfrak{g}'$ is semi-simple.

On the other hand $\mathfrak{P}' = \bigoplus_{p \geq 0} \mathfrak{g}_{pi}$ is a parabolic subalgebra of $\mathfrak{g}'$ which is maximal, again because the center of the Levi factor $\mathfrak{g}_0$ is one dimensional.

Then by the result cited at the beginning of the proof, the action of $\mathfrak{g}_0$ by ad on the first step (here $\mathfrak{g}_i$) is irreducible. $\square$

From the Lemma we can conclude that the representation by ad of $\mathfrak{g}_0$ on $\mathfrak{g}_{-2}$ and $\mathfrak{g}_2$ is irreducible. Recall that $\mathfrak{k}$ is $\mathfrak{g}_0$-stable because $\mathfrak{g}_0 \subseteq \mathfrak{k}$. Therefore one has

$$\mathfrak{g}_2 \cap \mathfrak{k} = \{0\} \quad \text{or} \quad \mathfrak{g}_2$$
$$\mathfrak{g}_{-2} \cap \mathfrak{k} = \{0\} \quad \text{or} \quad \mathfrak{g}_{-2}.$$

The case $\mathfrak{g}_{-2} \cap \mathfrak{k} = \mathfrak{g}_2 \cap \mathfrak{k} = \{0\}$ is impossible because (3-3) would imply that $\mathfrak{k} = \mathfrak{g}_0$ and here $\mathfrak{k}$ is semi-simple.

The case $\mathfrak{g}_2 \cap \mathfrak{k} = \{0\}$ and $\mathfrak{g}_{-2} \cap \mathfrak{k} = \mathfrak{g}_{-2}$ (as well as the symmetric case) is also impossible because it would imply that $\mathfrak{k} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0$ and hence not semi-simple.

Finally $\mathfrak{k} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$ and hence $\mathfrak{p} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$.

Conversely, suppose that one has a $\mathbb{Z}$-gradation of the following type:

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

Then it is easy to see that the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$ and $\mathfrak{p} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$, corresponds to an equi-rank subalgebra. The proof of Theorem 3.1. is completed. $\square$

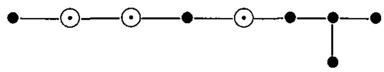
Definition 3.5 Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be an equi-rank algebra. Then any $\mathbb{Z}$-gradation $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ (if $\mathfrak{k} = \mathfrak{g}_0$ is reductive and not semi-simple) or $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ (if $\mathfrak{k} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$ is semi-simple) as described in Theorem 3.1. is called a parabolic realization of the equi-rank algebra. (It follows from the condition $\mathfrak{g}$ is simple that the parabolic subalgebras

$\mathfrak{g}_0 \oplus \mathfrak{g}_1$ or $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ are maximal).

4. The list of equi-rank complex simple Lie algebras

Let us first recall how parabolic subalgebras of a simple $\mathfrak{g}$ may be described by weighted Dynkin diagrams. Each parabolic subalgebra is uniquely defined (up to conjugacy) by a subset Γ of the set Ψ of simple roots. We make the convention that such a data is described by the Dynkin diagram of $\mathfrak{g}$ where the roots in $\Psi \setminus \Gamma$ are circled.

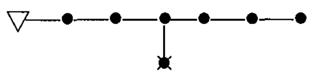
For example consider the following diagram:

D_9 

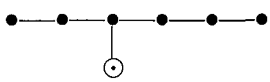
This diagram describes a parabolic subalgebra of $\mathfrak{g} \simeq D_9$ where the Levi factor is of type $A_1 \times A_1 \times D_4 \times \mathbb{C}^3$.

We will now give a diagrammatical description of an elementary operation. The vertex corresponding to $-\omega$ where ω is the highest root will be denoted by $\bigtriangledown$, and the vertex which corresponds to the removed root will be marked by a cross.

For example the diagram



describes an elementary operation in E_7 where the resulting algebra is of type A_7. Remember from the previous paragraph that this A_7-algebra is also obtained by considering the parabolic subalgebra associated to the diagram



More precisely as the circled root has coefficient 2 in the highest root the corresponding parabolic realization is of type

$$E_7 = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

and

$$A_7 = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2.$$

As we explained is the previous paragraph this gives E_7 the structure of an equi-rank algebra where $\mathfrak{k} = A_7$.

Theorem 3.1, now gives a very easy way to classify all equi-rank complex Lie algebras. They split into two distinct families.

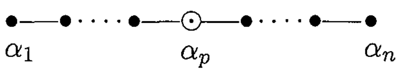
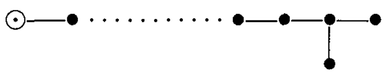
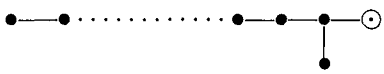
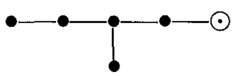
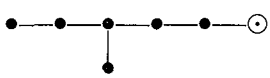
The first family is obtained by taking for $\mathfrak{k}$ the Levi factor of a maximal parabolic subalgebra with an abelian nilradical. This family corresponds to weighted Dynkin diagrams where the unique circled root has coefficient 1 in the highest root. This is the hermitian symmetric case.

The second family corresponds to elementary operations where the removed root (marked by a cross on the extended Dynkin diagram) has coefficient 2 in the highest root.

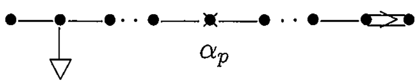
These remarks lead very easily to the following table. In this table we have already taken into account the isomorphisms of the (extended or not) Dynkin diagram.

Table 1

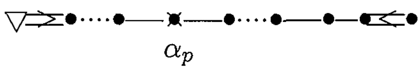
First Type (Hermitian Case)

(1-1)		A_n	$\mathfrak{k} = A_{p-1} \times A_{n-p} \times \mathbb{C}$
(1-2)		B_n	$\mathfrak{k} = B_{n-1} \times \mathbb{C}$
(1-3)		C_n	$\mathfrak{k} = A_{n-1} \times \mathbb{C}$
(1-4)		D_n	$\mathfrak{k} = D_{n-1} \times \mathbb{C}$
(1-5)		D_n	$\mathfrak{k} = A_{n-1} \times \mathbb{C}$
(1-6)		E_6	$\mathfrak{k} = D_5 \times \mathbb{C}$
(1-7)		E_7	$\mathfrak{k} = E_6 \times \mathbb{C}$

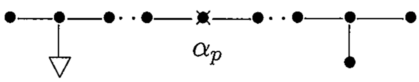
Second Type (Non Hermitian Case)

(2-1) 

B_n $\quad p \geq 2 \quad \mathfrak{k} = D_p \times B_{n-p}$

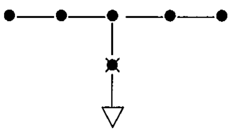
(2-2) 

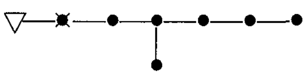
C_n $\quad p \leq n-1 \quad \mathfrak{k} = C_p \times C_{n-p}$

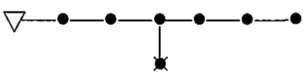
(2-3) 

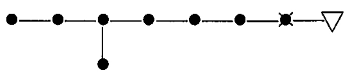
D_n $\quad 2 \leq p \leq n-2$

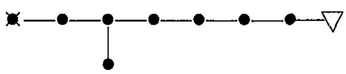
$\mathfrak{k} = D_p \times D_{n-p}$

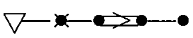
(2-4) 
E_6 $\quad \mathfrak{k} = A_1 \times A_5$

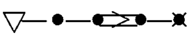
(2-5) 
E_7 $\quad \mathfrak{k} = A_1 \times D_6$

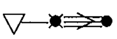
(2-6) 
E_7 $\quad \mathfrak{k} = A_7$

(2-7) 
E_8 $\quad \mathfrak{k} = A_1 \times E_7$

(2-8) 
E_8 $\quad \mathfrak{k} = D_8$

(2-9) 
F_4 $\quad \mathfrak{k} = A_1 \times C_3$

(2-10) 
F_4 $\quad \mathfrak{k} = B_4$

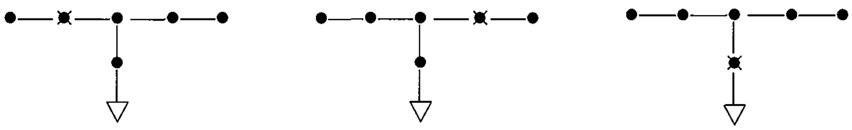
(2-11) 
G_2 $\quad \mathfrak{k} = A_1 \times A_1$

5. Distinct parabolic realizations of the same equi-rank algebra

In this paragraph we will give some examples of distinct parabolic realizations of the same equi-rank algebra.

5.1. This first example will show that the description by means of a maximal parabolic subalgebra and the associate $\mathbb{Z}$-gradation given in Theorem 3.1. may be rather different for the same equi-rank algebra.

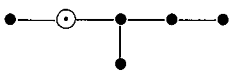
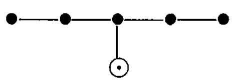
Let us consider the following three elementary operations in E_6



These three diagrams are obviously conjugate under an isomorphism of the extended Dynkin diagram, which induces an isomorphism of E_6. Therefore the corresponding equi-rank algebras are conjugate. In each case $\mathfrak{k} = A_1 \times A_5$ and the corresponding equi-rank Lie algebra is the case (2-4) of the table in the previous paragraph.

The first two diagrams are in fact conjugate under an automorphism of the Dynkin diagram itself and therefore the parabolic realizations basically only differ by a distinct numbering of the simple roots.

Let us consider the parabolic subalgebras corresponding to the first and third case:

(a)  (b) 

Of course in each case the circled root has coefficient 2 in the highest root.

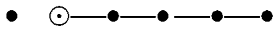
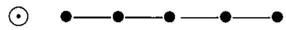
Consider now the associate parabolic realizations:

(a) $E_6 = \mathfrak{g}^1_{-2} \oplus \mathfrak{g}^1_{-1} \oplus \mathfrak{g}^1_0 \oplus \mathfrak{g}^1_1 \oplus \mathfrak{g}^1_2$ (b) $E_6 = \mathfrak{g}^2_{-2} \oplus \mathfrak{g}^2_{-1} \oplus \mathfrak{g}^2_0 \oplus \mathfrak{g}^2_1 \oplus \mathfrak{g}^2_2$

From Theorem 3.1, one knows that

$$\mathfrak{k} \simeq A_1 \times A_5 \simeq \mathfrak{g}^1_{-2} \oplus \mathfrak{g}^1_0 \oplus \mathfrak{g}^1_2 \simeq \mathfrak{g}^2_{-2} \oplus \mathfrak{g}^2_0 \oplus \mathfrak{g}^2_2$$

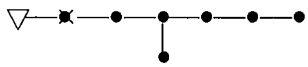
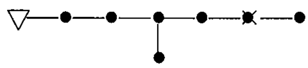
and the corresponding $\mathbb{Z}$-gradations of $\mathfrak{k}$ are described by the following diagrams:

(a)  (b) 

In the (a) case, one has $\dim \mathfrak{g}_2^1 = 5$ ("natural" representation of $A_4 = \mathfrak{sl}_5$) and $\mathfrak{g}_0^1 \simeq \mathbb{C} \times A_1 \times A_4$. Whereas in the (b) case, $\dim \mathfrak{g}_2^2 = 1$ (this is a general result if the removed root in the elementary operation is the one connected to $-\omega$, in fact the subalgebra $\mathfrak{g}_1^2 \oplus \mathfrak{g}_2^2$ is known to be a Heisenberg algebra) and $\mathfrak{g}_0^2 \simeq \mathbb{C} \times A_5$.

Of course the spaces $\mathfrak{g}_1^1$ and $\mathfrak{g}_1^2$ have the same dimension ($= \frac{1}{2} \dim \mathfrak{p} = 20$) but they are distinct in E_6.

5.2. Consider the following two elementary operations in E_7:

(a)  (b) 

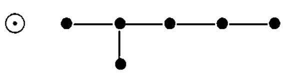
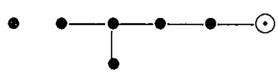
As the removed roots have coefficient 2 in the highest root, these diagrams correspond to parabolic realizations of two equi-rank algebra structures on E_7 which are obviously conjugate under an extended diagram automorphism.

As in 5.1. the associate parabolic realizations

(a) $E_7 = \mathfrak{g}_{-2}^1 \oplus \mathfrak{g}_{-1}^1 \oplus \mathfrak{g}_0^1 \oplus \mathfrak{g}_1^1 \oplus \mathfrak{g}_2^1$ (b) $E_7 = \mathfrak{g}_{-2}^2 \oplus \mathfrak{g}_{-1}^2 \oplus \mathfrak{g}_0^2 \oplus \mathfrak{g}_1^2 \oplus \mathfrak{g}_2^2$

are different.

For example, the corresponding gradation of $\mathfrak{k} \simeq A_1 \times D_6$ are given by the following diagrams

(a)  (b) 

This shows that in the case (a) $\dim \mathfrak{g}_2^1 = 1$, whereas in the case (b) $\dim \mathfrak{g}_2^2 = 10$.

5.3. The proof of the converse part of Theorem 3.1. 2) shows that if one has *any* $\mathbb{Z}$-gradation of the form

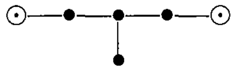
$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \tag{5-3-1}$$

then, even if the parabolic subalgebra $\mathfrak{P} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ *is not* maximal (that is even if (5-3-1) *is not* a parabolic realization), one can define an equi-rank algebra structure on $\mathfrak{g}$ by taking

$$\mathfrak{k} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2 \qquad \mathfrak{p} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1.$$

This situation occurs if $\Psi \setminus \Gamma = \{\alpha_1, \alpha_2\}$ (notations as in paragraph 3) where α_1 and α_2 have coefficient 1 in the highest root.

For example this is the case in the following diagram:



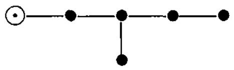
(5-3-2)

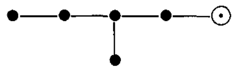
In this case it is easy to see that $\mathfrak{k} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2 \simeq \mathbb{C} \times D_5$ (think in terms of the extended diagram). Hence the Lie algebra $\mathfrak{k}$ is reductive and not semi-simple. Therefore by Theorem 3.1, there exists a parabolic realization of the type

$$E_6 = \mathfrak{g}'_{-1} \oplus \mathfrak{g}'_0 \oplus \mathfrak{g}'_1$$

where $\mathfrak{k} = \mathfrak{g}'_0$.

The only possibilities to get a maximal Levi factor of type $\mathbb{C} \times D_5$ correspond to the following diagrams





which are conjugate.

Of course the vector spaces $\mathfrak{g}_1$ and $\mathfrak{g}'_1$ are isomorphic (of dimension $2^4 = 16$), but $\mathfrak{g}'_1$ is $\mathfrak{k}$-stable (as a representation of D_5 it is the Spin representation) whereas $\mathfrak{g}_1$ is not $\mathfrak{k}$-stable.

Let Ω_1 (resp. Ω_2) be the set of positive roots α which can be expressed in the form $\alpha = \alpha_1 \bmod (\Gamma)$ (resp. $\alpha = \alpha_2 \bmod (\Gamma)$) where α_1 (resp. α_2) is the first (resp. second root) circled in (5-3-2).

Let

$$\mathfrak{g}^1_{\pm 1} = \sum_{\alpha \in \pm\Omega_1} \mathfrak{g}^\alpha \qquad \text{and} \qquad \mathfrak{g}^2_{\pm 1} = \sum_{\alpha \in \pm\Omega_2} \mathfrak{g}^\alpha.$$

Then

$$\mathfrak{g}_1 = \mathfrak{g}^1_1 \oplus \mathfrak{g}^2_1 \qquad \text{and} \qquad \mathfrak{g}_{-1} = \mathfrak{g}^1_{-1} \oplus \mathfrak{g}^2_{-1}$$

and in this picture there are the spaces $\mathfrak{g}^1_1 \oplus \mathfrak{g}^1_{-1}$ and $\mathfrak{g}^2_1 \oplus \mathfrak{g}^2_{-1}$ which are $\mathfrak{k}$-invariant (here $\mathfrak{k} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$).

6. Associated compact and semi-compact dual pairs

Let $\mathfrak{g}_{\mathbb{R}}$ be a real form of $\mathfrak{g}$ whose maximal compact subalgebra $\mathfrak{k}_{\mathbb{R}}$ is a real form of $\mathfrak{k}$. Recall from Proposition 2.1. that such a real form exists and is unique up to conjugacy by $\exp\mathfrak{k}$.

Let us also recall that a *dual pair* in $\mathfrak{g}_{\mathbb{R}}$ is a pair $(\mathfrak{a}_{\mathbb{R}}, \mathfrak{b}_{\mathbb{R}})$ of reductive subalgebras of $\mathfrak{g}_{\mathbb{R}}$, such that $\mathfrak{b}_{\mathbb{R}} = \mathcal{Z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{a}_{\mathbb{R}})$ and $\mathfrak{a}_{\mathbb{R}} = \mathcal{Z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{b}_{\mathbb{R}})$, where $\mathcal{Z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{a})$ (resp. $\mathcal{Z}_{\mathfrak{g}_{\mathbb{R}}}(\mathfrak{b})$) denotes the centralizer of $\mathfrak{a}$ (resp. $\mathfrak{b}$) in $\mathfrak{g}_{\mathbb{R}}$.

In this paragraph, we will describe three families of compact or semi-compact dual pairs (this means that at least one of the two algebras $\mathfrak{a}_{\mathbb{R}}$ and $\mathfrak{b}_{\mathbb{R}}$ is compact) inside $\mathfrak{g}_{\mathbb{R}}$ which are naturally associated to the corresponding equi-rank algebra.

6.1. This first family is associated to the first type of equi-rank algebras (described in the first part of Theorem 3.1, and listed as the first type in Table 1). The obvious remark to make is that as $\mathfrak{k}$ is a Levi subalgebra of $\mathfrak{g}$ then $(\mathfrak{k}, \mathcal{Z}(\mathfrak{k}))$ (where $\mathcal{Z}(\mathfrak{k})$ is the center of $\mathfrak{k}$) is a dual pair in $\mathfrak{g}$. Therefore $(\mathfrak{k}_{\mathbb{R}}, \mathcal{Z}(\mathfrak{k}_{\mathbb{R}}))$ is a dual pair in $\mathfrak{g}_{\mathbb{R}}$. Here $\mathcal{Z}(\mathfrak{k}_{\mathbb{R}})$ is one dimensional because $\mathfrak{k}$ is the Levi factor of a maximal parabolic subalgebra. Hence $\mathcal{Z}(\mathfrak{k}_{\mathbb{R}})$ is isomorphic to $\mathfrak{so}(2)$. This leads easily to the following table.

Table 2

(The numbering of the different cases is as in Table 1, the notations for the real simple Lie algebras are the same as in Helgason's book, Table V, page 518 [He])

(1-1) $(\mathfrak{so}(2) \times \mathfrak{su}(p) \times \mathfrak{su}(n+1-p), \mathfrak{so}(2)) \subset \mathfrak{su}(p, n+1-p)$

(1-2) $(\mathfrak{so}(2) \times \mathfrak{so}(2n-1), \mathfrak{so}(2)) \subset \mathfrak{so}(2, 2n-1)$

(1-3) $(\mathfrak{so}(2) \times \mathfrak{su}(n), \mathfrak{so}(2)) \subset \mathfrak{sp}(n, \mathbb{R})$

(1-4) $(\mathfrak{so}(2) \times \mathfrak{so}(2n-2), \mathfrak{so}(2)) \subset \mathfrak{so}(2, 2n-2)$

(1-5) $(\mathfrak{so}(2) \times \mathfrak{su}(n), \mathfrak{so}(2)) \subset \mathfrak{so}^*(2n)$

(1-6) $(\mathfrak{so}(2) \times \mathfrak{so}(10), \mathfrak{so}(2)) \subset \mathfrak{e}_{6(-14)}$

(1-7) $(\mathfrak{so}(2) \times \mathfrak{e}_{6(-78)}, \mathfrak{so}(2)) \subset \mathfrak{e}_{7(-25)}$

6.2. The second family is a family of semi-compact dual pairs, which is also associated to the first type of equi-rank algebras, or to be more precise to those equi-rank algebras which correspond to hermitian symmetric spaces of *tube type*.

The definition of these pairs is more involved than in the previous case. In fact these pairs are real forms of very basic complex dual pairs which have been used by the author as "primitive objects" from which all complex dual pairs in $\mathfrak{g}$ can be built ([Ru2]).

Here we shall consider only equi-rank algebras of the first type such that the underlying hermitian pair $(\mathfrak{g}_{\mathbb{R}}, \mathfrak{k}_{\mathbb{R}})$ is of *tube type*.

Let $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a parabolic realization of such an algebra. Let H be the element of $\mathfrak{g}_0$ which defines the gradation (i.e. which has eigenvalue i on $\mathfrak{g}_i$, see (3-1)). It is known that the tube type condition is equivalent to the fact that the element $2H$ can be put into an $\mathfrak{sl}_2$-triple. More precisely there exist $X \in \mathfrak{g}_1$ and $Y \in \mathfrak{g}_{-1}$ such that (Y, H, X) is an $\mathfrak{sl}_2$-triple ($[Y, X] = H$, $[H, X] = 2X$, $[H, Y] = -2Y$). For a proof see for example [KW]. Moreover, the elements H, X, Y can be described in the following way. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be a maximal set of strongly orthogonal roots such that the corresponding root spaces $\mathfrak{g}^{\alpha_i}$ are in $\mathfrak{g}_1$. Let $(X_{-\alpha_i}, H_{\alpha_i}, X_{\alpha_i})$ be a classical $\mathfrak{sl}_2$-triple associated to each root. Here H_{α_i} is the co-root of α_i, $X_{\alpha_i} \in \mathfrak{g}^{\alpha_i}$ and $X_{-\alpha_i} \in \mathfrak{g}^{-\alpha_i}$. Then $2H = \sum_{i=1}^{k} H_{\alpha_i}$ and one can take $X = \sum_{i=1}^{k} X_{\alpha_i}$ and $Y = \sum_{i=1}^{k} X_{-\alpha_i}$ (see for example [MRS]).

The algebra $\mathfrak{k}_{\mathbb{R}}$ has a non-trivial center, which is necessarily $i\mathbb{R}H$ because the restriction of the Killing form must be negative definite. As there exists a Cartan subalgebra contained in $\mathfrak{k}_{\mathbb{R}}$, on which all the roots take purely imaginary values, one has for any root $\overline{\mathfrak{g}^{\alpha}} = \mathfrak{g}^{-\alpha}$, where bar denotes the conjugation with respect to $\mathfrak{g}_{\mathbb{R}}$. Therefore there exists a non-zero constant c_i such that $\overline{X_{\alpha_i}} = c_i X_{-\alpha_i}$. As $X_{\alpha_i} \in \mathfrak{p}$, one has $B(X_{-\alpha_i}, X_{\alpha_i}) > 0$, where B is the Killing form. On the other hand, from the invariance of the Killing form, one gets $B(X_{-\alpha_i}, X_{\alpha_i}) = -\frac{1}{2}B(H_{\alpha_i}, H_{\alpha_i})$. Hence $B(\overline{X_{\alpha_i}}, X_{\alpha_i}) = c_i B(X_{-\alpha_i}, X_{\alpha_i}) = -\frac{1}{2}B(H_{-\alpha_i}, H_{\alpha_i})c_i > 0$. This implies $c_i < 0$.

Let $E_{\alpha_i} = |c_i|^{-\frac{1}{2}} X_{\alpha_i}$. Then $\overline{E_{\alpha_i}} = |c_i|^{-\frac{1}{2}}\overline{X_{\alpha_i}}$, and $(-\overline{E_{\alpha_i}}, H_{\alpha_i}, E_{\alpha_i})$ is an $\mathfrak{sl}_2$-triple. Define $\widetilde{X} = \sum_{i=1}^{k} E_{\alpha_i}$ and $\widetilde{Y} = \sum_{i=1}^{k} -\overline{E_{\alpha_i}}$. Then $(\widetilde{Y}, 2H, \widetilde{X})$ is again an $\mathfrak{sl}_2$-triple. This triple generates an $\mathfrak{sl}_2$ subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ which is stable under conjugation with respect to $\mathfrak{g}_{\mathbb{R}}$; therefore, $\mathfrak{a}_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{a}$ is a real form of $\mathfrak{a}$. One can take as a basis of $\mathfrak{a}_{\mathbb{R}}$ the elements

$$U = \sum_{i=1}^{k}(E_{\alpha_i} + \overline{E_{\alpha_i}}), \quad V = -i\sum_{i=1}^{k}(E_{\alpha_i} - \overline{E_{\alpha_i}}), \quad T = i\sum_{i=1}^{k} H_{\alpha_i}.$$

The relations are then $[T,U] = -2V$, $[T,V] = 2U$ and $[V,U] = -2T$. This proves that $\mathfrak{a}_{\mathbb{R}}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. The existence of this $\mathfrak{sl}_2(\mathbb{R})$ subalgebra in the tube type case is due to Koranyi and Wolf ([KW], Prop. 3.12.)

Recall now the following result.

Theorem 6.2.1 ([Ru2], Th. 4.3.) *Let $\mathfrak{b} = \mathcal{Z}_{\mathfrak{g}}(\mathfrak{a})$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. Then $(\mathfrak{a}, \mathfrak{b})$ is a dual pair in $\mathfrak{g}$.*

As $\mathfrak{a}$ is split relative to $\mathfrak{g}_{\mathbb{R}}$, the Lie algebra $\mathfrak{b}$ is also split relative to $\mathfrak{g}_{\mathbb{R}}$, i.e. $\mathfrak{b}_{\mathbb{R}} = \mathfrak{b} \cap \mathfrak{g}$ is a real form of $\mathfrak{b}$.

The following corollary is then straightforward.

Corollary 6.2.2 *The pair $(\mathfrak{a}_{\mathbb{R}}, \mathfrak{b}_{\mathbb{R}})$ is a dual pair in $\mathfrak{g}_{\mathbb{R}}$. The Lie algebra $\mathfrak{a}_{\mathbb{R}}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{b}_{\mathbb{R}}$ is a subalgebra of $\mathfrak{k}_{\mathbb{R}}$, hence is compact.*

This leads to the following table of dual pairs.

Table 3

(The numbering of the different cases is as in Table 1; the notations for the real simple Lie algebras are the same as in Helgason's book, Table V, page 518 [He]; the type of the compact algebra $\mathfrak{b}_{\mathbb{R}}$ is deduced from [Ru2], 6.13. Table 5.).

(1-1) $(n = 2p-1)$ $(\mathfrak{sl}_2(\mathbb{R}), \mathfrak{su}(p)) \subset \mathfrak{su}(p,p)$

(1-2) $(\mathfrak{sl}_2(\mathbb{R}), \mathfrak{so}(2n-2)) \subset \mathfrak{so}(2, 2n-1)$

(1-3) $(\mathfrak{sl}_2(\mathbb{R}), \mathfrak{so}(n)) \subset \mathfrak{sp}(n, \mathbb{R})$

(1-4) $(\mathfrak{sl}_2(\mathbb{R}), \mathfrak{so}(2n-3)) \subset \mathfrak{so}(2, 2n-2)$

(1-5) (n even) $(\mathfrak{sl}_2(\mathbb{R}), \mathfrak{sp}(n)) \subset \mathfrak{so}^*(2n)$

(1-7) $(\mathfrak{sl}_2(\mathbb{R}), \mathfrak{f}_{4(-52)}) \subset \mathfrak{e}_{7(-25)}$

Remark 6.2.3 The dual pair $(\mathfrak{sl}_2(\mathbb{R}), \mathfrak{so}(n)) \subset \mathfrak{sp}(n, \mathbb{R})$ from the preceeding table was of great historical importance. It gives rise to the correspondence between spherical harmonics of $\mathfrak{so}(n)$ and some highest weight modules (discrete series) of $\mathfrak{sl}_2(\mathbb{R})$. (see [H], and [KV] for a more general situation). This pair belongs also to another family of dual pairs of type $(\mathfrak{sl}_2(\mathbb{R}), \mathfrak{so}(p,q)) \subset \mathfrak{sp}(p+q, \mathbb{R})$ for which the so-called Howe correspondence was investigated by S. Rallis and G. Schiffmann [RS].

6.3. The dual pairs in the third family are compact dual pairs associated to the second type of equi-rank algebras.

Recall the following result.

Proposition 6.3.1 ([Ru2], Prop. 5.15.) *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two semi-simple subalgebras of $\mathfrak{g}$ such that $\mathfrak{a} \cap \mathfrak{b} = \{0\}$ and such that $\mathfrak{a} \times \mathfrak{b}$ is a regular maximal subalgebra of $\mathfrak{g}$ (this implies that $\mathfrak{a} \times \mathfrak{b}$ is obtained by a single elementary operation). Then $(\mathfrak{a}, \mathfrak{b})$ is a dual pair in $\mathfrak{g}$.*

Remark 6.3.2 As noted in the previous proposition, all maximal semisimple regular subalgebras are obtained by a single elementary operation, the converse is true only if the coefficient of the root which is removed has a prime coefficient in the highest root (see the remark following Theorem 3.5, in [T], exercise 4, par. 4, p. 229 in [Bo1] and exercise 2, par. 3, p. 222 in [Bo2]).

Corollary 6.3.3 *In all equi-rank algebras of the second type where the subalgebra $\mathfrak{k}$ can be written as a product $\mathfrak{k} = \mathfrak{a} \times \mathfrak{b}$ with $\mathfrak{a}$ and $\mathfrak{b}$ simple algebras, the pair $(\mathfrak{a}, \mathfrak{b})$ is a dual pair in $\mathfrak{g}$.*

Proof. As explained in paragraph 3 and 4, the equi-rank algebras of the second type are obtained by a single elementary operation where the removed root has coefficient 2 in the highest root. $\square$

Corollary 6.3.4 *Let $\mathfrak{g}$ be an equi-rank algebra of the second type where $\mathfrak{k}$ can be written as a product $\mathfrak{k} = \mathfrak{a} \times \mathfrak{b}$ with $\mathfrak{a}$ and $\mathfrak{b}$ simple algebras. Let $\mathfrak{a}_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{a}$ and $\mathfrak{b}_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{b}$, then $(\mathfrak{a}_{\mathbb{R}}, \mathfrak{b}_{\mathbb{R}})$ is a dual pair in $\mathfrak{g}_{\mathbb{R}}$.*

This leads easily to the following table.

Table 4

(The numbering of the different cases is as in Table 1; the notations for the real simple Lie algebras are the same as in Helgason's book, Table V, page 518 [He].)

(2-1) $p \geq 2$ $(\mathfrak{so}(2p), \mathfrak{so}(2n-2p+1)) \subset \mathfrak{so}(2p, 2n-2p+1)$

(2-2) $p \leq n-1$ $(\mathfrak{sp}(p), \mathfrak{sp}(n-p)) \subset \mathfrak{sp}(p, n-p)$

(2-3) $2 \leq p \leq n-2$ $(\mathfrak{so}(2p), \mathfrak{so}(2n-2p)) \subset \mathfrak{so}(2p, 2n-2p)$

(2-4) $(\mathfrak{su}(2), \mathfrak{su}(6)) \subset \mathfrak{e}_{6(2)}$

(2-5) $(\mathfrak{su}(2), \mathfrak{so}(12)) \subset \mathfrak{e}_{7(-5)}$

(2-7) $(\mathfrak{su}(2), \mathfrak{e}_{7(-133)}) \subset \mathfrak{e}_{8(-24)}$

(2-9) $(\mathfrak{su}(2), \mathfrak{sp}(3)) \subset \mathfrak{f}_{4(4)}$

(2-11) $(\mathfrak{su}(2), \mathfrak{su}(2)) \subset \mathfrak{g}_{2(2)}$

Remark 6.3.5 In the cases from the preceeding table where $\mathfrak{k}_{\mathbb{R}}$ has an $\mathfrak{su}(2)$ factor, Gross and Wallach have studied the discrete series of some groups corresponding to $\mathfrak{g}_{\mathbb{R}}$ ([GW]).

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IRMA et Département de Mathématiques
Université Louis Pasteur et C.N.R.S.
7, rue René Descartes
67084 Strasbourg Cedex
France
E-mail: rubenth@math.u-strasbg.fr