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On non-symmetric relative difference sets

Yutaka HIRAMINE

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Abstract. Let D be a (m, u, k, λ)-difference set in a group G relative to a subgroup U of G. We say D is symmetric if $D^{(-1)}$ is also a (m, u, k, λ)-difference set. By a result of [7] D is symmetric if U is a normal subgroup of G. In general, D is non-symmetric when U is not normal in G. In this paper we study a condition under which D is symmetric and show that if D is semiregular then D is symmetric if and only if the dual of $\mathrm{dev}(D)$ is a divisible design. We also give a modification of Davis' product construction of relative difference sets and as an application we give a class of non-symmetric semiregular relative difference sets.

Key words: relative difference set, non-symmetric transversal designs.

1. Introduction

Let G be a group of order mu and U a subgroup of G of order u. A k-subset D of G is called a (m, u, k, λ)-*difference set* in G with respect to U if the list of quotients $d_1 d_2^{-1}$ with $d_1, d_2 \in D$ $(d_1 \neq d_2)$ contains each element in $G \setminus U$ exactly λ times and no element in U. The definition yields the group ring equation

$$DD^{(-1)} = k + \lambda(G - U) \tag{1}$$

where we identify a subset X of G with a group ring element $\hat{X} = \sum_{x \in X} x \in \mathbb{C}[G]$ and set $X^{(-1)} = \sum_{x \in X} x^{-1}$. D is also called *a relative difference set* relative to U and U is called a *forbidden subgroup*. By definition $m \geq k$ and $k^2 = k + \lambda(mu - u)$. We note that $D^{(-1)}$ is not always a relative difference set. For a (m, u, k, λ)-difference set D in a group G relative to U, $\mathrm{dev}(D)$ $(= (\mathbb{P}, \mathbb{B}))$ is an incidence structure with a set of points $\mathbb{P} = \{g \mid g \in G\}$ and a set of blocks $\mathbb{B} = \{Dg \mid g \in G\}$. Then $\mathrm{dev}(D)$ is a (m, u, k, λ)-divisible design ([7]). If $m = k$ then $(m, u, k, \lambda) = (u\lambda, u, u\lambda, \lambda)$ and D is said to be *semiregular*.

A (m, u, k, λ)-difference set D is called *symmetric* if $D^{(-1)}$ is also a (m, u, k, λ)-difference set. In Section 2 we study a semiregular relative

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difference set D and show that D is symmetric if and only if the dual of $\mathrm{dev}(D)$ is a divisible design (Corollary 2.7).

In Sections 3 and 4 we give a construction for relative difference sets D such that $\mathrm{dev}(D)$ is non-symmetric. To construct such difference sets we present the following lemma on products of semiregular relative difference sets, which is a modification of Theorem 2.1 of [4] or Result 2.1 of [8].

Lemma 3.1 *Let $X = G \times H$ be a group, where G is a group of order $u^2\lambda$ and H is a group of order $u\lambda'$. Let D be a $(u\lambda, u, u\lambda, \lambda)$-difference set in G relative to a subgroup U of G of order u and let C be a $(u\lambda', u, u\lambda', \lambda')$-difference set in $G' = U \times H$ relative to U. Then*
(i) *CD is a $(u^2\lambda\lambda', u, u^2\lambda\lambda', u\lambda\lambda')$-difference set in X relative to $U \times 1$.*
(ii) *CD is symmetric if and only if D is symmetric.*

We note that U is not assumed to be normal in G in Lemma 3.1. Roughly speaking, Lemma 3.1(ii) implies that a non-symmetric and a splitting semiregular relative difference sets that share a forbidden subgroup give us another non-symmetric one. By a recursive construction applying Lemma 3.1 we obtain a class of non-symmetric semiregular relative difference sets (see Theorem 3.2 and Proposition 4.4).

2. Divisible designs and relative difference sets

Let D be a relative difference set in a group G relative to U. Then $\mathrm{dev}(D)$ is a divisible design. However, the dual of $\mathrm{dev}(D)$ is not always a divisible design. In this section we study a condition under which the dual of $\mathrm{dev}(D)$ is a divisible design when D is semiregular.

Definition 2.1 An incidence structure $(\mathbb{P}, \mathbb{B})$ is called a *square (m, u, k, λ)-divisible design* if the following conditions are satisfied.
(i) $|\mathbb{P}| = |\mathbb{B}| = mu$.
(ii) There exists a partition $\mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2 \cup \cdots \cup \mathbb{P}_m$ of $\mathbb{P}$ such that $|\mathbb{P}_1| = \cdots = |\mathbb{P}_m| = u$ and for any distinct points $p, q \in \mathbb{P}$ the number of blocks $B \in \mathbb{B}$ containing p and q is 0 if $p, q \in \mathbb{P}_i$ for some $i \in \{1, \ldots, m\}$ and λ otherwise. (Each P_i is called a *point class* of $(\mathbb{P}, \mathbb{B})$.)
(iii) $|B| = k$ $(\forall B \in \mathbb{B})$.

Counting all triples (p, q, B) $(p, q \in \mathbb{B}, p \neq q \in \mathbb{P}, B \in \mathbb{B})$ in two ways we obtain the following fundamental equation.

$$k(k-1) = \lambda(mu - u) \tag{2}$$

Definition 2.2 (i) A square (m, u, k, λ)-divisible design is said to be *symmetric* if its dual is also a square (m, u, k, λ)-divisible design. In other words, there is a partition $\mathbb{B} = \mathbb{B}_1 \cup \cdots \cup \mathbb{B}_m$ of $\mathbb{B}$ such that for any two distinct blocks $B, C \in \mathbb{B}$

$$|B \cap C| = \begin{cases} 0 & \text{if } B, C \in \mathbb{B}_i \text{ for some } i \in \{1, \ldots, m\}, \\ \lambda & \text{otherwise.} \end{cases}$$

(ii) A (m, u, k, λ)-difference set D is said to be *symmetric* if $D^{(-1)}$ is also a (m, u, k, λ)-difference set.

Remark 2.3 (i) In Theorem 6.2 of [3], W.S. Connor showed that $(\mathbb{P}, \mathbb{B})$ is symmetric whenever $k > u\lambda$ and $(k, \lambda) = 1$.
(ii) If a (m, u, k, λ)-difference set D in a group G relative to U satisfies $DD^{(-1)} = D^{(-1)}D$, then $\mathrm{dev}(D)$ is symmetric.
(iii) By Jungnickel's result in [7], $DD^{(-1)} = D^{(-1)}D$ whenever $G \rhd U$. Hence, if $G \rhd U$, then D is symmetric.

Definition 2.4 A square (m, u, k, λ)-divisible design $(\mathbb{P}, \mathbb{B})$ is called *a transversal design* and denoted by $\mathrm{TD}_\lambda[k; u]$ if $|B \cap \mathbb{P}_i| = 1$ for any $B \in \mathbb{B}$ and any point class P_i of $(\mathbb{P}, \mathbb{B})$.

Hence, a square (m, u, k, λ)-divisible design is a transversal design if and only if $k = m$ $(= u\lambda)$.

Lemma 2.5 *Let $\mathcal{D}$ be a transversal design $\mathrm{TD}_\lambda[u\lambda; u]$. If the dual of $\mathcal{D}$ is a (m, n, k, μ)-divisible design for some $m, n, k, \mu \in \mathbb{N}$, then $(m, n, k, \mu) = (u\lambda, u, u\lambda, \lambda)$ and $\mathcal{D}$ is symmetric.*

Proof. Clearly $mn = u^2\lambda$, $k = u\lambda$. By Theorem 3 of [2], $k \geq n\mu$. Hence

$$u\lambda \geq n\mu \tag{3}$$

By (2), $u\lambda(u\lambda - 1) = k(k-1) = \mu(u^2\lambda - n) = u\lambda(u\mu) - \mu n$. From this $\mu n \equiv 0 \pmod{u\lambda}$. Applying (3), we have $u\lambda = n\mu$ and so $(n\mu)(n\mu - 1) = \mu(mn - n) = n\mu(m-1)$. It follows that $m = n\mu$. As $n^2\mu = mn = u^2\lambda$, we have $n = u$ and so $\lambda = \mu$. Therefore the lemma holds. □

A (m, u, k, λ)-difference set is called *semiregular* if $m = k$ (or equivalently $m = k = u\lambda$). Then, clearly $|G| = u^2\lambda$. In this case $\mathrm{dev}(D)$ is a transversal design $\mathrm{TD}_\lambda[u\lambda; u]$.

As an application of Lemma 2.5, we can show the following.

Proposition 2.6 *Let D be a semiregular relative difference set in a group G relative to a subgroup U of G. Then the following conditions are equivalent.*

(i) $\mathrm{dev}(D)$ *is symmetric.*
(ii) $D^{(-1)}$ *is a relative difference set in G relative to a subgroup of G.*
(iii) *The dual of* $\mathrm{dev}(D)$ *is a divisible design.*

Proof. Set $(\mathbb{P}, \mathbb{B}) = \mathrm{dev}(D)$ and assume (i). Then, there exists a partition $\mathbb{B} = \mathbb{B}_1 \cup \cdots \cup \mathbb{B}_{u\lambda}$ of $\mathbb{B}$ such that for any two distinct blocks B, $C \in \mathbb{B}$,

$$|B \cap C| = \begin{cases} 0 & \text{if } B,\, C \in \mathbb{B}_i \text{ for some } i \in \{1,\, \ldots,\, u\lambda\}, \\ \lambda & \text{otherwise.} \end{cases} \tag{4}$$

Set $\mathbb{B}_1 = \{Dg_1, Dg_2, \ldots, Dg_u\}$, where $g_1 = 1$. As $Dg_i \cap Dg_j = \emptyset$ for any distinct i, $j \in \{1, 2, \ldots, u\}$, for each $\mathbb{B}_k$ there is an element $g \in G$ so that $\mathbb{B}_k = \{Dg_1g, Dg_2g, \ldots, Dg_ug\}$.

We note that

$$Dg_i \cap Dg_j = \emptyset \iff \{(d_1, d_2) \mid d_1, d_2 \in D, d_1g_i = d_2g_j\} = \emptyset.$$

Hence

$$Dg_i \cap Dg_j = \emptyset \iff \{(d_1, d_2) \mid d_1, d_2 \in D, d_1^{-1}d_2 = g_ig_j^{-1}\} = \emptyset. \tag{5}$$

Set $V = \{g_1(= 1), g_2, \ldots, g_u\}$. Let g_i, $g_j \in V$. Then, by (5), $Dg_ig_j^{-1} \cap D = \emptyset$. Hence $Dg_ig_j^{-1} \in \mathbb{B}_1$ and so $g_ig_j^{-1} \in V$. Thus V is a subgroup of G of order u. Assume $a \in G \setminus V$. Then $Da \notin \mathbb{B}_1$. As $D \in \mathbb{B}_1$, we have $|D \cap Da| = \lambda$ by (4). Then $|\{(d_1, d_2) \mid a = d_1^{-1}d_2\}| = \lambda$. Thus $D^{(-1)}D = u\lambda + \lambda(G - V)$. Therefore (ii) holds. Clearly (ii) implies (iii). By Lemma 2.5, (iii) implies (i). □

As a corollary of Proposition 2.6, we have

Corollary 2.7 *A semiregular relative difference set D is symmetric if and only if the dual of* $\mathrm{dev}(D)$ *is a divisible design.*

Under the above assumption, $DD^{(-1)} \neq D^{(-1)}D$ in general. To our knowledge, transversal designs obtained from semiregular relative difference sets and previously known were symmetric. In Section 3 and 4 we will give examples of semiregular relative difference sets D with $\mathrm{dev}(D)$ non-symmetric. Then they give us examples of non-symmetric semiregular relative difference sets.

Concerning the case $m > k$ we would like to ask the following.

Question 2.8 Let D be a (m, u, k, λ)-difference set in a group G such that $m > k$. Is D symmetric whenever the dual of $\mathrm{dev}(D)$ is a divisible design ?

3. Non-symmetric relative difference sets

In this section we construct non-symmetric relative difference sets. To do this we need the following lemma.

Lemma 3.1 *Let $X = G \times H$ be a group, where G is a group of order $u^2\lambda$ and H is a group of order $u\lambda'$. Let D be a $(u\lambda, u, u\lambda, \lambda)$-difference set in G relative to a subgroup U of G of order u and let C be a $(u\lambda', u, u\lambda', \lambda')$-difference set in $G' = U \times H$ relative to U. Then*

(i) *CD is a $(u^2\lambda\lambda', u, u^2\lambda\lambda', u\lambda\lambda')$-difference set in X relative to U.*

(ii) *CD is symmetric if and only if D is symmetric.*

Proof. Let $c_1, c_2 \in C$ and $d_1, d_2 \in D$ and assume $c_1d_1 = c_2d_2$. Then $c_1^{-1}c_2 = d_1d_2^{-1} \in UH \cap G = U$. Thus $d_1 = d_2$ and so $c_1 = c_2$. Therefore CD is a subset of X.

Let S and T be subsets of G and H, respectively. We identify S and T with $S \times \{1\}$ $(\subset X)$ and $\{1\} \times T$ $(\subset X)$, respectively. Then, by assumption, the following hold.

$$DD^{(-1)} = u\lambda + \lambda(G - U) \tag{6}$$

$$CC^{(-1)} = u\lambda' + \lambda'(UH - U) \tag{7}$$

$$G = UD, \quad UC = UH \tag{8}$$

Hence, substituting (6) and (7) we have

$$\begin{aligned}(CD)(CD)^{(-1)} &= C(DD^{(-1)})C^{(-1)}\\ &= C(u\lambda + \lambda(G - U))C^{(-1)}\\ &= u\lambda CC^{(-1)} + \lambda CGC^{(-1)} - \lambda CUC^{(-1)}.\end{aligned}$$

As $C, U \subset UH$ and $U \lhd UH$, we have $CU = UC$. Similarly $GC = CG$. It follows that

$$\begin{aligned}(CD)(CD)^{(-1)} &= u\lambda(u\lambda' + \lambda'(UH - U)) + \lambda GCC^{(-1)} - \lambda UCC^{(-1)}\\ &= u^2\lambda'\lambda + u\lambda'\lambda UH - u\lambda'\lambda U\\ &\quad + \lambda G(u\lambda' + \lambda' UH - \lambda' U) - \lambda U(u\lambda' + \lambda' UH - \lambda' U)\\ &= u^2\lambda'\lambda + u\lambda'\lambda(X - U).\end{aligned}$$

Thus we have (i).

Since $UH \triangleright U$, we obtain $C^{(-1)}C = CC^{(-1)} = u\lambda' + \lambda' UH - \lambda' U$. Hence $(CD)^{(-1)}CD = D^{(-1)}(CC^{(-1)})D = D^{(-1)}(u\lambda' + \lambda' UH - \lambda' U)D$. By (8), we have

$$(CD)^{(-1)}CD = u\lambda' D^{(-1)}D + u\lambda'\lambda X - u\lambda'\lambda G. \tag{9}$$

Assume CD is symmetric. Then $(CD)^{(-1)}CD = u^2\lambda'\lambda + u\lambda'\lambda(X - V)$ for a subgroup V of X of order u. By (9), $u\lambda' D^{(-1)}D - u\lambda'\lambda G = u^2\lambda'\lambda - u\lambda'\lambda V$. Thus $D^{(-1)}D = u\lambda + \lambda(G - V)$. In particular, V is a subgroup of G of order u and so D is symmetric. Conversely, assume D is symmetric. Then $D^{(-1)}D = u\lambda + \lambda(G - V)$ for a subgroup V of G of order u. Then, by (9), $(CD)^{(-1)}CD = u\lambda'(u\lambda + \lambda(G - V)) + u\lambda'\lambda X - u\lambda'\lambda G = u^2\lambda'\lambda + u\lambda'\lambda(X - V)$. Therefore CD is symmetric. Thus we have (ii). □

We note that Lemma 3.1(i) is a modification of Result 2.4 of [8], where N is assumed to be normal in G.

We now prove the following theorem on a recursive construction of non-symmetric semiregular relative difference sets.

Theorem 3.2 *Let D be a $(u\lambda_0,\, u,\, u\lambda_0,\, \lambda_0)$-difference set in a group G relative to a subgroup U of G. Let H_i be a group of order $u\lambda_i$ and assume the existence of a splitting $(u\lambda_i,\, u,\, u\lambda_i,\, \lambda_i)$-difference set, say D_i, in $U \times H_i$ relative to $U \times 1$ for each $i \in \{1, 2, \ldots, n-1\}$. Set $\lambda = \lambda_0\lambda_1\lambda_2 \cdots \lambda_{n-1}$. Then,*

(i) *$D_1D_2 \cdots D_{n-1}D$ is a $(u^n\lambda,\, u,\, u^n\lambda,\, u^{n-1}\lambda)$-difference set in $G \times H_{n-1} \times H_{n-2} \times \cdots \times H_1$ relative to $U \times 1 \times \cdots \times 1$.*

(ii) *$D_1D_2 \cdots D_{n-1}D$ is non-symmetric if and only if D is non-symmetric.*

Proof. Set $X = G \times H_{n-1}$. Since $U \times H_{n-1}$ contains a $(u\lambda_{n-1}, u, u\lambda_{n-1}, \lambda_{n-1})$-difference set D_{n-1} relative to $U \times 1$, applying Lemma 3.1 we have that $D_{n-1}D$ is a $(u^2\lambda\lambda_{n-1},\, u,\, u^2\lambda\lambda_{n-1},\, u\lambda\lambda_{n-1})$-difference set in X relative to $U \times 1$.

Set $X' = (G \times H_{n-1}) \times H_{n-2}$ and let ψ be the natural projection from $U \times H_{n-2}$ to X'. Then we can regard D_{n-2} as a $(u\lambda_{n-2}, u, u\lambda_{n-2}, \lambda_{n-2})$-difference set relative to $(U \times 1) \times 1$. Applying Lemma 3.1 again, we obtain a $(u^3\lambda\lambda_{n-1}\lambda_{n-2}, u, u^3\lambda\lambda_{n-1}\lambda_{n-2}, u^2\lambda\lambda_{n-1}\lambda_{n-2})$-difference set $C_{n-2}C_{n-1}D$ in X' relative to $U \times 1 \times 1$. Repeating the procedure we have the theorem. $\square$

4. Examples of non-symmetric relative difference sets

We denote by m^* the square free part of a positive integer m.

Proposition 4.1 *Assume the existence of a splitting* $(3\lambda, 3, 3\lambda, \lambda)$*-difference set. Then*
(i) $p \equiv 1 \pmod 3$ *for each prime divisor* p $(\neq 3)$ *of* λ^*.
(ii) *The congruence* $x^2 \equiv -12 \pmod{4\lambda^*}$ *has a solution in integers.*

Proof. Let D be a $(3\lambda, 3, 3\lambda, \lambda)$-difference set in a group $G = H \times U$ relative to $U \simeq \mathbb{Z}_3$. Let χ be a linear character of G such that $\chi\mid_H$ is principal, while $\chi\mid_U$ is not. Then, as $U \simeq \mathbb{Z}_3$, $\chi(D) = a + b\omega + c\omega^2$, $a + b + c = |D| = 3\lambda$ for non-negative integers a, b, c. Hence $\chi(D)\overline{\chi(D)} = a^2 + b^2 + c^2 - ab - bc - ca$. On the other hand, $\chi(D)\overline{\chi(D)} = 3\lambda$ by (1). From this, $(2a + b - 3\lambda)^2 + 3(b - \lambda)^2 = 4\lambda$. Thus an equation $x^2 + 3y^2 = 4\lambda$ has an integral solution $(x, y) = (2a + b - 3\lambda, b - \lambda)$. In particular, $2 \nmid \lambda^*$. By Theorem 7 in Section 7.6 of Chapter 2 in [1], the congruence $x^2 \equiv -12 \pmod{4\lambda^*}$ is solvable.

Let p $(\neq 3)$ be an odd prime dividing λ^*. Assume $p \equiv 2 \pmod 3$. Then, by Theorem 2 in Section 2.2 of Chapter 5 in [1], (p) is a prime ideal in the ring of algebraic integers in $\mathbb{Q}(\omega)$. This is contrary to the fact that $\chi(D)\overline{\chi(D)} = 3\lambda$. Thus $p \equiv 1 \pmod 3$. Therefore the proposition holds. $\square$

Example 4.2 By Proposition 4.1, there are no splitting $(3\lambda, 3, 3\lambda, \lambda)$-difference sets for $\lambda = 2, 5, 6, 8, 10, 11$. On the other hand, here exist splitting a $(3\lambda, 3, 3\lambda, \lambda)$-difference set for $\lambda = 1, 3, 4, 7, 9$ (for $\lambda = 7$, see [9]). Also there exists a splitting $(3 \cdot 2^{2s}3^t, 3, 3 \cdot 2^{2s}3^t, 2^{2s}3^t)$-difference set for any $s, t \geq 0$ by Corollary 4.4 of [5].

We now show that a relative difference set in $G = S_3 \times \mathbb{Z}_6$ construcred in [6] is non-symmetric.

Example 4.3 Let $G = \langle a, b, c \mid a^3 = b^2 = c^6 = 1, b^{-1}ab = a^{-1}, ac = ca, bc = cb\rangle$ and set $D = \{1, c, c^2, c^3, a, ac, b, a^2bc^5, abc^4, a^2bc, bc^4, abc\}$. Then D is a (12, 3, 12, 4)-difference set relative to $U = \langle ac^2\rangle \simeq Z_3$. We can easily check that $DD^{(-1)} = 12 + 4(G - U)$, while $D^{(-1)}D = 12 + 4a + 4a^2 + 4b + 4ab + 4a^2b + 4c + 3ac + 5a^2c + 3bc + 5abc + 4a^2bc + 4c^2 + 2ac^2 + 2a^2c^2 + 4bc^2 + 4abc^2 + 4a^2bc^2 + 4c^3 + 4ac^3 + 4a^2c^3 + 6bc^3 + 2abc^3 + 4a^2bc^3 + 4c^4 + 2ac^4 + 2a^2c^4 + 4bc^4 + 4abc^4 + 4a^2bc^4 + 4c^5 + 5ac^5 + 3a^2c^5 + 3bc^5 + 5abc^5 + 4a^2bc^5$. Thus $D^{(-1)}$ is not a relative difference set. Thus D is a non-symmetric relative difference set.

By Theorem 3.2 and Examples 4.2 and 4.3 we have the following.

Proposition 4.4 *There exists a non-symmetric* $(2^2 3^{m+1}\lambda, 3, 2^2 3^{m+1}\lambda, 2^2 3^m\lambda)$*-difference set D for any* $\lambda = 2^{2sm_2} 3^{tm_1} 7^{m_2}$, m $(\geq m_1 + m_2)$ *and* s, t $(m_1, m_2, s, t \in \mathbb{N} \cup \{0\})$. *Under this condition,* $\mathrm{dev}(D)$ *is a non-symmetric* $TD_{2^2 3^m\lambda}[2^2 3^{m+1}\lambda; 3]$.

Example 4.5 Let $G = \langle a, b\rangle \times \langle c\rangle \simeq S_3 \times \mathbb{Z}_6$, where $a^3 = b^2 = 1$, $bab = a^{-1}$ and let $H = \langle d\rangle \times \langle e\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_6$. Set $X = G \times H$. Then one can verify that $C = \{1, e, e^2, e^3, ac^2e^4, ac^2e^5, a^2c^4d, de, ac^2de^2, a^2c^4de^3, ac^2de^4, de^5\}$ is a (12, 3, 12, 4)-difference set in $\langle a^2c^4d\rangle \times \langle e\rangle \simeq \mathbb{Z}_6 \times \mathbb{Z}_6$ relative to $\langle ac^2\rangle \simeq \mathbb{Z}_3$. By Example 4.3, $D = \{1, c, c^2, c^3, a, ac, b, a^2bc^5, abc^4, a^2bc, bc^4, abc\}$ is a non-symmetric (12, 3, 12, 4)-difference set in G relative to $\langle ac^2\rangle \simeq \mathbb{Z}_3$. Applying Lemma 3.1, CD is a non-symmetric (144, 3, 144, 48)-difference set.

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Department of Mathematics
Faculty of Education
Kumamoto University
Kurokami, Kumamoto, Japan
E-mail: hiramine@gpo.kumamoto-u.ac.jp