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Commensurability between once-punctured torus groups and once-punctured Klein bottle groups

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Abstract. The once-punctured torus and the once-punctured Klein bottle are topologically commensurable, in the sense that both of them are doubly covered by the twice-punctured torus. In this paper, we give a condition for a faithful type-preserving $\mathrm{PSL}(2, \mathbf{C})$-representation of the fundamental group of the once-punctured Klein bottle to be "commensurable" with that of the once-punctured torus. We also show that such a pair of $\mathrm{PSL}(2, \mathbf{C})$-representations extend to a representation of the fundamental group of a common quotient orbifold. Finally, we give an application to the study of the Ford domains.

1. Introduction

The combinatorial structures of the Ford domains of quasi-fuchsian once-punctured torus groups are characterized by Jorgensen [6] (cf. [1]). It is natural to expect that there is an analogue of Jorgensen's theory for quasi-fuchsian once-punctured Klein bottle groups, because its deformation space also has complex dimension 2. In fact, for fuchsian once-punctured Klein bottle groups, we can completely describe the structures of their Ford domains (see [3, Theorem 5.7]). However, as shown in [3, Section 6], the Ford domains of general quasi-fuchsian once-punctured Klein bottle groups seem to have much more complicated structures than those of quasi-fuchsian once-punctured torus groups.

On the other hand, the once-punctured torus, $\Sigma_{1,1}$, and the once-punctured Klein bottle, $N_{2,1}$, are topologically commensurable, in the sense that they are doubly covered by the twice-punctured torus, $\Sigma_{1,2}$. Thus we can introduce a notion of commensurability between type-preserving $\mathrm{PSL}(2, \mathbf{C})$-representations of $\pi_1(\Sigma_{1,1})$ and $\pi_1(N_{2,1})$ (see Definitions 2.1 and 2.2). Moreover, we can easily observe that mutually commensurable discrete $\mathrm{PSL}(2, \mathbf{C})$-representations of $\pi_1(\Sigma_{1,1})$ and $\pi_1(N_{2,1})$ have the same Ford domain (see Proposition 6.3).

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Hence a natural problem now arises: which type-preserving $\mathrm{PSL}(2, \mathbf{C})$-representation of $\pi_1(N_{2,1})$ is commensurable with a type-preserving $\mathrm{PSL}(2, \mathbf{C})$-representation of $\pi_1(\Sigma_{1,1})$ (see Problem 2.3)?

The main purpose of this paper is to give a partial answer to this problem (see Theorem 5.1). This enable us to understand the Ford domains of the Kleinian groups obtained as the images of discrete faithful type-preserving representations of $\pi_1(N_{2,1})$ which are commensurable with those of $\pi_1(\Sigma_{1,1})$ (see Example 6.4).

The rest of this paper is organized as follows. In Section 2, we recall relation among the once-punctured torus, the once-punctured Klein bottle, the twice-punctured torus and their quotient orbifolds $\mathcal{O}_{\Sigma_{1,1}}$, $\mathcal{O}_{N_{2,1}}$, $\mathcal{O}_{\Sigma_{1,2}}$. We also recall type-preserving representations of their fundamental groups (see Definition 2.1). Then we introduce the concept of commensurability between type-preserving representations of their fundamental groups (see Definition 2.2). In Section 3, we recall the definition and basic properties of "geometric" generator systems of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ which are called elliptic generator triples. We also introduce geometric generator systems of $\pi_1(\mathcal{O}_{N_{2,1}})$, which are also called elliptic generator triples, and describe their basic properties. In Section 4, we study type-preserving representations. In particular, we recall the definition of the complex probabilities of type-preserving representations of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ and describe a conceptual geometric construction of a type-preserving representation from a given complex probability (see Proposition 4.8). We also introduce the concept of complex probabilities of type-preserving representations of $\pi_1(\mathcal{O}_{N_{2,1}})$ and establish a similar geometric construction of a type-preserving representation from a given complex probability (see Proposition 4.11). At the end of Section 4, we study type-preserving $\mathrm{PSL}(2, \mathbf{C})$-representations of $\pi_1(\Sigma_{1,2})$ extending to those of $\pi_1(\Sigma_{1,1})$ or $\pi_1(N_{2,1})$ (see Lemma 4.15). In Section 5, we give a partial answer to the commensurability problem for representations of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ and $\pi_1(\mathcal{O}_{N_{2,1}})$ in terms of complex probabilities (see Theorem 5.1). We also study what happens if we drop the assumption in Theorem 5.1. In Section 6, we give an application to the study of Ford domains.

2. Once-punctured torus, once-punctured Klein bottle and their friends

Let $\Sigma_{1,1}$, $N_{2,1}$ and $\Sigma_{1,2}$, respectively, be the once-punctured torus, the once-punctured Klein bottle and the twice-punctured torus. Their fundamental groups have the following presentations:

$$\pi_1(\Sigma_{1,1}) = \langle X_1, X_2 \,|\, - \rangle,$$

$$\pi_1(N_{2,1}) = \langle Y_1, Y_2 \,|\, - \rangle,$$

$$\pi_1(\Sigma_{1,2}) = \langle Z_1, Z_2, Z_3 \,|\, - \rangle.$$

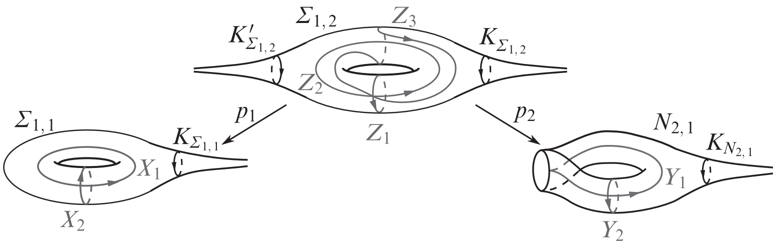

Fig. 1. $\Sigma_{1,1}$, $N_{2,1}$ and $\Sigma_{1,2}$.

Here the generators are represented by the based simple loops in Figure 1. It should be noted that Y_2 is represented by the unique non-separating simple orientable loop in $N_{2,1}$. Set $K_{\Sigma_{1,1}} = [X_1, X_2] = X_1 X_2 X_1^{-1} X_2^{-1}$, $K_{N_{2,1}} = (Y_1 Y_2 Y_1^{-1} Y_2)^{-1}$, $K_{\Sigma_{1,2}} = Z_1 Z_2 Z_3$ and $K'_{\Sigma_{1,2}} = Z_2 Z_1 Z_3$. Then they are represented by the punctures of the surfaces.

The once-punctured torus and the once-punctured Klein bottle are topologically commensurable, in the sense that both of them are doubly covered by the twice-punctured torus. To be precise, the following hold.

(1) There are three double coverings $p_1 : \Sigma_{1,2} \to \Sigma_{1,1}$ up to equivalence. In fact, there are three epimorphisms from $\pi_1(\Sigma_{1,1})$ to $\mathbf{Z}/2\mathbf{Z}$, and the double covering corresponding to each of them is homeomorphic to $\Sigma_{1,2}$.

(2) There is a unique orientation double covering $p_2 : \Sigma_{1,2} \to N_{2,1}$ up to equivalence. This corresponds to the epimorphism to $\mathbf{Z}/2\mathbf{Z}$ which maps the generator Y_1 of $\pi_1(N_{2,1})$ to the generator 1 of $\mathbf{Z}/2\mathbf{Z}$ and maps the generator Y_2 of $\pi_1(N_{2,1})$ to the identity element 0 of $\mathbf{Z}/2\mathbf{Z}$.

For each $F = \Sigma_{1,1}$, $N_{2,1}$ or $\Sigma_{1,2}$, let $\iota_F : F \to F$ be the involution illustrated in Figure 2. We denote the quotient orbifold F/ι_F by the symbol $\mathcal{O}_F$ and denote the covering projection from F to $\mathcal{O}_F$ by the symbol p_F. Then we have the following under the notation of [8] (see Figure 2).

(1) $\mathcal{O}_{\Sigma_{1,1}} = (2, 2, 2, \infty)$ is the orbifold with underlying space a punctured sphere and with three cone points of cone angle π, and $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ has the following presentation:

$$\pi_1(\mathcal{O}_{\Sigma_{1,1}}) = \langle P_0, P_1, P_2 \,|\, P_0^2 = P_1^2 = P_2^2 = 1 \rangle.$$

(2) $\mathcal{O}_{N_{2,1}} = (2, 2; \infty]$ is the orbifold with underlying space a disk and with two cone points of cone angle π and a corner reflector of order ∞, and $\pi_1(\mathcal{O}_{N_{2,1}})$ has the following presentation:

$$\pi_1(\mathcal{O}_{N_{2,1}}) = \langle Q_0, Q_1, Q_2 \,|\, Q_0^2 = Q_1^2 = Q_2^2 = 1 \rangle.$$

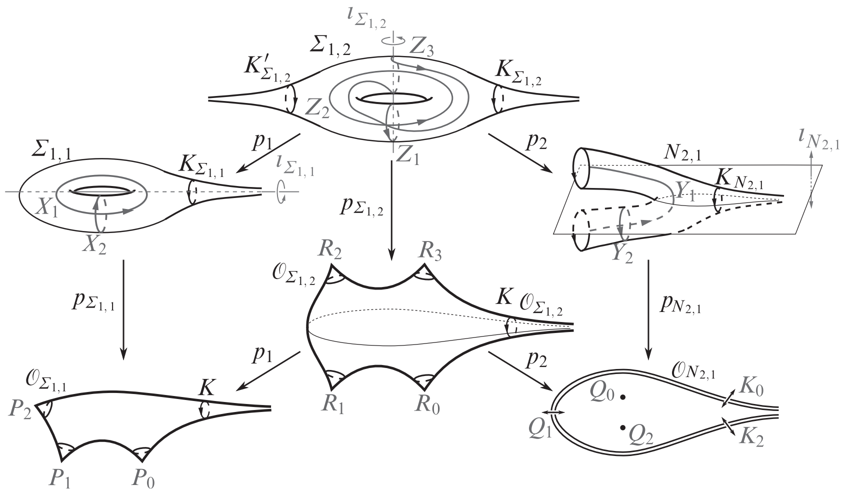

Fig. 2. The commutative diagram of the fundamental groups and coverings.

(3) $\mathcal{O}_{\Sigma_{1,2}} = (2,2,2,2,\infty)$ is the orbifold with underlying space a punctured sphere and with four cone points of cone angle π, and $\pi_1(\mathcal{O}_{\Sigma_{1,2}})$ has the following presentation:

$$\pi_1(\mathcal{O}_{\Sigma_{1,2}}) = \langle R_0, R_1, R_2, R_3 \mid R_0^2 = R_1^2 = R_2^2 = R_3^2 = 1 \rangle.$$

For each $F = \Sigma_{1,1}$, $N_{2,1}$ or $\Sigma_{1,2}$, the orbifold $\mathcal{O}_F$ admits a complete hyperbolic structure and hence $\pi_1(\mathcal{O}_F)$ is identified with a discrete subgroup of $\mathrm{Isom}(\mathbf{H}^2)$ (if we fix a hyperbolic structure). Then the generator Q_1 is a reflection and the other generators are order 2 elliptic transformations. Set $K = (P_0P_1P_2)^{-1}$, $K_0 = Q_1^{Q_0}$, $K_2 = Q_1^{Q_2}$ and $K_{\mathcal{O}_{\Sigma_{1,2}}} = R_0R_1R_2R_3$, where $A^B = BAB^{-1}$. Then K and $K_{\mathcal{O}_{\Sigma_{1,2}}}$, respectively, are represented by the punctures of $\mathcal{O}_{\Sigma_{1,1}}$ and $\mathcal{O}_{\Sigma_{1,2}}$, and K_0 and K_2 are represented by the reflector lines which generate the corner reflector of order ∞. We identify $\pi_1(F)$ with the image of the inclusion $\pi_1(F) \to \pi_1(\mathcal{O}_F)$ induced by the projection p_F. Then we have the following relations among the generators of the fundamental groups:

$$X_1 = P_2P_1, \qquad X_2 = P_0P_1, \qquad K_{\Sigma_{1,1}} = K^2,$$

$$Y_1 = Q_0Q_1, \qquad Y_2 = Q_0Q_2, \qquad K_{N_{2,1}} = K_2K_0,$$

$$Z_1 = R_0R_1, \qquad Z_2 = R_2R_1, \qquad Z_3 = R_1R_3,$$

$$K_{\Sigma_{1,2}} = K_{\mathcal{O}_{\Sigma_{1,2}}}, \qquad K'_{\Sigma_{1,2}} = (K_{\mathcal{O}_{\Sigma_{1,2}}}^{-1})^{R_3}.$$

Note that $\mathcal{O}_{\Sigma_{1,1}}$ and $\mathcal{O}_{N_{2,1}}$ are also topologically commensurable, namely, both of them are doubly covered by $\mathcal{O}_{\Sigma_{1,2}}$. To be precise, the following hold.

(1) There are three double coverings $p_1 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{\Sigma_{1,1}}$ up to equivalence. Each of such covering corresponds to an epimorphism from $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ to $\mathbf{Z}/2\mathbf{Z}$ which maps one of the generators P_0, P_1, P_2 to 1 and maps the remaining generators to 0. Each double covering $p_1 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{\Sigma_{1,1}}$ uniquely determines a double covering $p_1 : \Sigma_{1,2} \to \Sigma_{1,1}$ such that the diagram in the left hand side of Figure 2 is commutative.

(2) There is a unique orientation double covering $p_2 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{N_{2,1}}$ up to equivalence. This corresponds to the epimorphism from $\pi_1(\mathcal{O}_{N_{2,1}})$ to $\mathbf{Z}/2\mathbf{Z}$ which maps the generators Q_0, Q_1 and Q_2 to 0, 1 and 0, respectively. For the orientation double coverings $p_2 : \Sigma_{1,2} \to N_{2,1}$ and $p_2 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{N_{2,1}}$, the diagram in the right hand side of Figure 2 is commutative.

The assertion (2) is obvious and the assertion (1) is proved as follows. For a given double covering $p_1 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{\Sigma_{1,1}}$, we can check, by the relations among the generators of the fundamental groups, that $(p_1 \circ p_{\Sigma_{1,2}})_*(\pi_1(\Sigma_{1,2})) \subset (p_{\Sigma_{1,1}})_*(\pi_1(\Sigma_{1,1}))$. Hence, by the unique lifting property, there is a unique double covering $\tilde{p}_1 : \Sigma_{1,2} \to \Sigma_{1,1}$ such that $p_{\Sigma_{1,1}} \circ \tilde{p}_1 = p_1 \circ p_{\Sigma_{1,2}}$, modulo post composition of $\imath_{\Sigma_{1,1}}$. Since $\imath_{\Sigma_{1,1}} \circ \tilde{p}_1 = \tilde{p}_1 \circ \imath_{\Sigma_{1,2}}$, the coverings $\tilde{p}_1$ and $\imath_{\Sigma_{1,1}} \circ \tilde{p}_1$ are equivalent. Thus the double covering p_1 uniquely determines the double covering $\tilde{p}_1$.

Conversely, for the double covering $p_1 : \Sigma_{1,2} \to \Sigma_{1,1}$ associated with an epimorphism $\phi : \pi_1(\Sigma_{1,1}) \to \mathbf{Z}/2\mathbf{Z}$, we can see, by the relations among the generators of the fundamental groups, that there is a unique epimorphism $\check{\phi} : \pi_1(\mathcal{O}_{\Sigma_{1,1}}) \to \mathbf{Z}/2\mathbf{Z}$ such that it maps only one of generators P_0, P_1 and P_2 of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ to the generator 1 of $\mathbf{Z}/2\mathbf{Z}$ and satisfies $\phi = \check{\phi} \circ (p_{\Sigma_{1,1}})_*$. Hence there is a unique double covering $\check{p}_1 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{\Sigma_{1,1}}$ such that $p_{\Sigma_{1,1}} \circ p_1 = \check{p}_1 \circ p_{\Sigma_{1,2}}$. Thus we obtain the assertion (1).

The orbifolds $\mathcal{O}_{\Sigma_{1,1}}$ and $\mathcal{O}_{N_{2,1}}$ have two distinct common quotient orbifolds, $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$, as described in the following (see Figure 3).

(1) $\mathcal{O}_\alpha = (2; 2, \infty]$ is the orbifold with underlying space a disk and with a cone point of cone angle π and with a corner reflector of order 2 and a corner reflector of order ∞, and $\pi_1(\mathcal{O}_\alpha)$ has the following presentation:

$$\pi_1(\mathcal{O}_\alpha) = \langle S_0, S_1, S_2 \mid S_0^2 = S_1^2 = S_2^2 = 1, (S_1 S_2)^2 = 1 \rangle.$$

Here S_0 is an order 2 elliptic transformation, and S_1 and S_2 are reflections.

(1-1) There is a unique double covering $p_{\Sigma_{1,1}}^{(\alpha)} : \mathcal{O}_{\Sigma_{1,1}} \to \mathcal{O}_\alpha$. This corresponds to the epimorphism from $\pi_1(\mathcal{O}_\alpha)$ to $\mathbf{Z}/2\mathbf{Z}$ which

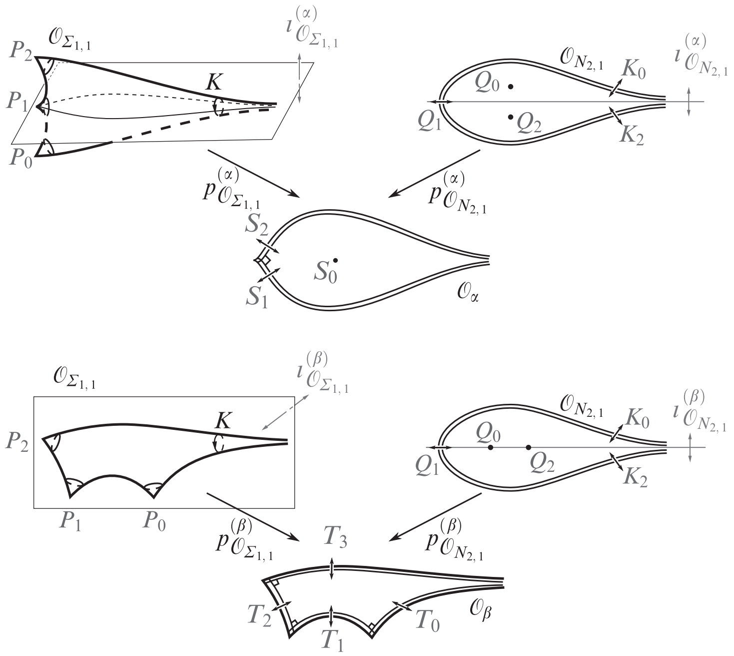

Fig. 3. Involutions of $\mathcal{O}_{\Sigma_{1,1}}$ and $\mathcal{O}_{N_{2,1}}$.

maps the generators S_1, S_2 to 1 and maps S_0 to 0. Then we have the following identities:

$$P_0 = S_0^{S_2}, \qquad P_1 = S_1 S_2, \qquad P_2 = S_0.$$

(1-2) There is a unique double covering $p^{(\alpha)}_{N_{2,1}} : \mathcal{O}_{N_{2,1}} \to \mathcal{O}_\alpha$. This corresponds to the epimorphism from $\pi_1(\mathcal{O}_\alpha)$ to $\mathbf{Z}/2\mathbf{Z}$ which maps the generator S_2 to 1 and maps S_0, S_1 to 0. Then we have the following identities:

$$Q_0 = S_0^{S_2}, \qquad Q_1 = S_1, \qquad Q_2 = S_0.$$

(2) $\mathcal{O}_\beta = [2,2,2,\infty]$ is the orbifold with underlying space a disk and with three corner reflectors of order 2 and a corner reflector of order ∞, and $\pi_1(\mathcal{O}_\beta)$ has the following presentation:

$$\pi_1(\mathcal{O}_\beta) = \left\langle T_0, T_1, T_2, T_3 \;\middle|\; \begin{array}{c} T_0^2 = T_1^2 = T_2^2 = T_3^2 = 1, \\ (T_0T_1)^2 = (T_1T_2)^2 = (T_2T_3)^2 = 1 \end{array} \right\rangle.$$

Here the generators T_0, T_1, T_2, T_3 are reflections.

(2-1) There is a unique double covering $p^{(\beta)}_{\Sigma_{1,1}} : \mathcal{O}_{\Sigma_{1,1}} \to \mathcal{O}_\beta$. This corresponds to the epimorphism from $\pi_1(\mathcal{O}_\beta)$ to $\mathbf{Z}/2\mathbf{Z}$ which maps the generators T_0, T_1, T_2, T_3 to 1. Then we have the following identities:

$$P_0 = T_0 T_1, \qquad P_1 = T_1 T_2, \qquad P_2 = T_2 T_3.$$

(2-2) There is a unique double covering $p_{N_{2,1}}^{(\beta)} : \mathcal{O}_{N_{2,1}} \to \mathcal{O}_\beta$. This corresponds to the epimorphism from $\pi_1(\mathcal{O}_\beta)$ to $\mathbf{Z}/2\mathbf{Z}$ which maps the generators T_0, T_1, T_2 to 1 and maps T_3 to 0. Then we have the following identities:

$$Q_0 = T_1 T_2, \qquad Q_1 = T_3^{T_1}, \qquad Q_2 = T_0 T_1.$$

In summary, we have the commutative diagram of double coverings as shown in Figure 4. Every arrow represents a double covering. There are three types of coverings p_1 from $\Sigma_{1,2}$ (resp. $\mathcal{O}_{\Sigma_{1,2}}$) to $\Sigma_{1,1}$ (resp. $\mathcal{O}_{\Sigma_{1,1}}$) up to equivalence, and the other coverings are unique up to equivalence.

Definition 2.1. (1) For $F = \Sigma_{1,1}$, $N_{2,1}$ $\Sigma_{1,2}$, $\mathcal{O}_{\Sigma_{1,1}}$, $\mathcal{O}_{N_{2,1}}$, $\mathcal{O}_{\Sigma_{1,2}}$, $\mathcal{O}_\alpha$ or $\mathcal{O}_\beta$, a representation $\rho : \pi_1(F) \to \mathrm{PSL}(2, \mathbf{C})$ is *type-preserving* if it is irreducible (equivalently, it does not have a common fixed point in $\partial \mathbf{H}^3$) and sends peripheral elements to parabolic transformations.

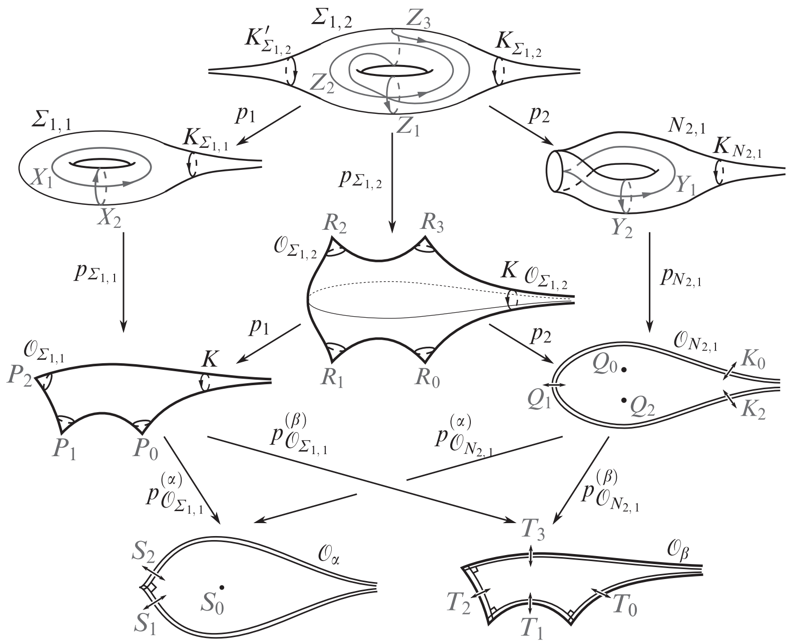

Fig. 4

(2) Type-preserving $\mathrm{PSL}(2,\mathbf{C})$-representations ρ and ρ' are *equivalent* if $i_g \circ \rho = \rho'$, where i_g is the inner automorphism, $i_g(h) = ghg^{-1}$, of $\mathrm{PSL}(2,\mathbf{C})$ determined by g.

In the above definition, if F is an orbifold with reflector lines, an element of $\pi_1(F)$ is said to be *peripheral* if it is (the image of) a peripheral element of $\pi_1(\tilde{F})$, where $\tilde{F}$ is the orientation double covering of F.

Definition 2.2. Let ρ_1 be a type-preserving $\mathrm{PSL}(2,\mathbf{C})$-representation of $\pi_1(\Sigma_{1,1})$ (resp. $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$). Let ρ_2 be a type-preserving $\mathrm{PSL}(2,\mathbf{C})$-representation of $\pi_1(N_{2,1})$ (resp. $\pi_1(\mathcal{O}_{N_{2,1}})$). The representations ρ_1 and ρ_2 are *commensurable* if there exist a double covering p_1 from $\Sigma_{1,2}$ (resp. $\mathcal{O}_{\Sigma_{1,2}}$) to $\Sigma_{1,1}$ (resp. $\mathcal{O}_{\Sigma_{1,1}}$) and a double covering p_2 from $\Sigma_{1,2}$ (resp. $\mathcal{O}_{\Sigma_{1,2}}$) to $N_{2,1}$ (resp. $\mathcal{O}_{N_{2,1}}$) such that $\rho_1 \circ (p_1)_*$ and $\rho_2 \circ (p_2)_*$ are equivalent, namely $i_g \circ \rho_1 \circ (p_1)_* = \rho_2 \circ (p_2)_*$ for some $g \in \mathrm{PSL}(2,\mathbf{C})$. After replacing ρ_1 with $i_g \circ \rho_1$, without changing the equivalence class, the last identity can be replaced with the identity $\rho_1 \circ (p_1)_* = \rho_2 \circ (p_2)_*$.

In this paper, we study the following problem:

Problem 2.3. For a given type-preserving $\mathrm{PSL}(2,\mathbf{C})$-representation ρ_2 of $\pi_1(N_{2,1})$ (resp. $\pi_1(\mathcal{O}_{N_{2,1}})$), when does there exist a type-preserving $\mathrm{PSL}(2,\mathbf{C})$-representation ρ_1 of $\pi_1(\Sigma_{1,1})$ (resp. $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$) which is commensurable with ρ_2?

We will give a partial answer to this problem for a certain family of type-preserving $\mathrm{PSL}(2,\mathbf{C})$-representations of $\pi_1(N_{2,1})$ in terms of "complex probabilities" introduced in Section 4 (see Theorem 5.1).

Remark 2.4. Recall that there are three equivalence classes of double coverings $\Sigma_{1,2} \to \Sigma_{1,1}$ and there is a unique equivalence class of double coverings $\Sigma_{1,2} \to N_{2,1}$. The three classes of double coverings $\Sigma_{1,2} \to \Sigma_{1,1}$ become equivalent after a post composition of a self-homeomorphism of $\Sigma_{1,1}$. Hence, by considering compositions of ρ_1 with the automorphism of $\pi_1(\Sigma_{1,1})$ induced by a self-homeomorphism of $\Sigma_{1,1}$, we may arbitrarily fix the equivalence classes of the coverings $\Sigma_{1,2} \to \Sigma_{1,1}$. However, we must be careful in the choices of a representative $p_1 : \Sigma_{1,2} \to \Sigma_{1,1}$ and a representative $p_2 : \Sigma_{1,2} \to N_{2,1}$ of the equivalence classes of the coverings, by the following reason. Assume that $\rho_1 : \pi_1(\Sigma_{1,1}) \to \mathrm{PSL}(2,\mathbf{C})$ and $\rho_2 : \pi_1(N_{2,1}) \to \mathrm{PSL}(2,\mathbf{C})$ are commensurable via coverings $p_1 : \Sigma_{1,2} \to \Sigma_{1,1}$ and $p_2 : \Sigma_{1,2} \to N_{2,1}$, i.e., $\rho_1 \circ (p_1)_*$ and $\rho_2 \circ (p_2)_*$ are equivalent. Pick a self-homeomorphism f of $\Sigma_{1,2}$ and replace p_1 with another covering $p_1' := p_1 \circ f : \Sigma_{1,2} \to \Sigma_{1,1}$. Then the representation $\rho_1 \circ (p_1')_*$ is not necessarily equivalent to the representation $\rho_1 \circ (p_1)_*$, and hence it is not necessarily equivalent to the representation

$\rho_2 \circ (p_2)_*$. In fact, we also need to replace p_2 with another covering $p_2' := p_2 \circ f : \Sigma_{1,2} \to N_{2,1}$, which is equivalent to p_2.

3. Elliptic generators

In this section, we first recall the definition and basic properties of elliptic generators of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ (see [1, Section 2] for details). We also introduce the concept of elliptic generators of $\pi_1(\mathcal{O}_{N_{2,1}})$, and then we establish similar basic properties.

Recall that the (orbifold) fundamental group of $\mathcal{O}_{\Sigma_{1,1}}$ has the following presentation:

$$\pi_1(\mathcal{O}_{\Sigma_{1,1}}) = \langle P_0, P_1, P_2 \mid P_0^2 = P_1^2 = P_2^2 = 1 \rangle,$$

and that $K = (P_0 P_1 P_2)^{-1}$ is represented by the puncture of $\mathcal{O}_{\Sigma_{1,1}}$.

Definition 3.1. An ordered triple (P_0, P_1, P_2) of elements of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ is called an *elliptic generator triple* of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ if its members generate $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ and satisfy $P_0^2 = P_1^2 = P_2^2 = 1$ and $(P_0 P_1 P_2)^{-1} = K$. A member of an elliptic generator triple of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ is called an *elliptic generator* of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$.

Remark 3.2. In the above definition, the condition that the members of the triple generate $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ is actually a consequence of the other conditions. This can be seen from the proof of [1, Lemma 2.1.7].

Proposition 3.3 ([1, Proposition 2.1.6]). *The elliptic generator triples of* $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ *are characterized as follows.*

(1) *For any elliptic generator triple* (P_0, P_1, P_2) *of* $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$, *the following hold:*

(1.1) *The triple of any three consecutive elements in the following bi-infinite sequence is also an elliptic generator triple of* $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$.

$$\ldots, P_2^{K^{-2}}, P_0^{K^{-1}}, P_1^{K^{-1}}, P_2^{K^{-1}}, P_0, P_1, P_2, P_0^K, P_1^K, P_2^K, P_0^{K^2}, \ldots$$

(1.2) $(P_0, P_2, P_1^{P_2})$ *and* $(P_1^{P_0}, P_0, P_2)$ *are also elliptic generator triples of* $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$.

(2) *Conversely, any elliptic generator triple of* $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ *is obtained from a given elliptic generator triple of* $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ *by successively applying the operations in* (1).

Definition 3.4. For an elliptic generator triple (P_0, P_1, P_2) of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$, the bi-infinite sequence $\{P_j\}$ in Proposition 3.3(1.1) is called the *sequence of elliptic generators* of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ (associated with (P_0, P_1, P_2)).

Recall that the (orbifold) fundamental group of $\mathcal{O}_{N_{2,1}}$ has the following presentation:

$$\pi_1(\mathcal{O}_{N_{2,1}}) = \langle Q_0, Q_1, Q_2 \mid Q_0^2 = Q_1^2 = Q_2^2 = 1 \rangle,$$

and that $K_0 = Q_1^{Q_0}$ and $K_2 = Q_1^{Q_2}$ are represented by the reflections in the lines which generate the corner reflector of order ∞. It should be noted that Q_0 and Q_2 act on the universal cover of $\mathcal{O}_{N_{2,1}}$ orientation preservingly, and Q_1 acts on the universal cover of $\mathcal{O}_{N_{2,1}}$ orientation reversingly.

Definition 3.5. An ordered triple (Q_0, Q_1, Q_2) of elements of $\pi_1(\mathcal{O}_{N_{2,1}})$ is called an *elliptic generator triple* of $\pi_1(\mathcal{O}_{N_{2,1}})$ if its members generate $\pi_1(\mathcal{O}_{N_{2,1}})$ and satisfy $Q_0^2 = Q_1^2 = Q_2^2 = 1$ and $Q_1^{Q_2} Q_1^{Q_0} = K_2 K_0$. A member of an elliptic generator triple of $\pi_1(\mathcal{O}_{N_{2,1}})$ is called an *elliptic generator* of $\pi_1(\mathcal{O}_{N_{2,1}})$.

Remark 3.6. In the above definition, the condition that the members of the triple generate $\pi_1(\mathcal{O}_{N_{2,1}})$ is actually a consequence of the other conditions. This can be seen from the proof of Proposition 3.7 (see [4]).

Proposition 3.7. *The elliptic generator triples of $\pi_1(\mathcal{O}_{N_{2,1}})$ are characterized as follows.*

(1) *For any elliptic generator triple (Q_0, Q_1, Q_2) of $\pi_1(\mathcal{O}_{N_{2,1}})$, the following hold:*

(1.1) *The triples in the following bi-infinite sequence are also elliptic generator triples of $\pi_1(\mathcal{O}_{N_{2,1}})$.*

$$\dots, (Q_0^{K_0K_2}, Q_1^{K_0K_2}, Q_2^{K_0K_2}), (Q_2^{K_0}, Q_1^{K_0}, Q_0^{K_0}), (Q_0, Q_1, Q_2),$$
$$(Q_2^{K_2}, Q_1^{K_2}, Q_0^{K_2}), (Q_0^{K_2K_0}, Q_1^{K_2K_0}, Q_2^{K_2K_0}), \dots$$

To be precise, the following holds. Let $\{Q_j\}$ be the sequence of elements of $\pi_1(\mathcal{O}_{N_{2,1}})$ obtained from (Q_0, Q_1, Q_2) by applying the following rule:

$$Q_j^{K_0} = Q_{-j-1}, \qquad Q_j^{K_2} = Q_{-j+5}.$$

Then the triple $(Q_{3k}, Q_{3k+1}, Q_{3k+2})$ is also an elliptic generator triple of $\pi_1(\mathcal{O}_{N_{2,1}})$ for any $k \in \mathbf{Z}$.

(1.2) *$(Q_2, Q_1^{Q_2Q_0}, Q_0^{Q_2})$ is also an elliptic generator triple of $\pi_1(\mathcal{O}_{N_{2,1}})$.*

(2) *Conversely, any elliptic generator triple of $\pi_1(\mathcal{O}_{N_{2,1}})$ is obtained from a given elliptic generator triple of $\pi_1(\mathcal{O}_{N_{2,1}})$ by successively applying the operations in* (1).

The proof of (1) is obvious, and the proof of (2) is given in [4]. In this paper, we need only (1).

Definition 3.8. For an elliptic generator triple (Q_0, Q_1, Q_2) of $\pi_1(\mathcal{O}_{N_{2,1}})$, the bi-infinite sequence $\{Q_j\}$ in Proposition 3.7(1.1) is called the *sequence of elliptic generators* of $\pi_1(\mathcal{O}_{N_{2,1}})$ (associated with (Q_0, Q_1, Q_2)).

It should be noted that Q_j is conjugate to the following element (cf. Proposition 4.11(1.1)):

$$\begin{aligned} &Q_0 \text{ if } j \equiv 0 \text{ or } 5 \quad &&(\mathrm{mod}\ 6), \\ &Q_1 \text{ if } j \equiv 1 &&(\mathrm{mod}\ 3), \\ &Q_2 \text{ if } j \equiv 2 \text{ or } 4 &&(\mathrm{mod}\ 6). \end{aligned}$$

In particular, Q_j acts on the universal cover of $\mathcal{O}_{N_{2,1}}$ orientation reversingly or orientation preservingly according to whether $j \equiv 1 \pmod 3$ or not.

4. Type-preserving representations

Let ρ_1 be a type-preserving $\mathrm{PSL}(2,\mathbf{C})$-representation of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$. Fix a sequence of elliptic generators $\{P_j\}$ of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$. Set

$$(x_1, x_{12}, x_2) = (\mathrm{tr}(\rho_1(X_1)), \mathrm{tr}(\rho_1(X_1X_2)), \mathrm{tr}(\rho_1(X_2))),$$

where $X_1 = P_2P_1$ and $X_2 = P_0P_1$. As the trace of an element in $\mathrm{PSL}(2,\mathbf{C})$ is only defined up to sign, we are free to choose the signs of x_1 and x_2 independently. Once we have done this though, the sign of x_{12} is determined. It is well-known that the triple (x_1, x_{12}, x_2) is a Markoff triple, namely, it satisfies the Markoff identity (see [2], [1]):

$$x_1^2 + x_{12}^2 + x_2^2 = x_1x_{12}x_2$$

and that the triple (x_1, x_{12}, x_2) is *non-trivial*, namely, it is different from (0,0,0). Moreover, the equivalence class of the triple (x_1, x_{12}, x_2) is uniquely determined by the equivalence class of the type-preserving representation ρ_1, and vice versa (see [1, Proposition 2.3.6 and 2.4.2] for details). Here, two triples (x_1, x_{12}, x_2) and (x_1', x_{12}', x_2') are said to be *equivalent* if the latter is equal to (x_1, x_{12}, x_2), $(x_1, -x_{12}, -x_2)$, $(-x_1, x_{12}, -x_2)$ or $(-x_1, -x_{12}, x_2)$. We call the triple $(x_1, x_{12}, x_2) = (\mathrm{tr}(\rho_1(X_1)), \mathrm{tr}(\rho_1(X_1X_2)), \mathrm{tr}(\rho_1(X_2)))$ the *Markoff triple* associated with $\{\rho_1(P_j)\}$.

Let ρ_2 be a type-preserving $\mathrm{PSL}(2,\mathbf{C})$-representation of $\pi_1(\mathcal{O}_{N_{2,1}})$. Fix a sequence of elliptic generators $\{Q_j\}$ of $\pi_1(\mathcal{O}_{N_{2,1}})$. Set

$$(y_1, y_{12}, y_2) = (\mathrm{tr}(\rho_2(Y_1))/i, \mathrm{tr}(\rho_2(Y_1Y_2))/i, \mathrm{tr}(\rho_2(Y_2))),$$

where $Y_1 = Q_0Q_1$ and $Y_2 = Q_0Q_2$ and $i = \sqrt{-1}$. Note that $\rho_2(K_{N_{2,1}}) = \rho_2((Y_1Y_2Y_1^{-1}Y_2)^{-1})$ is a parabolic element of $\mathrm{PSL}(2,\mathbf{C})$ with a trace that has a well defined sign (independent of the signs chosen for the traces of $\rho(Y_1)$ and $\rho(Y_2)$), which is equal to $y_1^2 + y_{12}^2 - y_1 y_{12} y_2 + 2$. Hence (y_1, y_{12}, y_2) satisfies one of the following identities:

$$\begin{aligned} y_1^2 + y_{12}^2 + 4 &= y_1 y_{12} y_2 \qquad \text{if } \operatorname{tr}(\rho_2(K_{N_{2,1}})) = -2, \\ y_1^2 + y_{12}^2 &= y_1 y_{12} y_2 \qquad \text{if } \operatorname{tr}(\rho_2(K_{N_{2,1}})) = +2. \end{aligned} \tag{Eq1}$$

In addition, the triple (y_1, y_{12}, y_2) is *non-trivial*, namely, it is different from (0,0,0) (see [3, Remark 4.3]). It is well-known that any two generator subgroup $\langle A, B \rangle$ of $\mathrm{PSL}(2, \mathbf{C})$ is irreducible if and only if $\operatorname{tr}([A, B]) \neq 2$ (see, for example [5, Proposition 2.3.1]). Since ρ_2 is irreducible, it satisfies one of the following identities:

$$\begin{aligned} y_2 &\neq 0 \qquad \text{if } \operatorname{tr}(\rho_2(K_{N_{2,1}})) = -2, \\ y_2 &\neq \pm 2 \qquad \text{if } \operatorname{tr}(\rho_2(K_{N_{2,1}})) = +2. \end{aligned} \tag{Eq2}$$

Moreover, the equivalence class of the triple (y_1, y_{12}, y_2) is uniquely determined by the equivalence class of the type-preserving representation ρ_2, and vice versa (see [3, Propositions 4.4 and 4.6] for details). Here, two triples (y_1, y_{12}, y_2) and (y_1', y_{12}', y_2') are said to be *equivalent* if the latter is equal to (y_1, y_{12}, y_2), $(y_1, -y_{12}, -y_2)$, $(-y_1, y_{12}, -y_2)$ or $(-y_1, -y_{12}, y_2)$. We call the triple $(y_1, y_{12}, y_2) = (\operatorname{tr}(\rho_2(Y_1))/i, \operatorname{tr}(\rho_2(Y_1 Y_2))/i, \operatorname{tr}(\rho_2(Y_2)))$ the *pseudo-Markoff triple* associated with $\{\rho_2(Q_j)\}$.

Proposition 4.1. (1) *The restriction of any type-preserving* $\mathrm{PSL}(2, \mathbf{C})$-*representation of* $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ *(resp.* $\pi_1(\mathcal{O}_{N_{2,1}})$*) to* $\pi_1(\Sigma_{1,1})$ *(resp.* $\pi_1(N_{2,1})$*) is type-preserving.*

(2) *Conversely, every type-preserving* $\mathrm{PSL}(2, \mathbf{C})$-*representation* ρ_1 *(resp.* ρ_2*) of* $\pi_1(\Sigma_{1,1})$ *(resp.* $\pi_1(N_{2,1})$*) extends to a unique type-preserving* $\mathrm{PSL}(2, \mathbf{C})$-*representation* $\tilde{\rho}_1$ *(resp.* $\tilde{\rho}_2$ *of* $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ *(resp.* $\pi_1(\mathcal{O}_{N_{2,1}})$*). Moreover, if* ρ_1 *(resp.* ρ_2*) is faithful, then* $\tilde{\rho}_1$ *(resp.* $\tilde{\rho}_2$*) is also faithful.*

Proof. The assertion (1) is obvious from the definition. The first assertion in (2) is well-known (cf. [10, Section 5.4] and [1, Proposition 2.2.2]). The second assertion in (2) is proved as follows. Suppose to the contrary that ρ_1 is faithful but that $\tilde{\rho}_1$ is not faithful. Pick a nontrivial element γ of $\operatorname{Ker} \tilde{\rho}_1$. Since $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ is the free product of three cyclic groups and since $\pi_1(\Sigma_{1,1})$ is an index 2 subgroup of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$, we can see that the normal closure of γ in $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ has a nontrivial intersection with $\pi_1(\Sigma_{1,1})$. This means that ρ_1 is not faithful, a contradiction. The same argument works for the pair of representations ρ_2 and $\tilde{\rho}_2$. $\square$

By this proposition, the following are well-defined.

Definition 4.2. (1) For $F = \Sigma_{1,1}$ or $\mathcal{O}_{\Sigma_{1,1}}$, the symbol $\Omega(\Sigma_{1,1})$ denotes the space of all type-preserving $\mathrm{PSL}(2,\mathbf{C})$-representations ρ_1 of $\pi_1(F)$.

(2) For $F = N_{2,1}$ or $\mathcal{O}_{N_{2,1}}$, the symbol $\Omega(N_{2,1})$ (resp. $\Omega'(N_{2,1})$) denotes the space of all type-preserving $\mathrm{PSL}(2,\mathbf{C})$-representations ρ_2 of $\pi_1(F)$ such that $\mathrm{tr}(\rho_2(K_{N_{2,1}})) = -2$ (resp. $\mathrm{tr}(\rho_2(K_{N_{2,1}})) = +2$).

Remark 4.3. For any $\rho_2 \in \Omega'(N_{2,1})$, the isometries $\rho_2(Q_0Q_2) = \rho_2(Y_2)$ and $\rho_2(K_{N_{2,1}})$ have a common fixed point (see [3, Lemma 4.5(ii)]), and hence ρ_2 is indiscrete or non-faithful (see [3, Lemma 4.7]).

The following lemma gives a (local) section of the projection from $\Omega(\Sigma_{1,1})$ (resp. $\Omega(N_{2,1})$) to the space of the equivalence classes of the non-trivial Markoff triples (resp. pseudo-Markoff triple) (cf. [6, Section 2], [9, Section 3], [1, Lemma 2.3.7] and [3, Lemma 4.5]).

Lemma 4.4. (1) *Let* $(x_1, x_{12}, x_2) \in \mathbf{C}^3$ *be a triple satisfying* $x_1^2 + x_{12}^2 + x_2^2 = x_1x_{12}x_2$ *and* $x_{12} \neq 0$, *and let* $\{P_j\}$ *be a sequence of elliptic generators of* $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$.

(1.1) *Let* $\rho_1 : \pi_1(\Sigma_{1,1}) \to \mathrm{PSL}(2,\mathbf{C})$ *be a representation defined by*

$$\rho_1(X_1) = \begin{pmatrix} x_1 - x_2/x_{12} & x_1/x_{12}^2 \\ x_1 & x_2/x_{12} \end{pmatrix}, \qquad \rho_1(X_1X_2) = \begin{pmatrix} x_{12} & -1/x_{12} \\ x_{12} & 0 \end{pmatrix},$$

$$\rho_1(X_2) = \begin{pmatrix} x_2 - x_1/x_{12} & -x_2/x_{12}^2 \\ -x_2 & x_1/x_{12} \end{pmatrix}, \qquad \rho_1(K_{\Sigma_{1,1}}) = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix},$$

where $X_1 = P_2P_1$ *and* $X_2 = P_0P_1$. *Then* $\rho_1 \in \Omega(\Sigma_{1,1})$ *such that the Markoff triple associated with* $\{\rho_1(P_j)\}$ *is equal to* (x_1, x_{12}, x_2) *up to equivalence.*

(1.2) *The above representation* ρ_1 *extends to a type-preserving representation of* $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ *satisfying the following identities:*

$$\rho_1(P_0) = \begin{pmatrix} x_2/x_{12} & (x_{12}x_2 - x_1)/x_{12}^2 \\ -x_1 & -x_2/x_{12} \end{pmatrix}, \qquad \rho_1(P_1) = \begin{pmatrix} 0 & -1/x_{12} \\ x_{12} & 0 \end{pmatrix},$$

$$\rho_1(P_2) = \begin{pmatrix} -x_1/x_{12} & (x_1x_{12} - x_2)/x_{12}^2 \\ -x_2 & x_1/x_{12} \end{pmatrix}, \qquad \rho_1(K) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(2) *Let* $(y_1, y_{12}, y_2) \in \mathbf{C}^3$ *be a triple satisfying* $y_1^2 + y_{12}^2 + 4 = y_1y_{12}y_2$ *and* $y_2 \neq 0$, *and let* $\{Q_j\}$ *be a sequence of elliptic generators of* $\pi_1(\mathcal{O}_{N_{2,1}})$.

(2.1) *Let* $\rho_2 : \pi_1(N_{2,1}) \to \mathrm{PSL}(2,\mathbf{C})$ *be a representation defined by*

$$\rho_2(Y_1) = \begin{pmatrix} y_1 i/2 & -y_{12}i/2y_2 \\ -(y_1y_2 - y_{12})y_2 i/2 & y_1 i/2 \end{pmatrix},$$

$$\rho_2(Y_1Y_2) = \begin{pmatrix} y_{12}i/2 & -(y_{12}y_2 - y_1)i/2y_2 \\ -y_1y_2 i/2 & y_{12}i/2 \end{pmatrix},$$

$$\rho_2(Y_2) = \begin{pmatrix} 0 & 1/y_2 \\ -y_2 & y_2 \end{pmatrix}, \qquad \rho_2(K_{N_{2,1}}) = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix},$$

where $Y_1 = Q_0Q_1$ *and* $Y_2 = Q_0Q_2$. *Then* $\rho_2 \in \Omega(N_{2,1})$ *such that the pseudo-Markoff triple associated with* $\{\rho_2(Q_j)\}$ *is equal to* (y_1, y_{12}, y_2) *up to equivalence.*

(2.2) *The above representation* ρ_2 *extends to a type-preserving representation of* $\pi_1(\mathcal{O}_{N_{2,1}})$ *satisfying the following identities:*

$$\rho_2(Q_0) = \begin{pmatrix} y_1/2 & -y_{12}/2y_2 \\ (y_1y_2 - y_{12})y_2/2 & -y_1/2 \end{pmatrix},$$

$$\rho_2(Q_1) = \begin{pmatrix} -(y_1^2+2)i/2 & y_1y_{12}i/2y_2 \\ -y_1y_2(y_1y_2 - y_{12})i/2 & (y_1^2+2)i/2 \end{pmatrix},$$

$$\rho_2(Q_2) = \begin{pmatrix} -y_{12}/2 & (y_{12}y_2 - y_1)/2y_2 \\ -y_1y_2/2 & y_{12}/2 \end{pmatrix},$$

$$\rho_2(K_0) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \rho_2(K_2) = \begin{pmatrix} i & -2i \\ 0 & -i \end{pmatrix}.$$

CONVENTION 4.5. (1) For any element $\rho_1 \in \Omega(\Sigma_{1,1})$, after taking conjugate of ρ_1 by some element of $\mathrm{PSL}(2, \mathbf{C})$, we always assume that ρ_1 is normalized so that $\rho_1(K)$ is given by the identity in Lemma 4.4(1.2) without changing the equivalence class.

(2) For any element $\rho_2 \in \Omega(N_{2,1})$, after taking conjugate of ρ_2 by some element of $\mathrm{PSL}(2, \mathbf{C})$, we always assume that ρ_2 is normalized so that $\rho_2(K_0)$ and $\rho_2(K_2)$ are given by the identities in Lemma 4.4(2.2) without changing the equivalence class.

Pick an element $\rho_1 \in \Omega(\Sigma_{1,1})$ and a sequence of elliptic generators $\{P_j\}$ of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$. Let $(x_1, x_{12}, x_2) \in \mathbf{C}^3$ be the Markoff triple associated with $\{\rho_1(P_j)\}$. Suppose $x_1x_{12}x_2 \neq 0$. Then the identity $x_1^2 + x_{12}^2 + x_2^2 = x_1x_{12}x_2$ implies the following identity:

$$a_0 + a_1 + a_2 = 1, \qquad \text{where } a_0 = \frac{x_1}{x_{12}x_2},\ a_1 = \frac{x_{12}}{x_2x_1},\ a_2 = \frac{x_2}{x_1x_{12}}.$$

We call the triple $(a_0, a_1, a_2) \in (\mathbf{C}^*)^3$ the *complex probability associated with* $\{\rho_1(P_j)\}$, where $\mathbf{C}^* = \mathbf{C} - \{0\}$. We note that the Markoff triple (x_1, x_{12}, x_2) with $x_1x_{12}x_2 \neq 0$ up to sign (that is, up to equivalence) is recovered from the complex probability by the following identities:

$$x_1^2 = \frac{1}{a_1 a_2}, \qquad x_{12}^2 = \frac{1}{a_2 a_0}, \qquad x_2^2 = \frac{1}{a_0 a_1}.$$

Moreover, there is a nice geometric construction of a type-preserving representation from the corresponding complex probability.

To introduce the geometric construction of the representations, we prepare some notations. Throughout this paper, $\mathbf{H}^3 = \mathbf{C} \times \mathbf{R}_+$ denotes the upper half space model of the 3-dimensional hyperbolic space.

Definition 4.6. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of $\mathrm{PSL}(2, \mathbf{C})$ such that $A(\infty) \neq \infty$, namely $c \neq 0$. Then the *isometric hemisphere* $I(A)$ of A is the hyperplane of the upper half space $\mathbf{H}^3$ bounded by

$$\{z \in \mathbf{C} \mid |A'(z)| = 1\} = \{z \in \mathbf{C} \mid |cz + d| = 1\}.$$

Thus $I(A)$ is a Euclidean hemisphere orthogonal to $\mathbf{C} = \partial \mathbf{H}^3$ with center $c(A) = A^{-1}(\infty) = -d/c$ and radius $r(A) = 1/|c|$. We denote by $E(A)$ the closed half space of $\mathbf{H}^3$ with boundary $I(A)$ which is of infinite diameter with respect to the Euclidean metric.

Lemma 4.7 ([1, Lemma 4.1.1]). *Let A be an element of* $\mathrm{PSL}(2, \mathbf{C})$ *which does not fix ∞ and let W be an element of* $\mathrm{PSL}(2, \mathbf{C})$ *which preserves ∞ and acts on $\mathbf{C} = \partial \mathbf{H}^3$ as a Euclidean isometry. Then*

$$I(AW) = W^{-1}(I(A)), \qquad I(WA) = I(A).$$

In particular, $I(WAW^{-1}) = WI(A)$.

Now we introduce a nice geometric construction of a type-preserving representation from the corresponding complex probability (cf. [1, Proposition 2.4.4]).

Proposition 4.8. *Under Convention* 4.5*, the following hold:*

(1) *For any triple $(a_0, a_1, a_2) \in (\mathbf{C}^*)^3$ such that $a_0 + a_1 + a_2 = 1$ and for any sequence of elliptic generators $\{P_j\}$ of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$, there is an element $\rho_1 \in \Omega(\Sigma_{1,1})$ such that the complex probability associated with $\{\rho_1(P_j)\}$ is equal to (a_0, a_1, a_2). Moreover, ρ_1 satisfies the following conditions (see Figure* 5*).*

(1.1) *The centers of isometric hemispheres of $\rho_1(P_j)$ satisfy the following conditions.*

- $c(\rho_1(P_{3k+2})) - c(\rho_1(P_{3k+1})) = a_0$.
- $c(\rho_1(P_{3k+3})) - c(\rho_1(P_{3k+2})) = a_1$.
- $c(\rho_1(P_{3k+4})) - c(\rho_1(P_{3k+3})) = a_2$.

(1.2) *The isometries $\rho_1(P_j)$ satisfy the following conditions.*

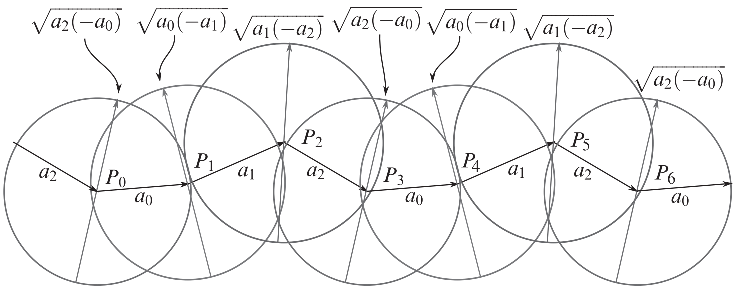

Fig. 5. Isometric hemispheres of elliptic generators of $\pi_1(\Sigma_{1,1})$.

- *The isometry* $\rho_1(P_{3k+2})$ *is the* π*-rotation about the geodesic with endpoints* $c(\rho_1(P_{3k+2})) \pm \sqrt{a_0(-a_1)}$.
- *The isometry* $\rho_1(P_{3k})$ *is the* π*-rotation about the geodesic with endpoints* $c(\rho_1(P_{3k})) \pm \sqrt{a_1(-a_2)}$.
- *The isometry* $\rho_1(P_{3k+1})$ *is the* π*-rotation about the geodesic with endpoints* $c(\rho_1(P_{3k+1})) \pm \sqrt{a_2(-a_0)}$.

(2) *Conversely, under Convention* 4.5, *any element* $\rho_1 \in \Omega(\Sigma_{1,1})$ *with the complex probability* (a_0, a_1, a_2) *associated with* $\{\rho_1(P_j)\}$ *for some sequence of elliptic generators* $\{P_j\}$ *of* $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ *satisfies the above conditions.*

Notation 4.9. Under Convention 4.5, let ρ_1 be an element of $\Omega(\Sigma_{1,1})$ and let $\{P_j\}$ be a sequence of elliptic generators of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$. Let ξ be the automorphism of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ given by the following (cf. Proposition 3.3):

$$(\xi(P_0), \xi(P_1), \xi(P_2)) = (P_2^{P_1}, P_1, P_0^K).$$

If the complex probability associated with $\{\rho_1(\xi^k(P_j))\}$ is well-defined, then we denote it by $(a_0^{(k)}, a_1^{(k)}, a_2^{(k)})$.

The following lemma can be verified by simple calculation (cf. [1, Lemma 2.4.1]).

Lemma 4.10. *Under Convention* 4.5, *let* ρ_1 *be an element of* $\Omega(\Sigma_{1,1})$ *and let* $\{P_j\}$ *be a sequence of elliptic generators of* $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$. *Suppose that the complex probability* $(a_0^{(k)}, a_1^{(k)}, a_2^{(k)})$ *associated with* $\{\rho_1(\xi^k(P_j))\}$ *is well-defined for any* $k \in \mathbf{Z}$. *Then we have the following identities (cf. Figure* 6*):*

$$a_0^{(k+1)} = 1 - a_2^{(k)}, \qquad a_1^{(k+1)} = \frac{a_1^{(k)} a_2^{(k)}}{1 - a_2^{(k)}}, \qquad a_2^{(k+1)} = \frac{a_2^{(k)} a_0^{(k)}}{1 - a_2^{(k)}},$$

$$a_0^{(k-1)} = \frac{a_2^{(k)} a_0^{(k)}}{1 - a_0^{(k)}}, \qquad a_1^{(k-1)} = \frac{a_0^{(k)} a_1^{(k)}}{1 - a_0^{(k)}}, \qquad a_2^{(k-1)} = 1 - a_0^{(k)}.$$

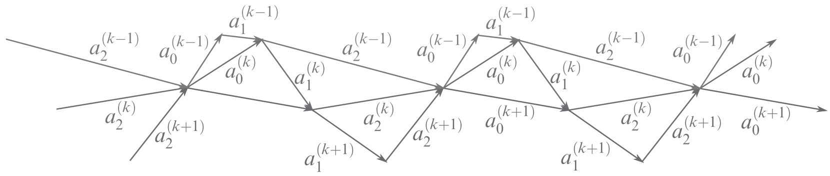

Fig. 6. Adjacent complex probabilities of $\rho_1 \in \Omega(\Sigma_{1,1})$.

Next we give a geometric description of (normalized) type-preserving representations of $\pi_1(\mathcal{O}_{N_{2,1}})$. Pick an element $\rho_2 \in \Omega(N_{2,1})$ and a sequence of elliptic generators $\{Q_j\}$ of $\pi_1(\mathcal{O}_{N_{2,1}})$. Let $(y_1, y_{12}, y_2) \in \mathbf{C}^3$ be the pseudo-Markoff triple associated with $\{\rho_2(Q_j)\}$. Suppose $y_1 y_2 y_{12}' \neq 0$, where $y_{12}' = \mathrm{tr}(\rho_2(Y_1 Y_2^{-1}))/i = y_1 y_2 - y_{12}$. Then the identity $y_1^2 + y_{12}^2 + 4 = y_1 y_{12} y_2$ implies the following identity:

$$b_0 + b_1 + b_2 = 1, \qquad \text{where } b_0 = \frac{y_1}{y_2 y_{12}'},\ b_1 = \frac{4}{y_1 y_2 y_{12}'},\ b_2 = \frac{y_{12}'}{y_1 y_2}.$$

We call the triple $(b_0, b_1, b_2) \in (\mathbf{C}^*)^3$ the *complex probability associated with* $\{\rho_2(Q_j)\}$. We note that the pseudo-Markoff triple (y_1, y_{12}, y_2) with $y_1 y_2 y_{12}' \neq 0$ up to sign (that is, up to equivalence) is recovered from the complex probability by the following identities:

$$y_1^2 = \frac{4b_0}{b_1}, \qquad (y_{12}')^2 = \frac{4b_2}{b_1}, \qquad y_2^2 = \frac{1}{b_2 b_0}.$$

Moreover, we have the following proposition.

Proposition 4.11. *Under Convention* 4.5, *the following hold:*

(1) *For any triple $(b_0, b_1, b_2) \in (\mathbf{C}^*)^3$ such that $b_0 + b_1 + b_2 = 1$ and for any sequence of elliptic generators $\{Q_j\}$ of $\pi_1(\mathcal{O}_{N_{2,1}})$, there is an element $\rho_2 \in \Omega(N_{2,1})$ such that the complex probability associated with $\{\rho_2(Q_j)\}$ is equal to (b_0, b_1, b_2). Moreover, ρ_2 satisfies the following conditions (see Figure* 7*).*

(1.1) *The centers of isometric hemispheres of $\rho_2(Q_j)$ satisfy the following conditions.*

- $c(\rho_2(Q_{6k})) - c(\rho_2(Q_{6k-3} Q_{6k-1})) = b_0$.
- $c(\rho_2(Q_{6k+2})) - c(\rho_2(Q_{6k})) = b_1$.
- $c(\rho_2(Q_{6k} Q_{6k+2})) - c(\rho_2(Q_{6k+2})) = b_2$.
- $c(\rho_2(Q_{6k+3})) - c(\rho_2(Q_{6k} Q_{6k+2})) = b_2$.
- $c(\rho_2(Q_{6k+5})) - c(\rho_2(Q_{6k+3})) = b_1$.
- $c(\rho_2(Q_{6k+3} Q_{6k+5})) - c(\rho_2(Q_{6k+5})) = b_0$.
- $c(\rho_2(Q_{3k+1})) = \frac{1}{2}(c(\rho_2(Q_{3k})) + c(\rho_2(Q_{3k+2})))$.

(1.2) *The isometries $\rho_2(Q_j)$ satisfy the following conditions.*

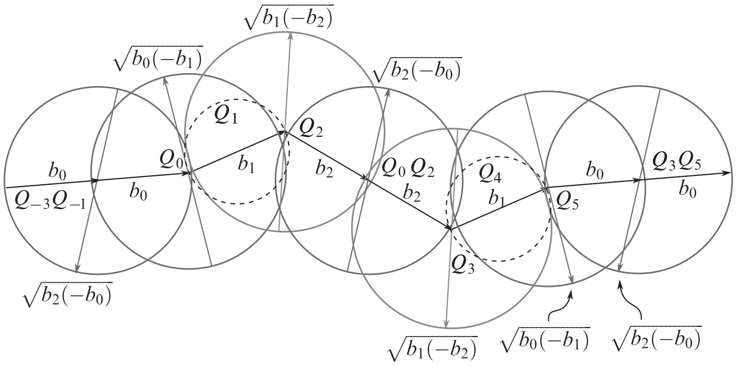

Fig. 7. Isometric hemispheres of elliptic generators of $\pi_1(N_{2,1})$.

- *For any j with $j \equiv 0$ or $5 \pmod 6$, the isometry $\rho_2(Q_j)$ is the π-rotation about the geodesic with endpoints $c(\rho_2(Q_j)) \pm \sqrt{b_0(-b_1)}$.*
- *For any j with $j \equiv 2$ or $3 \pmod 6$, the isometry $\rho_2(Q_j)$ is the π-rotation about the geodesic with endpoints $c(\rho_2(Q_j)) \pm \sqrt{b_1(-b_2)}$.*
- *For any k, the isometry $\rho_2(Q_{3k}Q_{3k+2})$ is the composition of the π-rotation about the geodesic with endpoints $c(\rho_2(Q_{3k}Q_{3k+2})) \pm \sqrt{b_2(-b_0)}$ and the horizontal translation $z \mapsto z-1$. In particular, the isometry $\rho_2(Q_{3k+2}Q_{3k})$ is the composition of the π-rotation about the geodesic with endpoints $c(\rho_2(Q_{3k+2}Q_{3k})) \pm \sqrt{b_2(-b_0)}$ and the horizontal translation $z \mapsto z+1$.*
- *For any k, the isometry $\rho_2(Q_{3k+1})$ is the π-rotation about the geodesic with endpoints $c(\rho_2(Q_{3k}))$ and $c(\rho_2(Q_{3k+2}))$.*

(2) *Conversely, under Convention 4.5, any element $\rho_2 \in \Omega(N_{2,1})$ with the complex probability (b_0, b_1, b_2) associated with $\{\rho_2(Q_j)\}$ for some sequence of elliptic generators $\{Q_j\}$ of $\pi_1(\mathcal{O}_{N_{2,1}})$ satisfies the above conditions.*

PROOF. (1) Pick a triple $(b_0, b_1, b_2) \in (\mathbf{C}^*)^3$ satisfying $b_0 + b_1 + b_2 = 1$ and fix a sequence of elliptic generators $\{Q_j\}$ of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$. Let $(y_1, z, y_2) \in (\mathbf{C}^*)^3$ be a triple of a root of the following polynomial equation:

$$y_1^2 = \frac{4b_0}{b_1}, \qquad z^2 = \frac{4b_2}{b_1}, \qquad y_2^2 = \frac{1}{b_2 b_0}.$$

Replacing y_2 by $-y_2$ if necessary, the triple $(y_1, z, y_2) \in (\mathbf{C}^*)^3$ satisfies

$$y_1^2 + z^2 + 4 = y_1 z y_2$$

and y_2 is not equal to 0. Hence the triple $(y_1, z, y_2) \in (\mathbf{C}^*)^3$ is a pseudo-Markoff triple. Set $y_{12} = y_1 y_2 - z$. By direct calculation, we can see that the triple $(y_1, y_{12}, y_2) \in (\mathbf{C}^*)^3$ is also a pseudo-Markoff triple, namely, the triple satisfies (Eq1) and (Eq2). Hence, for the triple (y_1, y_{12}, y_2), we have an element $\rho_2 \in \Omega(N_{2,1})$ which is as in Lemma 4.4(2.2). By the formula in Lemma 4.4(2.2), we have the following (cf. Figure 7):

- $c(\rho_2(Q_0)) - c(\rho_2(Q_2Q_0)) = b_0$.
- $c(\rho_2(Q_2)) - c(\rho_2(Q_0)) = b_1$.
- $c(\rho_2(Q_0Q_2)) - c(\rho_2(Q_2)) = b_2$.
- $c(\rho_2(Q_1)) = \frac{1}{2}(c(\rho_2(Q_2)) + c(\rho_2(Q_0)))$.
- $\rho_2(Q_0)$ is the π-rotation about the geodesic with endpoints $c(\rho_2(Q_0)) \pm \sqrt{b_0(-b_1)}$.
- $\rho_2(Q_2)$ is the π-rotation about the geodesic with endpoints $c(\rho_2(Q_2)) \pm \sqrt{b_1(-b_2)}$.
- $\rho_2(Q_0Q_2)$ is the composition of the π-rotation about the geodesic with endpoints $c(\rho_2(Q_0Q_2)) \pm \sqrt{b_2(-b_0)}$ and the translation $z \mapsto z-1$.
- $\rho_2(Q_1)$ is the π-rotation about the geodesic with endpoints $c(\rho_2(Q_0))$ and $c(\rho_2(Q_2))$.

Recall that the sequence of elliptic generators $\{Q_j\}$ satisfies the following:

$$Q_j^{K_0} = Q_{-j-1}, \qquad Q_j^{K_2} = Q_{-j+5}, \qquad \text{where } K_0 = Q_1^{Q_0},\ K_2 = Q_1^{Q_2}.$$

Note that the isometry $\rho_2(K_0)$ (resp. $\rho_2(K_2)$) is the π-rotation about the vertical geodesic above 0 (resp. 1). Here a *vertical geodesic* above a point $z \in \mathbf{C}$ means the geodesic $\{z\} \times \mathbf{R}_+$ in $\mathbf{H}^3 = \mathbf{C} \times \mathbf{R}_+$. Hence, by Lemma 4.7, we have $I(\gamma^{\rho_2(K_0)}) = \rho_2(K_0)(I(\gamma))$ and $I(\gamma^{\rho_2(K_2)}) = \rho_2(K_2)(I(\gamma))$ for any $\gamma \in \mathrm{PSL}(2, \mathbf{C})$ such that $\gamma(\infty) \neq \infty$. Thus we obtain the desired result.

(2) Let ρ_2 be an element of $\Omega(N_{2,1})$. Since ρ_2 is normalized, the representation ρ_2 is conjugate to a representation as in Lemma 4.4(2.2) by some Euclidean translation. Since the properties in Proposition 4.11(1) are invariant by Euclidean translations, we have the desired result by the above proof. $\square$

Notation 4.12. Under Convention 4.5, let ρ_2 be an element of $\Omega(N_{2,1})$ and let $\{Q_j\}$ be a sequence of elliptic generators of $\pi_1(\mathcal{O}_{N_{2,1}})$. Let σ be the automorphism of $\pi_1(\mathcal{O}_{N_{2,1}})$ given by Proposition 3.7(1.2), namely,

$$(\sigma(Q_0), \sigma(Q_1), \sigma(Q_2)) = (Q_2, Q_1^{Q_2Q_0}, Q_0^{Q_2}).$$

If the complex probability associated with $\{\rho_2(\sigma^k(Q_j))\}$ is well-defined, then we denote it by $(b_0^{(k)}, b_1^{(k)}, b_2^{(k)})$.

The following lemma can be verified by simple calculation.

Lemma 4.13. *Under Convention 4.5, let ρ_2 be an element of $\Omega(N_{2,1})$ and let $\{Q_j\}$ be a sequence of elliptic generators of $\pi_1(\mathcal{O}_{N_{2,1}})$. Suppose that the*

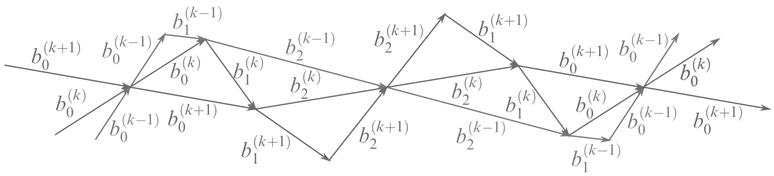

Fig. 8. Adjacent complex probabilities of $\rho_2 \in \Omega(N_{2,1})$.

complex probability $(b_0^{(k)}, b_1^{(k)}, b_2^{(k)})$ *associated with* $\{\rho_2(\sigma^k(Q_j))\}$ *is well-defined for any* $k \in \mathbf{Z}$. *Then we have the following identities (cf. Figure* 8*):*

$$b_0^{(k+1)} = 1 - b_2^{(k)}, \qquad b_1^{(k+1)} = \frac{b_1^{(k)} b_2^{(k)}}{1 - b_2^{(k)}}, \qquad b_2^{(k+1)} = \frac{b_2^{(k)} b_0^{(k)}}{1 - b_2^{(k)}},$$

$$b_0^{(k-1)} = \frac{b_2^{(k)} b_0^{(k)}}{1 - b_0^{(k)}}, \qquad b_1^{(k-1)} = \frac{b_0^{(k)} b_1^{(k)}}{1 - b_0^{(k)}}, \qquad b_2^{(k-1)} = 1 - b_0^{(k)}.$$

As a consequence of Propositions 4.8, 4.11 and Lemmas 4.10, 4.13, we have the following corollary.

Corollary 4.14. *Under Convention* 4.5, *let* ρ_1 *and* ρ_2 *be elements of* $\Omega(\Sigma_{1,1})$ *and* $\Omega(N_{2,1})$, *respectively. Let* $\{P_j\}$ *and* $\{Q_j\}$ *be sequences of elliptic generators of* $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ *and* $\pi_1(\mathcal{O}_{N_{2,1}})$, *respectively. Suppose that the complex probabilities* (a_0, a_1, a_2) *and* (b_0, b_1, b_2) *associated with* $\{\rho_1(P_j)\}$ *and* $\{\rho_2(Q_j)\}$, *respectively, are well-defined. Then the following hold.*

(1) $(a_0, a_1, a_2) = (b_0, b_1, b_2)$ *and* $c(\rho_1(P_1)) = c(\rho_2(Q_2Q_0))$ *if and only if* $(\rho_1(P_{6j+2}), \rho_1(P_{6j+3})) = (\rho_2(Q_{6j}), \rho_2(Q_{6j+2}))$ *for some* $j \in \mathbf{Z}$. *Moreover, if these conditions hold, then the following identities hold for any* $j, k \in \mathbf{Z}$*:*

$$(a_0^{(k)}, a_1^{(k)}, a_2^{(k)}) = (b_0^{(k)}, b_1^{(k)}, b_2^{(k)}),$$

$$(\rho_1(\xi^k(P_{6j+2})), \rho_1(\xi^k(P_{6j+3}))) = (\rho_2(\sigma^k(Q_{6j})), \rho_2(\sigma^k(Q_{6j+2}))).$$

(2) $(a_0, a_1, a_2) = (b_2, b_1, b_0)$ *and* $c(\rho_1(P_1)) = c(\rho_2(Q_2Q_0))$ *if and only if* $(\rho_1(P_{6j+5}), \rho_1(P_{6j+6})) = (\rho_2(Q_{6j+3}), \rho_2(Q_{6j+5}))$ *for some* $j \in \mathbf{Z}$. *Moreover, if these conditions hold, then the following identities hold for any* $j, k \in \mathbf{Z}$*:*

$$(a_0^{(k)}, a_1^{(k)}, a_2^{(k)}) = (b_2^{(-k)}, b_1^{(-k)}, b_0^{(-k)}),$$

$$(\rho_1(\xi^k(P_{6j+5})), \rho_1(\xi^k(P_{6j+6}))) = (\rho_2(\sigma^{-k}(Q_{6j+3})), \rho_2(\sigma^{-k}(Q_{6j+5}))).$$

At the end of this section, we prove the following lemma.

Lemma 4.15. *Let ρ_1 and ρ_2 be type-preserving* $\mathrm{PSL}(2, \mathbf{C})$*-representations of $\pi_1(\Sigma_{1,1})$ and $\pi_1(N_{2,1})$, respectively. Let $\tilde{\rho}_1$ and $\tilde{\rho}_2$, respectively, be the unique extensions of ρ_1 and ρ_2 given by Proposition* 4.1. *Then ρ_1 and ρ_2 are commensurable if and only if $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are commensurable.*

Proof. We first show the if part. Suppose that $\tilde{\rho}_1$ and $\tilde{\rho}_2$ are commensurable, i.e., there exist double coverings $p_1 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{\Sigma_{1,1}}$ and $p_2 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{N_{2,1}}$ such that $\tilde{\rho}_1 \circ (p_1)_* = \tilde{\rho}_2 \circ (p_2)_*$. By the correspondence between double coverings described in Section 2 (see Figure 2), there exist double coverings $\tilde{p}_1 : \Sigma_{1,2} \to \Sigma_{1,1}$ and $\tilde{p}_2 : \Sigma_{1,2} \to N_{2,1}$ such that $p_{\Sigma_{1,1}} \circ \tilde{p}_1 = p_1 \circ p_{\Sigma_{1,2}}$ and $p_{N_{2,1}} \circ \tilde{p}_2 = p_2 \circ p_{\Sigma_{1,2}}$. Hence we have the following identity:

$$\begin{aligned}
\rho_1 \circ (\tilde{p}_1)_* &= \tilde{\rho}_1 \circ (p_{\Sigma_{1,1}})_* \circ (\tilde{p}_1)_* \\
&= \tilde{\rho}_1 \circ (p_1) \circ (p_{\Sigma_{1,2}})_* \\
&= \tilde{\rho}_2 \circ (p_2) \circ (p_{\Sigma_{1,2}})_* \\
&= \tilde{\rho}_2 \circ (p_{N_{2,1}})_* \circ (\tilde{p}_2)_* = \rho_2 \circ (\tilde{p}_2)_*.
\end{aligned}$$

Next we show the only if part. Suppose that ρ_1 and ρ_2 are commensurable, namely there exist double coverings $p_1 : \Sigma_{1,2} \to \Sigma_{1,1}$ and $p_2 : \Sigma_{1,2} \to N_{2,1}$ such that $\rho_1 \circ (p_1)_* = \rho_2 \circ (p_2)_*$. By the correspondence between double coverings described in Section 2 (see Figure 2), we have double coverings $\check{p}_1 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{\Sigma_{1,1}}$ and $\check{p}_2 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{N_{2,1}}$ such that $p_{\Sigma_{1,1}} \circ p_1 = \check{p}_1 \circ p_{\Sigma_{1,2}}$ and $p_{N_{2,1}} \circ p_2 = \check{p}_2 \circ p_{\Sigma_{1,2}}$. Hence we have the following identity (see Figure 9):

$$\tilde{\rho}_1 \circ (\check{p}_1)_* \circ (p_{\Sigma_{1,2}})_* = \rho_1 \circ (p_1)_* = \rho_2 \circ (p_2)_* = \tilde{\rho}_2 \circ (\check{p}_2)_* \circ (p_{\Sigma_{1,2}})_*.$$

$$\begin{tikzcd}
 & & \pi_1(\Sigma_{1,2}) \arrow[ddll, "(p_1)_*"'] \arrow[d, "(p_{\Sigma_{1,2}})_*"] \arrow[ddrr, "(p_2)_*"] & & \\
 & & \pi_1(\mathcal{O}_{\Sigma_{1,2}}) \arrow[dl, "(\check{p}_1)_*"'] \arrow[dr, "(\check{p}_2)_*"] & & \\
\pi_1(\Sigma_{1,1}) \arrow[r, "(p_{\Sigma_{1,1}})_*"] \arrow[ddrr, "\rho_1"'] & \pi_1(\mathcal{O}_{\Sigma_{1,1}}) \arrow[ddr, "\tilde{\rho}_1"] & & \pi_1(\mathcal{O}_{N_{2,1}}) \arrow[ddl, "\tilde{\rho}_2"'] & \pi_1(N_{2,1}) \arrow[l, "(p_{N_{2,1}})_*"'] \arrow[ddll, "\rho_2"] \\
 & & & & \\
 & & \mathrm{PSL}(2, \mathbf{C}) & &
\end{tikzcd}$$

Fig. 9

This means that both $\tilde{\rho}_1 \circ (\check{p}_1)_*$ and $\tilde{\rho}_2 \circ (\check{p}_2)_*$ are extensions of $\rho := \rho_1 \circ (p_1)_* = \rho_2 \circ (p_2)_*$ of $\pi_1(\Sigma_{1,2})$ to $\pi_1(\mathcal{O}_{\Sigma_{1,2}})$. Note that $\pi_1(\mathcal{O}_{\Sigma_{1,2}})$ is generated by $\pi_1(\Sigma_{1,2}) = \langle Z_1, Z_2, Z_3 \rangle$ and the element R_1 and that the generators satisfy the following identities (see Section 2):

$$R_1 Z_j R_1^{-1} = Z_j^{-1} \qquad \text{for } j = 1, 2, 3.$$

Hence both $\tilde{\rho}_1 \circ (\check{p}_1)_*(R_1)$ and $\tilde{\rho}_2 \circ (\check{p}_2)_*(R_1)$ are solutions of the following system of equation in $\mathrm{PSL}(2, \mathbf{C})$.

$$g\rho(Z_j)g^{-1} = \rho(Z_j)^{-1} \qquad \text{for } j = 1, 2, 3.$$

On the other hand, since ρ is irreducible, the system of equations have at most one solution. Hence we have $\tilde{\rho}_1 \circ (\check{p}_1)_*(R_1) = \tilde{\rho}_2 \circ (\check{p}_2)_*(R_1)$, and therefore we have $\tilde{\rho}_1 \circ (\check{p}_1)_* = \tilde{\rho}_2 \circ (\check{p}_2)_*$. □

Remark 4.16. Let ρ_1, ρ_2, $\tilde{\rho}_1$ and $\tilde{\rho}_2$ be as in Lemma 4.15 and assume that ρ_1 and ρ_2 (and so $\tilde{\rho}_1$ and $\tilde{\rho}_2$) are commensurable. Then we can easily see, as in the proof of Proposition 4.1, that if one of the representations ρ_1, ρ_2, $\tilde{\rho}_1$ and $\tilde{\rho}_2$ is faithful, then all of them are faithful.

5. Main theorem

In this section, we give a partial answer to Problem 2.3. By Lemma 4.15, we may only consider the problem for the quotient orbifolds. Our partial answer to the commensurability problem for representations of the fundamental groups of the orbifolds $\mathcal{O}_{\Sigma_{1,1}}$ and $\mathcal{O}_{N_{2,1}}$ is given as follows.

Theorem 5.1. *Under Convention* 4.5*, the following hold:*

(1) *Let ρ_2 be an element of $\Omega(N_{2,1})$. Suppose that ρ_2 is faithful. Then the following conditions are equivalent.*

(i) *There exists a faithful representation $\rho_1 \in \Omega(\Sigma_{1,1})$ which is commensurable with ρ_2.*

(ii) *There exist a sequence of elliptic generators $\{Q_j\}$ of $\pi_1(\mathcal{O}_{N_{2,1}})$ and an integer k_0 such that the complex probability (b_0, b_1, b_2) associated with $\{\rho_2(Q_j)\}$ satisfies the following identity under Notation* 4.12 *(cf. Figure* 10*):*

$$(b_0^{(k_0)}, b_1^{(k_0)}, b_2^{(k_0)}) = (b_2, b_1, b_0).$$

(iii) *There exists a sequence of elliptic generators $\{Q_j\}$ of $\pi_1(\mathcal{O}_{N_{2,1}})$ such that the complex probability (b_0, b_1, b_2) associated with $\{\rho_2(Q_j)\}$ satisfies one of the following identities:*

(α) $(b_0^{(0)}, b_1^{(0)}, b_2^{(0)}) = (b_2, b_1, b_0)$,

(β) $(b_0^{(1)}, b_1^{(1)}, b_2^{(1)}) = (b_2, b_1, b_0)$.

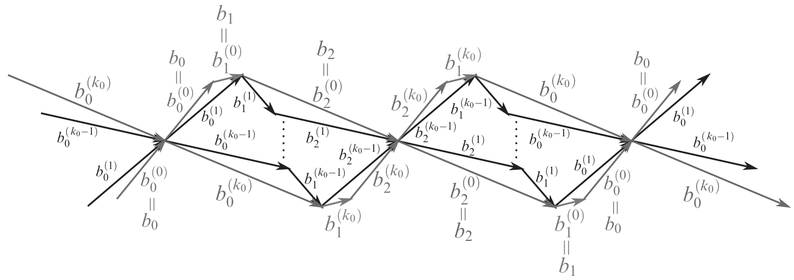

Fig. 10. $(b_0^{(k_0)}, b_1^{(k_0)}, b_2^{(k_0)}) = (b_2, b_1, b_0)$.

(2) *If the conditions in* (1) *hold, the representation* ρ_1 *is unique up to precomposition by an automorphism of* $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ *preserving* K.

(3) *Moreover, the following hold:*

(α) ρ_2 *extends to a type-preserving* $\mathrm{PSL}(2, \mathbf{C})$*-representation of* $\pi_1(\mathcal{O}_\alpha)$ *if and only if* ρ_2 *satisfies the condition* (iii)-(α). *Moreover, if these conditions are satisfied, the extension is unique.*

(β) ρ_2 *extends to a type-preserving* $\mathrm{PSL}(2, \mathbf{C})$*-representation of* $\pi_1(\mathcal{O}_\beta)$ *if and only if* ρ_2 *satisfies the condition* (iii)-(β). *Moreover, if these conditions are satisfied, the extension is unique.*

Remark 5.2. By using this theorem, we can prove the "converse" condition, namely, we can give a condition for a faithful type-preserving $\mathrm{PSL}(2, \mathbf{C})$-representation of $\pi_1(\Sigma_{1,1})$ to be commensurable with that of $\pi_1(N_{2,1})$ (see [4]).

Proof. We prove (1) by proving the implications (iii) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (ii).

(iii) $\Rightarrow$ (ii). This is obvious.

(ii) $\Rightarrow$ (iii). Suppose that there exist a sequence of elliptic generators $\{Q_j\}$ of $\pi_1(\mathcal{O}_{N_{2,1}})$ and an integer k_0 such that the complex probability (b_0, b_1, b_2) associated with $\{\rho_2(Q_j)\}$ satisfies the following identity:

$$(b_0^{(k_0)}, b_1^{(k_0)}, b_2^{(k_0)}) = (b_2, b_1, b_0).$$

Recall that the triple $(b_0^{(k_0)}, b_1^{(k_0)}, b_2^{(k_0)})$ is the complex probability associated with $\{\rho_2(\sigma^{k_0}(Q_j))\}$, where σ is the automorphism of $\pi_1(\mathcal{O}_{N_{2,1}})$ given as in Notation 4.12. Since ρ_2 is faithful, we have $\mathrm{tr}(\rho_2(\gamma)) \neq 0$ for any $\gamma \in \pi_1(N_{2,1})$. Hence the complex probability $(b_0^{(k)}, b_1^{(k)}, b_2^{(k)})$ associated with $\{\rho_2(\sigma^k(Q_j))\}$ is well-defined for any $k \in \mathbf{Z}$. By the assumption $(b_0^{(k_0)}, b_1^{(k_0)}, b_2^{(k_0)}) = (b_2, b_1, b_0)$ and Lemma 4.13, we have

$$(b_0^{(k_0 \pm l)}, b_1^{(k_0 \pm l)}, b_2^{(k_0 \pm l)}) = (b_2^{(\mp l)}, b_1^{(\mp l)}, b_0^{(\mp l)})$$

for any $l \in \mathbf{Z}$. In particular we have

$$(b_0^{(k_0-k_0/2)}, b_1^{(k_0-k_0/2)}, b_2^{(k_0-k_0/2)}) = (b_2^{(k_0/2)}, b_1^{(k_0/2)}, b_0^{(k_0/2)}) \qquad \text{if } k_0 \text{ is even,}$$

$$(b_0^{(k_0-(k_0-1)/2)}, b_1^{(k_0-(k_0-1)/2)}, b_2^{(k_0-(k_0-1)/2)})$$
$$= (b_2^{((k_0-1)/2)}, b_1^{((k_0-1)/2)}, b_0^{((k_0-1)/2)}) \qquad \text{if } k_0 \text{ is odd.}$$

Hence, by replacing $\{Q_j\}$ with $\{\sigma^{k_0/2}(Q_j)\}$ or $\{\sigma^{(k_0-1)/2}(Q_j)\}$ according to whether k_0 is even or odd, we obtain the desired result.

(ii) $\Rightarrow$ (i). Suppose that there exist a sequence of elliptic generators $\{Q_j\}$ of $\pi_1(\mathcal{O}_{N_{2,1}})$ and an integer k_0 such that the complex probability (b_0, b_1, b_2) associated with $\{\rho_2(Q_j)\}$ satisfies the following identity:

$$(b_0^{(k_0)}, b_1^{(k_0)}, b_2^{(k_0)}) = (b_2, b_1, b_0).$$

Pick a sequence of elliptic generators $\{P_j\}$ of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$. Then by Proposition (1), there exists a (normalized) type-preserving representation ρ_1 of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ such that the complex probability associated with $\{\rho_1(P_j)\}$ is equal to (b_0, b_1, b_2). After taking conjugate of ρ_1 by a parallel translation, we may assume that $c(\rho_1(P_1)) = c(\rho_2(Q_2Q_0))$. Then, by Corollary 4.14(1), we see that

$$(\rho_1(P_2), \rho_1(P_3)) = (\rho_2(Q_0), \rho_2(Q_2)).$$

By Lemmas 4.10 and 4.13, we see that the complex probability associated with $\{\rho_1(\xi^{k_0}(P_j))\}$ is equal to the complex probability $(b_0^{(k_0)}, b_1^{(k_0)}, b_2^{(k_0)})$ associated with $\{\rho_2(\sigma^{k_0}(Q_j))\}$, where ξ and σ are, respectively, the automorphisms of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ and $\pi_1(\mathcal{O}_{N_{2,1}})$ given by Notations 4.9 and 4.12. Hence, by the assumption $(b_0^{(k_0)}, b_1^{(k_0)}, b_2^{(k_0)}) = (b_2, b_1, b_0)$ and Corollary 4.14(2), we have

$$(\rho_1(\xi^{k_0}(P_5)), \rho_1(\xi^{k_0}(P_6))) = (\rho_2(Q_2^{K_2}), \rho_2(Q_0^{K_2})).$$

Hence we have

$$(\rho_1(P_2), \rho_1(P_3), \rho_1(\xi^{k_0}(P_5)), \rho_1(\xi^{k_0}(P_6))) = (\rho_2(Q_0), \rho_2(Q_2), \rho_2(Q_2^{K_2}), \rho_2(Q_0^{K_2})).$$

CLAIM 5.3. *Let* (R_0, R_1, R_2, R_3) *be the generator system of* $\pi_1(\mathcal{O}_{\Sigma_{1,2}})$ *given in Section* 2.

(1) *There is a double covering* $p_1 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{\Sigma_{1,1}}$ *such that*

$$((p_1)_*(R_0), (p_1)_*(R_1), (p_1)_*(R_2), (p_1)_*(R_3)) = (P_2, P_3, \xi^{k_0}(P_5), \xi^{k_0}(P_6)).$$

(2) *There is a double covering* $p_2 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{N_{2,1}}$ *such that*

$$((p_2)_*(R_0), (p_2)_*(R_1), (p_2)_*(R_2), (p_2)_*(R_3)) = (Q_0, Q_2, Q_2^{K_2}, Q_0^{K_2}).$$

Proof. (2) can be seen by choosing p_2 to be the covering corresponding to the epimorphism $\phi_2 : \pi_1(\mathcal{O}_{N_{2,1}}) \to \mathbf{Z}/2\mathbf{Z}$ defined by the following formula (see Figure 2):

$$\phi_2(Q_j) = \begin{cases} 0 & \text{if } j = 0 \text{ or } 2, \\ 1 & \text{if } j = 1. \end{cases}$$

To prove (1), let $q_1 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{\Sigma_{1,1}}$ be the double covering such that the following holds (see Figure 2):

$$((q_1)_*(R_0), (q_1)_*(R_1), (q_1)_*(R_2), (q_1)_*(R_3)) = (P_0, P_1, P_0^K, P_1^K).$$

Let f be a self-homeomorphism of $\mathcal{O}_{\Sigma_{1,1}}$ such that f_* maps (P_0, P_1, P_2) to (P_2, P_3, P_1^K), and consider the double covering $p_1^{(0)} := f \circ q_1 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{\Sigma_{1,1}}$. Then we have

$$((p_1^{(0)})_*(R_0), (p_1^{(0)})_*(R_1), (p_1^{(0)})_*(R_2), (p_1^{(0)})_*(R_3)) = (P_2, P_3, P_5, P_6).$$

Let $\tilde{\xi}$ be the self-homeomorphism of $\mathcal{O}_{\Sigma_{1,2}}$ such that

$$((\tilde{\xi})_*(R_0), (\tilde{\xi})_*(R_1), (\tilde{\xi})_*(R_2), (\tilde{\xi})_*(R_3)) = (R_0, R_1, R_3, R_2^{R_3}).$$

Then the double covering $p_1 := p_1^{(0)} \circ \tilde{\xi} : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{\Sigma_{1,1}}$ satisfies the desired condition.

By Claim 5.3, there are double coverings $p_1 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{\Sigma_{1,1}}$ and $p_2 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{N_{2,1}}$ satisfying the following identity:

$$\begin{aligned}
&(\rho_1 \circ (p_1)_*(R_0), \rho_1 \circ (p_1)_*(R_1), \rho_1 \circ (p_1)_*(R_2), \rho_1 \circ (p_1)_*(R_3)) \\
&\quad = (\rho_1(P_2), \rho_1(P_3), \rho_1(\xi^{k_0}(P_5)), \rho_1(\xi^{k_0}(P_6))) \\
&\quad = (\rho_2(Q_0), \rho_2(Q_2), \rho_2(Q_2^{K_2}), \rho_2(Q_0^{K_2})) \\
&\quad = (\rho_2 \circ (p_2)_*(R_0), \rho_2 \circ (p_2)_*(R_1), \rho_2 \circ (p_2)_*(R_2), \rho_2 \circ (p_2)_*(R_3)).
\end{aligned}$$

Hence $\rho_1 \circ (p_1)_* = \rho_2 \circ (p_2)_*$, namely, the representation ρ_2 is commensurable with ρ_1. By Remark 4.16, ρ_1 is faithful. Thus we obtain the desired representation ρ_1.

(i) $\Rightarrow$ (ii). Suppose that there exists a faithful (normalized) type-preserving $\mathrm{PSL}(2, \mathbf{C})$-representation ρ_1 of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ which is commensurable with ρ_2, i.e., there exist double coverings $p_1 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{\Sigma_{1,1}}$ and $p_2 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{N_{2,1}}$ such that $\rho_1 \circ (p_1)_* = \rho_2 \circ (p_2)_*$. Recall that $(p_2)_*(\pi_1(\mathcal{O}_{\Sigma_{1,2}}))$ is equal to the kernel of the epimorphism $\phi_2 : \pi_1(\mathcal{O}_{\Sigma_{1,1}}) \to \mathbf{Z}/2\mathbf{Z}$ and that the kernel of ϕ_2 is equal to the subgroup of $\pi_1(\mathcal{O}_{N_{2,1}})$ generated by the quadruple $(Q_0, Q_2, Q_2^{K_2}, Q_0^{K_2})$. Hence we have $Q_0, Q_2 \in (p_2)_*(\pi_1(\mathcal{O}_{\Sigma_{1,2}}))$. Set $P^{(0)} = (p_1 \circ p_2^{-1})_*(Q_0)$ and $P^{(1)} = (p_1 \circ p_2^{-1})_*(Q_2)$.

Claim 5.4. *The ordered triple* $(K^{-1}P^{(1)}P^{(0)}, P^{(0)}, P^{(1)})$ *is an elliptic generator triple of* $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$.

Proof. Note that $P^{(0)}$ and $P^{(1)}$ have order 2, because

(1) $(p_1 \circ p_2^{-1})_* : (p_2)_*(\pi_1(\mathcal{O}_{\Sigma_{1,2}})) \to (p_1)_*(\pi_1(\mathcal{O}_{\Sigma_{1,2}}))$ is an isomorphism and

(2) Q_0 and Q_2 have order 2.

By using the third assertion of Proposition 4.11(1.2) and the fact that $\rho_1(K)$ is the horizontal translation $z \mapsto z+1$, we see that $\rho_1(K^{-1})\rho_2(Q_2Q_0)$ has order 2. By the definition of $P^{(0)}$ and $P^{(1)}$ and by the identity $\rho_1 \circ (p_1)_* = \rho_2 \circ (p_2)_*$, we have $\rho_1(P^{(1)}P^{(0)}) = \rho_2(Q_2Q_0)$. Hence $\rho_1(K^{-1}P^{(1)}P^{(0)})$ has order 2. Since ρ_1 is faithful, this implies that $K^{-1}P^{(1)}P^{(0)}$ has order 2. Hence, by Remark 3.2, the triple $(K^{-1}P^{(1)}P^{(0)}, P^{(0)}, P^{(1)})$ is an elliptic generator triple of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$. □

By Claim 5.4 and Proposition 3.3(1.1), $((P^{(1)})^{K^{-1}}, K^{-1}P^{(1)}P^{(0)}, P^{(0)})$ is also an elliptic generator triple of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$. Let $\{P_j\}$ be the sequence of elliptic generators of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ associated with this triple. Then $\rho_1(P_2) = \rho_1(P^{(0)}) = \rho_2(Q_0)$ and $\rho_1(P_3) = \rho_1(P^{(1)}) = \rho_2(Q_2)$. This implies, together with Corollary 4.14(1), that the complex probability associated with $\{\rho_1(P_j)\}$ is equal to (b_0, b_1, b_2). Set $(P^{(2)}, P^{(3)}) = ((p_1 \circ p_2^{-1})_*(Q_2^{K_2}), (p_1 \circ p_2^{-1})_*(Q_0^{K_2}))$. Then, by a parallel argument, the triple $((P^{(3)})^{K^{-2}}, (K^{-1}P^{(3)}P^{(2)})^{K^{-1}}, (P^{(2)})^{K^{-1}})$ is an elliptic generator triple of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$. Let $\{P_j'\}$ be a sequence of elliptic generators of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ associated with this triple. Then $\rho_1(P_5') = \rho_1(P^{(2)}) = \rho_2(Q_2^{K_2})$ and $\rho_1(P_6') = \rho_1(P^{(3)}) = \rho_2(Q_0^{K_2})$. This implies, together with Corollary 4.14(2), that the complex probability associated with $\{\rho_1(P_j')\}$ is equal to (b_2, b_1, b_0). Since $\rho_1(K)\rho_2(Q_0Q_2) = \rho_1(K^{-1})\rho_2(Q_0^{K_2}Q_2^{K_2})$ by Proposition 4.11(1.2), we have

$$\begin{aligned}\rho_1(P_4) = \rho_1(KP_2P_3) &= \rho_1(K)\rho_2(Q_0Q_2)\\ &= \rho_1(K^{-1})\rho_2(Q_0^{K_2}Q_2^{K_2}) = \rho_1(K^{-1}P_6'P_5') = \rho_1(P_4').\end{aligned}$$

Since ρ_1 is faithful, this implies $P_4 = P_4'$. Hence, by Proposition 3.3, there is an integer k_0 such that $P_j' = \xi^{k_0}(P_j)$. By Lemmas 4.10 and 4.13, the complex probability associated with $\{\rho_1(\xi^k(P_j))\}$ is equal to the complex probability $(b_0^{(k)}, b_1^{(k)}, b_2^{(k)})$ associated with $\{\rho_2(\sigma^k(Q_j))\}$ for any $k \in \mathbf{Z}$. Hence we have

$$(b_0^{(k_0)}, b_1^{(k_0)}, b_2^{(k_0)}) = (b_2, b_1, b_0).$$

Thus the proof of the assertion (1) is complete.

Next we prove the assertion (2). Let ρ_1 and ρ_1' be type-preserving $\mathrm{PSL}(2,\mathbf{C})$-representations of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ such that they are commensurable with ρ_2. Then there exist coverings $p_1, p_1' : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{\Sigma_{1,1}}$ and $p_2 : \mathcal{O}_{\Sigma_{1,2}} \to$

$\mathcal{O}_{N_{2,1}}$ such that $\rho_1 \circ (p_1)_* = \rho_2 \circ (p_2)_*$ and $\rho_1' \circ (p_1')_* = \rho_2 \circ (p_2)_*$. By Claim 5.4, the following triples are elliptic generator triples of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$:

$$(P_0, P_1, P_2) := (K^{-1}(p_1 \circ p_2^{-1})_*(Q_2Q_0), (p_1 \circ p_2^{-1})_*(Q_0), (p_1 \circ p_2^{-1})_*(Q_2)),$$

$$(P_0', P_1', P_2') := (K^{-1}(p_1' \circ p_2^{-1})_*(Q_2Q_0), (p_1' \circ p_2^{-1})_*(Q_0), (p_1' \circ p_2^{-1})_*(Q_2)).$$

Since ρ_1 and ρ_1' are commensurable with ρ_2, we have the following identity

$$\begin{aligned}(\rho_1(P_0), \rho_1(P_1), \rho_1(P_2)) &= (\rho_1(K^{-1})\rho_2(Q_2Q_0), \rho_2(Q_0), \rho_2(Q_2)) \\ &= (\rho_1'(K^{-1})\rho_2(Q_2Q_0), \rho_2(Q_0), \rho_2(Q_2)) \\ &= (\rho_1'(P_0'), \rho_1'(P_1'), \rho_1'(P_2')).\end{aligned}$$

By Proposition 3.3(2), there is an automorphism f of $\pi_1(\mathcal{O}_{\Sigma_{1,1}})$ preserving K which maps (P_0, P_1, P_2) to (P_0', P_1', P_2'). Hence we have $\rho_1 = \rho_1' \circ f$.

Finally we prove the assertion (3).

The if part of (α). Suppose that there exists a sequence of elliptic generators $\{Q_j\}$ of $\pi_1(\mathcal{O}_{N_{2,1}})$ such that the complex probability (b_0, b_1, b_2) associated with $\{\rho_2(Q_j)\}$ satisfies the following identity:

$$(b_0^{(0)}, b_1^{(0)}, b_2^{(0)}) = (b_2, b_1, b_0), \qquad \text{namely}, \qquad b_0 = b_2.$$

Let $\tilde{K}$ be the horizontal translation $z \mapsto z + 1$. For simplicity of notation, we write (g_0, g_1, g_2) instead of $(\rho_2(Q_2), \rho_2(Q_1), \tilde{K}\rho_2(K_0))$. We first show that there is a representation ρ_2^* from $\pi_1(\mathcal{O}_\alpha) = \langle S_0, S_1, S_2 \mid S_0^2 = S_1^2 = S_2^2 = 1, (S_1S_2)^2 = 1\rangle$ to $\mathrm{PSL}(2, \mathbf{C})$ sending (S_0, S_1, S_2) to (g_0, g_1, g_2). Since $g_0 = \rho_2(Q_2)$ and $g_1 = \rho_2(Q_1)$, we have $g_0^2 = g_1^2 = 1$. Thus the existence of the representation ρ_2^* is guaranteed by the following claim.

Claim 5.5. (1) *g_2 is the π-rotation about the axis which is the image of the vertical geodesic* $\mathrm{Axis}(\rho_2(K_0))$ *by the translation $z \mapsto z + \frac{1}{2}$, where* $\mathrm{Axis}(A)$ *denotes the axis of $A \in \mathrm{PSL}(2, \mathbf{C})$. In particular, $g_2^2 = 1$.*

(2) *The axes of the π-rotations g_1 and g_2 intersect orthogonally and hence g_1g_2 is also a π-rotation. In particular, $(g_1g_2)^2 = 1$.*

Proof. (1) Since $\tilde{K}(z) = z + 1$ and since $\rho_2(K_0)$ is the π-rotation about the vertical geodesic $\mathrm{Axis}(\rho_2(K_0))$, the isometry $g_2 = \tilde{K}\rho_2(K_0)$ is also the π-rotation about the vertical geodesic, which is the image of $\mathrm{Axis}(\rho_2(K_0))$ by the translation $z \mapsto z + \frac{1}{2}$.

(2) Note that $\rho_2(K_0)$ is the π-rotation about the vertical geodesic above $c(\rho_2(Q_2Q_0))$, because we have the following identity by Lemma 4.7:

$$\rho_2(K_0)I(\rho_2(Q_2Q_0)) = I(\rho_2(Q_2Q_0K_0)) = I(\rho_2(K_2Q_2Q_0)) = I(\rho_2(Q_2Q_0)).$$

Thus the axis of g_2 is the vertical geodesic above $c(\rho_2(Q_2Q_0)) + \frac{1}{2}$ by Claim 5.5(1). Moreover, we have the following identity:

$$
\begin{aligned}
&c(\rho_2(Q_2Q_0)) + \frac{1}{2} \\
&\quad = c(\rho_2(Q_2Q_0)) + \frac{1}{2}(c(\rho_2(Q_0Q_2)) - c(\rho_2(Q_2Q_0))) && \text{by Proposition 4.11(1.1)} \\
&\quad = \frac{1}{2}(c(\rho_2(Q_2Q_0)) + c(\rho_2(Q_0Q_2))) \\
&\quad = \frac{1}{2}(c(\rho_2(Q_2Q_0)) + b_0 - b_2 + c(\rho_2(Q_0Q_2))) && \text{by the assumption } b_0 = b_2 \\
&\quad = \frac{1}{2}(c(\rho_2(Q_0)) + c(\rho_2(Q_2))) && \text{by Proposition (1.1)} \\
&\quad = c(\rho_2(Q_1)) && \text{by Proposition (1.1).}
\end{aligned}
$$

Hence g_2 is the π-rotation about the vertical geodesic above $c(\rho_2(Q_1)) = c(g_1)$ and hence the axes of g_1 and g_2 intersect orthogonally. $\square$

Recall that $\pi_1(\mathcal{O}_{N_{2,1}})$ is identified with a subgroup of $\pi_1(\mathcal{O}_\alpha)$ and their generators satisfy the following identities:

$$Q_0 = S_0^{S_2}, \qquad Q_1 = S_1, \qquad Q_2 = S_0.$$

Since g_2 is the π-rotation about the vertical geodesic above $c(\rho_2(Q_1))$, we have $c(\rho_2(Q_0)) = c(\rho_2(Q_2)^{g_2}) = c(\rho_2^*(S_0^{S_2}))$. This together with the assumption $b_0 = b_2$ implies that $\rho_2(Q_0) = \rho_2(Q_2)^{g_2} = \rho_2^*(S_0^{S_2})$ by Proposition 4.11. Hence the restriction of ρ_2^* to $\pi_1(\mathcal{O}_{N_{2,1}})$ is equal to the original representation ρ_2.

The only if part of (α). Suppose that ρ_2 extends to a type-preserving representation $\tilde{\rho}_2$ of $\pi_1(\mathcal{O}_\alpha)$. Pick a sequence of elliptic generators $\{Q_j\}$ of $\pi_1(\mathcal{O}_{N_{2,1}})$. Since ρ_2 is faithful, we have $\operatorname{tr}(\rho_2(Y_1)) \operatorname{tr}(\rho_2(Y_2)) \operatorname{tr}(\rho_2(Y_1 Y_2^{-1})) \neq 0$, where $Y_1 = Q_0Q_1$ and $Y_2 = Q_0Q_2$. Thus the complex probability (b_0, b_1, b_2) associated with $\{\rho_2(Q_j)\}$ is well-defined. Since $\pi_1(\mathcal{O}_{N_{2,1}})$ is identified with a subgroup of $\pi_1(\mathcal{O}_\alpha)$, the isometry $\tilde{\rho}_2(S_2)$ satisfies the following identities:

$$(\tilde{\rho}_2(S_2))^2 = 1, \qquad \tilde{\rho}_2(Q_0^{S_2}) = \rho_2(Q_2), \qquad (\tilde{\rho}_2(Q_1S_2))^2 = 1.$$

Claim 5.6. *The isometry $\tilde{\rho}_2(S_2)$ is the π-rotation about the vertical geodesic above $\frac{1}{2}(c(\rho_2(Q_2Q_0)) + c(\rho_2(Q_0Q_2))) = c(\rho_2(Q_1))$.*

Proof. Since $(\tilde{\rho}_2(S_2))^2 = 1$, $\tilde{\rho}_2(S_2)$ is either the identity or a π-rotation. If $\tilde{\rho}_2(S_2) = 1$, then $\rho_2(K_{N_{2,1}}) = \tilde{\rho}_2((S_1^{S_0}S_2)^2) = \tilde{\rho}_2((S_1^{S_0})^2) = 1$, a contradiction. Hence $\tilde{\rho}_2(S_2)$ is a π-rotation. By $\tilde{\rho}_2(Q_0^{S_2}) = \rho_2(Q_2)$ and $(\tilde{\rho}_2(Q_1S_2))^2 = 1$, we have $\tilde{\rho}_2(K_0^{S_2}) = \rho_2(K_2)$. Hence $\tilde{\rho}_2(S_2)$ maps $\operatorname{Fix}(\rho_2(K_0)) = \{c(\rho_2(Q_2Q_0)), \infty\}$

to $\mathrm{Fix}(\rho_2(K_2)) = \{c(\rho_2(Q_0Q_2)), \infty\}$. Since $\tilde{\rho}_2(S_2)$ has order 2, the isometry $\tilde{\rho}_2(S_2)$ must fix ∞. (Otherwise $c(\rho_2(Y_2^{-1})) = c(\rho_2(Q_2Q_0)) = c(\rho_2(Q_0Q_2)) = c(\rho_2(Y_2))$ and hence $\mathrm{tr}(\rho_2(Y_2^{-1})) = 0$, a contradiction to (Eq2).) Hence we have $\mathrm{Fix}(\tilde{\rho}_2(S_2)) = \{\frac{1}{2}(c(\rho_2(Q_2Q_0)) + c(\rho_2(Q_0Q_2))), \infty\}$. By the faithfulness of ρ_2, the isometry $\rho_2(Q_1)$ does not fix ∞. In fact, if $\rho_2(Q_1)$ fixes ∞, then $\mathrm{tr}(\rho_2(Y_1))\,\mathrm{tr}(\rho_2(Y_2))\,\mathrm{tr}(\rho_2(Y_1Y_2^{-1})) = 0$ by Lemma 4.4(2.2). Since $\tilde{\rho}_2(S_2)$ fixes ∞, the axes of $\rho_2(Q_1)$ and $\tilde{\rho}_2(S_2)$ intersect orthogonally by $(\tilde{\rho}_2(Q_1S_2))^2 = 1$. Hence the isometry $\tilde{\rho}_2(S_2)$ is the π-rotation about the vertical geodesic above $\frac{1}{2}(c(\rho_2(Q_2Q_0)) + c(\rho_2(Q_0Q_2))) = c(\rho_2(Q_1))$. $\square$

Hence we have

$$\begin{aligned} b_0 &= c(\rho_2(Q_0)) - c(\rho_2(Q_2Q_0)) && \text{by Proposition 4.11(1.1)}\\ &= -c(\rho_2(Q_2)) + 2c(\rho_2(Q_1)) - c(\rho_2(Q_2Q_0)) && \text{by Proposition 4.11(1.1)}\\ &= -c(\rho_2(Q_2)) + c(\rho_2(Q_0Q_2)) && \text{by Claim 5.6}\\ &= b_2 && \text{by Proposition 4.11(1.1).} \end{aligned}$$

To show the uniqueness of the extensions of ρ_2, let $\tilde{\rho}_2$ and $\tilde{\rho}_2'$ be extensions of ρ_2 to $\pi_1(\mathcal{O}_\alpha)$. Then we have the following identity:

$$\begin{aligned} (\tilde{\rho}_2(S_0^{S_2}), \tilde{\rho}_2(S_1), \tilde{\rho}_2(S_0)) &= (\tilde{\rho}_2(Q_0), \tilde{\rho}_2(Q_1), \tilde{\rho}_2(Q_2))\\ &= (\rho_2(Q_0), \rho_2(Q_1), \rho_2(Q_2))\\ &= (\tilde{\rho}_2'(Q_0), \tilde{\rho}_2'(Q_1), \tilde{\rho}_2'(Q_2))\\ &= (\tilde{\rho}_2'(S_0^{S_2}), \tilde{\rho}_2'(S_1), \tilde{\rho}_2'(S_0)). \end{aligned}$$

By Claim 5.6, we have $\tilde{\rho}_2(S_2) = \tilde{\rho}_2'(S_2)$. Hence we have $\tilde{\rho}_2 = \tilde{\rho}_2'$.

The if part of (β). Suppose that there exists a sequence of elliptic generators $\{Q_j\}$ of $\pi_1(\mathcal{O}_{N_{2,1}})$ such that the complex probability (b_0, b_1, b_2) associated with $\{\rho_2(Q_j)\}$ satisfies the following identity:

$$(b_0^{(1)}, b_1^{(1)}, b_2^{(1)}) = (b_2, b_1, b_0).$$

Let $\tilde{K}$ be the horizontal translation $z \mapsto z + 1$. For simplicity of notation, we write (g_0, g_1, g_2, g_3) instead of $(\tilde{K}\rho_2(K_0), g_0^{-1}\rho_2(Q_2), g_1^{-1}\rho_2(Q_0), \rho_2(K_0))$. We first show that there is a representation ρ_2^* from $\pi_1(\mathcal{O}_\beta) = \langle T_0, T_1, T_2, T_3 \mid T_0^2 = T_1^2 = T_2^2 = T_3^2 = 1, (T_0T_1)^2 = (T_1T_2)^2 = (T_2T_3)^2 = 1\rangle$ to $\mathrm{PSL}(2, \mathbf{C})$ sending (T_0, T_1, T_2, T_3) to (g_0, g_1, g_2, g_3). Since $g_3 = \rho_2(K_0)$, $g_0g_1 = \rho_2(Q_2)$ and $g_1g_2 = \rho_2(Q_0)$, we have $g_3^2 = (g_0g_1)^2 = (g_1g_2)^2 = 1$. By Convention 4.5, $g_0 = \tilde{K}\rho_2(K_0)$ are π-rotations and hence $g_0^2 = 1$. Thus the existence of the representation ρ_2^* is guaranteed by the following claim.

CLAIM 5.7. (1) *g_0 is a π-rotation satisfying $\rho_2(Q_2Q_0)^{g_0} = \rho_2(Q_0Q_2)$. In particular, $g_2 = g_0\rho_2(Q_2Q_0)$ has order 2, and hence $g_2^2 = 1$.*

(2) *The axes of g_0 and $\rho_2(Q_2)$ intersect orthogonally and hence $g_0^{-1}\rho_2(Q_2) = g_1$ is also a π-rotation. In particular, $g_1^2 = 1$.*

(3) *g_2g_3 is a π-rotation and hence $(g_2g_3)^2 = 1$.*

PROOF. (1) By the proof of Claim 5.5(1), the isometry $g_0 = \rho_2(K_2)\tilde{K}$ is the π-rotation about the vertical geodesic above $\frac{1}{2}(c(\rho_2(Q_0Q_2)) + c(\rho_2(Q_2Q_0)))$. This together with Proposition 4.11 implies that $\rho_2(Q_2Q_0)^{g_0} = \rho_2(Q_0Q_2)$.

(2) By Lemma 4.13, we have

$$b_0^{(1)} = 1 - b_2, \qquad b_1^{(1)} = \frac{b_1b_2}{1-b_2}, \qquad b_2^{(1)} = \frac{b_2b_0}{1-b_2}.$$

This together with the assumption $(b_0^{(1)}, b_1^{(1)}, b_2^{(1)}) = (b_2, b_1, b_0)$ implies that $b_2 = b_0^{(1)} = 1/2$. In particular, $\frac{1}{2}(c(\rho_2(Q_0Q_2)) + c(\rho_2(Q_2Q_0))) = c(\rho_2(Q_2))$ by Proposition 4.11(1.1). Hence g_0 is the π-rotation about the vertical geodesic above $c(\rho_2(Q_2))$ and hence g_0 and $\rho_2(Q_2)$ intersect orthogonally.

(3) Since $g_0g_3 = \tilde{K}$, we have $g_0g_3(z) = z+1$. By Proposition 4.11(1.2), the isometry $g_3g_2 = g_3g_0\rho_2(Q_2Q_0)$ is a π-rotation. $\square$

Recall that $\pi_1(\mathcal{O}_{N_{2,1}})$ is identified with a subgroup of $\pi_1(\mathcal{O}_\beta)$ and their generators satisfy the following identities:

$$Q_0 = T_1T_2, \qquad Q_1 = T_3^{T_1}, \qquad Q_2 = T_0T_1.$$

Since

$$\begin{aligned}(\rho_2^*(T_0), \rho_2^*(T_1), \rho_2^*(T_2), \rho_2^*(T_3)) &= (g_0, g_1, g_2, g_3) \\ &= (\tilde{K}\rho_2(K_0), g_0^{-1}\rho_2(Q_2), g_1^{-1}\rho_2(Q_0), \rho_2(K_0)),\end{aligned}$$

we have

$$\begin{aligned}(\rho_2(Q_0), \rho_2(Q_1), \rho_2(Q_2)) &= (\rho_2^*(T_1T_2), \rho_2^*(T_3^{T_1T_2}), \rho_2^*(T_0T_1)) \\ &= (\rho_2^*(T_1T_2), \rho_2^*(T_3^{T_1}), \rho_2^*(T_0T_1)).\end{aligned}$$

Thus the restriction of ρ_2^* to $\pi_1(\mathcal{O}_{N_{2,1}})$ is equal to the original representation ρ_2.

The only if part of (β). Suppose that ρ_2 extends to a type-preserving representation $\tilde{\rho}_2$ of $\pi_1(\mathcal{O}_\beta)$. Pick a sequence of elliptic generators $\{Q_j\}$ of $\pi_1(\mathcal{O}_{N_{2,1}})$. Since ρ_2 is faithful, we have $\mathrm{tr}(\rho_2(Y_1))\,\mathrm{tr}(\rho_2(Y_2))\,\mathrm{tr}(\rho_2(Y_1Y_2^{-1})) \neq 0$ and $\mathrm{tr}(\rho_2(Y_1))\,\mathrm{tr}(\rho_2(Y_2))\,\mathrm{tr}(\rho_2(Y_1Y_2)) \neq 0$, where $Y_1 = Q_0Q_1$ and $Y_2 = Q_0Q_2$. Thus the complex probability (b_0, b_1, b_2) associated with $\{\rho_2(Q_j)\}$ and the complex probability $(b_0^{(1)}, b_1^{(1)}, b_2^{(1)})$ associated with $\{\rho_2(\sigma(Q_j))\}$ are well-defined. Since $\pi_1(\mathcal{O}_{N_{2,1}})$ is identified with a subgroup of $\pi_1(\mathcal{O}_\beta)$, the isometry

$\tilde{\rho}_2(T_0)$ satisfies the following identities:

$$(\tilde{\rho}_2(T_0))^2 = 1, \qquad (\tilde{\rho}_2(T_0 Q_2))^2 = 1,$$

$$(\tilde{\rho}_2(T_0 Q_2 Q_0))^2 = 1, \qquad (\tilde{\rho}_2(T_0 Q_2 Q_0 K_0))^2 = 1.$$

Claim 5.8. *The isometry $\tilde{\rho}_2(T_0)$ is the π-rotation about the vertical geodesic above $\frac{1}{2}(c(\rho_2(Q_2 Q_0)) + c(\rho_2(Q_0 Q_2))) = c(\rho_2(Q_2))$. Moreover, we have $\tilde{\rho}_2(T_0)(c(\rho_2(Q_0))) = c(\rho_2(Q_0^{Q_2}))$.*

Proof. Since $(\tilde{\rho}_2(T_0))^2 = 1$, $\tilde{\rho}_2(T_0)$ is either the identity or a π-rotation. If $\tilde{\rho}_2(T_0) = 1$, then $\rho_2(K_{N_{2,1}}) = \tilde{\rho}_2((T_0 T_3)^2) = \tilde{\rho}_2(T_3^2) = 1$, a contradiction. Hence $\tilde{\rho}_2(T_0)$ is a π-rotation. By $(\tilde{\rho}_2(T_0 Q_2 Q_0))^2 = 1$ and $(\tilde{\rho}_2(T_0 Q_2 Q_0 K_0))^2 = 1$, we have $\tilde{\rho}_2(K_2^{T_0}) = \rho_2(K_0)$. Hence $\tilde{\rho}_2(T_0)$ maps $\mathrm{Fix}(\rho_2(K_2)) = \{c(\rho_2(Q_0 Q_2)), \infty\}$ to $\mathrm{Fix}(\rho_2(K_0)) = \{c(\rho_2(Q_2 Q_0)), \infty\}$. Since $\tilde{\rho}_2(T_0)$ has order 2, the isometry $\tilde{\rho}_2(T_0)$ must fix ∞. (Otherwise $c(\rho_2(Y_2^{-1})) = c(\rho_2(Q_2 Q_0)) = c(\rho_2(Q_0 Q_2)) = c(\rho_2(Y_2))$ and hence $\mathrm{tr}(\rho_2(Y_2^{-1})) = 0$, a contradiction to (Eq2).) Hence we have $\mathrm{Fix}(\tilde{\rho}_2(T_0)) = \{\frac{1}{2}(c(\rho_2(Q_2 Q_0)) + c(\rho_2(Q_0 Q_2))), \infty\}$. By Lemma 4.4(2.2), if $\rho_2(Q_2)$ fixes ∞, then $\mathrm{tr}(\rho_2(Y_1))\,\mathrm{tr}(\rho_2(Y_2)) = 0$. This contradicts the identities $\mathrm{tr}(\rho_2(Y_1))\,\mathrm{tr}(\rho_2(Y_2))\,\mathrm{tr}(\rho_2(Y_1 Y_2^{-1})) \neq 0$ and $\mathrm{tr}(\rho_2(Y_1))\,\mathrm{tr}(\rho_2(Y_2))\,\mathrm{tr}(\rho_2(Y_1 Y_2)) \neq 0$. Hence $\rho_2(Q_2)$ does not fix ∞. Since $\tilde{\rho}_2(T_0)$ fixes ∞, the axes of $\tilde{\rho}_2(T_0)$ and $\rho_2(Q_2)$ intersect orthogonally by $(\tilde{\rho}_2(T_0 Q_2))^2 = 1$. Hence $\tilde{\rho}_2(T_0)$ is the π-rotation about the vertical geodesic above $\frac{1}{2}(c(\rho_2(Q_2 Q_0)) + c(\rho_2(Q_0 Q_2))) = c(\rho_2(Q_2))$.

By $(\tilde{\rho}_2(T_0 Q_2))^2 = 1$ and $(\tilde{\rho}_2(T_0 Q_2 Q_0))^2 = 1$, we have $\tilde{\rho}_2(Q_0^{T_0}) = \rho_2(Q_0^{Q_2})$. By the above argument, the isometry $\tilde{\rho}_2(T_0)$ is a Euclidean isometry preserving ∞. Hence, by Lemma 4.7, we have $\tilde{\rho}_2(T_0)(c(\rho_2(Q_0))) = c(\rho_2(Q_0^{Q_2}))$. □

Then we have

$$\begin{aligned}
b_0 &= c(\rho_2(Q_0)) - c(\rho_2(Q_2 Q_0)) && \text{by Proposition 4.11(1.1)}\\
&= c(\rho_2(Q_0 Q_2)) - c(\rho_2(Q_0^{Q_2})) && \text{by Claim 5.8}\\
&= b_2^{(1)} && \text{by Proposition 4.11(1.1) and Notation 4.12,}\\
b_1 &= c(\rho_2(Q_2)) - c(\rho_2(Q_0)) && \text{by Proposition 4.11(1.1)}\\
&= c(\rho_2(Q_0^{Q_2})) - c(\rho_2(Q_2)) && \text{by Claim 5.8}\\
&= b_1^{(1)} && \text{by Proposition 4.11(1.1) and Notation 4.12,}\\
b_2 &= c(\rho_2(Q_0 Q_2)) - c(\rho_2(Q_2)) && \text{by Proposition 4.11(1.1)}\\
&= c(\rho_2(Q_2)) - c(\rho_2(Q_2 Q_0)) && \text{by Claim 5.8}\\
&= b_0^{(1)} && \text{by Proposition 4.11(1.1) and Notation 4.12.}
\end{aligned}$$

To show the uniqueness of the extension of ρ_2, let $\tilde{\rho}_2$ and $\tilde{\rho}'_2$ be extensions of ρ_2 to $\pi_1(\mathcal{O}_\beta)$. Then we have the following identity:

$$\begin{aligned}(\tilde{\rho}_2(T_1T_2), \tilde{\rho}_2(T_3^{T_1}), \tilde{\rho}_2(T_0T_1)) &= (\tilde{\rho}_2(Q_0), \tilde{\rho}_2(Q_1), \tilde{\rho}_2(Q_2))\\ &= (\tilde{\rho}'_2(Q_0), \tilde{\rho}'_2(Q_1), \tilde{\rho}'_2(Q_2))\\ &= (\tilde{\rho}'_2(T_1T_2), \tilde{\rho}'_2(T_3^{T_1}), \tilde{\rho}'_2(T_0T_1)).\end{aligned}$$

By Claim 5.8, we have $\tilde{\rho}_2(T_0) = \tilde{\rho}'_2(T_0)$. Hence we have $\tilde{\rho}_2 = \tilde{\rho}'_2$. □

In the remainder of this section, we study what happens if we drop the faithfulness condition in Theorem 5.1.

PROPOSITION 5.9. *Under Convention* 4.5, *the following hold for every* $\rho_2 \in \Omega(N_{2,1})$.

(1) *For the conditions* (ii) *and* (iii) *in Theorem* 5.1(1) *and the condition* (i)$'$ *defined below, the implication* (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i)$'$ *holds.*

  (i)$'$ *There exists a (possibly non-faithful) representation* $\rho_1 \in \Omega(\Sigma_{1,1})$ *which is commensurable with* ρ_2.

(2) *The assertion* (α) *in Theorem* 5.1(3) *holds.*

(3) *The if part of* (β) *in Theorem* 5.1(3) *holds.*

PROOF. (1) In the proof of the implication (iii) $\Rightarrow$ (ii) in Theorem 5.1(1), we do not use the faithfulness of ρ_2. In the proof of the implication (ii) $\Rightarrow$ (i) in Theorem 5.1(1), we do not use the faithfulness of ρ_2 to show the existence of the representation $\rho_1 \in \Omega(\Sigma_{1,1})$ which is commensurable with ρ_2. Hence we have the desired results.

(2) The proof of the if part of (α) in Theorem 5.1(3) does not use the faithfulness. In the proof of the only if part of (α) in Theorem 5.1(3), we use the faithfulness of ρ_2 only to guarantee the existence of a sequence of elliptic generators $\{Q_j\}$ of $\pi_1(\mathcal{O}_{N_{2,1}})$ such that

- $\operatorname{tr}(\rho_2(Y_1))\operatorname{tr}(\rho_2(Y_2))\operatorname{tr}(\rho_2(Y_1Y_2^{-1})) \neq 0$ with $Y_1 = Q_0Q_1$ and $Y_2 = Q_0Q_2$ and
- $\rho_2(Q_1)$ does not fix ∞.

On the other hand, Lemma 4.4(2.2) implies that the above two conditions are equivalent. Hence, we have only to show that $\rho_2(Q_1)$ does not fix ∞ without the faithfulness of ρ_2.

Let $\tilde{\rho}_2$ be the extension of ρ_2 to $\pi_1(\mathcal{O}_\alpha)$. Since $\pi_1(\mathcal{O}_{N_{2,1}}) = \langle Q_0, Q_1, Q_2 \mid Q_0^2 = Q_1^2 = Q_2^2 = 1\rangle$ is identified with a subgroup of $\pi_1(\mathcal{O}_\alpha) = \langle S_0, S_1, S_2 \mid S_0^2 = S_1^2 = S_2^2 = 1, (S_1S_2)^2 = 1\rangle$, we have $(\tilde{\rho}_2(Q_1S_2))^2 = (\tilde{\rho}_2(S_1S_2))^2 = 1$. By the proof of Claim 5.6, we have $\operatorname{Fix}(\tilde{\rho}_2(S_2)) = \left\{\frac{1}{2}(c(\rho_2(Q_2Q_0)) + c(\rho_2(Q_0Q_2))), \infty\right\}$. Suppose to the contrary that $\rho_2(Q_1)$ fixes ∞. Then $\rho_2(Q_1)$ is equal to $\rho_2(K_0)$ or $\rho_2(K_2)$ by Lemma 4.4(2.2). Hence $\operatorname{Fix}(\rho_2(Q_1)) = \{c(\rho_2(Q_2Q_0)), \infty\}$

or $\{c(\rho_2(Q_0Q_2)), \infty\}$. Since $(\tilde{\rho}_2(Q_1S_2))^2 = 1$, we have $\rho_2(Q_1) = \tilde{\rho}_2(S_2)$. Hence $c(\rho_2(Q_2Q_0)) = c(\rho_2(Q_0Q_2))$, and therefore $\mathrm{tr}(\rho_2(Q_0Q_2)) = \mathrm{tr}(\rho_2(Y_2)) = 0$. This contradicts (Eq2). Hence $\rho_2(Q_1)$ does not fix ∞.

(3) The proof of the if part of (β) in Theorem 5.1(3) does not use the faithfulness. $\square$

Definition 5.10. An element ρ_2 of $\Omega(N_{2,1})$ is *strongly non-faithful* if there exists an elliptic generator Q_j of $\pi_1(\mathcal{O}_{N_{2,1}})$ with $j \not\equiv 1 \pmod 3$ such that $\mathrm{tr}(\rho_2(K_0Q_j)) = 0$.

Proposition 5.11. *Under Convention* 4.5, *let ρ_2 be an element of $\Omega(N_{2,1})$. Then the following conditions are equivalent.*

(1) *ρ_2 is strongly non-faithful.*

(2) *The conditions* (i)$'$ *in Proposition* 5.9(1) *and* (ii) *in Theorem* 5.1(1) *hold, but the condition* (iii) *in Theorem* 5.1(1) *does not hold.*

(3) *ρ_2 extends to a type-preserving representation $\tilde{\rho}_2$ of $\pi_1(\mathcal{O}_\beta)$ such that $\tilde{\rho}_2(T_1) = 1$, where T_1 is the generator of $\pi_1(\mathcal{O}_\beta)$ as in Figure* 3.

Proof. We prove this proposition by proving the implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (1), (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Suppose that ρ_2 is strongly non-faithful, i.e., there is an elliptic generator Q_j of $\pi_1(\mathcal{O}_{N_{2,1}})$ with $j \not\equiv 1 \pmod 3$ such that $\mathrm{tr}(\rho_2(K_0Q_j)) = 0$. We may assume without losing generality that $j = 0$. Then the pseudo-Markoff triple associated with $\{\rho_2(Q_j)\}$ is equal to $(0, 2i, r)$ for some $r \in \mathbf{C}^*$. In particular, the complex probability associated with $\{\rho_2(Q_j)\}$ is not defined. Hence ρ_2 does not satisfy the condition (iii) in Theorem 5.1(1). However, the complex probability $(b_0^{(2)}, b_1^{(2)}, b_2^{(2)})$ associated with $\{\rho_2(\sigma^2(Q_j))\}$ is equal to $(1, -1/r^2, 1/r^2)$, and the complex probability $(b_0^{(-1)}, b_1^{(-1)}, b_2^{(-1)})$ associated with $\{\rho_2(\sigma^{-1}(Q_j))\}$ is equal to $(1/r^2, -1/r^2, 1)$. Thus we have $(b_0^{(2)}, b_1^{(2)}, b_2^{(2)}) = (b_2^{(-1)}, b_1^{(-1)}, b_0^{(-1)})$. By replacing $\{Q_j\}$ with $\{\sigma^{-1}(Q_j)\}$, the representation ρ_2 (together with $\{Q_j\}$) satisfies the condition (ii) in Theorem 5.1(1), and hence ρ_2 satisfies the condition (i)$'$ in Proposition 5.9(1) by Proposition 5.9(1).

(2) $\Rightarrow$ (1) Suppose that the conditions (i)$'$ in Proposition 5.9(1) and (ii) in Theorem 5.1(1) hold, but the condition (iii) in Theorem 5.1(1) does not hold. Then, by the proof of (ii) $\Rightarrow$ (iii) in Theorem 5.1(1), for some sequence of elliptic generators $\{Q_j\}$ and some integer k, the complex probability associated with $\{\rho_2(\sigma^k(Q_j))\}$ is not defined. This implies $y_1^{(k)} y_2^{(k)} (y_1^{(k)} y_2^{(k)} - y_{12}^{(k)}) = 0$ for the pseudo-Markoff triple $(y_1^{(k)}, y_{12}^{(k)}, y_2^{(k)})$ associated with $\{\rho_2(\sigma^k(Q_j))\}$. Hence $\mathrm{tr}(\rho_2(K_0\sigma^k(Q_0)))$ or $\mathrm{tr}(\rho_2(K_0\sigma^{k-1}(Q_0)))$ is equal to 0, and therefore ρ_2 is strongly non-faithful.

(1) $\Rightarrow$ (3) Suppose that ρ_2 is strongly non-faithful, namely, there is an elliptic generator Q_j of $\pi_1(\mathcal{O}_{N_{2,1}})$ with $j \not\equiv 1 \pmod 3$ such that $\mathrm{tr}(\rho_2(K_0Q_j))$

$= 0$. We may assume $j = 0$ without losing generality. Then there is a representation ρ_2^* from $\pi_1(\mathcal{O}_\beta) = \langle T_0, T_1, T_2, T_3 \mid T_0^2 = T_1^2 = T_2^2 = T_3^2 = 1, (T_0T_1)^2 = (T_1T_2)^2 = (T_2T_3)^2 = 1\rangle$ to $\mathrm{PSL}(2, \mathbf{C})$ sending (T_0, T_1, T_2, T_3) to $(g_0, g_1, g_2, g_3) := (\tilde{K}\rho_2(K_0), g_0^{-1}\rho_2(Q_2), g_1^{-1}\rho_2(Q_0), \rho_2(K_0))$, where $\tilde{K}$ is the horizontal translation $z \mapsto z + 1$ and ρ_2^* is an extension of ρ_2 to $\pi_1(\mathcal{O}_\beta)$. This can be seen as follows. By the proof of the if part of (β) in Theorem 5.1(3), we have $g_3^2 = (g_0g_1)^2 = (g_1g_2)^2 = g_0^2 = 1$. Hence we have only to show that $g_1^2 = 1$, $g_2^2 = 1$ and $(g_2g_3)^2 = 1$. Since $y_1 = \mathrm{tr}(\rho_2(Y_1)) = \mathrm{tr}(\rho_2(K_0Q_0)) = 0$, we have $c(\rho_2(Q_0)) = 0 \in \mathrm{Fix}(\rho_2(K_0))$ and $\rho_2(Q_2) = \tilde{K}\rho_2(K_0)$ by Lemma 4.4(2.2). By $\rho_2(Q_2) = \tilde{K}\rho_2(K_0)$, we have $g_1 = g_0^{-1}\rho_2(Q_2) = 1$, and hence $g_1^2 = 1$. Since $g_2 = g_1^{-1}\rho_2(Q_0) = \rho_2(Q_0)$, we have $g_2^2 = 1$. By $c(\rho_2(Q_0)) \in \mathrm{Fix}(\rho_2(K_0))$, the axes of $\rho_2(Q_0)$ and $\rho_2(K_0)$ intersect orthogonally, and hence $g_2g_3 = g_1^{-1}\rho_2(Q_0)\rho_2(K_0) = \rho_2(Q_0)\rho_2(K_0)$ is a π-rotation. In particular, $(g_2g_3)^2 = 1$. Moreover, we have $\rho_2^*(T_1) = g_1 = 1$. Thus we obtain the desired representation ρ_2^*.

$(3) \Rightarrow (1)$ Suppose that the representation ρ_2 extends to a type-preserving $\mathrm{PSL}(2, \mathbf{C})$-representation $\tilde{\rho}_2$ of $\pi_1(\mathcal{O}_\beta)$ such that $\tilde{\rho}_2(T_1) = 1$. Note that $\pi_1(\mathcal{O}_{N_{2,1}})$ is identified with a subgroup of $\pi_1(\mathcal{O}_\beta)$ and $T_1 = T_0Q_2$. By the proof of Claim 5.8, the isometry $\tilde{\rho}_2(T_0)$ fixes ∞. Since $\tilde{\rho}_2(T_1) = \tilde{\rho}_2(T_0Q_2) = 1$, the isometry $\rho_2(Q_2)$ fixes ∞. By Lemma 4.4(2.2), we have $\mathrm{tr}(\rho_2(Y_1))\,\mathrm{tr}(\rho_2(Y_2)) = 0$. By (Eq2), we have $\mathrm{tr}(\rho_2(Y_1)) = \mathrm{tr}(\rho_2(K_0Q_0)) = 0$. Hence ρ_2 is strongly non-faithful. □

6. Application to Ford domains

In this section, we give an application to the study of the Ford domains.

DEFINITION 6.1. Let Γ be a non-elementary Kleinian group such that the stabilizer Γ_∞ of ∞ contains parabolic transformations. Then the *Ford domain* $P(\Gamma)$ of Γ is the polyhedron in $\mathbf{H}^3$ defined below:

$$P(\Gamma) := \bigcap \{E(\gamma) \mid \gamma \in \Gamma - \Gamma_\infty\}.$$

LEMMA 6.2. *Under Convention* 4.5, *the following hold:*

(1) *Let* ρ_1 *be an element of* $\Omega(\Sigma_{1,1})$. *Suppose that* ρ_1 *is discrete. Then* $P(\rho_1(\pi_1(\mathcal{O}_{\Sigma_{1,1}}))) = P(\rho_1(\pi_1(\Sigma_{1,1})))$.

(2) *Let* ρ_2 *be an element of* $\Omega(N_{2,1})$. *Suppose that* ρ_2 *is discrete. Then* $P(\rho_2(\pi_1(\mathcal{O}_{N_{2,1}}))) = P(\rho_2(\pi_1(N_{2,1})))$.

PROOF. The assertion (1) is well-known (see [1, Proposition 2.2.8]). The assertion (2) follows from the fact that $\pi_1(\mathcal{O}_{N_{2,1}}) = \langle \pi_1(N_{2,1}), K_2\rangle$ and that $\rho_2(K_2)$ is a Euclidean transformation preserving ∞ by Convention 4.5. □

Proposition 6.3. *Under Convention* 4.5, *let* ρ_1 *and* ρ_2 *be elements of* $\Omega(\Sigma_{1,1})$ *and* $\Omega(N_{2,1})$, *respectively. Suppose that they are discrete and commensurable. Then* $P(\rho_1(\pi_1(\Sigma_{1,1}))) = P(\rho_1(\pi_1(\mathcal{O}_{\Sigma_{1,1}}))) = P(\rho_2(\pi_1(\mathcal{O}_{N_{2,1}}))) = P(\rho_2(\pi_1(N_{2,1})))$.

Proof. We prove only the second equality, because the remaining equalities can be proved by Lemma 6.2.

Since ρ_1 and ρ_2 are commensurable, there exist a double covering $p_1 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{\Sigma_{1,1}}$ and a double covering $p_2 : \mathcal{O}_{\Sigma_{1,2}} \to \mathcal{O}_{N_{2,1}}$ such that $\rho_1 \circ (p_1)_* = \rho_2 \circ (p_2)_*$. Then we can easily observe that

$$\pi_1(\mathcal{O}_{\Sigma_{1,1}}) = \langle (p_1)_*(\pi_1(\mathcal{O}_{\Sigma_{1,2}})), K\rangle,$$

$$\pi_1(\mathcal{O}_{N_{2,1}}) = \langle (p_2)_*(\pi_1(\mathcal{O}_{\Sigma_{1,2}})), K_2\rangle.$$

Since $\rho_1(K)$ and $\rho_2(K_2)$ are Euclidean transformations preserving ∞, we see

$$\begin{aligned} P(\rho_1(\pi_1(\mathcal{O}_{\Sigma_{1,1}}))) &= P(\rho_1((p_1)_*(\pi_1(\mathcal{O}_{\Sigma_{1,2}}))) \\ &= P(\rho_2((p_2)_*(\pi_1(\mathcal{O}_{\Sigma_{1,2}}))) = P(\rho_2(\pi_1(\mathcal{O}_{N_{2,1}}))). \end{aligned}$$ □

Example 6.4. Jorgensen and Marden [7] constructed complete hyperbolic structures of the punctured torus bundles over the circle with monodromy matrices $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ by explicitly constructing the fiber groups G_1 and G_2 and their Ford domains. The groups G_1 and G_2, respectively, are the images of (faithful) representations ρ_1 and ρ_1' in $\Omega(\Sigma_{1,1})$ constructed by Proposition 4.8(1) from the following triples:

$$(a_0, a_1, a_2) = \left(\frac{1}{\sqrt{3}}e^{(\pi/6)i}, \frac{1}{\sqrt{3}}e^{-(\pi/2)i}, \frac{1}{\sqrt{3}}e^{(\pi/6)i}\right),$$

$$(a_0', a_1', a_2') = \left(-\frac{1}{2}, \frac{1}{\sqrt{2}}e^{(\pi/4)i}, \frac{1}{2}\right).$$

Let ρ_2 and ρ_2', respectively, be elements of $\Omega(N_{2,1})$ constructed by Proposition 4.11(1) from the above triples. Then ρ_2 and ρ_2', respectively, satisfy the conditions (iii)-(α) and (iii)-(β) in Theorem 5.1(1). Hence, by Proposition 5.9(1), for each of ρ_2 and ρ_2', there is an element of $\Omega(\Sigma_{1,1})$ which is commensurable with it. In fact, we can easily check that ρ_1 (resp. ρ_1') is commensurable with ρ_2 (resp. ρ_2'). Hence ρ_2 and ρ_2' are faithful by Remark 4.16, and $P(\rho_1(\pi_1(\mathcal{O}_{\Sigma_{1,1}}))) = P(\rho_2(\pi_1(\mathcal{O}_{N_{2,1}})))$ and $P(\rho_1'(\pi_1(\mathcal{O}_{\Sigma_{1,1}}))) = P(\rho_2'(\pi_1(\mathcal{O}_{N_{2,1}})))$ by Proposition 6.3. The Ford domain $P(\rho_2(\pi_1(\mathcal{O}_{N_{2,1}})))$ (resp. $P(\rho_2'(\pi_1(\mathcal{O}_{N_{2,1}})))$) is illustrated in the left (resp. right) of Figure 11 (compare with [7, FIG. 1 and 2]).

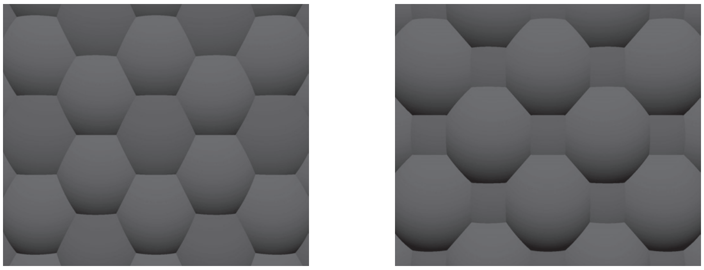

Fig. 11. Left: The Ford domain of $\rho_2(\pi_1(N_{2,1}))$. Right: The Ford domain of $\rho_2'(\pi_1(N_{2,1}))$.

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