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Hardy's inequality in Musielak-Orlicz-Sobolev spaces

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Abstract. Our aim in this paper is to treat Hardy's inequalities for Musielak-Orlicz-Sobolev functions on proper open subsets of $\mathbf{R}^N$.

1. Introduction

The higher dimensional Hardy's inequality of the form

$$\int_\Omega |u(x)|^p \delta(x)^{-p+\beta} dx \le C \int_\Omega |\nabla u(x)|^p \delta(x)^\beta dx, \qquad u \in C_0^\infty(\Omega)$$

appeared in [12] for bounded Lipschitz domains $\Omega \subset \mathbf{R}^N$, $1 < p < \infty$ and $\beta < p - 1$, where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. For related results, we refer to [1], [2], [6], [7], [8] and [13].

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth conditions. Harjulehto-Hästö-Koskenoja [4] proved Hardy's inequality for Sobolev functions $u \in W_0^{1,p(\cdot)}(\Omega)$ when Ω is bounded and $p(\cdot)$ is a variable exponent satisfying the log-Hölder conditions on Ω, as an extension of [2]. In fact they proved the following:

Theorem A. *Let Ω be an open and bounded subset of $\mathbf{R}^N$. Suppose $1 < p^- \le p^+ < \infty$, where $p^- := \inf_{x\in\mathbf{R}^N} p(x)$ and $p^+ := \sup_{x\in\mathbf{R}^N} p(x)$. Assume that Ω satisfies the measure density condition, that is, there exists a constant $k > 0$ such that*

$$|B(z,r) \cap \Omega^c| \ge k|B(z,r)| \tag{1}$$

for every $z \in \partial\Omega$ and $r > 0$ (see [3]). *Then there exist positive constants C and b_0 such that the inequality*

$$\|\delta^{b-1} u\|_{L^{p(\cdot)}(\Omega)} \le C \|\delta^b |\nabla u|\,\|_{L^{p(\cdot)}(\Omega)} \tag{2}$$

holds for all $u \in W_0^{1,p(\cdot)}(\Omega)$ and all $0 \le b < b_0$, where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$.

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In the case when $b = 0$, Hästö [5, Theorem 3.2] proved Theorem A without the assumption that Ω is bounded. It is also shown in [4] that if $p^- > N$ then (2) holds without the measure density condition (1).

Recently, these results have been extended to the two variable exponents Sobolev spaces $W_0^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$ in [10], where $\Phi_{p(\cdot),q(\cdot)}(x,t) = (t(\log(c_0+t))^{q(x)})^{p(x)}$ with $p(\cdot)$ as above and a measurable bounded function $q(\cdot)$. In fact, the following results are shown in [10]:

THEOREM B ([10, Theorem 1.1]). *Let $\Omega \neq \mathbf{R}^N$ be an open set. Suppose $1 < p^- \le p^+ < \infty$ and Ω satisfies the measure density condition* (1). *Then, for $0 < A < N/p^+$, $A \le 1$, there exist positive constants C and b_0 such that the inequality*

$$\|\delta^{\alpha+b-1}u\|_{\Phi_{p_\alpha(\cdot),q(\cdot)}(\Omega)} \le C\|\delta^b|\nabla u|\,\|_{\Phi_{p(\cdot),q(\cdot)}(\Omega)}$$

holds for all $u \in W_0^{1,\Phi_{p(\cdot),q(\cdot)}}(\Omega)$, $0 \le \alpha \le A$ and $0 \le b \le b_0$, where $1/p_\alpha(x) = 1/p(x) - \alpha/N$.

THEOREM B′ ([10, Theorem 1.2]). *If $N < p^- \le p^+ < \infty$, then the same conclusion as in Theorem B holds without the measure density condition* (1).

Our aim in this paper is to extend these results to functions in general Musielak-Orlicz-Sobolev spaces $W_0^{1,\Phi}(\Omega)$ defined by a general function $\Phi(x,t)$ satisfying certain conditions (see Section 2 for the definitions of Φ and $W_0^{1,\Phi}(\Omega)$). Corresponding to the functions $\Phi_{p_\alpha(\cdot),q(\cdot)}(x,t)$ in [10], we shall introduce functions $\Psi_\alpha(x,t)$ to state our main Theorems 1 and 2, which are extensions of Theorem B and Theorem B′, respectively.

2. Preliminaries

Throughout this paper, let C denote various constants independent of the variables in question and $C(a,b,\dots)$ be a constant that depends on $a,b,\dots$.

We consider a function

$$\Phi(x,t) = t\phi(x,t) : \mathbf{R}^N \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions (Φ1)–(Φ4):

(Φ1) $\phi(\cdot,t)$ is measurable on $\mathbf{R}^N$ for each $t \ge 0$ and $\phi(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in \mathbf{R}^N$;

(Φ2) there exists a constant $A_1 \ge 1$ such that

$$A_1^{-1} \le \phi(x,1) \le A_1 \qquad \text{for all } x \in \mathbf{R}^N;$$

(Φ3) $\phi(x,\cdot)$ is uniformly almost increasing, namely there exists a constant $A_2 \ge 1$ such that

$$\phi(x,t) \le A_2\phi(x,s) \qquad \text{for all } x \in \mathbf{R}^N \qquad \text{whenever } 0 \le t < s;$$

($\Phi 4$) there exists a constant $A_3 \geq 1$ such that

$$\phi(x, 2t) \leq A_3 \phi(x, t) \qquad \text{for all } x \in \mathbf{R}^N \text{ and } t > 0.$$

Note that ($\Phi 2$), ($\Phi 3$) and ($\Phi 4$) imply

$$0 < \inf_{x \in \mathbf{R}^N} \phi(x, t) \leq \sup_{x \in \mathbf{R}^N} \phi(x, t) < \infty$$

for each $t > 0$.

If $\Phi(x, \cdot)$ is convex for each $x \in \mathbf{R}^N$, then ($\Phi 3$) holds with $A_2 = 1$; namely $\phi(x, \cdot)$ is non-decreasing for each $x \in \mathbf{R}^N$.

Let $\bar{\phi}(x, t) = \sup_{0 \leq s \leq t} \phi(x, s)$ and

$$\bar{\Phi}(x, t) = \int_0^t \bar{\phi}(x, r) dr$$

for $x \in \mathbf{R}^N$ and $t \geq 0$. Then $\bar{\Phi}(x, \cdot)$ is convex and

$$\frac{1}{2A_3} \Phi(x, t) \leq \bar{\Phi}(x, t) \leq A_2 \Phi(x, t)$$

for all $x \in \mathbf{R}^N$ and $t \geq 0$.

By ($\Phi 3$), we see that

$$\Phi(x, at) \begin{cases} \leq A_2 a \Phi(x, t) & \text{if } 0 \leq a \leq 1 \\ \geq A_2^{-1} a \Phi(x, t) & \text{if } a \geq 1. \end{cases} \tag{3}$$

We shall also consider the following conditions:

($\Phi 5$) for every $\gamma > 0$, there exists a constant $B_\gamma \geq 1$ such that

$$\phi(x, t) \leq B_\gamma \phi(y, t)$$

whenever $|x - y| \leq \gamma t^{-1/N}$ and $t \geq 1$;

($\Phi 6$) there exist a function $g \in L^1(\mathbf{R}^N)$ and a constant $B_\infty \geq 1$ such that $0 \leq g(x) < 1$ for all $x \in \mathbf{R}^N$ and

$$B_\infty^{-1} \phi(x, t) \leq \phi(x', t) \leq B_\infty \phi(x, t)$$

whenever $|x'| \geq |x|$ and $g(x) \leq t \leq 1$.

Example 1. Let $p(\cdot)$ and $q_j(\cdot)$, $j = 1, \ldots, k$, be measurable functions on $\mathbf{R}^N$ such that

(P1) $1 \leq p^- := \inf_{x \in \mathbf{R}^N} p(x) \leq \sup_{x \in \mathbf{R}^N} p(x) =: p^+ < \infty$

and

(Q1) $-\infty < q_j^- := \inf_{x \in \mathbf{R}^N} q_j(x) \leq \sup_{x \in \mathbf{R}^N} q_j(x) =: q_j^+ < \infty$

for all $j = 1, \ldots, k$.

Set $L_c(t) = \log(c+t)$ for $c \ge e$ and $t \ge 0$, $L_c^{(1)}(t) = L_c(t)$, $L_c^{(j+1)}(t) = L_c(L_c^{(j)}(t))$ and

$$\Phi(x,t) = t^{p(x)} \prod_{j=1}^{k} (L_c^{(j)}(t))^{q_j(x)}.$$

Then, $\Phi(x,t)$ satisfies ($\Phi1$), ($\Phi2$) and ($\Phi4$). It satisfies ($\Phi3$) if there is a constant $K \ge 0$ such that $K(p(x)-1) + q_j(x) \ge 0$ for all $x \in \mathbf{R}^N$ and $j = 1, \dots, k$; in particular if $p^- > 1$ or $q_j^- \ge 0$ for all $j = 1, \dots, k$.

Moreover, we see that $\Phi(x,t)$ satisfies ($\Phi5$) if

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \le \frac{C_p}{L_e(1/|x-y|)}$$

with a constant $C_p \ge 0$ and

(Q2) $q_j(\cdot)$ is $(j+1)$-log-Hölder continuous, namely

$$|q_j(x) - q_j(y)| \le \frac{C_{q_j}}{L_e^{(j+1)}(1/|x-y|)}$$

with constants $C_{q_j} \ge 0$, $j = 1, \dots k$.

Finally, we see that $\Phi(x,t)$ satisfies ($\Phi6$) with $g(x) = 1/(1+|x|)^{N+1}$ if $p(\cdot)$ is log-Hölder continuous at ∞, namely if it satisfies

(P3) $|p(x) - p(x')| \le \dfrac{C_{p,\infty}}{L_e(|x|)}$ whenever $|x'| \ge |x|$ with a constant $C_{p,\infty} \ge 0$.

In fact, if $1/(1+|x|)^{N+1} < t \le 1$, then $t^{-|p(x)-p(x')|} \le e^{(N+1)C_{p,\infty}}$ for $|x'| \ge |x|$ and $L_c^{(j)}(t)^{|q_j(x)-q_j(x')|} \le L_c^{(j)}(1)^{q_j^+ - q_j^-}$.

EXAMPLE 2. Let $p_1(\cdot)$, $p_2(\cdot)$, $q_1(\cdot)$ and $q_2(\cdot)$ be measurable functions on $\mathbf{R}^N$ satisfying (P1) and (Q1). Then,

$$\Phi(x,t) = (1+t)^{p_1(x)}(1+1/t)^{-p_2(x)} L_c(t)^{q_1(x)} L_c(1/t)^{-q_2(x)}$$

satisfies ($\Phi1$), ($\Phi2$) and ($\Phi4$). It satisfies ($\Phi3$) if $p_j^- > 1$, $j = 1, 2$ or $q_j^- \ge 0$, $j = 1, 2$. As a matter of fact, it satisfies ($\Phi3$) if and only if $p_j(\cdot)$ and $q_j(\cdot)$ satisfy the following conditions:

(1) $q_j(x) \ge 0$ at points x where $p_j(x) = 1$, $j = 1, 2$;

(2) $\sup_{x: p_j(x) > 1} \{\min(q_j(x), 0) \log(p_j(x) - 1)\} < \infty$, $j = 1, 2$.

Moreover, we see that $\Phi(x,t)$ satisfies ($\Phi5$) if $p_1(\cdot)$ is log-Hölder continuous and $q_1(\cdot)$ is 2-log-Hölder continuous.

Finally, we see that $\Phi(x,t)$ satisfies ($\Phi6$) with $g(x) = 1/(1+|x|)^{N+1}$ if $p_2(\cdot)$ is log-Hölder continuous at ∞ and

(Q3) $q_2(\cdot)$ is 2-log-Hölder continuous at ∞, namely

$$|q_2(x) - q_2(x')| \le \frac{C_{q_2,\infty}}{L_c^{(2)}(|x|)} \qquad \text{whenever } |x'| \ge |x|$$

with a constant $C_{q_2,\infty} \ge 0$.

In fact, if $1/(1+|x|)^{N+1} < t \le 1$, then $(1+t)^{|p_1(x)-p_1(x')|} \le 2^{p_1^+-1}$, $(1+1/t)^{|p_2(x)-p_2(x')|} \le e^{(N+1)C_{p_2,\infty}}$, $(\log(e+t))^{|q_1(x)-q_1(x')|} \le (\log(e+1))^{q_1^+-q_1^-}$ and $(\log(e+1/t))^{|q_2(x)-q_2(x')|} \le C(N, C_{q_2,\infty})$ for $|x'| \ge |x|$.

Let Ω be an open set in $\mathbf{R}^N$. Given $\Phi(x,t)$ as above, the associated Musielak-Orlicz space

$$L^{\Phi}(\Omega) = \left\{ f \in L^1_{loc}(\Omega); \int_{\Omega} \Phi(y, |f(y)|)dy < \infty \right\}$$

is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}(\Omega)} = \inf\left\{ \lambda > 0; \int_{\Omega} \overline{\Phi}(y, |f(y)|/\lambda)dy \le 1 \right\}$$

(cf. [11]). Further, we define the Musielak-Orlicz-Sobolev space by

$$W^{1,\Phi}(\Omega) = \{u \in L^{\Phi}(\Omega) : |\nabla u| \in L^{\Phi}(\Omega)\}.$$

The norm

$$\|u\|_{W^{1,\Phi}(\Omega)} = \|u\|_{L^{\Phi}(\Omega)} + \| \, |\nabla u| \, \|_{L^{\Phi}(\Omega)}$$

makes $W^{1,\Phi}(\Omega)$ a Banach space. We denote the closure of $C_0^{\infty}(\Omega)$ in $W^{1,\Phi}(\Omega)$ by $W_0^{1,\Phi}(\Omega)$. As usual, let $W_{loc}^{1,\Phi}(\mathbf{R}^N)$ denote the set of functions u on $\mathbf{R}^N$ such that $u|_{\Omega} \in W^{1,\Phi}(\Omega)$ for every bounded open set Ω. By $(\Phi 2)$ and $(\Phi 3)$, $W_{loc}^{1,\Phi}(\mathbf{R}^N) \subset W_{loc}^{1,1}(\mathbf{R}^N)$.

3. Lemmas

We denote by $B(x,r)$ the open ball centered at x of radius r. For a measurable set E, we denote by $|E|$ the Lebesgue measure of E.

For a locally integrable function f on Ω, the Hardy-Littlewood maximal function Mf is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap\Omega} |f(y)|dy.$$

We know the following boundedness of the maximal operator on $L^{\Phi}(\Omega)$.

LEMMA 1 ([9, Corollary 4.4]). *Suppose that $\Phi(x,t)$ satisfies (Φ5), (Φ6) and further assume:*

($\Phi3^*$) *$t \mapsto t^{-\varepsilon_0}\phi(x,t)$ is uniformly almost increasing on $(0,\infty)$ for some $\varepsilon_0 > 0$, namely there is a constant $A_{2,\varepsilon_0} \ge 1$ such that*

$$t^{-\varepsilon_0}\phi(x,t) \le A_{2,\varepsilon_0} s^{-\varepsilon_0}\phi(x,s) \qquad \text{for all } x \in \mathbf{R}^N \text{ whenever } 0 < t < s.$$

Then the maximal operator M is bounded from $L^{\Phi}(\Omega)$ into itself, namely, there is a constant $C > 0$ such that

$$\|Mf\|_{L^{\Phi}(\Omega)} \le C\|f\|_{L^{\Phi}(\Omega)}$$

for all $f \in L^{\Phi}(\Omega)$.

For $\lambda \ge 1$, $x \in \mathbf{R}^N$ and $t \ge 0$, set

$$\Phi_\lambda(x,t) = \Phi(x, t^{1/\lambda}) = t\phi_\lambda(x,t),$$

where $\phi_\lambda(x,t) = t^{1/\lambda - 1}\phi(x, t^{1/\lambda})$.

LEMMA 2. (1) *$\Phi_\lambda(x,t)$ satisfies the conditions (Φ2) and (Φ4).*

(2) *Suppose $\Phi(x,t)$ satisfies ($\Phi3^*$). Then $\Phi_\lambda(x,t)$ satisfies (Φ1) and (Φ3) when $\lambda \le 1 + \varepsilon_0$, and it satisfies ($\Phi3^*$) when $\lambda < 1 + \varepsilon_0$ (with ε_0 replaced by $(1 + \varepsilon_0 - \lambda)/\lambda$).*

(3) *If $\Phi(x,t)$ satisfies (Φ5), then so does $\Phi_\lambda(x,t)$.*

(4) *If $\Phi(x,t)$ satisfies (Φ6), then so does $\Phi_\lambda(x,t)$.*

PROOF. (1) (Φ2) for Φ immediately implies that for Φ_λ. For (Φ4), note that $\phi_\lambda(x,2t) \le 2^{1/\lambda - 1}A_2A_3\phi_\lambda(x,t)$.

(2) The assertions of (2) follow from ($\Phi3^*$) and the equality

$$\phi_\lambda(x,t) = t^{(1+\varepsilon_0)/\lambda - 1}(t^{1/\lambda})^{-\varepsilon_0}\phi(x, t^{1/\lambda}).$$

(3) It is enough to note that $t^{-\lambda/N} \le t^{-1/N}$ for $t \ge 1$.

(4) It is enough to note that $g(x) \le g(x)^{1/\lambda}$ when $0 \le g(x) < 1$. □

From Lemma 1 and the above lemma, we obtain

COROLLARY 1. *Suppose that $\Phi(x,t)$ satisfies (Φ5), (Φ6) and ($\Phi3^*$). Then the maximal operator M is bounded from $L^{\Phi_\lambda}(\Omega)$ into itself for $1 \le \lambda < 1 + \varepsilon_0$.*

Set

$$\Phi^{-1}(x,s) = \sup\{t > 0; \Phi(x,t) < s\}$$

for $x \in \mathbf{R}^N$ and $s > 0$.

LEMMA 3 (cf. [9, Lemma 5.1]). *$\Phi^{-1}(x,\cdot)$ is non-decreasing,*

$$\Phi(x, \Phi^{-1}(x,t)) = t$$

and

$$A_2^{-1} t \le \Phi^{-1}(x, \Phi(x,t)) \le A_2^2 t \tag{4}$$

for all $x \in \mathbf{R}^N$ *and* $t > 0$.

We shall consider the following condition:
($\Phi 6^*$) $\Phi(x,t)$ satisfies ($\Phi 6$) with $g(x) \le (1+|x|)^{-\beta}$ for some $\beta > N$.

Lemma 4. *If* $\Phi(x,t)$ *satisfies* ($\Phi 6^*$), *then there exists* $0 < \lambda < 1$ *such that*

$$\Phi(x, \lambda g^*(x)) \le (2|x|)^{-N} \qquad \textit{for all } x \in \mathbf{R}^N,$$

where $g^*(x) = \max(g(x), Mg(x))$.

Proof. Since $g(x) \le (1+|x|)^{-\beta}$ with $\beta > N$, $Mg(x) \le C(1+|x|)^{-N}$, so that $g^*(x) \le C(1+|x|)^{-N}$. Hence

$$\Phi(x, \lambda g^*(x)) \le \lambda C(1+|x|)^{-N} A_2 \phi(x, \lambda C) \le 2^N \lambda C A_2 (2|x|)^{-N} \phi(x, \lambda C).$$

Thus, the required inequality holds if $\lambda \le (2^N C A_1 A_2^2)^{-1}$. □

Lemma 5. $r \mapsto r^{\sigma_0} \Phi^{-1}(x, r^{-N})$ *is uniformly almost decreasing on* $(0, \infty)$, *where* $\sigma_0 = N/(1 + (\log A_3)/(\log 2))$.

Proof. By ($\Phi 4$), we see that

$$\Phi^{-1}\left(x, \frac{1}{2A_3} s\right) \le \frac{1}{2} \Phi^{-1}(x, s) \tag{5}$$

for all $x \in \mathbf{R}^N$ and $s > 0$. If $0 < \lambda < 1$, then choosing $k \in \mathbf{N}$ such that $(2A_3)^{-k} \le \lambda < (2A_3)^{-k+1}$ and applying (5), we have

$$\Phi^{-1}(x, \lambda s) \le 2^{-k+1} \Phi^{-1}(x, s) \le 2\lambda^{1/(1+\sigma)} \Phi^{-1}(x, s),$$

where $\sigma = (\log A_3)/(\log 2)$. Note that $\sigma_0 = N/(1+\sigma)$. Thus, for $a > 1$, we have

$$\begin{aligned} (ar)^{\sigma_0} \Phi^{-1}(x, (ar)^{-N}) &\le (ar)^{\sigma_0} 2 (a^{-N})^{1/(1+\sigma)} \Phi^{-1}(x, r^{-N}) \\ &= 2 r^{\sigma_0} \Phi^{-1}(x, r^{-N}), \end{aligned}$$

which shows the assertion of the lemma. □

Lemma 6. *Suppose that* $\Phi(x,t)$ *satisfies* ($\Phi 5$) *and* ($\Phi 6^*$). *Let* $0 < \alpha < \sigma_0$ *for* σ_0 *given in Lemma* 5. *Then there exists a constant* $C > 0$ *such that*

$$\int_{B(x, 2|x|) \setminus B(x, r)} |x-y|^{\alpha - N} f(y) dy \le C r^{\alpha} \Phi^{-1}(x, r^{-N}) \tag{6}$$

and

$$\int_{B(x,r)} f(y)dy \le Cr^N \Phi^{-1}(x, r^{-N}) \tag{7}$$

for all $x \in \mathbf{R}^N$, $0 < r \le 2|x|$, *and* $f \ge 0$ *satisfying* $\|f\|_{L^\Phi(\mathbf{R}^N)} \le 1$.

Proof. Condition $(\Phi\kappa J)$ in [9] with $\kappa(x,r) = r^N$ and $J(x,r) = r^{\alpha-N}$ is satisfied by Lemma 5, if $0 < \alpha < \sigma_0$. Hence, (6) follows from [9, Lemma 6.3] in view of Lemma 4. (7) follows from [9, Lemma 5.3] and Lemma 4. □

Hereafter, let Ω is an open set in $\mathbf{R}^N$ such that $\Omega \ne \mathbf{R}^N$, and let $\delta(x) = \mathrm{dist}(x, \partial\Omega)$.

The following is a key lemma:

Lemma 7. (1) *If* Ω *satisfies*

$$|B(z,r) \cap \Omega^c| \ge k|B(z,r)| \tag{8}$$

for every $z \in \partial\Omega$ *and* $r > 0$ *with a constant* $k > 0$ $(k \le 1)$, *then there exists a constant* $C = C(N,k) > 0$ *such that*

$$|u(x)| \le C \int_{B(x,2\delta(x))} |x-y|^{1-N} |\nabla u(y)| dy$$

for almost every $x \in \Omega$, *whenever* $u \in W^{1,1}_{loc}(\mathbf{R}^N)$ *and* $u = 0$ *outside* Ω.

(2) *Let* $\lambda > N$. *Then there exists a constant* $C > 0$ *such that*

$$|v(x)| \le C \left(\delta(x)^{\lambda-N} \int_{B(x,2\delta(x))} |\nabla v(y)|^\lambda dy \right)^{1/\lambda}$$

for every $x \in \Omega$, *whenever* $v \in W^{1,\lambda}_{loc}(\mathbf{R}^N)$ *and* $v = 0$ *outside* Ω.

For (1) see [10, Lemma 2.1]; for (2) see e.g. [6, (3.1)] (also cf. [2, Proposition 1]). Here note that (2) holds without the assumption (8).

We consider

$$H(f; x, \alpha) = \delta(x)^{\alpha-1} \int_{B(x,2\delta(x))} |x-y|^{1-N} f(y) dy$$

for $x \in \Omega$, $0 \le \alpha \le 1$ and $f \in L^1_{loc}(\mathbf{R}^N)$ such that $f \ge 0$, $f = 0$ outside Ω.

We know (by integration by parts)

$$H(f; x, 0) \le CMf(x) \tag{9}$$

for all $x \in \Omega$.

Lemma 8. *Let $\Omega \neq \mathbf{R}^N$ be an open set and suppose that $\Phi(x,t)$ satisfies* ($\Phi 5$) *and* ($\Phi 6^*$).

(1) *Let $\alpha \in [0, \sigma_0) \cap [0, 1]$. Then there exists a constant $C > 0$ such that*

$$H(f; x, \alpha) \leq CMf(x)\Phi(x, Mf(x))^{-\alpha/N} \tag{10}$$

for all $x \in \Omega$ and $f \geq 0$ such that $f = 0$ outside Ω and $\|f\|_{L^\Phi(\Omega)} \leq 1$.

(2) *Let $\alpha \in [0, \sigma_0]$. Then there exists a constant $C > 0$ such that*

$$\delta(x)^{\alpha-N} \int_{B(x, 2\delta(x))} f(y)dy \leq CMf(x)\Phi(x, Mf(x))^{-\alpha/N} \tag{11}$$

for all $x \in \Omega$ and $f \geq 0$ such that $f = 0$ outside Ω and $\|f\|_{L^\Phi(\Omega)} \leq 1$.

Proof. We have only to consider the case $\alpha > 0$. Without loss of generality, we may assume that $0 \in \partial\Omega$, so that $\delta(x) \leq |x|$. Let $f \geq 0$, $f = 0$ outside Ω and $\|f\|_{L^\Phi(\Omega)} \leq 1$.

(1) For $0 < r \leq \delta(x)$, we have by (6) in Lemma 6

$$\begin{aligned} H(f; x, \alpha) &\leq C\left\{\delta(x)^{\alpha-1} rMf(x) + \int_{B(x, 2\delta(x)) \setminus B(x, r)} |x - y|^{\alpha-N} f(y)dy\right\} \\ &\leq C\{r^\alpha Mf(x) + r^\alpha \Phi^{-1}(x, r^{-N})\}. \end{aligned}$$

Suppose $\Phi(x, Mf(x))^{-1/N} > \delta(x)$. Then we have by (9)

$$H(f; x, \alpha) = \delta(x)^\alpha H(f; x, 0) \leq C\delta(x)^\alpha Mf(x) \leq CMf(x)\Phi(x, Mf(x))^{-\alpha/N},$$

which is (10).

Next, if $\Phi(x, Mf(x))^{-1/N} \leq \delta(x)$, then take $r = \Phi(x, Mf(x))^{-1/N}$. Then, in view of (4) in Lemma 3, we obtain (10).

(2) By (7),

$$\delta(x)^{\alpha-N} \int_{B(x, 2\delta(x))} f(y)dy \leq C\delta(x)^\alpha \Phi^{-1}(x, \delta(x)^{-N}).$$

If $\alpha \leq \sigma_0$, then $r \mapsto r^\alpha \Phi^{-1}(x, r^{-N})$ is uniformly almost decreasing in view of Lemma 5. Hence

$$\delta(x)^{\alpha-N} \int_{B(x, 2\delta(x))} f(y)dy \leq Cr^\alpha \Phi^{-1}(x, r^{-N})$$

for $0 < r \leq \delta(x)$. Thus, by the same arguments as above we obtain (11). □

4. Hardy's inequality I

Lemma 9. *Let $\Omega \neq \mathbf{R}^N$ be an open set satisfying* (8). *Suppose $\Phi(x,t)$ satisfies* (Φ5), (Φ6) *and* ($\Phi 3^*$). *Then there exist constants $C > 0$ and $0 < b_0 < 1$ such that*

$$\|\delta^{b-1}u\|_{L^\Phi(\Omega)} \le C\|\delta^b|\nabla u|\,\|_{L^\Phi(\Omega)} \tag{12}$$

for all $u \in W_0^{1,\Phi}(\Omega)$ and $0 \le b \le b_0$. If $u \in W_0^{1,\Phi}(\Omega)$ and $\delta^b|\nabla u| \in L^\Phi(\Omega)$ for $0 \le b \le b_0$, then $\delta^b u$ extended by 0 outside Ω belongs to $W^{1,\Phi}(\mathbf{R}^N)$.

Proof. Without loss of generality, we may assume that $0 \in \partial\Omega$. For $u \in W_0^{1,\Phi}(\Omega)$ and $b \ge 0$, let

$$u_b(x) = \begin{cases} \delta(x)^b u(x), & \text{if } x \in \Omega \\ 0, & \text{if } x \in \Omega^c. \end{cases}$$

We first treat $u \in C_0^\infty(\Omega)$. Note that δ and $1/\delta$ are bounded on the support of u and $\delta \in W^{1,\infty}(\Omega)$. Hence $u_b \in W^{1,\Phi}(\mathbf{R}^N) \subset W_{loc}^{1,1}(\mathbf{R}^N)$ for every $b \ge 0$. Applying Lemma 7 (1) to this function, we have

$$\delta(x)^b|u(x)| \le C\int_{B(x,2\delta(x))\cap\Omega} |x-y|^{1-N}\{b\delta(y)^{b-1}|u(y)| + \delta(y)^b|\nabla u(y)|\}dy, \tag{13}$$

so that

$$\delta(x)^{b-1}|u(x)| \le C\{bM(\delta^{b-1}u)(x) + M(\delta^b|\nabla u|)(x)\}$$

for a.e. $x \in \Omega$ with a constant C independent of b. In view of Lemma 1, we find

$$\|\delta^{b-1}u\|_{L^\Phi(\Omega)} \le C_0\{b\|\delta^{b-1}u\|_{L^\Phi(\Omega)} + \|\delta^b|\nabla u|\,\|_{L^\Phi(\Omega)}\},$$

which gives

$$(1 - C_0 b)\|\delta^{b-1}u\|_{L^\Phi(\Omega)} \le C_0\|\delta^b|\nabla u|\,\|_{L^\Phi(\Omega)}.$$

Hence, taking b_0 such that $1 - C_0 b_0 > 0$, we have (12) for $0 \le b \le b_0$.

We next treat $u \in W_0^{1,\Phi}(\Omega)$ such that $u = 0$ outside $B(0,R)$ for some $R > 0$. Then we can find a sequence $\varphi_j \in C_0^\infty(\Omega)$ such that $\varphi_j \to u$ in $W_0^{1,\Phi}(\Omega)$ and $\varphi_j = 0$ outside $B(0,2R)$ for each j. By the above discussions, for $0 < b \le b_0$, we have

$$\|\delta^{b-1}\varphi_j\|_{L^\Phi(\Omega)} \le C\|\delta^b|\nabla\varphi_j|\,\|_{L^\Phi(\Omega)} \tag{14}$$

for all j and

$$\|\delta^{b-1}(\varphi_j - \varphi_{j'})\|_{L^\Phi(\Omega)} \le C\|\delta^b|\nabla\varphi_j - \nabla\varphi_{j'}|\,\|_{L^\Phi(\Omega)} \tag{15}$$

for all j, j'. Since δ is bounded on $B(0,2R)$, we see that

$$\|\delta^b|\nabla\varphi_j|\,\|_{L^\Phi(\Omega)} \to \|\delta^b|\nabla u|\,\|_{L^\Phi(\Omega)}$$

as $j \to \infty$. Similarly

$$\|\delta^b|\nabla\varphi_j - \nabla\varphi_{j'}|\,\|_{L^\Phi(\Omega)} \to 0$$

as $j, j' \to \infty$. Hence by (15), $\{\delta^{b-1}\varphi_j\}$ is a Cauchy sequence in $L^\Phi(\Omega)$, which implies that $\delta^{b-1}\varphi_j \to \delta^{b-1}u$ in $L^\Phi(\Omega)$. Thus, letting $j \to \infty$ in (14), we obtain (12). Further, $(\varphi_j)_b \to u_b$ in $L^\Phi(\mathbf{R}^N)$ and

$$\begin{aligned}\nabla(\varphi_j)_b &= \begin{cases} b\delta^{b-1}\varphi_j\nabla\delta + \delta^b\nabla\varphi_j & \text{on } \Omega \\ 0 & \text{on } \Omega^c \end{cases} \\ &\to \begin{cases} b\delta^{b-1}u\nabla\delta + \delta^b\nabla u & \text{on } \Omega \\ 0 & \text{on } \Omega^c \end{cases}\end{aligned}$$

in $L^\Phi(\mathbf{R}^N)$ as $j \to \infty$. It then follows that

$$\nabla u_b = \begin{cases} b\delta^{b-1}u\nabla\delta + \delta^b\nabla u & \text{on } \Omega \\ 0 & \text{on } \Omega^c, \end{cases}$$

which belongs to $L^\Phi(\mathbf{R}^N)$, and hence $u_b \in W^{1,\Phi}(\mathbf{R}^N)$.

Finally we treat a general $u \in W_0^{1,\Phi}(\Omega)$. For each $n \in \mathbf{N}$, we consider a C^1-function H_n on $[0,\infty)$ such that $0 \le H_n \le 1$ on $[0,\infty)$, $H_n = 1$ on $[0,n]$, $H_n = 0$ on $[3n,\infty)$, $0 \le -H_n'(t) \le t^{-1}$ for $t \in (n,3n)$. The existence of such H_n is assured since $\int_n^{3n} t^{-1}\,dt = \log 3 > 1$. Set $u_n(x) = H_n(|x|)u(x)$, $n = 1,2,\dots$. Then we know by the above that

$$\|\delta^{b-1}u_n\|_{L^\Phi(\Omega)} \le C\|\delta^b|\nabla(u_n)|\,\|_{L^\Phi(\Omega)}. \tag{16}$$

Since $\delta^{b-1}|u_n| \uparrow \delta^{b-1}|u|$ $(n \to \infty)$,

$$\|\delta^{b-1}u_n\|_{L^\Phi(\Omega)} \to \|\delta^{b-1}u\|_{L^\Phi(\Omega)} \qquad (n \to \infty).$$

On the other hand,

$$\begin{aligned}|\nabla u_n(x)| &\le |H_n{}'(|x|)|\,|u(x)| + H_n(|x|)|\nabla u(x)| \\ &\le \frac{1}{|x|}|u(x)|\chi_{B(0,3n)\setminus B(0,n)}(x) + |\nabla u(x)|.\end{aligned}$$

Since $\delta(x)^b/|x| \le |x|^{b-1} \le n^{b-1}$ for $|x| \ge n$ and $b < 1$,

$$\delta(x)^b|\nabla u_n(x)| \le n^{b-1}|u(x)| + \delta(x)^b|\nabla u(x)|,$$

so that

$$\|\delta^b|\nabla u_n|\,\|_{L^\Phi(\Omega)} \le n^{b-1}\|u\|_{L^\Phi(\Omega)} + \|\delta^b|\nabla u|\,\|_{L^\Phi(\Omega)}$$
$$\to \|\delta^b|\nabla u|\,\|_{L^\Phi(\Omega)} \qquad (n \to \infty).$$

Therefore, by letting $n \to \infty$ in (16), we obtain (12), which also implies that $u_b \in W^{1,\Phi}(\mathbf{R}^N)$. □

For $\alpha \ge 0$, we consider a function $\Psi_\alpha(x,t) : \mathbf{R}^N \times [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(Ψ1) $\Psi_\alpha(\cdot,t)$ is measurable on $\mathbf{R}^N$ for each $t \ge 0$ and $\Psi_\alpha(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in \mathbf{R}^N$;

(Ψ2) $\Psi_\alpha(x,\cdot)$ is uniformly almost increasing on $[0,\infty)$, namely there is a constant $A_4 \ge 1$ such that $\Psi_\alpha(x,t) \le A_4\Psi_\alpha(x,s)$ for all $x \in \mathbf{R}^N$, whenever $0 \le t < s$;

(Ψ3) there exists a constant $A_5 \ge 1$ such that

$$\Psi_\alpha(x, t\Phi(x,t)^{-\alpha/N}) \le A_5\Phi(x,t)$$

for all $x \in \mathbf{R}^N$ and $t > 0$.

Note that we may take $\Psi_0(x,t) = \Phi(x,t)$.

Example 3. Let $\Phi(x,t)$ be as in Example 1. Set

$$\Psi_\alpha(x,t) = \left(t\prod_{j=1}^{k}(L_e^{(j)}(t))^{q_j(x)/p(x)}\right)^{p^\#(x)},$$

where $1/p^\#(x) = 1/p(x) - \alpha/N$. If $0 \le \alpha < N/p^+$, then Ψ_α satisfies (Ψ1), (Ψ2) and (Ψ3).

Example 4. Let $\Phi(x,t)$ be as in Example 2. Set

$$\Psi_\alpha(x,t) = ((1+t)L_c(t)^{q_1(x)/p_1(x)})^{p_1^\#(x)}((1+1/t)L_c(1/t)^{-q_2(x)/p_2(x)})^{p_2^\#(x)}.$$

If $0 \le \alpha < \min\{N/p_1^+, N/p_2^+\}$, then Ψ_α satisfies (Ψ1), (Ψ2) and (Ψ3).

Theorem 1. *Let $\Omega \ne \mathbf{R}^N$ be an open set satisfying* (8). *Suppose $\Phi(x,t)$ satisfies* (Φ5), ($\Phi 3^*$) *and* ($\Phi 6^*$) *and let $\alpha \in [0,\sigma_0) \cap [0,1]$ for σ_0 given in Lemma* 5. *Then there exist constants $C^* > 0$ and $0 < b_0 < 1$ such that*

$$\int_\Omega \Psi_\alpha(x, \delta(x)^{\alpha+b-1}|u(x)|/C^*)dx \le 1$$

for all $u \in W_0^{1,\Phi}(\Omega)$ with $\|\delta^b|\nabla u|\,\|_{L^\Phi(\Omega)} \le 1$ and $0 \le b \le b_0$.

PROOF. Let b_0 be the number given in Lemma 9 and let $0 \le b \le b_0$. Let $u \in W_0^{1,\Phi}(\Omega)$ with $\|\delta^b|\nabla u|\,\|_{L^{\Phi}(\Omega)} \le 1$. By Lemma 9, $\delta^b u$ extended by 0 outside Ω belongs to $W_{loc}^{1,1}(\mathbf{R}^N)$, so that by Lemma 7 (1), (13) holds a.e. $x \in \Omega$. Hence

$$\delta(x)^{\alpha+b-1}|u(x)| \le C\delta(x)^{\alpha-1}\int_{B(x,2\delta(x))}|x-y|^{1-N}f_u(y)dy$$

for a.e. $x \in \Omega$, where $f_u(y) = b\delta(y)^{b-1}|u(y)| + \delta(y)^b|\nabla u(y)|$ for $y \in \Omega$ and $f_u(y) = 0$ for $y \in \Omega^c$. By Lemma 9, there is a constant $C_1 \ge 1$ such that $\|f_u\|_{L^{\Phi}(\Omega)} \le C_1$. Applying Lemma 8 (1) to f_u/C_1 and using ($\Phi 4$), we have

$$\delta(x)^{\alpha+b-1}|u(x)| \le C_2 Mf_u(x)\Phi(x, Mf_u(x))^{-\alpha/N}$$

a.e. $x \in \Omega$. Hence by ($\Psi 2$) and ($\Psi 3$) we have

$$\int_\Omega \Psi_\alpha(x, \delta(x)^{\alpha+b-1}|u(x)|/C_2)dx \le A_4A_5\int_\Omega \Phi(x, Mf_u(x))dx \tag{17}$$

whenever $\|\delta^b|\nabla u|\,\|_{L^{\Phi}(\Omega)} \le 1$. By Lemma 1, $\|Mf_u\|_{L^{\Phi}(\Omega)} \le C_3$, which implies $\int_\Omega \Phi(x, Mf_u(x))dx \le C_4$ $(C_4 \ge 1)$.

Now let $0 < \varepsilon \le 1$. Since

$$\Phi(x, Mf_{\varepsilon u}(x)) = \Phi(x, \varepsilon Mf_u(x)) \le A_2\varepsilon\Phi(x, Mf_u(x))$$

by (3), applying (17) to εu, we have

$$\begin{aligned}\int_\Omega \Psi_\alpha(x, \delta(x)^{\alpha+b-1}|\varepsilon u(x)|/C_2)dx &\le A_4A_5\int_\Omega \Phi(x, Mf_{\varepsilon u}(x))dx \\ &\le A_2A_4A_5\varepsilon\int_\Omega \Phi(x, Mf_u(x))dx \le A_2A_4A_5C_4\varepsilon.\end{aligned}$$

Thus, taking $\varepsilon = (A_2A_4A_5C_4)^{-1}$ and $C^* = C_2/\varepsilon$, we obtain the required result. $\square$

Applying Theorem 1 to special Φ and Ψ_α given in Examples 1 and 3, we obtain the following corollary, which is an extension of Theorem B.

COROLLARY 2. *Let Φ and Ψ_α be as in Examples* 1 *and* 3 *and let $\Omega \ne \mathbf{R}^N$ be an open set satisfying* (8). *Suppose $p^- > 1$ and let $\alpha \in [0, N/p^+) \cap [0, 1]$. Then there exist constants $C > 0$ and $0 < b_0 < 1$ such that*

$$\|\delta^{\alpha+b-1}u\|_{L^{\Psi_\alpha}(\Omega)} \le C\|\delta^b|\nabla u|\,\|_{L^{\Phi}(\Omega)}$$

for all $u \in W_0^{1,\Phi}(\Omega)$ and $0 \le b \le b_0$.

Similarly, applying Theorem 1 to special Φ and Ψ_α given in Examples 2 and 4, we obtain another extension of Theorem B:

Corollary 3. *Let* Φ *and* Ψ_α *be as in Examples* 2 *and* 4 *and let* $\Omega \neq \mathbf{R}^N$ *be an open set satisfying* (8). *Suppose* $\min(p_1^-, p_2^-) > 1$ *and let* $\alpha \in [0, \min(N/p_1^+, N/p_2^+)) \cap [0, 1]$. *Then there exist constants* $C > 0$ *and* $0 < b_0 < 1$ *such that*

$$\|\delta^{\alpha+b-1} u\|_{L^{\Psi_\alpha}(\Omega)} \le C \|\delta^b |\nabla u|\,\|_{L^\Phi(\Omega)}$$

for all $u \in W_0^{1,\Phi}(\Omega)$ *and* $0 \le b \le b_0$.

5. Hardy's inequality II

For a proof of Theorem 2 below, we prepare the following lemma instead of Lemma 9.

Lemma 10. *Let* $\Omega \neq \mathbf{R}^N$ *be an open set. Suppose that* $\Phi(x,t)$ *satisfies* (Φ5), (Φ6) *and* ($\Phi 3^*$) *for* $\varepsilon_0 > N-1$. *Then there exist constants* $C > 0$ *and* $0 < b_1 < 1$ *such that*

$$\|\delta^{b-1} u\|_{L^\Phi(\Omega)} \le C \|\delta^b |\nabla u|\,\|_{L^\Phi(\Omega)}$$

for all $u \in W_0^{1,\Phi}(\Omega)$ *and* $0 \le b \le b_1$. *If* $u \in W_0^{1,\Phi}(\Omega)$ *and* $\delta^b |\nabla u| \in L^\Phi(\Omega)$ *for* $0 \le b \le b_1$, *then* $\delta^b u$ *extended by* 0 *outside* Ω *belongs to* $W^{1,\Phi}(\mathbf{R}^N)$.

Proof. Take λ such that $N < \lambda < \varepsilon_0 + 1$. Then $W^{1,\Phi}(\mathbf{R}^N) \subset W^{1,\lambda}_{loc}(\mathbf{R}^N)$.

First, let $u \in C_0^\infty(\Omega)$ and $b \ge 0$. Let u_b be the function $\delta^b u$ extended by 0 outside Ω. Then $u_b \in W^{1,\Phi}(\mathbf{R}^N) \subset W^{1,\lambda}_{loc}(\mathbf{R}^N)$ and applying Lemma 7 (2) to $v = u_b$, we have

$$[\delta(x)^{b-1} |u(x)|]^\lambda \le C \delta(x)^{-N} \int_{B(x, 2\delta(x)) \cap \Omega} f_u(y)\,dy \le C M f_u(x) \tag{18}$$

for all $x \in \Omega$, where $f_u(y) = [b\delta(y)^{b-1}|u(y)| + \delta(y)^b |\nabla u(y)|]^\lambda$. In view of Corollary 1, we find

$$\|[\delta^{b-1}|u|]^\lambda\|_{L^{\Phi_\lambda}(\Omega)} \le C \|f_u\|_{L^{\Phi_\lambda}(\Omega)}.$$

Since $\|f\|_{L^{\Phi_\lambda}(\Omega)} = \|f^{1/\lambda}\|^\lambda_{L^\Phi(\Omega)}$ for every $f \in L^{\Phi_\lambda}(\Omega)$, we obtain

$$\|\delta^{b-1} u\|_{L^\Phi(\Omega)} \le C^{1/\lambda} \|f_u^{1/\lambda}\|_{L^\Phi(\Omega)} \le C_1 \{ b \|\delta^{b-1} u\|_{L^\Phi(\Omega)} + \|\delta^b |\nabla u|\,\|_{L^\Phi(\Omega)} \},$$

which gives

$$(1 - C_1 b) \|\delta^{b-1} u\|_{L^\Phi(\Omega)} \le C_1 \|\delta^b |\nabla u|\,\|_{L^\Phi(\Omega)}.$$

Take b_1 such that $1 - C_1 b_1 > 0$. Then, in the same way as the last half of the proof of Lemma 9, we obtain the required results for $u \in W_0^{1,\Phi}(\Omega)$ and $0 \le b \le b_1$. □

THEOREM 2. *Let $\Omega \neq \mathbf{R}^N$ be an open set. Suppose $\Phi(x,t)$ satisfies ($\Phi 5$), ($\Phi 6^*$) and ($\Phi 3^*$) with $\varepsilon_0 > N-1$. Let $\alpha \in [0,\sigma_0]$. Then there exist $C^* > 0$ and $0 < b_1 < 1$ such that*

$$\int_\Omega \Psi_\alpha(x, \delta(x)^{\alpha+b-1}|u(x)|/C^*)dx \leq 1$$

for all $u \in W_0^{1,\Phi}(\Omega)$ with $\|\delta^b|\nabla u|\,\|_{L^\Phi(\Omega)} \leq 1$ and $0 \leq b \leq b_1$.

PROOF. Let b_1 be as in the above lemma and let $0 \leq b \leq b_1$. Let $u \in W_0^{1,\Phi}(\Omega)$ with $\|\delta^b|\nabla u|\,\|_{L^\Phi(\Omega)} \leq 1$. Take λ such that $N < \lambda < \varepsilon_0 + 1$. By the above lemma, $\delta^b u$ extended by 0 outside Ω belongs to $W_{loc}^{1,\lambda}(\mathbf{R}^N)$, so that by (18) we have

$$[\delta(x)^{\alpha+b-1}|u(x)|]^\lambda \leq C\delta(x)^{\alpha\lambda-N}\int_{B(x,2\delta(x))} f_u(y)dy$$

for all $x \in \Omega$, where $f_u(y) = [b\delta(y)^{b-1}|u(y)| + \delta(y)^b|\nabla u(y)|]^\lambda$ for $y \in \Omega$ and $f_u(y) = 0$ for $y \in \Omega^c$. By Lemma 10, there is a constant $C_1 \geq 1$ such that $\|f_u^{1/\lambda}\|_{L^\Phi(\Omega)} \leq C_1$, so that $\|f_u\|_{L^{\Phi_\lambda}(\Omega)} \leq C_1^\lambda$.

Here we note that $\Phi_\lambda(x,t)$ satisfies ($\Phi 6^*$) with g^λ in place of g and that $r \mapsto r^{\lambda\sigma_0}\Phi_\lambda^{-1}(x, r^{-N})$ is uniformly almost decreasing on $(0,\infty)$. Since $\lambda\alpha \in [0, \lambda\sigma_0]$, we can apply Lemma 8 (2) to f_u/C_1^λ, $\lambda\alpha$ and Φ_λ in place of f, α and Φ respectively, and using ($\Phi 4$), we obtain

$$\begin{aligned}\delta(x)^{\alpha+b-1}|u(x)| &\leq C[Mf_u(x)]^{1/\lambda}\Phi_\lambda(x, Mf_u(x)/C_1^\lambda)^{-\alpha/N}\\ &\leq C_2[Mf_u(x)]^{1/\lambda}\Phi(x, [Mf_u(x)]^{1/\lambda})^{-\alpha/N}\end{aligned}$$

for all $x \in \Omega$. Hence by ($\Psi 2$) and ($\Psi 3$)

$$\begin{aligned}\int_\Omega \Psi_\alpha(x, \delta(x)^{\alpha+b-1}|u(x)|/C_2)dx &\leq A_4A_5\int_\Omega \Phi(x, [Mf_u(x)]^{1/\lambda})dx\\ &= A_4A_5\int_\Omega \Phi_\lambda(x, Mf_u(x))dx. \qquad (19)\end{aligned}$$

By Corollary 1, $\|Mf_u\|_{L^{\Phi_\lambda}(\Omega)} \leq C_3$, which implies $\int_\Omega \Phi_\lambda(x, Mf_u(x))dx \leq C_4$.

Let $0 < \varepsilon \leq 1$. Since

$$\begin{aligned}\Phi_\lambda(x, Mf_{\varepsilon u}(x)) &= \Phi_\lambda(x, \varepsilon^\lambda Mf_u(x)) = \Phi(x, \varepsilon[Mf_u(x)]^{1/\lambda})\\ &\leq A_2\varepsilon\Phi(x, [Mf_u(x)]^{1/\lambda}) = A_2\varepsilon\Phi_\lambda(x, Mf_u(x))\end{aligned}$$

by (3), applying (19) to εu, we have

$$\int_\Omega \Psi_\alpha(x, \delta(x)^{\alpha+b-1}|\varepsilon u(x)|/C_2)dx \le A_4 A_5 \int_\Omega \Phi_\lambda(x, Mf_{\varepsilon u}(x))dx$$

$$\le A_2 A_4 A_5 \varepsilon \int_\Omega \Phi_\lambda(x, Mf_u(x))dx \le A_2 A_4 A_5 C_4 \varepsilon.$$

Thus, taking $\varepsilon = (A_2 A_4 A_5 C_4)^{-1}$ and $C^* = C_2/\varepsilon$, we obtain the required result. □

Applying Theorem 2 to special Φ and Ψ_α given in Examples 1 and 3, we obtain the following corollary, which is an extension of Theorem B′.

Corollary 4. *Let Φ and Ψ_α be as in Examples* 1 *and* 3. *Suppose $p^- > N$ and let $0 \le \alpha < N/p^+$. Then there exist constants $C > 0$ and $0 < b_1 < 1$ such that*

$$\|\delta^{\alpha+b-1}u\|_{L^{\Psi_\alpha}(\Omega)} \le C\|\delta^b|\nabla u|\,\|_{L^\Phi(\Omega)}$$

for all $u \in W_0^{1,\Phi}(\Omega)$ and $0 \le b \le b_1$.

Similarly, applying Theorem 2 to special Φ and Ψ_α given in Examples 2 and 4, we obtain another extension of Theorem B′:

Corollary 5. *Let Φ and Ψ_α be as in Examples* 2 *and* 4. *Suppose $\min(p_1^-, p_2^-) > N$ and let $0 \le \alpha < \min(N/p_1^+, N/p_2^+)$. Then there exist constants $C > 0$ and $0 < b_1 < 1$ such that*

$$\|\delta^{\alpha+b-1}u\|_{L^{\Psi_\alpha}(\Omega)} \le C\|\delta^b|\nabla u|\,\|_{L^\Phi(\Omega)}$$

for all $u \in W_0^{1,\Phi}(\Omega)$ and $0 \le b \le b_1$.

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