Hiroshima Math. J.
40 (2010), 37–45

Moduli spaces for secondary Hopf surfaces

Noriaki Nakagawa

(Received December 18, 2008)
(Revised February 12, 2009)

Abstract. Hopf surfaces with the infinite cyclic fundamental group are said to be primary and their moduli spaces are studied by K. Dabrowski. We extend his results to the moduli spaces for the secondary Hopf surfaces with abelian fundamental groups which are diffeomorphic to $L(p,q) \times S^1$. The punctured disk D^* is the moduli space for the other cases.

1. Introduction

A Hopf surface is a compact complex surface whose universal covering is $W = \mathbf{C}^2 - \{(0,0)\}$. Hopf surfaces with the infinite cyclic fundamental group are said to be primary and the others secondary. The holomorphic automorphism groups of primary Hopf surfaces are determined by Namba [10] and Wehler [12]. In the same papers the versal deformations of primary Hopf surfaces are also given. Although the moduli space of all primary Hopf surfaces does not exist, Dabrowski [2] gave the fine moduli space for $h^0 = 2$ primary Hopf surfaces and the coarse moduli space for degree zero primary Hopf surfaces.

In the case of secondary Hopf surfaces with abelian fundamental groups, there are the same kinds of moduli spaces for the complex structures on $L(p,q) \times S^1$. As we studied in [9], they are connected if and only if $q^2 \equiv -1 \mod p$.

In the case of secondary Hopf surfaces with non-abelian fundamental groups, we review more explicitly the description of the covering transformations in [7] so that each family forms a fine moduli space. Li and Zhang [6] also studied this case, but they were not concerned with the cases (D2) and (D7).

2000 *Mathematics Subject Classification*. 32G05, 32J15.
Key words and phrases. Hopf surface, moduli space, versal deformation, lens spaces.

2. Versal families for secondary Hopf surfaces

For a compact complex surface X we denote $h^{p,q} = \dim H^q(X, \Omega^p)$. Then, $\sum_{p+q=k} h^{p,q} = \dim H^k(X, \mathbf{C})$. Now let X be a secondary Hopf surface. Then, we have

$$\dim H^1(X, \mathbf{C}) = 1$$

and hence $h^{0,1} = h^{1,0} + 1$. So, $h^{0,0} = h^{0,1} = h^{2,1} = h^{2,2} = 1$ and $h^{p,q} = 0$ for the other (p,q). Therefore, the statement of Lemma 1 of [12] is still valid for a secondary Hopf surface X:

$$h^0 = \dim H^0(X, \Theta) = \dim H^1(X, \Theta) \qquad \text{and} \qquad H^2(X, \Theta) = 0,$$

where Θ is the sheaf of holomorphic vector fields on X.

This means that for each secondary Hopf surface X there is a versal family for X whose parameter dimension at X is equal to h^0. We studied h^0 in [9], as we will state in the next section.

3. Moduli space for the complex structures on $L(p,q) \times S^1$

Let $p \geq 2$, $0 < q < p$, $(p,q) = 1$ and $\rho = \exp(2\pi i/p)$. Any complex structure on $L(p,q) \times S^1$ induces a complex structure on $S^3 \times S^1$ as a finite covering, which is a primary Hopf surface by Kodaira [5]. So, it is a quotient of a primary Hopf surface by a torsion element $g_r = g[\rho, r]$ with $r \in \mathbf{N}$ and $r \equiv \pm q^{\pm 1} \bmod p$ defined by

$$g_r(z_1, z_2) = (\rho^r z_1, \rho z_2).$$

Remember that a primary Hopf surface is a quotient of $W = \mathbf{C}^2 - \{(0,0)\}$ by $f = f[\alpha, \beta, \lambda, s]$, which is defined by

$$f(z_1, z_2) = (\alpha z_1 + \lambda z_2^s, \beta z_2)$$

with $\alpha, \beta, \lambda \in \mathbf{C}$, $0 < |\alpha|, |\beta| < 1$ and $s \in \mathbf{N}$ [12].

So, we know that the secondary Hopf surface homeomorphic to $L(p,q) \times S^1$ is given as $X = W/G$ where $\pi_1(X) = G$ is generated by a torsion element g_r with $r \equiv \pm q^{\pm 1} \bmod p$ and a contraction $f[\alpha, \beta, \lambda, s]$ which are commutative, that is, $f[\alpha, \beta, \lambda, s] \circ g_r = g_r \circ f[\alpha, \beta, \lambda, s]$. By a simple calculation the commutativity condition is equivalent to

$$(1)\ \lambda = 0, \qquad \text{or} \qquad (2)\ \lambda \neq 0 \text{ and } s \equiv r \bmod p.$$

We use the abbreviation $f_0[\alpha, \beta]$ for the diagonal contraction $f[\alpha, \beta, 0, *]$. We call $W/\langle f_0[\alpha, \beta], g_r \rangle$ a degree zero Hopf surface. A degree zero Hopf surface $W/\langle f_0[\beta^s, \beta], g_r \rangle$ has $h^0 = 4$ for $s = 1 \equiv r \bmod p$ and $h^0 = 3$ for

$s \equiv r \bmod p$ with $s \geq 2$ and $h^0 = 2$ for $s \not\equiv r \bmod p$. A degree zero Hopf surface $W/\langle f_0[\alpha,\beta], g_r\rangle$ with $\alpha \neq \beta^s$, $\beta \neq \alpha^s$ for any $s \equiv r \bmod p$ has $h^0 = 2$. The Hopf surface $X_s = W/\langle f[\gamma^s,\gamma,\lambda,s], g_r\rangle$ with $\lambda \neq 0$ has $h^0 = 2$ and is called a degree $s(\geq 1)$ or a non-diagonal Hopf surface. For the proof of the result on h^0 we refer to [9] Lemma 2.2.

Let $s > 0$ and $s \equiv r \bmod p$. The group $G_s = \langle f[\alpha,\beta,1,s], g_r\rangle$ operates freely and discontinuously on $W \times D^* \times D^*$ by $f(z_1,z_2) = (\alpha z_1 + z_2^s, \beta z_2)$ and $g_r(z_1,z_2) = (\rho^r z_1, \rho z_2)$ for $(z_1,z_2) \in W$ and $(\alpha,\beta) \in D^* \times D^*$. The complex manifold $V_s = (W \times D^* \times D^*)/G_s$ and the induced map $\pi_s : V_s \to D^* \times D^*$ from the projection form a versal family for $X_s = W/\langle f[\gamma^s,\gamma,1,s], g_r\rangle$ by [9] Theorem 2.5 (iii)(b).

We have also an analytic family $V_0 = W \times D^* \times D^*/G_0$ with $\pi_0 : V_0 \to D^* \times D^*$ by taking $G_0 = \langle f_0[\alpha,\beta], g_r\rangle$. Note that if $\alpha \neq \beta^s$ then $f[\alpha,\beta,\lambda,s]$ is conjugate to the diagonal contraction $f_0[\alpha,\beta] = f[\alpha,\beta,0,*]$; in fact, $f_0[\alpha,\beta] \circ h_s = h_s \circ f[\alpha,\beta,\lambda,s]$ for $h_s(z_1,z_2) = (z_1 + \lambda z_2^s/(\alpha - \beta^s), z_2)$. Let $N_1 = \{(\alpha,\beta) \in D^* \times D^* \mid \alpha \neq \beta^k$ for any $k \equiv r \bmod p\}$. Then, we can glue the family $\pi_0^{-1}(N_1) \to N_1$ of degree zero Hopf surfaces with the restriction of the above versal family for $X_s = W/\langle f[\gamma^s,\gamma,1,s], g_r\rangle$ on a sufficiently small neighborhood of $(\gamma^s,\gamma) \in D^* \times D^*$ for each $s > 1$ with $s \equiv r \bmod p$. This is a basic technique to construct the moduli space for $h^0 = 2$ Hopf surfaces.

Since there are still "jump" at $\lambda = 0$ in the family $\{X_\lambda\}_{\lambda \in \mathbf{C}}$ with $X_\lambda = W/\langle f[\gamma^s,\gamma,\lambda,s], g_r\rangle$ and $s \equiv r \bmod p$, there are no global moduli spaces. In fact, the degree s Hopf surfaces $W/\langle f[\gamma^s,\gamma,\lambda,s], g_r\rangle$ with $\lambda \neq 0$ are biholomorphic to each other.

Let $\mathfrak{M} = \kappa(D^* \times D^*)$ be a Stein domain in $\mathbf{C}^2$ given in [2], where $D^* = \{z \in \mathbf{C} \mid 0 < |z| < 1\}$ is the punctured open unit disk in the complex plane, and $\kappa(\alpha,\beta) = \left(\frac{1}{2}(\alpha+\beta), \frac{1}{4}(\alpha-\beta)^2\right)$.

Theorem 1. *We consider the complex structures on* $L(p,q) \times S^1$.

(1) *When* $q = 1$ *and* $p = 2$, *we can fix* $r = 1$. *There is an analytic family of Hopf surfaces over* $\mathfrak{M}$ *with non-diagonal Hopf surfaces at* $\kappa(\gamma^s,\gamma)$ *with* $s \equiv 1 \bmod 2$ *which forms a fine moduli space for* $h^0 = 2$ *Hopf surfaces.* $\mathfrak{M}$ *gives also a coarse moduli space for degree zero Hopf surfaces.*

(2) *When* $q^2 \equiv 1 \bmod p$ *and* $p \geq 3$, *we can take* $r \equiv \pm q \bmod p$ *because* $q^{-1} \equiv q \bmod p$. *There is no fine moduli space for* $h^0 = 2$ *Hopf surfaces.* $\mathfrak{M} \amalg \mathfrak{M}$ *gives a coarse moduli space for* $h^0 = 2$ *Hopf surfaces and also a coarse moduli space for degree zero Hopf surfaces.*

(3) *When* $q^2 \equiv -1 \bmod p$ *and* $p \geq 3$, *we can take* $r \equiv q^{\pm 1} \bmod p$ *because* $-q \equiv q^{-1} \bmod p$. *There is an analytic family of Hopf surfaces with* $r = q$ *over* $D^* \times D^*$ *with non-diagonal Hopf surfaces at* (γ^s,γ) *for* $s \equiv q \bmod p$ *and at* (γ,γ^s) *for* $s \equiv q^{-1} \bmod p$ *which forms a fine moduli space for* $h^0 = 2$ *Hopf*

surfaces. $D^* \times D^*$ *gives also a coarse moduli space for degree zero Hopf surfaces.*

(4) *When* $q^2 \not\equiv 1$, $q^2 \not\equiv -1 \bmod p$ *and* $p \geq 3$, *we have to take* $r \equiv \pm q^{\pm 1} \bmod p$. *There are an analytic family of Hopf surfaces with* $r = q$ *over* $D^* \times D^*$ *with non-diagonal Hopf surfaces at* (γ^s, γ) *for* $s \equiv q \bmod p$ *and at* (γ, γ^s) *for* $s \equiv q^{-1} \bmod p$ *and an analytic family of Hopf surfaces with* $r = p - q$ *over* $D^* \times D^*$ *with non-diagonal Hopf surfaces at* (γ^s, γ) *for* $s \equiv -q \bmod p$ *and at* (γ, γ^s) *for* $s \equiv -q^{-1} \bmod p$ *which form a fine moduli space for* $h^0 = 2$ *Hopf surfaces.* $D^* \times D^* \amalg D^* \times D^*$ *gives also a coarse moduli space for degree zero Hopf surfaces.*

Here, in (3) and (4) we have to consider $N_2 = \{(\alpha, \beta) \in D^* \times D^* \mid \alpha \neq \beta^k$ for any $k \equiv r \bmod p$ and $\alpha^\ell \neq \beta$ for any $\ell \equiv r^{-1} \bmod p\}$ and $u(z_1, z_2) = (z_2, z_1)$. We glue the family of degree zero Hopf surfaces $\pi_0^{-1}(N_2) \to N_2$ with the restriction of the versal family for $X_s = W/\langle f[\gamma^s, \gamma, 1, s], g_r \rangle$ on a sufficiently small neighborhood of $(\gamma^s, \gamma) \in D^* \times D^*$ for each $s > 1$ with $s \equiv r \bmod p$ and also with the restriction of the versal family for $W/\langle u \circ f[\gamma^s, \gamma, 1, s] \circ u, u \circ g_{r^{-1}} \circ u \rangle$ on a sufficiently small neighborhood of (γ, γ^s) for each $s > 1$ with $s \equiv r^{-1} \bmod p$.

PROOF. First we consider the moduli space for the degree zero Hopf surfaces. Recall that the coarse moduli space for the primary degree zero Hopf surfaces is given by $\mathfrak{M}$ by the map $W/\langle f_0[\alpha, \beta] \rangle \sim W/\langle f_0[\beta, \alpha] \rangle \mapsto \kappa(\alpha, \beta) \in \mathfrak{M}$. The degree zero Hopf surface diffeomorphic to $L(p, q) \times S^1$ is given by $W/\langle f_0[\alpha, \beta], g_r \rangle$ with $r \equiv \mp q^{\mp 1} \bmod p$. Since the conjugation by $u(z_1, z_2) = (z_2, z_1)$ gives a biholomorphic map of $W/\langle f_0[\alpha, \beta], g_r \rangle$ to $W/\langle f_0[\beta, \alpha], g_{r^{-1}} \rangle$, we mainly consider the cases $r \equiv q \bmod p$ and $r \equiv -q \bmod p$. They are contained in different connected components if and only if $q^2 \not\equiv -1 \bmod p$ as shown in [9]. In the cases (1) and (2) where $q^{-1} \equiv q \bmod p$ we have $r^{-1} \equiv r \bmod p$ for any $r \equiv \pm q^{\pm 1} \bmod p$. So, on each connected component the map $W/\langle f_0[\alpha, \beta], g_r \rangle \sim W/\langle f_0[\beta, \alpha], g_r \rangle \mapsto \kappa(\alpha, \beta) \in \mathfrak{M}$ gives a bijection between the set of biholomorphic classes of the secondary degree zero Hopf surfaces and $\mathfrak{M}$, because its composite with the quotient map from the set of biholomorphic classes of the primary degree zero Hopf surfaces is bijective and the quotient map is surjective. In the case (2) we consider the cases $r \equiv q \bmod p$ and $r \equiv -q \bmod p$ separately and get two connected components. In the case (3) we can fix $r \equiv q \bmod p$ and see $W/\langle f_0[\alpha, \beta], g_r \rangle \not\sim W/\langle f_0[\beta, \alpha], g_r \rangle$ if $\alpha \neq \beta$ by [9] Lemma 1.1. Hence, $D^* \times D^*$ gives a coarse moduli space. In the case (4) we consider the cases $r \equiv q \bmod p$ and $r \equiv -q \bmod p$ separately and get the result as in the case (2).

To get the moduli space for $h^0 = 2$ secondary Hopf surfaces we fix r at first and consider $W' = W/\langle g_r \rangle$. In the case (1) where $p = 2$ and $q = 1$ we can fix $r = 1$ and get $g_r = g_{r^{-1}} = g_{-r} = g_{-r^{-1}} = -I$. So, the family of $h^0 = 2$ Hopf surfaces over $\mathfrak{M}$ is constructed in almost the same way as in the primary case. We take a family of degree zero Hopf surfaces over $\mathfrak{M} - \{\kappa(\gamma^s, \gamma)$ for some $s > 0$ with $s \equiv 1 \mod 2\}$. For each $s > 1$ with $s \equiv 1 \mod 2$ we can glue the above family with the restriction of the versal family for the degree s Hopf surfaces $W'/\langle f[\gamma^s, \gamma, 1, s]\rangle$ on a sufficiently small neighborhood of $\kappa(\gamma^s, \gamma)$ by the basic technique mentioned above Theorem. Moreover, the argument of [2] §9 for gluing together the versal families for the degree one ($s = 1$) Hopf surfaces can apply because $g_r = -I$ commutes with any matrix. The constructed family contains every $h^0 = 2$ Hopf surface up to biholomorphic equivalence and is versal at each fiber. So, it is a fine moduli space for $h^0 = 2$ Hopf spaces. Only the difference from the primary case is that the family for the case (1) does not contain degree positive and even Hopf surfaces. Also, it is easy to see that Hopf surfaces $X = W'/\langle f_0[\beta^{2n}, \beta]\rangle$ have $h^0 = 2$ in this case.

In the case (2) we have $q^{-1} \equiv q \mod p$. So, for any $r \equiv \pm q^{\pm 1} \mod p$ we have $r^{-1} \equiv r \mod p$ and $W/\langle f_0[\alpha, \beta], g_r\rangle \sim W/\langle f_0[\beta, \alpha], g_r\rangle$. Hence, it is not difficult to see that $\mathfrak{M} \amalg \mathfrak{M}$ gives also a coarse moduli space for $h^0 = 2$ Hopf surfaces. Assume now that a fine moduli space for $h^0 = 2$ Hopf surfaces exists. The congruence $r \equiv 1 \mod p$ does not hold for at least one of $r = q$ or $r = p - q$. So, the degree zero Hopf surface $X_0 = W/\langle f_0[\gamma, \gamma], g_r\rangle$ with $r \not\equiv 1 \mod p$ must be contained in the family of the fine moduli space. Since the versal family for X_0 is $W \times D^* \times D^*/G_0$, the family over $\mathfrak{M} \amalg \mathfrak{M}$ cannot be fine at X_0 by the argument of the proof of [2] Theorem 10.2. This depends on the fact that $\kappa : D^* \times D^* \to \mathfrak{M}$ does not have any local section near $\kappa(\alpha, \alpha) = [(\alpha, 0)] \in \mathfrak{M}$.

Note again that in the cases (3) and (4) $W/\langle f_0[\alpha, \beta], g_r\rangle$ is not biholomorphic to $W/\langle f_0[\beta, \alpha], g_r\rangle$ if $\alpha \neq \beta$ by [9] Lemma 1.1. So, each connected component of the moduli space is not $\mathfrak{M}$ but $D^* \times D^*$. Also any degree s Hopf surfaces with $s \equiv \pm q^{\pm 1} \mod p$ should be considered. Since $W/\langle f_0[\beta, \alpha], g_{r^{-1}}\rangle$ is biholomorphic to $W/\langle f_0[\alpha, \beta], g_r\rangle$, the versal family $W \times D^* \times D^*/G_s$ with $G_s = \langle f[\beta, \alpha, 1, s], g_{r^{-1}}\rangle$ for the degree s Hopf surface $W/\langle f[\gamma^s, \gamma, 1, s], g_{r^{-1}}\rangle$ with $s \equiv r^{-1} \mod p$ can be glued together with the family of degree zero Hopf surfaces $\pi_0^{-1}(N_2) = W \times N_2/\langle f_0[\alpha, \beta], g_r\rangle$, where $N_2 = \{(\alpha, \beta) \in D^* \times D^* \mid \alpha \neq \beta^k$ for any $k \equiv r \mod p$ and $\alpha^\ell \neq \beta$ for any $\ell \equiv r^{-1} \mod p\}$ as before by the basic technique. Now, we get easily a fine moduli space by gluing the versal families for the $h^0 = 2$ Hopf surfaces together. The fiber over $(\alpha, \beta) \in D^* \times D^*$ is $W/\langle f_0[\alpha, \beta], g_r\rangle$ except $W/\langle f[\beta^s, \beta, 1, s], g_r\rangle$ over (β^s, β) and $W/\langle f[\alpha^s, \alpha, 1, s], g_{r^{-1}}\rangle$ over (α, α^s). In the case (4) we consider the cases $r \equiv q \mod p$ and $r \equiv -q \mod p$ separately and get two connected components.

Before ending this section we remark some misprints in [9]: p. 428 ℓ.9 $(id + f_*)\theta$ should read $(id - f_*)\theta$ and p. 430 ℓ.6 $\partial F/\partial s(H^{-1}(z_1, z_2, \alpha, 0, \alpha, 0))$ should read $\partial F/\partial s(H^{-1}(z_1, z_2, \alpha, 0, 0, \alpha))$.

4. Moduli spaces for Hopf surfaces with non-abelian fundamental groups

In the cases that the fundamental groups are not abelian, we know that the secondary Hopf surfaces, having the isomorphic fundamental group G, are diffeomorphic to each other essentially by [3] Theorem 10. We see moreover that the moduli space is D^* for each diffeomorphic class, where $D^* = \{z \in \mathbf{C} : 0 < |z| < 1\}$ is the punctured open unit disk in the complex plane.

The covering transformation groups G can be embedded in $GL(2, \mathbf{C})$ due to Kodaira [5]. We put $H = \{g \in G \mid |\det g| = 1\}$ and $K = \{g \in G \mid \det g = 1\}$. Recall that G is an extension of H by $\mathbf{Z}$ and is said to be decomposable if it is isomorphic to the product group $\mathbf{Z} \times H$ and indecomposable otherwise. We may assume that H is a finite subgroup of $U(2)$.

When G is decomposable and non-abelian we gave in [7] complete families (C1) to (C6) with the contraction parameter γ in the punctured disk D^*. They are actually the moduli spaces, since the different contractions have different determinants and hence their groups G are not conjugate each other in $GL(2, \mathbf{C})$.

When G is indecomposable we also gave in [7] the families (D1) to (D7) with the contraction parameter γ in the punctured disk D^*.

In both cases only the constant multiple of the identity matrix gives their holomorphic automorphisms and hence $h^0 = 1$. So, it is not difficult to see that the family is a versal family for each fiber (cf. [6]).

To be complete we review the explicit families of such Hopf surfaces over D^* which form fine moduli spaces with some corrections to [7] (cf. [8]).

Let a be a primitive m-th root of 1, $\varepsilon = \exp(2\pi i/5)$, $\rho_n = \exp(\pi i/n)$, $\zeta = \rho_4$, $\gamma \in \mathbf{C}$ with $0 < |\gamma| < 1$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We denote by $\langle h_1, \ldots, h_k \rangle$ the subgroup generated by $h_1, \ldots, h_k$.

(C) The case when G is decomposable and not abelian: $G = \langle \gamma I \rangle \times H$ where $H = H_i$ is a finite subgroup of $U(2)$ which operates freely on S^3 classified by Hopf and Threlfall-Seifert (Cf. [11] p. 111 Th. 1 or [1] p. 347).

(C1) $G = \langle \gamma I \rangle \times H_1$ where $H_1 = \langle aI \rangle \times B'_{2^k(2\ell+1)}$ and $K = A_{2(2\ell+1)}$ with $(2^k(2\ell+1), m) = 1$, $2\ell + 1 \geq 3$ and $k \geq 3$. Note that $B'_{2^k(2\ell+1)} = \left\langle h' = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}, h = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} \right\rangle$ such that b and d have finite orders $2\ell + 1$ and 2^k respectively. Note also that $K = A_{2(2\ell+1)}$ is generated by $-h'$.

(C2) $G = \langle \gamma I \rangle \times H_2$ where $H_2 = \langle aI \rangle \times B_n$ and $K = B_n$ with $(m, 4n) = 1$ and $n \geq 2$. Here $B_n = \left\langle \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} \rho_n & 0 \\ 0 & \rho_n^{-1} \end{pmatrix} \right\rangle$ is the binary dihedral group of order $4n$.

(C3) $G = \langle \gamma I \rangle \times H_3$ where $H_3 = \langle aI \rangle \times C$ and $K = C$ with $(m, 6) = 1$. Here $C = \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix} \right\rangle$ is the binary tetrahedral group of order 24.

(C4) $G = \langle \gamma I \rangle \times H_4$ where $H_4 = \langle aI \rangle \times C'_{8 \cdot 3^k}$ and $K = B_2$ with $(m, 6) = 1$ and $k \geq 2$. Here $C'_{8 \cdot 3^k} = \left\langle \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{\omega}{\sqrt{2}} \begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix} \right\rangle$ is a group of order $8 \cdot 3^k$ and ω is a primitive 3^k-th root of 1.

(C5) $G = \langle \gamma I \rangle \times H_5$ where $H_5 = \langle aI \rangle \times D$ and $K = D$ with $(m, 6) = 1$. Here $D = \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix} \right\rangle$ is the binary octahedral group of order 48.

(C6) $G = \langle \gamma I \rangle \times H_6$ where $H_6 = \langle aI \rangle \times E$ and $K = E$ with $(m, 30) = 1$. Here $E = \left\langle \begin{pmatrix} \varepsilon^3 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} \varepsilon^4 - \varepsilon & \varepsilon^2 - \varepsilon^3 \\ \varepsilon^2 - \varepsilon^3 & \varepsilon - \varepsilon^4 \end{pmatrix} \right\rangle$ is the binary icosahedral group of order 120.

(D) The case when G is indecomposable: G is given as

$$G = G_0 \cup gG_0, \qquad G_0 = \langle \gamma^2 I \rangle \times H \text{ and } g = \gamma u$$

in the following cases from (D1) to (D6) and in the exceptional case (D7)

$$G = G_0 \cup gG_0 \cup g^2 G_0, \qquad G_0 = \langle \gamma^3 I \rangle \times H \text{ and } g = \gamma u,$$

where H is a finite cyclic group or one of H in the case (C) and $u \in GL(2, \mathbf{C})$.

(D1) The case H is abelian and K is of order $m_K \geq 3$: We can take $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the case is divided into the following three cases.

(D1-1) $H = \langle aI \rangle \times K$ with $K = \left\langle \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \right\rangle$ where b has the finite order $m_K \geq 3$ and $(m, m_K) = 1$.

(D1-2) $H = \langle aI \rangle \times \left\langle \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \right\rangle \times \left\langle \begin{pmatrix} c & 0 \\ 0 & -c^{-1} \end{pmatrix} \right\rangle$ where b and c have finite orders $2\ell + 1 \geq 1$ and 2^{k_0} with $k_0 \geq 3$ respectively. Moreover, we have $(m, 2\ell + 1) = (m, 2) = 1$ and $m_K = 2^{k_0 - 1}(2\ell + 1)$.

(D1-3) $H = \langle aI \rangle \times \left\langle \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \right\rangle \times \left\langle \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \right\rangle$ where b and c have finite orders $2\ell + 1 \geq 3$ and 2^{k_0} with $k_0 \geq 3$ respectively. Moreover, $(m, 2\ell + 1) = (m, 2) = 1$ and $m_K = 2(2\ell + 1)$.

(D2) The case H is abelian and $K = \{\pm I\}$: $H = \langle aI \rangle \times \left\langle \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \right\rangle$ and $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where c is a primitive 2^k-th root of 1 with $k \geq 3$ and $(2, m) = 1$.

(D3-1) $H = H_1$ as in (C1) and $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(D3-2) $H = H_1$ as in (C1) and $u = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

(D4 & D5) $H = \langle aI \rangle \times B_n$, $K = B_n$ as in (C2) and $u = \begin{pmatrix} \rho_{2n} & 0 \\ 0 & \rho_{2n}^{-1} \end{pmatrix}$ where (D4) is the case $n \geq 3$ and (D5) is the case $n = 2$.

(D6-1) $H = \langle aI \rangle \times C$, $K = C$ as in (C3) and $u = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$.

(D6-2) $H = \langle aI \rangle \times C'_{8\cdot 3^k}$ with $k \geq 2$, $K = B_2$ as in (C4) and $u = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$.

(D7) $H = \langle aI \rangle \times B_2$, $K = B_2$ as in (D5) and $u = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta^3 & \zeta^3 \\ \zeta & -\zeta \end{pmatrix}$.

We will give an outline of proof for the indecomposable cases (D1) to (D7). We may assume $K \neq \{I\}$, since G is abelian otherwise.

Let H be a cyclic group of order m_H at first. We may assume that the generator is $\begin{pmatrix} d & 0 \\ 0 & d^n \end{pmatrix}$. The matrix u is also determined as $u = \begin{pmatrix} 0 & t^{-1} \\ t & 0 \end{pmatrix}$ in this case by [3] Lemma 6 and Proposition 8. We need the condition $n^2 \equiv 1 \mod m_H$ because the conjugation of the generator by u should be contained in H. Since u does not commute with H, we get the condition $n \not\equiv 1 \mod m_H$. Note that $n^2 \equiv 1 \mod p^k$ implies $n \equiv \pm 1 \mod p^k$ for odd prime p. In case $p = 2$, $n^2 \equiv 1 \mod 2^k$ implies $n \equiv \pm 1 \mod 2^k$ for $k = 1, 2$ and $n \equiv \pm 1$ or $n \equiv \pm 1 + 2^{k-1} \mod 2^k$ for $k \geq 3$. Let $m_H = p_0^{k_0} p_1^{k_1} \dots p_\ell^{k_\ell}$ be the prime decomposition with $p_0 = 2$.

(D1) Assume first that $n \equiv \pm 1 \mod 2^{k_0}$. Then, we get the case (D1-1). When $k_0 \geq 3$ and $n \equiv -1 + 2^{k_0 - 1} \mod 2^{k_0}$, we get the case (D1-2). When $n \equiv 1 + 2^{k_0 - 1} \mod 2^{k_0}$ with $k_0 \geq 3$, it is easy to see that $K = \{\pm I\}$ if and only if $n \equiv 1 \mod p_j^{k_j}$ for every odd prime p_j. Since we treat the case $K \neq \{\pm I\}$ in (D1), the case when $n \equiv 1 + 2^{k_0 - 1} \mod 2^{k_0}$ with $k_0 \geq 3$ and $n \not\equiv 1 \mod p_j^{k_j}$ for some odd prime p_j is named (D1-3). We see also that the matrix u above with any $t \in \mathbf{C}^*$ is conjugate to u with $t = 1$ in these three cases.

(D2) This is the remaining case when $n \equiv 1 + 2^{k_0 - 1} \mod 2^{k_0}$ with $k_0 \geq 3$ and $n \equiv 1 \mod p_j^{k_j}$ for every odd prime p_j. Note that the case $K = \{\pm I\}$ is studied separately in [3] p. 229 and u is determined as above in the indecomposable case. Also, $m_H = 2(2\ell + 1) \geq 6$ in the case (D2) in [7] should be corrected to $m_H = 2^{k_0}(2\ell+1)$ with $k_0 \geq 3$. For the matrix u we can take $t = 1$ in the same way as in the case (D1).

When K is not abelian, [3] Lemma 7 and [4] Lemma 7′ showed that K and u are uniquely determined in the cases (D4) to (D7). In [7] the case (D6-2) was missing. Remark that we have to take $k \geq 2$ for $C'_{8\cdot 3^k}$ in the cases (C4) and (D6-2) because the action has fixed points when $k = 1$.

Now, we have only to check the case where K is abelian and H is not abelian. So, we may assume that $H = H_1$ as in (C1). Since $K = A_{2(2\ell+1)}$, we are concerned only with Step 3 of [3] pp. 235–236. Hence, the group G is isomorphic to one of the non-isomorphic groups $\langle h', h, \gamma u_1 \rangle$ and $\langle h', h, \gamma u_2 \rangle$, where $h' = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$, $h = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$, $u_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. We get the case (D3) and no others.

References

[1] E. Brieskorn: Rationale Singularitäten komplexer Flächen, Invent. Math. **4** (1968), 336–358.

[2] K. Dabrowski: Moduli spaces for Hopf surfaces, Math. Ann. **259** (1982), 201–225.

[3] M. Kato: Topology of Hopf surfaces, J. Math. Soc. Japan **27** (1975), 222–238.

[4] M. Kato: Erratum to "Topology of Hopf surfaces", J. Math. Soc. Japan **41** (1989), 173–174.

[5] K. Kodaira: On the structure of compact complex analytic surfaces II, Amer. J. Math. **88** (1966), 682–721.

[6] Li Qingzhong and Zhang Jinhao: On the moduli space of non-primary Hopf surfaces, 数学進展 (Advances in Mathematics) **29** (2000), 245–252.

[7] T. Matumoto and N. Nakagawa: Explicit description of Hopf surfaces and their automorphism groups, Osaka J. Math. **37** (2000), 417–424.

[8] T. Matumoto and N. Nakagawa: Complement to explicit description of Hopf surfaces and their automorphism groups, Preprint.

[9] N. Nakagawa: Complex structures on $L(p,q) \times S^1$, Hiroshima Math. J. **25** (1995), 423–431.

[10] M. Namba: Automorphism groups of Hopf surfaces, Tôhoku Math. J. **26** (1974), 133–157.

[11] P. Orlik: Seifert Manifolds, Lecture Notes in Math. **291** (1972), Springer-Verlag.

[12] J. Wehler: Versal deformations for Hopf surfaces, J. Reine Angew. Math. **328** (1982), 22–32.

Noriaki Nakagawa
Tohnan High School
Minami-ku
Kyoto 601-8348, *Japan*
E-mail: wp-n-nkgw@s4.dion.ne.jp