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Some Containment Relations between Classes of Ideals in an Integral Domain

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I. Introduction: Throughout this paper D will denote an integral domain with $1\neq 0$ and quotient field K. An ideal A of D is called a valuation ideal provided there exists a valuation ring D_v such that $D\subset D_v\subset K$ and $AD_v\cap D=A$ ([10; 340]). Denote by $\mathcal{Q}$ the set of primary ideals of D, by $\mathcal{V}$ the set of valuation ideals of D, by $\mathcal{S}$ the set of semi-primary ideals of D, (i.e. ideals with prime radical) and by $\mathcal{PP}$ the set of prime powers of D. The significance of the various containment relations between these classes of ideals has been investigated in [1], [2], [5], [6] and [8]. The notion of valuation ideal can be generalized by replacing the valuation ring D_v in the above definition by other types of domains, e.g., Prüfer domains, almost Dedekind domains, Dedekind domains, rank one valuation rings, and rank one, discrete valuation rings (which are the generalizations considered in this paper). The purpose of this paper is to investigate certain of the containment relations between the classes of ideals obtained in this manner and the classes listed above. We will usually follow [9] and [10] in matters of notation and definitions. Containment will be denoted by $\subset$ and proper containment by $<$. An ideal A of D is proper provided $(0)<A<D$.

II. Preliminaries. A domain D is called a Prüfer domain provided D_p (the quotient ring of D with respect to the prime ideal P, [9; 228]) is a valuation ring for each proper prime ideal P in D, (see [1], [11; 554], [12; 127], and [13]) and is called an almost Dedekind domain provided D_p is a rank one, discrete valuation ring (i.e., a valuation ring which is a Dedekind domain) for each proper prime ideal P in D (see [3] and [7]). An ideal A of D is a Prüfer ideal if there exists a Prüfer domain J such that $D\subset J\subset K$ and $AJ\cap D=A$. Almost Dedekind ideals, Dedekind ideals, etc... are defined in an analogous manner. Denote by $\mathcal{P}$, $\mathcal{D}$, and $\mathcal{A}$, the set of Prüfer ideals, Dedekind ideals, and almost Dedekind ideals, respectively of the domain D.

In Section III necessary and sufficient conditions are given in order that $\mathcal{P}\subset\mathcal{Q}$, $\mathcal{P}=\mathcal{Q}$, $\mathcal{P}\subset\mathcal{S}$, $\mathcal{P}=\mathcal{S}$, $\mathcal{A}\subset\mathcal{S}$, $\mathcal{A}=\mathcal{S}$, $\mathcal{P}\subset\mathcal{PP}$. Furthermore, it is shown that D is Dedekind if and only if each proper ideal of D is a Dedekind ideal. In Section IV it is shown that D is almost Dedekind if and only if $\mathcal{Q}\subset\mathcal{D}$ and proper prime ideals of D are maximal. Also, the prime ideal structure of D is studied in case that $\mathcal{S}\subset\mathcal{V}$.

III. Some relationships between $\mathcal{Q}$, $\mathcal{P}$, $\mathcal{S}$, $\mathcal{A}$ and $\mathcal{PP}$.

Theorem 3.1: *$\mathcal{P} \subset \mathcal{Q}$ in a domain D if and only if there exists only one proper prime ideal in D.*

Proof: If there exists only one proper prime ideal in D, then every ideal in D is primary, hence Prüfer ideals are primary. Conversely, if Prüfer ideals are primary, then valuation ideals are primary and hence the proper prime ideals of D are maximal ([1; thm. 3.1]). Suppose P_1 and P_2 are distinct maximal ideals of D. There exist valuation rings V_1 and V_2 containing D and contained in K, with maximal ideals M_1 and M_2 respectively, such that $M_1 \cap D = P_1$ and $M_2 \cap D = P_2$. If $V_1 \subset V_2$, then $M_2 \cap V_1 \subset M_1$ since M_1 is maximal in V_1 and hence $P_2 \subset P_1$. Since $P_2 \not\subset P_1$ we see that $V_1 \not\subset V_2$ and in a similar manner $V_2 \not\subset V_1$. We set $V_1 \cap V_2 = J$, then $M_1 \cap J = Q_1$ and $M_2 \cap J = Q_2$ are the only maximal ideals in J and $J_{Q_1} = V_1$, $J_{Q_2} = V_2$ [14; 38], hence J is a Prüfer domain. Furthermore, Q_1 and Q_2 are distinct since neither V_1 nor V_2 is contained in the other. We have $P_1 \cap P_2 = (Q_1 \cap D) \cap (Q_2 \cap D) = (Q_1 \cap Q_2) \cap D$ is a Prüfer ideal, hence $\sqrt{P_1 \cap P_2} = P_1 \cap P_2$ is maximal. Therefore $P_1 \subset P_2$ or $P_2 \subset P_1$. This contradiction establishes the converse.

Theorem 3.2: *$\mathcal{P} = \mathcal{Q}$ if and only if D is a rank one valuation ring.*

Proof: Let D be a rank one valuation ring, then D has only one proper prime ideal, hence $\mathcal{P} \subset \mathcal{Q}$ by Theorem 1. Since every ideal of D is a Prüfer ideal, $\mathcal{Q} \subset \mathcal{P}$, whence $\mathcal{P} = \mathcal{Q}$. Convervely, suppose $\mathcal{P} = \mathcal{Q}$. By Theorem 3.1, there exists only one proper prime ideal M in D, therefore every ideal of D is primary, hence Prüfer. If A is an arbitrary proper ideal of D, let J be a Prüfer domain such that $A \cdot J \cap D = A$. Let C denote the set of maximal ideals in J, then $J = \bigcap_{M \in C} J_M$ and $A \cdot J = \bigcap_{M \in C} A \cdot J_M$ [10; 94], where each $A \cdot J_M$ is valuation ideal since J_M is a valuation ring for each proper prime ideal M of J. Therefore $A = A \cdot J \cap D = (\bigcap_{M \in C} A \cdot J_M) \cap D = \bigcap_{M \in C} (A \cdot J_M \cap D)$ where each $A \cdot J_M \cap D$ is a valuation ideal in D. Therefore every proper ideal of D can be expressed as the intersection of valuation ideals, hence D is a Prüfer domain [1; thm. 2.2]. Since $D = D_M$, D is a rank one valuation ring.

Theorem 3.3: *In a domain D, $\mathcal{P} \subset \mathcal{S}$ if and only if the prime ideals of D are chained.*

Proof: Suppose $\mathcal{P} \subset \mathcal{S}$ and let P_1 and P_2 denote arbitrary prime ideals of D. Let V_1 and V_2 be valuation rings, containing D and contained in K, with maximal ideals M_1 and M_2 respectively, such that $M_1 \cap D = P_1$ and $M_2 \cap D = P_2$. If $V_1 \subset V_2$, then $M_2 \cap V_1 \subset M_1$, hence $P_2 \subset P_1$. Similarly, $V_2 \subset V_1$

implies $P_1 \subset P_2$. If $V_1 \not\subset V_2$ and $V_2 \not\subset V_1$ then $J = V_1 \cap V_2$ is a Prüfer domain with maximal ideals $Q_1 = M_1 \cap J$ and $Q_2 = M_2 \cap J$ [14; 38). Now $(Q_1 \cap Q_2) \cap D = P_1 \cap P_2$ is a Prüfer ideal, thus $\sqrt{P_1 \cap P_2} = P_1 \cap P_2$ is prime. If $P_1 \not\subset P_2$ and $P_2 \not\subset P_1$, there is an element $x \in P_1$, $x \notin P_2$ and an element $y \in P_2$, $y \notin P_1$ such that $xy \in P_1 \cap P_2$. Since neither x nor y is an element of $P_1 \cap P_2$, we have contradicted the fact that $P_1 \cap P_2$ is Prime. Therefore $P_1 \subset P_2$ or $P_2 \subset P_1$ and hence prime ideals in D are chained. Conversely, if the prime ideals of D are chained, every ideal of D has prime radical, hence $\mathcal{P} \subset \mathcal{S}$.

Theorem 3.4: *In a domain D, $\mathcal{P} = \mathcal{S}$ if and only if D is a valuation ring.*

Proof: If $\mathcal{P} = \mathcal{S}$, the prime ideals of D are chained, by Theorem 3.3 Hence every ideal of D has prime radical and is therefore a Prüfer ideal, hence D is a Prüfer domain as shown in the proof of Theorem 3.2. Since the prime ideals of D are chained, D has only one maximal ideal M, hence $D = D_M$ is a valuation ring. The converse is obvious.

Theorem 3.5. *If the prime ideals of D are almost Dedekind ideals, then $\mathcal{A} \subset \mathcal{S}$ if and only if the prime ideals of D are chained.*

Proof. If the prime ideals of D are chained, then $\mathcal{P} \subset \mathcal{S}$ by Theorem 3.3, hence $\mathcal{A} \subset \mathcal{S}$. Conversely, if $\mathcal{A} \subset \mathcal{S}$, let $P \neq Q$ be arbitrary proper prime ideals of D. Let J and J' be almost Dedekind domains contained in K with the property that $P \cdot J \cap D = P$ and $Q \cdot J' \cap D = Q$. Let $M = D \setminus P$, then clearly $P \cdot D_M \subset P \cdot J_M \cap D_M$. Let $x \in P \cdot J_M \cap D_M$. Then $x = a\left(\dfrac{d}{m}\right) = \dfrac{r}{n}$ where $d \in J$, $a \in P$, $r \in D$, and $m, n \in M$. Therefore $n \cdot a \cdot d = r \cdot m$ and $n \cdot a \cdot d \in P \cdot J$, $r \cdot m \in D$, so $r \cdot m \in P \cdot J \cap D = P$. Since $m, n \in M$, then $\dfrac{1}{m \cdot n} \in D_M$, hence $r \cdot m\left(\dfrac{1}{m \cdot n}\right) = \dfrac{r}{n} = x \in P \cdot D_M$. Therefore $P \cdot D_M = P \cdot J_M \cap D_M$ and in a similar manner $Q \cdot D_N = Q \cdot J'_N \cap D_N$ where $N = D \setminus Q$. Now $P \cdot D_M$ is the unique maximal ideal in D_M so there exists a maximal ideal R of J_M, such that $P \cdot J_M \subset R$ and $R \cap D_M = P \cdot D_M$. Therefore $R \cap D = (R \cap D_M) \cap D = P \cdot D_M \cap D = P$. Similarly, there exists a maximal ideal S in J'_N such that $Q \cdot J'_N \subset S$ and $S \cap D = Q$. Since the domains J_M and J'_N contain J and J' respectively, they are almost Dedekind domains [7; thm. 1.3]. Therefore $(J_{M_R}$ and $(J'_N)_S$ are discrete rank one valuation rings and unequal since $P \neq Q$. If either $(J_M)_R \subset (J'_N)_S$ or $(J'_N)_S \subset (J_M)_R$, the theorem would be proved since then $Q \subset P$ or $P \subset Q$. We assume that neither containment holds, hence $T = (J_M)_R \cap (J'_N)_S$ is a Prüfer domain with exactly two maximal ideals [14; 38]. Furthermore, T has no other proper prime ideals since both $(J_M)_R$ and $(J'_N)_S$ are discrete, rank one valuation rings, hence T is an almost Dedekind domain. Therefore $(R \cap T) \cap D = P$, $(S \cap T) \cap D = Q$, and $\{(R \cap T) \cap (S \cap T)\} \cap D = P \cap Q$ are almost Dedekind ideals. Since $\mathcal{A} \subset \mathcal{S}$, then $P \cap Q \in \mathcal{A}$ implies $P \subset Q$ or $Q \subset P$. This completes the proof.

We remark that prime ideals are not necessarily almost Dedekind ideals- eg. the maximal ideal of a rank one, non-discrete valuation ring.

THEOREM 3.6: *If $\mathcal{A}=\mathcal{D}$ in a domain D, then D is a valuation ring.*

PROOF: The prime ideals of D are almost Dedekind ideals since $\mathcal{D}\subset\mathcal{A}$. Then $\mathcal{A}\subset\mathcal{D}$ implies prime ideals are chained, by Theorem 3.5. Therefore, every ideal in D has prime radical, in particular $\mathcal{P}\subset\mathcal{D}$. Now $\mathcal{A}=\mathcal{D}$ implies $\mathcal{D}\subset\mathcal{P}$, hence $\mathcal{P}=\mathcal{D}$. By Theorem 3.4, D is then a valuation ring.

THEOREM 3.7. *In a domain D, $\mathcal{P}\subset\mathcal{P}\mathcal{P}$ if and only if D is contained in only one valuation ring, it being P-adic for some prime ideal P of D.*

PROOF: If $\mathcal{P}\subset\mathcal{P}\mathcal{P}$, then $\mathcal{V}\subset\mathcal{P}\mathcal{P}$, hence $\mathcal{V}=\mathcal{P}\mathcal{P}$ and D is one-dimensional [2; cor. 1]. Furthermore, $\mathcal{P}\subset\mathcal{P}\mathcal{P}$ implies $\mathcal{P}\subset\mathcal{D}$, thus the prime ideals of D are chained, by Theorem 3.3. As a consequence, D has only one proper prime ideal, and since every valuation of K, finite on D, is P-adic for some prime ideal P of D [2; thm. 1] then there exists only one valuation ring V between D and K. Conversely, if D is contained in only one valuation ring V, it being P-adic for some prime ideal P of D, then D has a unique proper prime ideal. Therefore $\mathcal{V}\subset\mathcal{P}\mathcal{P}$ [2; thm. 1]. Now let A be any Prüfer ideal in D and let J be a Prüfer domain with the property that $A\cdot J\cap D=A$. Since J lies between D and K, there is only one valuation ring containing J, hence J_p is the same valuation ring for every prime ideal P in J. Therefore J is a valuation ring, hence A is also a valuation ideal, thus $\mathcal{P}=\mathcal{V}$. This equality with $\mathcal{V}\subset\mathcal{P}\mathcal{P}$ gives $\mathcal{P}\subset\mathcal{P}\mathcal{P}$ to complete the proof of the theorem.

THEOREM 3.8: *A necessary and sufficient condition that D be a discrete, rank one valuation ring is that D be integrally closed and $\mathcal{P}\subset\mathcal{P}\mathcal{P}$.*

PROOF: By Theorem 3.7 D is contained in only one valuation ring V, it being P-adic for some proper prime ideal P of D; hence V is discrete and rank one. The intersection of all valuation rings of K containing D is the integral closure of D in K, hence $D=V$ since D is integrally closed in K. Conversely, if D is a discrete, rank one valuation ring, it is clear that D is integrally closed and $\mathcal{P}\subset\mathcal{P}\mathcal{P}$.

THEOREM 3.9. *If every proper ideal of D is a Dedekind ideal (not necessarily for the same Dedekind domain), then D is a Dedekind domain (and conversely).*

PROOF: If A and B are proper ideals of D, let J_1 and J_2 be Dedekind domains such that $A\cdot J_1\cap D=A$ and $B\cdot J_2\cap D=B$. In J_1, we have $AJ_1=P_1^{e_1}\cdot\ldots\cdot P_n^{e_n}$

where P_i is a prime ideal of J_1 and e_i is a positive integer, for each i. Let $J_1^* = \bigcap_{i=1}^{n} (J_1)_{P_i}$. We wish to show that $A \cdot J_1 = (A \cdot J_1) \cdot J_1^* \cap J_1$. Clearly $A \cdot J_1 \subset (A \cdot J_1) \cdot J_1^* \cap J_1$. Now $A \cdot J_1 = P_1^{e_1} \cdots P_n^{e_n} = P_1^{e_1} \cap \cdots \cap P_n^{e_n} = \{(P_1^{e_1} \cdot \ldots \cdot P_n^{e_n})(J_1)_{P_1} \cap J_1\} \cap \cdots \cap \{(P_1^{e_1} \cdot \ldots \cdot P_n^{e_n}) \cdot (J_1)_{P_n} \cap J_1\} = \{(A \cdot J_1)(J_1)_{P_1} \cap J\} \cap \cdots \cap \{(A \cdot J_1) \cdot (J_1)_{P_n} \cap J_1\} = \bigcap_{i=1}^{n} \{(A \cdot J_1) \cdot (J_1)_{P_i} \cap J_1\} = \{\bigcap_{i=1}^{n} (A \cdot J_1) \cdot (J_1)_{P_i}\} \cap J_1 \supset (A \cdot J_1) \cdot \{\bigcap_{i=1}^{n} (J_1)_{P_i}\} \cap J_1 = (A \cdot J_1) \cdot J_1^* \cap J_1$. These two containments give $A \cdot J_1 = (A \cdot J_1) \cdot J_1^* \cap J_1$. In a similar manner we get a Dedekind domain J_2^* such that $B \cdot J_2 = (B \cdot J_2) \cdot J_2^* \cap J_2$. We will show that $J = J_1^* \cap J_2^*$ is a Dedekind domain by showing that the intersection of any finite number of discrete, rank one valuation rings containing D and contained in K is a Dedekind domain. The proof of this fact is by induction, we will give the proof for the intersection of two such valuation rings. Let V_1 and V_2 be distinct discrete, rank one valuation rings containing D and contained in K. As shown in the proof of Theorem 3.1, $V_1 \cap V_2 = R$ is a Prüfer domain with exactly two maximal ideals, namely $M_1 \cap R$ and $M_2 \cap R$ where M_1 and M_2 are the maximal ideals of V_1 and V_2 respectively, and $R_{(M_1 \cap R)} = V_1$ and $R_{(M_2 \cap R)} = V_2$, hence R is an almost Dedekind domain. There exists $x \in M_1 \cap R$, $x \notin M_2 \cap R$, hence $\sqrt{(x)} = M_1 \cap R$. Therefore, (x) is a power of $M_1 \cap R$ since R is an almost Dedekind domain [7; thm. 1.1]. Then $(x) = (M_1 \cap R)^n$ for some positive integer n, hence $(M_1 \cap R)$ is invertible since (x) is invertible [9; 272]. In an analogous manner, we can show $(M_2 \cap R)$ is invertible, hence R is a Dedekind domain since every proper prime ideal of R is invertible [15; 33]. Since both J_1^* and J_2^* are finite intersections of discrete, rank one valuation rings, then $J = J_1^* \cap J_2^*$ is a Dedekind domain. Furthermore $A = A \cdot J \cap D$ and $B = B \cdot J \cap D$. We have shown that any pair of ideals in D are Dedekind ideals for the same Dedekind domain. Now if $C \neq 0$ is an ideal of D such that $AC = BC$, then $(AC) \cdot J = (BC) \cdot J$, hence $(A \cdot J) \cdot (C \cdot J) = (B \cdot J)(C \cdot J)$, thus $(A \cdot J) = (B \cdot J)$ since the cancellation law for ideals is valid in a Dedekind domain. But then $A = A \cdot J \cap D = B \cdot J \cap D = B$, thus the cancellation law for ideals is valid in D, hence D is an almost Dedekind domain [4]. Let A be an arbitrary ideal of D, then if J_1 is a Dedekind domain such that $A \cdot J_1 \cap D = A$, we have $A \cdot J_1 = P_1^{e_1} \cap \cdots \cap P_n^{e_n}$ as stated earlier. Then $A = (P_1^{e_1} \cap D) \cap \cdots \cap (P_n^{e_n} \cap D)$ where each $(P_i^{e_i} \cap D)$ is a primary Dedekind ideal. We assume that this representation is reduced so that $(P_k^{e_k} \cap D) \not\subset (P_j^{e_j} \cap D)$ for any $k \neq j$. Now since D is almost Dedekind, each $(P_i^{e_i} \cap D)$ is a prime (maximal) power [7; thm. 1.1]. For each i, let $P_i^{e_i} \cap D = M_i^{f_i}$ where M_i is a maximal ideal of D and f_i is a positive integer, then $A = M_1^{f_1} \cap \cdots \cap M_n^{f_n} = M_1^{f_1} \cdot \ldots \cdot M_n^{f_n}$ since the $M_i^{f_i}$ are pairwise comaximal [9; 177]. Therefore A is the product of prime ideals, hence D is a Dedekind domain.

IV. Some consequences of $\mathcal{D} \supset \mathcal{Q}$, $\mathcal{PP} \subset \mathcal{M}$, and $\mathcal{S} \subset \mathcal{M}$.

THEOREM 4.1. *A domain D is an almost Dedekind domain if and only if*

$Q \subset \mathfrak{D}$ and proper prime ideals of D are maximal.

PROOF: Suppose $Q \subset \mathfrak{D}$ and proper prime ideals of D are maximal. Let P be an arbitrary proper prime ideal of D and form the quotient ring D_p. Then PD_p is the only proper prime of D_p since proper prime ideals of D are maximal. Furthermore, if Q is any P-primary ideal in D, then $Q \cdot D_p$ is primary in D_p. If R is a Dedekind domain such that $D \subset R \subset K$ and $Q \cdot R \cap D = Q$, we can prove, as in Theorem 3.5, that $Q \cdot R_p \cap D_p = Q \cdot D_p$. Therefore $Q \cdot D_p$ is also a Dedekind ideal, and hence every ideal in D_p is a Dedekind ideal, thus D_p is a Dedekind domain by Theorem 3.9. Now D_p has only one proper prime ideal, hence is a discrete rank one valuation ring. Therefore D is an almost Dedekind domain since P is arbitrary. Conversely, if D is an almost Dedekind domain, then proper prime ideals of D are maximal. If P is an arbitrary proper prime ideal of D, then D_p is a Dedekind domain, and if Q is any P-primary ideal in D, we have $Q = Q \cdot D_p \cap D$, hence $Q \subset \mathfrak{D}$.

LEMMA 4.2. *If $P_1 > P_2$ are prime ideals of D, then there exist prime ideals P and P^* such that $P_1 \supset P > P^* \supset P_2$ and there are no prime ideals properly between P and P^*.*

PROOF: The proof of this lemma is an easy application of Zorn's lemma; see [8; lem. 1].

LEMMA 4.3. *If R is a valuation ring and P is a proper prime ideal of R such that P is the only P-primary ideal of R, then $P = P^2$. Furthermore, if $\{P_\alpha\}$ denotes the set of prime ideals properly contained in P, then $P = \bigcup_\alpha P_\alpha$.*

PROOF: Suppose $P \neq P^2$, then $\bigcap_{n=1} P^n = P^*$ is a prime ideal since each P^n is a valuation ideal for each n [1; lem. 2.10]. Thus $P > P^*$ and if P_1 is any prime ideal with the property that $P \supset P_1 \supset P^*$, then either $P^n \subset P_1$ or $P^n \supset P_1$, for each positive integer n. We consider the following two cases; either 1) $P_1 \supset P^n$ for some n, or 2) $P^n \supset P_1$ for all n. In case 1), $P = \sqrt{P^n} \subset P_1$ implies $P = P_1$, and in case 2), $P_1 = P^*$ since then $P_1 \subset \bigcap_{n=1}^{\infty} P^n = P^*$. Therefore, if $P \supset P_1 > P^*$, then $P = P_1$ and there are no prime ideals properly between P and P^*. Then P^* is the intersection of all P-primary ideals in R [1; thm. 3.3], but this contradicts the hypothesis that P is the only P-primary ideal of R. Hence $P = P^2$. For the second part of the lemma, we consider two cases; either 1) there exists a prime ideal P_1 such that $P_1 < P$ and there are no prime ideals properly between P_1 and P, or 2) there exists no prime ideal statisfying case 1). If case 1) holds, then P_1 is the intersection of the P-primary ideals [1; thm. 3.3], hence $P_1 = P$ and therefore case 1) cannot hold. Since case 2) must hold, we let $\{P_\alpha\}$ denote the set of all prime ideals of R which are properly contained in P. All the ideals of R are chained, hence $\bigcup_{\alpha} P_\alpha$ is a prime ideal and contains each P_α, thus $P=\bigcup_{\alpha} P_\alpha$.

Theorem 4.4. *If $\mathcal{P}\mathcal{P}\subset\mathcal{V}$ and if P is any prime ideal of D, then $\bigcap_{n=1}^{\infty} P^n = P^*$ is a prime ideal. Furthermore, if $\{Q_\alpha\}$ denotes the set of P-primary ideals of D, then $P^*\supset\bigcap_{\alpha} Q_\alpha$.*

Proof: The first part of the theorem is a special case of [1; lem. 2.10]. If P is an idempotent prime ideal, then $\bigcap_{n=1}^{\infty} P^n = P^* = P \supset \bigcap_{\alpha} Q_\alpha$. If P is not idempotent, let n denote an arbitrary integer larger than one. Now P^n is a valuation ideal by hypothesis, so there exists a valuation ring $R_v \supset D$ such that $P^n\cdot R_v \cap D = P^n$. Furthermore, $\sqrt{P^n\cdot R_v}=P_v$ is a prime ideal of R_v and $P_v\cap D = P$. Thus every P_v-primary ideal of R_v contracts to a P-primary ideal of D. In R_v, we have either 1) $P^n\cdot R_v$ contains a P_v-primary ideal, or 2) $P^n\cdot R_v$ contains no P_v-primary ideal. If case 2) holds, then $P^n\cdot R_v \subset \bigcap_{\beta}\bar{Q}_\beta$, where $\{\bar{Q}_\beta\}$ denotes the set of all P_v-primary ideals of R_v. But $\bigcap_{\beta}\bar{Q}_\beta$ is a prime ideal [1; lem. 2.12], thus $P_v=\sqrt{P^n\cdot R_v}\subset\bigcap_{\beta}\bar{Q}_\beta$ implies $P_v=\bigcap_{\beta}\bar{Q}_\beta$. Then P_v is the only P_v-primary ideal in R_v and thus $P_v = P_v^2$, by the previous lemma. Since $\sqrt{P^n\cdot R_v}=P_v$, we have $P^n\cdot R_v$ contains every prime ideal which is properly contained in P_v. If $\{\bar{P}_\alpha\}$ denotes the set of prime ideals of R_v which are properly contained in P_v, then $P^n\cdot R_v\supset\bigcup_{\alpha}\bar{P}_\alpha$. But $P_v=\bigcup_{\alpha}\bar{P}_\alpha$ by the previous lemma, hence $P^n\cdot R_v = P_v$. Now $P^n = P^n\cdot R_v\cap D = P_v\cap D = P$ shows P is idempotent, thus case 2) cannot hold. Therefore case 1) holds and thus $P^n\cdot R_v\cap D = P^n$ contains a P-primary ideal. The integer n is arbitrary, thus P^n contains a P-primary ideal for every positive integer n, and hence $P^*\supset\bigcap_{\alpha} Q_\alpha$ where $\{Q_\alpha\}$ denotes the set of all P-primary ideals of D.

Theorem 4.5. *If $\mathcal{S}\subset\mathcal{V}$ and $P>P^*$ are prime ideals of D such that there are no prime ideals properly between them, then either $\bigcap_{n=1}^{\infty} P^n = P$ or $\bigcap_{n=1}^{\infty} P^n = P^*$.*

Proof: If P is an idempotent prime ideal, then $\bigcap_{n=1}^{\infty} P^n = P$. If P is not idempotent, then $\bigcap_{n=1}^{\infty} P^n$ is a prime ideal by the previous theorem. Furthermore, $P^* = \bigcap_{\alpha} Q_\alpha$ where $\{Q_\alpha\}$ denotes the set of all P-primary ideals in D [1; thm. 3.3]. But the previous theorem also states $\bigcap_{n=1}^{\infty} P^n \supset \bigcap_{\alpha} Q_\alpha$, so we have $P > \bigcap_{n=1} P^n \supset \bigcap_{\alpha} Q_\alpha = P^*$. Therefore $P^* = \bigcap_{n=1}^{\infty} P^n$.

COROLLARY 4.6. *If* $\mathcal{S} \subset \mathcal{Q}$ *and* $P_1 > P_2$ *are prime ideals of* D, *then* $P_1^n > P_2$ *for all positive integers* n.

PROOF: By Lemma 4.2, there exist prime ideals P and P^* such that $P_1 \supset P > P^* \supset P_2$, and such that there exist no prime ideals properly between P and P^*. By the previous theorem, we have either $\bigcap_{n=1}^{\infty} P^n = P$ or $\bigcap_{n=1}^{\infty} P^n = P^*$, thus $P_1^n \supset P^n > P^* \supset P_2$ for all n.

THEOREM 4.7. *If* $\mathcal{S} \subset \mathcal{Q}$ *and* P *is a prime ideal of* D *with the property that if* P' *is any prime ideal such that* $P > P'$ *then there is a prime ideal properly contained between* P *and* P', *then* $P = P^2$.

PROOF: By the previous corollary, $P^n > P'$ for all n, hence $\bigcap_{n=1}^{\infty} P^n \supset P'$. But $\bigcap_{n=1}^{\infty} P^n$ is a prime ideal, by Theorem 4.4, and contains every prime ideal P' which is properly contained in P. We have, therefore, $P \supset \bigcap_{n=1}^{\infty} P^n$ and there are no prime ideals properly contained between these two prime ideals, hence $P = \bigcap_{n=1}^{\infty} P^n$ and thus $P = P^2$.

PROPOSITION. 4.8. *Let* $\mathcal{S} \subset \mathcal{Q}$ *and suppose there are no proper idempotent prime ideals in* D. *Let* P *be a proper prime ideal such that there exists a prime ideal* P^* *with the property that* $P > P^*$ *and there are no prime ideals properly contained between* P *and* P^*. *Then* $\bigcap_{n=1}^{\infty} P^n = P^*$ *and if* $\bar{P}$ *is auy prime ideal such that* $P > \bar{P}$, *then* $P^* \supset \bar{P}$.

PROOF: The equality $\bigcap_{n=1}^{\infty} P^n = P^*$ follows from Theorem 4.5. By Corollary 4.6, we have $P^n > \bar{P}$ for all n, hence $P^* = \bigcap_{n=1}^{\infty} P^n \supset \bar{P}$.

THEOREM 4.9. *If* $\mathcal{S} \subset \mathcal{Q}$ *and there are no proper idempotent prime ideals in* D, *then the ascending chain condition for prime ideals is valid in* D.

PROOF: Let $P_1 < P_2 < P_3 < \cdots$ be an ascending chain of prime ideals in D; then $\bigcup_i P_i = P$ is also prime. If this chain is not finite, then $P > P_j$ for each j, hence $P^2 > P_j$ for each j, by Corollary 4.6. Therefore $P^2 \supset \bigcup_i P_i$ and thus $P = P^2$. This contradiction establishes the ascending chain condition for prime ideals in D.

THEOREM 4.10. *If* $\mathcal{S} \subset \mathcal{Q}$ *and if there are no idempotent proper prime ideals in* D, *then* D *is a Prüfer domain.*

PROOF: By the above theorem, the ascending chain condition for prime ideals in D is valid, hence the theorem follows from [1; thm. 3.8].

THEOREM 4.11. *If semi-primary ideals (i.e., ideals of $\mathcal{S}$) in D are rank one valuation ideals and if the ascending chain condition for prime ideals is valid, then D is a one-dimensional Prüfer domain (and conversely).*

PROOF: It follows that D is a Prüfer domain from [1; thm. 3.8]. Let A denote an ideal of D with prime radical P. Then there exists a rank one valuation ring R_v containing D such that $A\cdot R_v \cap D = A$ and $\sqrt{A\cdot R_v} \cap D = P$. Let P_v denote the unique proper prime ideal of R_v, then $\sqrt{A\cdot R_v} = P_v$, hence $A\cdot R_v$ is P_v-primary in R_v, and A is P-primary. Therefore every semi-primary ideal of D is primary, so $\mathcal{S} = Q$, and hence proper prime ideals of D are maximal [5; Cor. 3.2].

THEOREM 4.12. *If semi-primary ideals in D are rank one, discrete valuation ideals, then D is an almost Dedekind domain (and conversely).*

PROOF: Since prime ideals are rank one, discrete valuation ideals, there are no proper idempotent prime ideals in D and therefore the ascending chain condition for prime ideals is valid in D, by Theorem 4.9. By the previous theorem, D is a one-dimensional Prüfer domain, hence proper prime ideals are maximal. Since rank one, discrete valuation ideals are Dedekind ideals, the theorem follows by applying Theorem 4.1.

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