J. Sci. Hiroshima Univ. Ser. A-I
30 (1966), 151-181

On the Theory of the Multiplicative Products of Distributions

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(Received September 20, 1966)

Many attempts have been made to define a multiplication for distributions. H. König [9, 10] was the first to develop a systematic treatment of the subject in an abstract way, and showed that there are actually many possible multiplication theories if one gives up some of the requirements that L. Schwartz [14] has shown impossible to be satisfied. His theory is, however, rather complicated and mainly concerns with the one dimensional case. Some writers [1], [7] also worked out the theories designed for certain physical applications, where multiplication need not be commutative and the product may contain arbitrary constants.

Y. Hirata and H. Ogata [4] introduced the definition of the multiplicative product of two distributions in order to generalize the exchange formula concerning Fourier transformation. An equivalent definition of the product was given by J. Mikusiński [13]. Among the results of [18] and [6], it has been shown that given S, $T \in \mathcal{D}'(\Omega)$, where Ω is a non-empty open subset of an N-dimensional Euclidean space R^N, the product ST exists if and only if to every $\alpha \in \mathcal{D}(\Omega)$ there is a 0-neighbourhood of R^N so that $\alpha S*\check{T}$ is equivalent to a bounded measurable function continuous at 0. Here ST is defined to be a unique distribution $W \in \mathcal{D}'(\Omega)$ such that $<W, \alpha>=(\alpha S*\check{T})(0)$. The same result has been announced by J. Jelínek [8] incidentally.

The main purpose of this paper is to generalize the multiplication just considered above so as to maintain as many reasonable properties as possible. With this in mind, we reach the requirements I through IV (see Section 1 below) for multiplication between distributions. Especially the requirement IV states that multiplicative product of distributions is invariant under diffeomorphisms. The results of [6] constitute a basis and background for the present paper. With the same notations as above, if $\alpha S*\check{T}$ has the value $(\alpha S*\check{T})(0)$ at 0 in the sense of S. Łojasiewicz [12], the distribution $W \in \mathcal{D}'(\Omega)$ defined as before will be called the multiplicative product of S and T and denoted by $S\bigcirc T$. The multiplication thus defined will satisfy the requirements indicated above. In the case $N=1$, we can make further generalizations of the notion of the multiplication. Another purpose of this paper is to investigate these multiplications.

The presentation of the material is arranged as follows: In Section 1 we write down our requirements I-IV for multiplication. Any multiplication satisfying these requirements is called normal. Section 2 contains two lemmas concerning the value of a distribution at a point for our later purpose. In Section 3 some equivalent definitions of the product $S \bigcirc T$ are given. We show that the multiplication thus defined is normal. It is noticed that we can introduce a multiplication between distributions for any differentiable maniford, which is invariant under diffeomorphisms. Section 4 is devoted to the investigations of further properties of the multiplication. It contains also the results concerning simultaneous product of more than two distributions and partial multiplication. In the rest of this paper we shall consider only the case $N=1$. Section 5 deals with a generalization of $S \bigcirc T$ which will be denoted by $S \times_0 T$. We show that if $S \times_0 T$ makes sense, then so does $S \bigcirc T$ (the product in the strict sense of H. G. Tillmann) and they coincide. In the final section a further generalization is made so as to make the product of δ by δ significant. The section is closed with the comparison between the product and that in the general sense of Tillmann.

§ 1. Requirements for multiplicative product

Let R^N be an N-dimensional Euclidean space. If $x=(x_1, \cdots, x_N)$, $y=(y_1, \cdots, y_N) \in R^N$ and λ is a real number, we write $x+y=(x_1+y_1, \cdots, x_N+y_N)$, $\lambda x=(\lambda x_1, \cdots, \lambda x_N)$ and $|x|=(\sum_1^N |x_j|^2)^{\frac{1}{2}}$. If p is an N-tuple $(p_1, \cdots, p_N)$ of non-negative integers, the sum $\sum_1^N p_j$ will be denoted by $|p|$, the product $\prod_1^N p_j!$ by $p!$ and the product $\prod_1^N \binom{p_j}{q_j}$ by $\binom{p}{q}$, where $q=(q_1, \cdots, q_N)$ is such that $q \leqq p$, that is, $q_j \leqq p_j$ for $j=1, 2, \cdots, N$. With $D=(D_1, \cdots, D_N)$, $D_j=\frac{\partial}{\partial x_j}$, we put $D^p=D_1^{p_1} \cdots D_N^{p_N}$. Similarly we write $x^p=x_1^{p_1} \cdots x_N^{p_N}$.

Denoting by Ω a non-empty open subset of R^N, we shall consider the following spaces:

$C(\Omega)$: the space of the complex valued continuous functions in Ω;

$\mathcal{D}(\Omega)$: the space of the complex valued C^∞-functions in Ω with compact support, equipped with the usual topology;

$\mathcal{D}'(\Omega)$: the space of Schwartz distributions in Ω, the strong dual of $\mathcal{D}(\Omega)$;

$\mathcal{E}(\Omega)$: the space of the complex valued C^∞-functions in Ω, with the usual topology;

$\mathcal{E}'(\Omega)$: the subspace of distributions $\in \mathcal{D}'(\Omega)$ with compact support, the strong dual of $\mathcal{E}(\Omega)$.

Without explaining explicitly, $<S, \phi>$ will denote the scalar product of $S \in \mathcal{D}'(\Omega)$ and $\phi \in \mathcal{D}(\Omega)$, or $S \in \mathcal{E}'(\Omega)$ and $\phi \in \mathcal{E}(\Omega)$. We often use the symbol $S(x)$ instead of S. This does not mean that S is a function of x. For example, $<S(x), \phi(x)>$ means $<S, \phi>$. The restriction of $S \in \mathcal{D}'(\Omega)$ to a non-empty open subset $\Omega_1 \subset \Omega$, will be denoted by S_{Ω_1}. Let $x=\Phi(x')$ be a diffeomorphism of Ω' onto Ω, that is, Φ carries Ω' homeomorphically onto Ω and both Φ and Φ^{-1} are smooth. Given $S \in \mathcal{D}'(\Omega)$, we denote by $\tilde{S}(x')=S(\Phi(x'))$ the distribution in Ω' such that

$$<\tilde{S}(x'), \phi(x')> = <S(x), \ |J(x)|\phi(\Phi^{-1}(x))>, \quad \phi \in \mathcal{D}(\Omega'),$$

where $J(x)$ is the Jacobian of the mapping $x'=\Phi^{-1}(x)$.

Now let $\alpha \in \mathcal{E}(\Omega)$ and $S \in \mathcal{D}'(\Omega)$. According to Schwartz [15] the multiplicative product αS is a distribution in Ω defined by

$$<\alpha S, \phi> = <S, \alpha\phi>, \quad \phi \in \mathcal{D}(\Omega).$$

This multiplication has the usual properties, in particular bilinearity, associativity with multiplication in $\mathcal{E}(\Omega)$, and Leibniz' formula for the derivative of the product and so on. However, multiplication of arbitrary two distributions cannot be defined so that it may possess these reasonable properties. If we can associate a subset $\mathfrak{M}_\Omega \subset \mathcal{D}'(\Omega)\times\mathcal{D}'(\Omega)$ with each non-empty open subset $\Omega \subset R^N$ in such a way that $S \bigcirc T \in \mathcal{D}'(\Omega)$, called the multiplicative product of S and T, is defined for any $(S, T) \in \mathfrak{M}_\Omega$ with the following conditions I through IV, then the multiplication will be called *normal*:

I_1. if $(f, g) \in C(\Omega)\times C(\Omega)$, then $(f, g) \in \mathfrak{M}_\Omega$ and $f \bigcirc g$ coincides with the ordinary product fg;

I_2. if $(S, T) \in \mathfrak{M}_\Omega$, then $(T, S) \in \mathfrak{M}_\Omega$ and $S\bigcirc T = T \bigcirc S$;

I_3. if $(S_1, T), (S_2, T) \in \mathfrak{M}_\Omega$, then $(S_1+S_2, T) \in \mathfrak{M}_\Omega$ and $(S_1+S_2)\bigcirc T = S_1 \bigcirc T + S_2 \bigcirc T$;

I_4. if $\alpha \in \mathcal{E}(\Omega)$, $(S, T) \in \mathfrak{M}_\Omega$, then $(\alpha S, T) \in \mathfrak{M}_\Omega$ and $(\alpha S)\bigcirc T = \alpha(S \bigcirc T)$;

II. if $\left(\frac{\partial S}{\partial x_j}, T\right) \in \mathfrak{M}_\Omega$ for $j=1, 2, \cdots, N$, then (S, T), $\left(S, \frac{\partial T}{\partial x_j}\right) \in \mathfrak{M}_\Omega$ for $j=1, 2, \cdots, N$ and $\frac{\partial}{\partial x_j}(S \bigcirc T) = \frac{\partial S}{\partial x_j} \bigcirc T + S \bigcirc \frac{\partial T}{\partial x_j}$;

III_1. if $(S, T) \in \mathfrak{M}_\Omega$, then $(S_{\Omega_1}, T_{\Omega_1}) \in \mathfrak{M}_{\Omega_1}$ and $S_{\Omega_1} \bigcirc T_{\Omega_1} = (S \bigcirc T)_{\Omega_1}$ for any $\Omega_1 \subset \Omega$;

III_2. if $(S_{\Omega_\iota}, T_{\Omega_\iota}) \in \mathfrak{M}_{\Omega_\iota}$ for each ι, where $S, T \in \mathcal{D}'(\Omega)$ and $\Omega = \cup_\iota \Omega_\iota$, then $(S, T) \in \mathfrak{M}_\Omega$;

IV. if $(S, T) \in \mathfrak{M}_\Omega$ and $W = S \bigcirc T$, then $(\tilde{S}, \tilde{T}) \in \mathfrak{M}_{\Omega'}$ and $\tilde{W} = \tilde{S} \bigcirc \tilde{T}$ for any diffeomorphism $x=\Phi(x')$ of Ω' onto Ω.

One of our main objectives is to generalize the multiplication in the sense of Hirata-Ogata [4] or Mikusiński [13] so that it may be made normal. Before proceeding further we shall stay here to make a few remarks on normal multiplication between distributions.

Remark 1. Suppose a normal multiplication $\bigcirc$ is defined. Let $S, T \in \mathcal{D}'(\Omega)$ and let p be a multi-index with $p \geqq 0$. If $(D^q S, T) \in \mathfrak{M}_\Omega$ for any q such that $0 \leqq |q| \leqq |p|$, we see by II that Leibniz' formula for the derivative of the product

$$S\bigcirc D^p T=\sum_q(-1)^{|q|}\binom{p}{q}D^{p-q}(D^q S\bigcirc T)$$

remains true. Especially, in the case $S=\alpha\in\mathcal{E}(\Omega)$ and $T=f\in C(\Omega)$, I_1 implies that $D^q\alpha\bigcirc f$ exists for any multi-index $q\geqq 0$ and $D^q\alpha\bigcirc f=(D^q\alpha)f$. Thus $\alpha\bigcirc D^p f=\alpha(D^p f)$ in Ω. Furthermore, if $\alpha\in\mathcal{E}(\Omega)$ and $T\in\mathcal{D}'(\Omega)$, then $(\alpha, T)\in\mathfrak{M}_\Omega$ and $\alpha\bigcirc T=\alpha T$. In fact, T is locally represented as a derivative of a continuous function ([15], I, p. 82). For each point $x_\iota\in\Omega$, there exist a multi-index p_ι, an open neighbourhood $\Omega_\iota(\subset\Omega)$ of x_ι and a continuous function f_ι in Ω_ι, for which $T_{\Omega_\iota}=D^{p_\iota}f_\iota$. Thus $(\alpha_{\Omega_\iota}, T_{\Omega_\iota})\in\mathfrak{M}_{\Omega_\iota}$ and $\alpha_{\Omega_\iota}\bigcirc T_{\Omega_\iota}=\alpha_{\Omega_\iota}T_{\Omega_\iota}$ and then $(\alpha\bigcirc T)_{\Omega_\iota}=(\alpha T)_{\Omega_\iota}$ by III_1 and III_2. Consequently we see that $(\alpha, T)\in\mathfrak{M}_\Omega$ and $\alpha\bigcirc T=\alpha T$, completing the proof.

REMARK 2. Let $N=1$. $\mathrm{Pf}\frac{1}{x}\bigcirc\delta$ cannot be defined because otherwise we would obtain

$$x\left(\mathrm{Pf}\frac{1}{x}\bigcirc\delta\right)=\left(x\mathrm{Pf}\frac{1}{x}\right)\bigcirc\delta=1\bigcirc\delta=\delta,$$

$$x\left(\mathrm{Pf}\frac{1}{x}\bigcirc\delta\right)=\mathrm{Pf}\frac{1}{x}\bigcirc(x\delta)=0.$$

REMARK 3. If $S\bigcirc T$ exists, then $\alpha(S\bigcirc T)=(\alpha S)\bigcirc T=S\bigcirc(\alpha T)$ by I_2 and I_4, and hence the support theorem

$$\mathrm{supp}\,(S\bigcirc T)\subset\mathrm{supp}\,S\cap\mathrm{supp}\,T$$

is valid.

For our later purpose we shall show in the case $N=1$ the following

LEMMA 1. *Let T, $\delta\in\mathcal{D}'(\Omega)$. Assume that $T\bigcirc\delta$ exists, then $T\bigcirc\delta=c\delta$ with a constant c. If $\delta^{(p_0)}\bigcirc\delta^{(q_0)}$ exists, then $\delta^{(p)}\bigcirc\delta^{(q)}$ exists and is equal to 0 for $0\leqq p+q\leqq p_0+q_0$ ($\delta^{(0)}=\delta$).*

PROOF. By Remark 3 we can write

$$T\bigcirc\delta=a_0\delta+a_1\delta'+\cdots+a_n\delta^{(n)}$$

with constants a_j, $j=0, 1, \cdots, n$. Since $x\delta=0$ and $x\delta^{(j)}=-j\delta^{(j-1)}$ for $j\geqq 1$, we have

$$0=T\bigcirc(x\delta)=x(T\bigcirc\delta)=-\sum_1^n ja_j\delta^{(j-1)}.$$

This means that $a_1=a_2=\cdots=a_n=0$. Thus we have $T\bigcirc\delta=a_0\delta$.

If $\delta^{(p_0)}\bigcirc\delta^{(q_0)}$ exists, then we see by Remark 1 that $\delta^{(p)}\bigcirc\delta^{(q)}$ exists for $0\leqq p+q\leqq p_0+q_0$ and Leibniz' formula

$$\delta^{(p)}\bigcirc\delta^{(q)}=\sum_r(-1)^r\binom{q}{r}(\delta^{(p+r)}\bigcirc\delta)^{(q-r)}$$

remains true. For our end, it suffices to show that $\delta^{(k)}\bigcirc\delta=0$ for each $k\geqq 0$. There exists a constant c such that $\delta^{(k)}\bigcirc\delta=c\delta$. Let us now consider the diffeomorphism $x=ax'$ of Ω' onto Ω with a positive number a, and we shall have

$$\widetilde{\delta^{(k)}}=\frac{1}{a^{k+1}}\delta^{(k)}.$$

Thus $\frac{1}{a^{k+1}}\delta^{(k)}\bigcirc\frac{1}{a}\delta=\frac{c}{a}\delta$, that is, $\delta^{(k)}\bigcirc\delta=a^{k+1}c\delta$. This implies $c=0$, which completes the proof.

§ 2. The value of a distribution at a point

We shall first recall some basic facts and definitions concerning the value of a distribution at a point introduced by Łojasiewicz [11, 12] which will play an important rôle in studying our multiplication theory.

Let T be a distribution defined in a neighbourhood of $x_0 \in R^N$ and λ a positive real number. If the distributional limit

$$\lim_{\lambda\to+0} T(x_0+\lambda x)$$

exists in a neighbourhood of 0 and is a constant function, then the value $T(x_0)$ of T at $x=x_0$ is defined as the value of this constant function.

It is known that T has the value c at $x=x_0$ if and only if there exist a multi-index $p\geqq 0$, a neighbourhood U of x_0 and a continuous function $F(x)$ in U, for which

$$T=c+D^pF$$

in U, where $F(x)=o(|x-x_0|^{|p|})$ as $|x-x_0|\to 0$.

Łojasiewicz has also introduced a notion of the section of a distribution, extending the notion of the value of a distribution at a point.

Consider a non-empty open subset Ω of $R^N=R^m\times R^n$. A point of R^N will be denoted by (x, y), where $x=(x_1, \cdots, x_m)$, $y=(y_1, \cdots, y_n)$. Put

$$\Omega_{y_0}=\{x \in R^m;\ (x, y_0) \in \Omega\}.$$

Let $T \in \mathcal{D}'(\Omega)$ and let y_0 be such that Ω_{y_0} is not empty. If there exists a distribution $S \in \mathcal{D}'(\Omega_{y_0})$ such that

$$\lim_{\lambda\to+0} T(x,\ y_0+\lambda y)=S(x),$$

or equivalently

$$\lim_{\lambda\to+0} <T,\ \frac{1}{\lambda^n}\phi(x)\psi\Big(\frac{y-y_0}{\lambda}\Big)> = <S, \phi>\int_{R^n}\psi(y)dy$$

for any $\phi \in \mathcal{D}(\Omega_{y_0})$, $\psi \in \mathcal{D}(R^n)$, then according to Lojasiewicz [12] (p. 15) we shall say that $y=y_0$ can be fixed in $T(x, y)$ and that S is the section of T for $y=y_0$ which will be denoted by $T(x, y_0)$.

It is known that a necessary and sufficient condition for S to be the section of T for $y=y_0$ is that for any non-empty open subset $G \subset\subset \Omega_{y_0}$ ($A \subset\subset B$ means that A is relatively compact in B), there exist a multi-order (p, q), a neighbourhood Δ of y_0 and a continuous function $F(x, y)$ in $G\times\Delta$, for which

$$T(x, y)=S(x)\otimes 1_y+D_x^p D_y^q F(x, y)$$

in $G\times\Delta$, where $F(x, y)=o(|y-y_0|^{|q|})$ as $|y-y_0|\to 0$. We may take here p, q so that $p\geqq p_0$, $q\geqq q_0$ for any given $p_0\geqq 0$, $q_0\geqq 0$. As an immediate consequence of these considerations, we obtain the following

LEMMA 2. *If $T(x, y_0)$ exists, then*

$$\lim_{\lambda\to+0} <T, \chi_\lambda(x, y)> = <S, \phi>$$

for any $\chi_\lambda \in \mathcal{D}(\Omega_{y_0}\times R^n)$ depending on $\lambda>0$ such that

(i) $\operatorname{supp}\chi_\lambda \subset G\times\{y: |y-y_0|<\lambda\}$;

(ii) $\sup|D_x^p D_y^q \chi_\lambda(x, y)| = O\left(\dfrac{1}{\lambda^{n+|q|}}\right)$;

(iii) $\int_{R^n}\chi_\lambda(x, y)dy \to \phi(x)$ *in* $\mathcal{D}(\Omega_{y_0})$ *as* $\lambda\to 0$.

Now we shall show

LEMMA 3. *Let S be a distribution in a neighbourhood of 0 in R^N. If the values $\dfrac{\partial S}{\partial x_j}(0)=c_j$ exist for $j=1, 2, \dots, N$, then the same is also true of S.*

PROOF. Let $S_j=\dfrac{\partial S}{\partial x_j}$, $j=1, 2, \dots, N$ and $P_\lambda=\{x;\ |x|_\infty=\max|x_j|<\lambda\}$ for any positive number $\lambda\leqq 1$. We may assume that $c_j=0$ and $\operatorname{supp} S_j\subset P_1$, $j=1, 2, \dots, N$. In fact, otherwise setting $S'=S-\sum_1^N c_j x_j$ we shall have $\dfrac{\partial(\alpha S')}{\partial x_j}=\dfrac{\partial S}{\partial x_j}-c_j$ in a neighbourhood of 0, where α is taken from $\mathcal{D}(P_1)$ and equal to 1 in a neighbourhood of 0. Then there exist a multi-index p with $|p|\geqq N$ and continuous functions $F_j(x)$ in P_1, for which

$$S_j=D^p F_j$$

in P_1, where $F_j(x)=o(|x|^{|p|})$ as $|x|\to 0$. Let $\operatorname{supp} S_j\subset P_{\lambda_0}$, $0<\lambda_0<1$, and let $\beta\in\mathcal{D}(P_1)$ be chosen so that $\beta=1$ in P_{λ_0}. If we put $G_j=\beta F_j$, then we can write S_j in the form

$$S_j=D^p G_j+T_j,$$

where $|G_j|\leqq\varepsilon(|x|_\infty)|x|^{|p|}$ and $\varepsilon(|x|_\infty)\downarrow 0$ as $|x|_\infty\downarrow 0$. Then $\operatorname{supp} G_j\subset P_1$ and

$\operatorname{supp} T_j \subset P_1 \setminus P_{\lambda_0}$.

Now, let E be a fundamental solution for Laplacian Δ. Then we can write with a constant C

$$\frac{\partial E}{\partial x_j} = \frac{\phi_j(x)}{|x|^{N-1}}, \qquad |\phi_j(x)| \leqq C$$

in $P_1 \setminus \{0\}$.

Now

$$S = S * \delta = S * \Delta E = \sum_1^N \frac{\partial S}{\partial x_j} * \frac{\partial E}{\partial x_j}$$

$$= \sum_1^N D^p\Big(G_j * \frac{\partial E}{\partial x_j}\Big) + \sum_1^N T_j * \frac{\partial E}{\partial x_j}.$$

Since $\sum_1^N T_j * \frac{\partial E}{\partial x_j}$ is continuous at 0, we only have to show that $D^p\Big(G_j * \frac{\partial E}{\partial x_j}\Big)(0)$ exists for each $j = 1, 2, \cdots, N$.

Let $x \in P_{\theta^\nu}$ with $\theta = \frac{1}{2}$ $(\nu = 1, 2, \cdots)$. Put

$$\tilde{G}_j(x) = \int_{P_1} \frac{\phi_j(x-y)}{|x-y|^{N-1}} G_j(y) dy$$

$$= \Big(\int_{P_{\theta^{\nu-1}}} + \int_{P_1 \setminus P_{\theta^{\nu-1}}}\Big) \frac{\phi_j(x-y)}{|x-y|^{N-1}} G_j(y) dy = I_{j,\nu}^1 + I_{j,\nu}^2.$$

We have an estimate for $I_{j,\nu}^1$

$$|I_{j,\nu}^1| = \left| \int_{P_{\theta^{\nu-1}}} \frac{\phi_j(x-y)}{|x-y|^{N-1}} G_j(y) dy \right| \leqq \varepsilon_\nu \theta^{|p|\nu},$$

where ε_ν is a constant such that $\varepsilon_\nu \downarrow 0$ as $\nu \uparrow \infty$.

On the other hand,

$$D_x^p I_{j,\nu}^2 = \int_{P_1 \setminus P_{\theta^{\nu-1}}} \Big(D_x^p \frac{\phi_j(x-y)}{|x-y|^{N-1}} \Big) G_j(y) dy$$

and we can write with a constant C_1

$$D^p \frac{\phi_j(x)}{|x|^{N-1}} = \frac{\tilde{\phi}_j(x)}{|x|^{N+|p|-1}}, \qquad |\tilde{\phi}_j| \leqq C_1.$$

Putting

$$a_j = \int_{P_1} \frac{\tilde{\phi}_j(-y)}{|y|^{N+|p|-1}} G_j(y) dy,$$

we shall estimate $D^p I_{j,\nu}^2 - a_j$ in P_{θ^ν}.

$$D^p I^2_{j,\nu} - a_j = \int_{P_1 \setminus P_{\theta^{\nu-1}}} \left(\frac{\tilde{\phi}_j(x-y)}{|x-y|^{N+|p|-1}} - \frac{\tilde{\phi}_j(-y)}{|y|^{N+|p|-1}} \right) G_j(y) dy$$

$$- \int_{P_{\theta^{\nu-1}}} \frac{\tilde{\phi}_j(-y)}{|y|^{N+|p|-1}} G_j(y) dy$$

$$= J^1_{j,\nu} + J^2_{j,\nu}.$$

Then we have with $\varepsilon'_\nu \downarrow 0$

$$|J^2_{j,\nu}| = \left| \int_{P_{\theta^{\nu-1}}} \frac{\tilde{\phi}_j(-y)}{|y|^{N+|p|-1}} G_j(y) dy \right| \leqq \varepsilon'_\nu.$$

When $x \in P_{\theta^\nu}$ and $y \in P_1 \setminus P_{\theta^{\nu-1}}$ we can write with a constant C'_1 and a positive $t<1$

$$\left| \frac{\tilde{\phi}_j(x-y)}{|x-y|^{N+|p|-1}} - \frac{\tilde{\phi}_j(-y)}{|y|^{N+|p|-1}} \right| \leqq \frac{C'_1 |x|}{|tx-y|^{N+|p|}}.$$

Since

$$|tx-y| \geqq |tx-y|_\infty \geqq \frac{1}{2} |y|_\infty \geqq \frac{1}{2\sqrt{N}} |y|,$$

we have with constants C_2, C_3

$$|J^1_{j,\nu}| \leqq C_2 |x| \int_{\theta^{\nu-1} \leqq |y| \leqq \sqrt{N}} \frac{1}{|y|^N} dy \leqq C_3 \theta^\nu \log \frac{\sqrt{N}}{\theta^{\nu-1}}.$$

Thus we obtain

$$|D^p I^2_{j,\nu} - a_j| \leqq \varepsilon''_\nu,$$

where ε''_ν is a constant such that $\varepsilon''_\nu \downarrow 0$ as $\nu \uparrow \infty$.

Consequently there exist continuous functions $H_{j,\nu}(x)$ in $\overline{P_{\theta^\nu}}$ such that we can write

$$D^p I^2_{j,\nu} - a_j = D^p H_{j,\nu}(x), \qquad x \in P_{\theta^\nu},$$

where $|H_{j,\nu}| \leqq \varepsilon'''_\nu \theta^{|p|\nu}$ and $\varepsilon'''_\nu \downarrow 0$ as $\nu \uparrow \infty$.

Combining these results we can find continuous functions $K_{j,\nu}(x)$ in $\overline{P_{\theta^\nu}}$ such that

$$D^p \tilde{G}_j(x) = a_j + D^p K_{j,\nu}(x), \qquad x \in P_{\theta^\nu},$$

where $|K_{j,\nu}| \leqq \eta_\nu \theta^{|p|\nu}$ and $\eta_\nu \downarrow 0$ as $\nu \uparrow \infty$.

If we put

$$q_{j,\nu} = K_{j,\nu} - K_{j,\nu+1}, \qquad j=1, 2, \cdots, N, \ \nu=1, 2, \cdots,$$

then the functions $q_{j,\nu}$ are continuous in $\overline{P_{\theta^{\nu+1}}}$ and $D^p q_{j,\nu}=0$ in $P_{\theta^{\nu+1}}$. By virtue of Lemma 2 in [12] (p. 12), there exist continuous functions $\tilde{q}_{j,\nu}$ in $\overline{P_\theta}$ such that

(i) $\tilde{q}_{j,\nu}|P_{\theta^{\nu+1}}=\tilde{q}_{j,\nu}$;

(ii) $D^p\tilde{q}_{j,\nu}=0$;

(iii) $|\tilde{q}_{j,\nu}|\leqq K\eta_\nu\theta^{\nu N}(\theta^{\nu(|p|-N)}+|x|^{|p|-N})$;

where K is a constant independent of ν and j.
Consider the continuous functions $\tilde{K}_{j,\nu}$ in P_θ defined by

$$\tilde{K}_{j,1}=K_{j,1},\quad \tilde{K}_{j,\nu}=K_{j,1}-\tilde{q}_{j,1}-\cdots-\tilde{q}_{j,\nu-1}\qquad (\nu\geqq 2),$$

then owing to (i) through (iii) we obtain

(iv) $\tilde{K}_{j,\nu}|P_{\theta^\nu}=K_{j,\nu}$;

(v) $D^p\tilde{K}_{j,\nu}=D^pK_{j,1}$;

(vi) $|\tilde{K}_{j,\nu+\sigma}-\tilde{K}_{j,\nu}|=|\tilde{q}_{j,\nu}+\cdots+\tilde{q}_{j,\nu+\sigma-1}|$

$$\leqq K\eta_\nu\frac{\theta^{\nu N}}{1-\theta^N}(\theta^{\nu(|p|-N)}+|x|^{|p|-N}).$$

This shows that $\tilde{K}_{j,\nu}$ converges uniformly for each j as $\nu\uparrow\infty$. Hence if we put for each j

$$K_j=K_{j,1}-\sum_{i=1}^{\infty}\tilde{q}_{j,i},$$

then K_j is a continuous function in P_θ satisfying

$$|K_j|=|\tilde{K}_{j,\nu}-\sum_{i=\nu}^{\infty}\tilde{q}_{j,i}|\leqq\eta'_\nu\theta^{|p|\nu},\qquad x\in P_{\theta^\nu},$$

where η'_ν is a constant such that $\eta'_\nu\downarrow 0$ as $\nu\uparrow\infty$. This means that $K_j=o(|x|^{|p|})$.

Consequently we can write

$$D^p\tilde{G}_j(x)=a_j+D^pK_j(x),\qquad x\in P_\theta,$$

where $|K_j|=o(|x|^{|p|})$ as $|x|\downarrow 0$. This shows that $D^p\tilde{G}_j$ has the value a_j at 0 for each j. Thus the proof is completed.

REMARK 4. Let $P(D)$ be any hypoelliptic differential operator of degree $m\neq 0$ and $E(x)$ a fundamental solution for $P(D)$. There exist then constants α $(0<\alpha\leqq 1)$ and C such that $P(\sigma+i\tau)=0$, $|\tau|\geqq 2NC|\sigma|^\alpha$ when $|\sigma|$ is large. V.V. Grušin [3] obtained the following estimate:

$$|D^rE|=\begin{cases} O(1) & \text{for } m-\dfrac{N+|r|}{\alpha}>0,\\[2ex] O(|\log|x||) & \text{for } m-\dfrac{N+|r|}{\alpha}=0,\\[2ex] O(|x|^{m-\frac{N+|r|}{\alpha}}) & \text{for } m-\dfrac{N+|r|}{\alpha}<0,\end{cases}$$

as $|x|\to 0$. If $P(D)S(0)$ exists, then $D^r S(0)$ exists for r such that $|r|\leqq\alpha(m-1)-N(1-\alpha)$. In fact, as in the proof of Lemma 3, we may assume that $P(D)S(0)=0$ and supp $S\subset P_{\lambda_0}\subset P_1$ with a contant λ_0. Then there exist a multi-index p with $|p|\geqq N$ and a continuous function $F(x)$ in P_1, for which $P(D)S=D^p F$ in P_1, where $F(x)=o(|x|^{|p|})$ as $|x|\to 0$. Let $\beta\in\mathcal{D}(P_1)$ be equal to 1 in P_{λ_0} and put $G=\beta F$. Then $P(D)S=D^p G+T$, where $G=o(|x|^{|p|})$ as $|x|\to 0$. We can write $D^r S$ in the form

$$
\begin{aligned}
D^r S &= D^r S*\delta = D^r S*P(D)E \\
&= P(D)S*D^r E \\
&= D^p(G*D^r E)+T*D^r E,
\end{aligned}
$$

where $T*D^r E$ is continuous at 0 and $|D^r E|=O\left(\frac{1}{|x|^{N-1}}\right)$ as $|x|\to 0$. Applying the method in the proof of Lemma 3, we can show that $(D^r S)(0)$ exists. Since we can choose $\alpha=1$ if $P(D)$ is elliptic, $D^r S(0)$ exists for r such that $|r|\leqq m-1$. For example, if $\Delta S(0)$ exists for Laplacian Δ, then S, $\dfrac{\partial S}{\partial x_j}$ have the values at 0.

§ 3. The multiplicative product $S\circ T$

Let S, $T\in\mathcal{D}'(\Omega)$. When $S*\check{T}$ exists in a neighbourhood of 0 and has the value at 0, we shall define the scalar product $<S, T>$ of S and T by the formula ([11], p. 34)

$$
<S,\ T>=(S*\check{T})(0),
$$

which is a generalization of the notion of the scalar product between $\mathcal{D}'(\Omega)$ and $\mathcal{D}(\Omega)$.

We shall now define normal multiplication between distributions basing on this generalized scalar product.

Assuming that the scalar product $<\alpha S, T>$ exists for any $\alpha\in\mathcal{D}(\Omega)$, the linear form $\alpha\to<\alpha S, T>$ will be continuous on $\mathcal{D}(\Omega)$. In fact, if we put

$$
\phi_\lambda(x)=\frac{1}{\lambda^N}\phi\left(\frac{x}{\lambda}\right),\ \lambda \text{ being a positive real number,}
$$

for any $\phi\in\mathcal{D}(R^N)$ such that $\phi\geqq 0$ and $\int\phi(x)dx=1$, then $<\alpha S*\check{T}, \phi_\lambda>$ is well defined for small $\lambda>0$. The mapping $\alpha\to<\alpha S*\check{T}, \phi_\lambda>$ being continuous, the linear form $\mathcal{D}(\Omega)\ni\alpha\to<\alpha S, T>$ will be continuous by the Banach-Steinhaus theorem, and so there exists a unique distribution $W\in\mathcal{D}'(\Omega)$ such that

$$
<W,\ \alpha>=<\alpha S,\ T>,\qquad \alpha\in\mathcal{D}(\Omega).
$$

We shall now introduce

Definition I. *Let $S, T \in \mathcal{D}'(\Omega)$. If there exists $<\alpha S, T>$ for any $\alpha \in \mathcal{D}(\Omega)$, then $W \in \mathcal{D}'(\Omega)$ defined by the formula*

$$<W, \alpha> = <\alpha S, T> = ((\alpha S)*\check{T})(0), \qquad \alpha \in \mathcal{D}(\Omega),$$

will be called the multiplicative product of S and T and denoted by $S \bigcirc T$.

The identity

$$<(\alpha S)*\check{T}, \phi_\lambda> = <S(T*\phi_\lambda), \alpha>, \qquad \alpha \in \mathcal{D}(\Omega),$$

makes it possible to restate this definition in the following

Definition II. *Let $S, T \in \mathcal{D}'(\Omega)$. If the distributional limit*

$$\lim_{\lambda \to +0} S(T*\phi_\lambda)$$

exists in $\mathcal{D}'(\Omega)$ and does not depend on the choice of ϕ, then the limit will be called the multiplicative product of S and T and denoted by $S \bigcirc T$.

For the sake of convenience we shall use the symbol $S(x) \otimes T(x-y)$ to denote the distribution obtained from the tensor product $S(x') \otimes T(y')$ by the change of variables, $x = x'$, $y = x' - y'$, and use similar notations in later consideration. Owing to the discussion in Section 2, this definition will prove to be equivalent to the following

Definition III. *Let $S, T \in \mathcal{D}'(\Omega)$. If $S(x) \otimes T(x-y)$ admits a section W for $y=0$, then W will be called the multiplicative product of S and T and denoted by $S \bigcirc T$.*

In what follows we shall show that the multiplication just defined is normal; that is, if we take $\mathfrak{M}_\Omega$ as the set of $(S, T) \in \mathcal{D}'(\Omega) \times \mathcal{D}'(\Omega)$ such that $S \bigcirc T$ exists, the requirements I through IV in Section 1 are fulfilled.

I_1 and I_3 are obvious from the definition of our multiplicative product.

As for I_2, assume that $S \bigcirc T$ exists. We have for any $\alpha \in \mathcal{D}(\Omega)$

$$\begin{aligned} <(\alpha T)*\check{S}, \phi_\lambda> &= <(S*\phi_\lambda)T, \alpha> \\ &= <S(x-y) \otimes T(x), \alpha(x)\phi_\lambda(y)> \\ &= <S(x) \otimes T(x-y), \alpha(x-y)\check{\phi}_\lambda(y)>. \end{aligned}$$

Here $\int \alpha(x-y)\check{\phi}_\lambda(y)dy$ tends to $\alpha(x)$ in $\mathcal{D}(\Omega)$ as $\lambda \to 0$ and $\sup|D_x^p D_y^q \alpha(x-y)\check{\phi}_\lambda(y)| = O\left(\frac{1}{\lambda^{N+|q|}}\right)$. By Lemma 2 we obtain

$$\lim_{\lambda \to +0} <(\alpha T)*\check{S}, \phi_\lambda> = <S \bigcirc T, \alpha>.$$

Consequently $T \bigcirc S$ exists and is equal to $S \bigcirc T$.

I_2 means that, if $y=0$ can be fixed in $S(x) \otimes T(x-y)$, the same is true of

$S(x-y)\otimes T(x)$ and they have the same section for $y=0$. This will also be proved by a change of variable. The method will be applied to a similar question in Section 5.

As for I_4, assume that $S\bigcirc T$ exists. Let $\alpha \in \mathcal{E}(\Omega)$. Since $\phi\alpha \in \mathcal{D}(\Omega)$ for any $\phi \in \mathcal{D}(\Omega)$, the value $((\phi\alpha S)*\check{T})(0)$ exists, which implies the existence of $(\alpha S)\bigcirc T$. In addition, we have

$$\begin{aligned} <(\alpha S)\bigcirc T, \phi> &= ((\phi\alpha S)*\check{T})(0) \\ &= <S\bigcirc T, \alpha\phi> = <\alpha(S\bigcirc T), \phi>. \end{aligned}$$

Thus $(\alpha S)\bigcirc T=\alpha(S\bigcirc T)$, which completes the proof.

As for II, assume that $\frac{\partial S}{\partial x_j}\bigcirc T$ exists for each $j=1, 2, \cdots, N$. Then we have for any $\phi \in \mathcal{D}(\Omega)$

$$<\frac{\partial S}{\partial x_j}\bigcirc T, \phi> = \Big(\frac{\partial S}{\partial x_j}*(\phi T)^{\vee}\Big)(0) = \frac{\partial}{\partial x_j}(S*(\phi T)^{\vee})(0), \qquad j=1, 2, \cdots, N.$$

Owing to Lemma 3 the value $(S*(\phi T)^{\vee})(0)$ exists, and so does $S\bigcirc T$. From the equations

$$\begin{aligned} \phi S*\Big(\frac{\partial T}{\partial x_j}\Big)^{\vee} &= -\frac{\partial}{\partial x_j}(\phi S)*\check{T} \\ &= -\frac{\partial \phi}{\partial x_j}S*\check{T} - \phi\frac{\partial S}{\partial x_j}*\check{T} \end{aligned}$$

we can conclude that $S\bigcirc\frac{\partial T}{\partial x_j}$ exists since $S\bigcirc T$ and $\frac{\partial S}{\partial x_j}\bigcirc T$ exist. Furthermore, we have

$$\begin{aligned} <S\bigcirc\frac{\partial T}{\partial x_j}, \phi> &= -<S\bigcirc T, \frac{\partial \phi}{\partial x_j}> - <\frac{\partial S}{\partial x_j}\bigcirc T, \phi> \\ &= <\frac{\partial}{\partial x_j}(S\bigcirc T) - \frac{\partial S}{\partial x_j}\bigcirc T, \phi>, \end{aligned}$$

that is,

$$\frac{\partial}{\partial x_j}(S\bigcirc T) = \frac{\partial S}{\partial x_j}\bigcirc T + S\bigcirc\frac{\partial T}{\partial x_j}, \qquad j=1, 2, \cdots, N,$$

completing the proof.

III_1 is obvious from the definition of our multiplicative product.

As for III_2, we can choose a partition of unity subordinate to the covering $\{\Omega_\iota\}$, that is, we can choose functions ρ_j and $\iota(j)$, $j=1, 2, \cdots$ so that

(i) $\rho_j \in \mathcal{D}(\Omega_{\iota(j)})$;

(ii) all but a finite number of functions ρ_j vanish identically on any compact subset of Ω;

(iii) $\sum_j \rho_j = 1$ on Ω.

We can now write for any $\phi \in \mathcal{D}(\Omega)$

$$(\phi S)*\check{T} = (\sum_j \phi\rho_j S)*\check{T} = \sum_j ((\phi\rho_j S)*\check{T}),$$

where each summand has the value at 0 equal to $< S_{\Omega_{\iota(j)}} \bigcirc T_{\Omega_{\iota(j)}}, \phi\rho_j >$ respectively. Then the distribution $W \in \mathcal{D}'(\Omega)$ defined by

$$< W, \phi > = \sum_j < S_{\Omega_{\iota(j)}} \bigcirc T_{\Omega_{\iota(j)}}, \phi\rho_j >,$$

will yield the multiplicative product $S \bigcirc T$. This completes the proof.

Finally, we shall show that $\bigcirc$ has the property IV. By Definition III, if we put $W = S \bigcirc T$ the distribution $S(x) \otimes T(x-y)$ admits W as the section for $y=0$: Namely, for any open subset $\Delta \subset\subset \Omega$ there exist a multi-order (p, q), a neighbourhood U of 0 in R_y^N and a continuous function $F(x, y)$ in $\Delta \times U$, for which the relation

$$S(x) \otimes T(x-y) = W(x) \otimes 1_y + D_x^p D_y^q F(x, y)$$

remains true in $\Delta \times U$, where $F(x, y) = o(|y|^{|q|})$ as $|y| \to 0$. Let $x = \Phi(x')$ be a diffeomorphism of Ω' onto Ω. The distributions corresponding to S, T and W will be denoted by $\tilde{S}$, $\tilde{T}$ and $\tilde{W} \in \mathcal{D}'(\Omega')$ respectively. Since the distribution $\tilde{S}(x') \otimes \tilde{T}(x'-y') = S(\Phi(x')) \otimes T(\Phi(x'-y'))$ is obtained from $S(x) \otimes T(x-y)$ by the change of variables

$$(*) \qquad x = \Phi(x'), \quad y = \Phi(x') - \Phi(x'-y')$$

and

$$\tilde{S}(x') \otimes \tilde{T}(x'-y') = W(\Phi(x')) \otimes 1_{y'} + (D_x^p D_y^q F)(\Phi(x'), \Phi(x') - \Phi(x'-y')),$$

we only have to show that $\tilde{W}$ is the section of $S(x') \otimes T(x'-y')$, in other words, $(D_x^p D_y^q F)(\Phi(x'), \Phi(x') - \Phi(x'-y'))$ has 0 as the section for $y'=0$. Let

$$x' = g(x), \quad y' = h(x, y)$$

be the inverse of the transformation $(*)$. Differentiating with respect to x_j, y_j we obtain

$$(D_{x_j} F)(\Phi(x'), \Phi(x') - \Phi(x'-y')) = \sum_k (D_{x'_k} \tilde{F}) g_j^k(x') + \sum_k (D_{y'_k} \tilde{F}) h_j^k(x', y'),$$

$$(D_{y_j} F)(\Phi(x'), \Phi(x') - \Phi(x'-y')) = \sum_k (D_{y'_k} \tilde{F}) \tilde{h}_j^k(x', y'),$$

$$(j = 1, 2, \cdots, N)$$

where we have written

$$\tilde{F}(x', y') = F(\Phi(x'), \Phi(x') - \Phi(x'-y')),$$

$$g_j^k(x')=\frac{\partial g_k}{\partial x_j}(\Phi(x')),\quad h_j^k(x',\ y')=\frac{\partial h_k}{\partial x_j}(\Phi(x'),\ \Phi(x')-\Phi(x'-y')),$$

$$\tilde{h}_j^k(x',\ y')=\frac{\partial h_k}{\partial y_j}(\Phi(x'),\ \Phi(x')-\Phi(x'-y')).$$

Since $y'=0$ corresponds to $y=0$, these expressions yield

$$|h_j^k|=O(|y'|),\quad |\tilde{h}_j^k|=O(1),\qquad j,\ k=1,\ 2,\ \cdots,\ N.$$

Let α', α'', β' and β'' be multi-orders of the distributional derivatives relative to the variables x, y, x' and y' respectively. Put $\alpha=(\alpha',\ \alpha'')$, $\beta=(\beta',\ \beta'')$.

Now by induction over the order of differentiation we can show that

$$(D^\alpha F)(\Phi(x'),\ \Phi(x')-\Phi(x'-y'))=\sum_{|\beta|\leqq|\alpha|}a_\beta(x',\ y')(D^\beta\tilde{F})(x',\ y'),$$

where a_β are indefinitely differentiable and $|\beta''|\geqq|\alpha''|$ implies $|a_\beta|=O(|y'|^{|\beta''|-|\alpha''|})$.

If $|\alpha|=1$ the result is already shown above. Next, assume that the result is valid for any α with $|\alpha|\leqq l$. From our hypothesis we have

$$\begin{aligned}&((D_{x_j}D^\alpha)F)(\Phi(x'),\ \Phi(x')-\Phi(x'-y'))\\&=\sum_k g_j^k D_{x'_k}(\widetilde{D^\alpha F})+\sum_k h_j^k(D_{y'_k}(\widetilde{D^\alpha F}))\\&=\sum_k\sum_{|\beta|\leqq|\alpha|}g_j^k(D_{x'_k}a_\beta)(D^\beta\tilde{F})+\sum_k\sum_{|\beta|\leqq|\alpha|}g_j^k a_\beta(D_{x'_k}D^\beta\tilde{F})\\&\quad+\sum_k\sum_{|\beta|\leqq|\alpha|}h_j^k(D_{y'_k}a_\beta)(D^\beta\tilde{F})+\sum_k\sum_{|\beta|\leqq|\alpha|}h_j^k a_\beta(D_{y'_k}D^\beta\tilde{F}).\end{aligned}$$

With regard to the first two sums of the last expression the desired estimates are valid by the hypothesis. Consider the third sum. $|\beta''|\geqq|\alpha''|$ implies that $|h_j^k|=O(|y'|)$ and $|D_{y'_k}a_\beta|=O(|y'|^{|\beta''|-|\alpha''|-1})$, and therefore $|h_j^k(D_{y'_k}a_\beta)|=O(|y'|^{|\beta''|-|\alpha''|})$. In the last sum, if $|\beta''|+1\geqq|\alpha''|$, then $|h_j^k a_\beta|=O(|y'|^{|\beta''|+1-|\alpha''|})$, since $|h_j^k|=O(|y'|)$ and $|a_\beta|=O(|y'|^{|\beta''|-|\alpha''|})$ hold for $|\beta''|\geqq|\alpha''|$. Combining these together we obtain the desired result.

Consider also the equations:

$$\begin{aligned}&((D_{y_j}D^\alpha)F)(\Phi(x'),\ \Phi(x')-\Phi(x'-y'))\\&=\sum_k\tilde{h}_j^k D_{y'_k}\Big(\sum_{|\beta|\leqq|\alpha|}a_\beta(D^\beta\tilde{F})\Big)\\&=\sum_k\sum_{|\beta|\leqq|\alpha|}\tilde{h}_j^k(D_{y'_k}a_\beta)(D^\beta\tilde{F})+\sum_k\sum_{|\beta|\leqq|\alpha|}\tilde{h}_j^k a_\beta(D_{y'_k}D^\beta\tilde{F}).\end{aligned}$$

In the first sum of the last expression, $|\beta''|\geqq|\alpha''|+1$ implies that $|\tilde{h}_j^k|=O(1)$ and $|D_{y'_k}a_\beta|=O(|y'|^{|\beta''|-|\alpha''|-1})$, and therefore $|\tilde{h}_j^k(D_{y'_k}a_\beta)|=O(|y'|^{|\beta''|-|\alpha''|-1})$. In the second sum, $|\beta''|+1\geqq|\alpha''|+1$ implies that $|\tilde{h}_j^k|=O(1)$ and $|a_\beta|=O(|y|^{|\beta''|-|\alpha''|})$, and so $|\tilde{h}_j^k a_\beta|=O(|y'|^{|\beta''|-|\alpha''|})$.

Combining these together, we obtain also the desired result. This completes the induction step.

Thus we can write

$$(D_x^p D_y^q F)(\Phi(x'),\ \Phi(x')-\Phi(x'-y'))=\sum_{|p'|+|q'|\leqq|p|+|q|} a_{p',q'} D_{x'}^{p'} D_{y'}^{q'} \tilde{F},$$

where $\tilde{F}=o(|y'|^{|q|})$, and $|a_{p',q'}|=O(|y'|^{|q'|-|q|})$ for $|q'|\geqq|q|$. By Leibniz' formula, we have

$$a_{p',q'} D_{x'}^{p'} D_{y'}^{q'} \tilde{F}=\sum_{r\leqq p',s\leqq q'}(-1)^{|r|+|s|}\binom{p'}{r}\binom{q'}{s} D_{x'}^{p'-r} D_{y'}^{q'-s}((D_{x'}^r D_{y'}^s a_{p',q'})\tilde{F}).$$

For the conclusion of the proof of IV, it is sufficient to show that

$$(D_{x'}^r D_{y'}^s a_{p',q'})\tilde{F}=o(|y'|^{|q'|-|s|}).$$

It is trivially true of the case $|q'|<|q|$, otherwise it follows from the estimates $\tilde{F}=o(|y'|^{|q|})$ and $|a_{p',q'}|=O(|y'|^{|q'|-|q|})$ already proved.

Thus we have shown

THEOREM 1. *The multiplication given by Definition* I (II, III) *is normal.*

REMARK 5. In Definition I, if $(\alpha S)*\check{T}$ is a bounded function in a neighbourhood of 0 and continuous at 0, then according to [18] W will be written ST instead of $S\bigcirc T$. By definition, if ST exists, $S\bigcirc T$ also exists and coincides with ST. But the converse is not always true; $\sin\frac{1}{x}\bigcirc\delta=0$ but $\left(\sin\frac{1}{x}\right)\delta$ does not exist.

REMARK 6. Let M be a differenciable manifold of dimension N and $\{\kappa\}$ its coordinate systems ([5], p. 25). κ is a homeomorphism of an open set $\Omega_\kappa\subset M$ onto an open set $\tilde{\Omega}_\kappa\subset R^N$, and the mapping

$$\kappa\kappa'^{-1}:\quad \kappa'(\Omega_\kappa\cap\Omega_{\kappa'})\rightarrow\kappa(\Omega_\kappa\cap\Omega_{\kappa'})$$

is a diffeomorphism for any two coordinate systems κ, κ'. If to every coordinate system κ in M we are given a distribution $S_\kappa\in\mathcal{D}'(\tilde{\Omega}_\kappa)$ such that $S_{\kappa'}(x')=S_\kappa(\kappa\kappa'^{-1}(x'))$ in $\kappa'(\Omega_\kappa\cap\Omega_{\kappa'})$, then the system $\{S_\kappa\}$ is called a distribution S in M and the set of all distributions in M will be denoted by $\mathcal{D}'(M)$. Let S, $T\in\mathcal{D}'(M)$. Assume that $W_\kappa=S_\kappa\bigcirc T_\kappa$ exists for every coordinate system κ. It follows from IV that $W_{\kappa'}(x')=W_\kappa(\kappa\kappa'^{-1}(x'))$ for any κ and κ', so there exists a unique distribution $W\in\mathcal{D}'(M)$ determined by the system $\{W_\kappa\}$. We shall define W as the multiplicative product $S\bigcirc T$ of S and T. The requirements I through IV for manifold will have an obvious meaning and are fulfilled by the multiplication just considered.

§ 4. Further properties of the multiplicative product $S\bigcirc T$

This section is devoted to a discussion about multiplication considered in the preceding section. Otherwise explicitely stated, we shall assume that S, T are distributions in a non-empty open subset $\Omega \subset R^N$.

LEMMA 4. *$S(0)$ exists if and only if $S\bigcirc\delta$ exists. Then we can write $S\bigcirc\delta = S(0)\delta$.*

This is clear from the identity

$$S*(\alpha\delta)^{\vee}=\alpha(0)S, \qquad \alpha \in \mathcal{D}(\Omega).$$

Consequently we have

COROLLARY. *The multiplicative product $S\bigcirc T$ exists if and only if $(\alpha S*\check{T})\bigcirc\delta$ exists for every $\alpha \in \mathcal{D}(\Omega)$. If this is the case, we can write $(\alpha S*\check{T})\bigcirc\delta = <S\bigcirc T, \alpha>\delta$.*

PROPOSITION 1. *$S\bigcirc T$ exists for every $T \in \mathcal{D}'(\Omega)$ if and only if $S \in \mathcal{E}(\Omega)$.*

PROOF. The "if" part is evident. We only have to show the "only if" part. Let $\phi \in \mathcal{D}(\Omega)$ be such that $\phi \geqq 0$ and $\int \phi(x)dx=1$. Then by definition

$$S\bigcirc T = \lim_{\lambda\to+0} S(T*\phi_\lambda).$$

Owing to the Banach-Steinhaus theorem, the mapping $T\to S\bigcirc T$ is continuous from $\mathcal{D}'(\Omega)$ into itself, so there exists for any $\psi \in \mathcal{D}(\Omega)$ a unique $\chi \in \mathcal{D}(\Omega)$ such that

$$<S\bigcirc T, \psi> = <T, \chi>.$$

Replacing T by $\alpha \in \mathcal{D}(\Omega)$, we can conclude that $\psi S=\chi$ and therefore $S \in \mathcal{E}(\Omega)$. This completes the proof.

The method of proof just given is also applied in proving that S is a locally bounded function if and only if $S\bigcirc T$ exists for every locally summable function T in Ω.

If we put $R^N=R^m_x\times R^n_y$, $N=m+n$, then we have

LEMMA 5. *Let $W_\lambda \in \mathcal{D}'(\Omega)$, $0<\lambda<1$. If*

$$\lim_{\lambda\to+0} <W_\lambda, \alpha\otimes\beta>$$

exists for any $\alpha \in \mathcal{D}(R^m_x)$, $\beta \in \mathcal{D}(R^n_y)$ with $\operatorname{supp}\alpha\times\operatorname{supp}\beta\subset\Omega$, *then there exists a unique $W \in \mathcal{D}'(\Omega)$ such that*

$$\lim_{\lambda\to+0} <W_\lambda, \phi> = <W, \phi>, \qquad \phi \in \mathcal{D}(\Omega).$$

Proof. Let K, L be compact cubes in R_x^m and R_y^n respectively such that $K\times L\subset\Omega$. $\mathcal{D}_{K\times L}=\mathcal{D}_K\hat{\otimes}_\pi\mathcal{D}_L$. Since $\mathcal{D}_K$ and $\mathcal{D}_L$ are spaces of type (**F**) $\lim\limits_{\lambda\to+0}<W_\lambda, \alpha\otimes\beta>$ is separately continuous, and so continuous in $\mathcal{D}_K\times\mathcal{D}_L$. Hence there exists a unique $W\in\mathcal{D}'_{K\times L}$ such that

$$\lim_{\lambda\to+0}<W_\lambda, \alpha\otimes\beta>=<W, \alpha\otimes\beta>, \qquad \alpha\in\mathcal{D}_K,\ \beta\in\mathcal{D}_L.$$

We can therefore assume that $W=0$. We only have to show that

$$\lim_{\lambda\to+0}<W_\lambda, \phi>=0, \qquad \phi\in\mathcal{D}_{K\times L}.$$

Using the Banach-Steinhaus theorem and the fact that $\mathcal{D}_K$ and $\mathcal{D}_L$ are nuclear, we see that the convergence $\lim\limits_{\lambda\to+0}<W_\lambda, \alpha\otimes\beta>$ is uniform on some 0-neighbourhoods of $\mathcal{D}_K$ and $\mathcal{D}_L$. Namely, if $\varepsilon>0$ is given, there are norms $\alpha\to\|\alpha\|_k$ and $\beta\to\|\beta\|_l$ so that for small λ

$$|<W_\lambda, \alpha\otimes\beta>|\leqq\varepsilon\|\alpha\|_k\|\beta\|_l.$$

On the other hand, ϕ may be written in the form

$$\phi=\textstyle\sum_j\alpha_j\otimes\beta_j, \qquad \alpha_j\in\mathcal{D}_K,\ \beta_j\in\mathcal{D}_L,$$

where $\sum_j\|\alpha_j\|_k\|\beta_j\|_l<\infty$. It follows therefore that

$$|<W_\lambda, \phi>|=|\textstyle\sum_j<W_\lambda, \alpha_j\otimes\beta_j>|\leqq\varepsilon\sum_j\|\alpha_j\|_k\|\beta_j\|_l,$$

which completes the proof.

Applying Lemma 5, it is clear that we can reformulate Definition I with α replaced by $\alpha\otimes\beta$ indicated above. Another application of the lemma gives

Lemma 6. *If $S_1(0)$ and $S_2(0)$ exist, so does $(S_1\otimes S_2)(0)$. Then $(S_1\otimes S_2)(0)=S_1(0)S_2(0)$.*

From Lemmas 5, 6 we have immediately

Proposition 2. *Let S_1, $T_1\in\mathcal{D}'(\Omega^1)$ and S_2, $T_2\in\mathcal{D}'(\Omega^2)$, Ω^1, Ω^2 being any non-empty open subsets of R_x^m, R_y^n respectively. If $S_1\bigcirc T_1$ and $S_2\bigcirc T_2$ exist, then $(S_1\otimes S_2)\bigcirc(T_1\otimes T_2)$ exists and is equal to $(S_1\bigcirc T_1)\otimes(S_2\bigcirc T_2)$.*

Proof. Let $\alpha\in\mathcal{D}(\Omega^1)$ and $\beta\in\mathcal{D}(\Omega^2)$. Then

$$((\alpha\otimes\beta)(S_1\otimes S_2))*(T_1\otimes T_2)^\vee=(\alpha S_1*\check{T}_1)\otimes(\beta S_2*\check{T}_2).$$

Since $(\alpha S_1*\check{T}_1)(0)$ and $(\beta S_2*\check{T}_2)(0)$ exist, $(S_1\otimes S_2)\bigcirc(T_1\otimes T_2)$ exists and is equal to $(S_1\bigcirc T_1)\otimes(S_2\bigcirc T_2)$, which was to be proved.

Lemma 7, *$((\alpha\otimes\beta)S*\check{T})(0)$ exists for every $\alpha\otimes\beta\in\mathcal{D}(\Omega)$ if and only if $(\alpha S*(\beta T)^\vee)(0)$ exists for every $\alpha\otimes\beta\in\mathcal{D}(\Omega)$.
Then $((\alpha\otimes\beta)S*\check{T})(0)=(\alpha S*(\beta T)^\vee)(0)$.*

PROOF. Recall that if $((\alpha\otimes\beta)S*\check{T})(0)$ and $(\alpha S*(\beta T)^{\vee})(0)$ make sense, they are respectively the sections of $\alpha(x)\beta(y)S(x, y)\otimes T(x-x^1, y-y^1)$ and $\alpha(x)S(x, y)\otimes\beta(y-y^1)T(x-x^1, y-y^1)$ for $x^1=y^1=0$. Take $\gamma\in\mathcal{D}(R^n)$ with value 1 in a small neighbourhood of $\operatorname{supp}\beta$. We can write for small $|y^1|$

$$\alpha(x)\beta(y)S(x, y)\otimes T(x-x^1, y-y^1)-\alpha(x)S(x, y)\otimes\beta(y-y^1)T(x-x^1, y-y^1)$$
$$=(\beta(y)-\beta(y-y^1))(\alpha(x)\gamma(y)S(x, y)\otimes T(x-x^1, y-y^1)),$$

which has the section 0 for $x^1=y^1=0$ when $((\alpha\otimes\beta)S*\check{T})(0)$ exists, because $\beta(y)-\beta(y-\lambda y^1)$ converges to 0 in $\mathcal{D}(R^n\times R^n)$ as $\lambda\downarrow 0$. This shows the "only if" part. The "if" part will follow with $\gamma(y)$ replaced by $\gamma(y-y^1)$. This completes the proof.

As a result we obtain immediately

COROLLARY. *$S\bigcirc T$ exists if and only if $(\alpha S*(\beta T)^{\vee})(0)$ exists for every $\alpha\otimes\beta\in\mathcal{D}(\Omega)$. In this case*

$$<S\bigcirc T, \alpha\otimes\beta>=(\alpha S*(\beta T)^{\vee})(0).$$

Basing on this corollary we shall show

PROPOSITION 3. *If the multiplicative products $\frac{\partial S}{\partial y_j}\bigcirc T$, $j=1, 2, \cdots, n$, and $S\bigcirc\frac{\partial T}{\partial x_i}$, $i=1, 2, \cdots, m$, exist, then $S\bigcirc T$, $\frac{\partial S}{\partial x_i}\bigcirc T$ and $S\bigcirc\frac{\partial T}{\partial y_j}$, $i=1, 2, \cdots, m$, $j=1, 2, \cdots, n$, also exist and*

$$\frac{\partial}{\partial x_i}(S\bigcirc T)=\frac{\partial S}{\partial x_i}\bigcirc T+S\bigcirc\frac{\partial T}{\partial x_i},\ \frac{\partial}{\partial y_j}(S\bigcirc T)=\frac{\partial S}{\partial y_j}\bigcirc T+S\bigcirc\frac{\partial T}{\partial y_j}.$$

PROOF. We can write

$$\frac{\partial}{\partial x_i}(\alpha(x)S*(\beta(y)T)^{\vee})=-\left(\alpha(x)S*\left(\beta(y)\frac{\partial T}{\partial x_i}\right)^{\vee}\right),$$

and

$$\frac{\partial}{\partial y_j}(\alpha(x)S*(\beta(y)T)^{\vee})=\alpha(x)\frac{\partial S}{\partial y_j}*(\beta(y)T)^{\vee}.$$

Owing to the hypotheses it follows from Lemma 3 that $(\alpha(x)S*(\beta(y)T)^{\vee})(0)$ exists, and so does $S\bigcirc T$ from the preceding corollary. We have for any $\phi\in\mathcal{D}(\Omega)$

$$\frac{\partial S}{\partial x_i}*(\phi T)^{\vee}=-S*\left(\frac{\partial\phi}{\partial x_i}T\right)^{\vee}-S*\left(\phi\frac{\partial T}{\partial x_i}\right)^{\vee},\qquad i=1, 2, \cdots, m.$$

Each term of the right side has the value at 0, so $\frac{\partial S}{\partial x_i}\bigcirc T$ exists for each i. In addition, it follows from the above equations that

$$<\frac{\partial S}{\partial x_i}\bigcirc T, \phi> = -<S\bigcirc T, \frac{\partial \phi}{\partial x_i}> - <S\bigcirc \frac{\partial T}{\partial x_i}, \phi>$$

$$= <\frac{\partial}{\partial x_i}(S\bigcirc T), \phi> - <S\bigcirc \frac{\partial T}{\partial x_i}, \phi>.$$

Consequently we obtain

$$\frac{\partial}{\partial x_i}(S\bigcirc T) = \frac{\partial S}{\partial x_i}\bigcirc T + S\bigcirc \frac{\partial T}{\partial x_i}, \qquad i=1, 2, \cdots, m.$$

Similarly we can show the existence of $S\bigcirc \frac{\partial T}{\partial x_j}$, $j=1, 2, \cdots, n$, and the relations

$$\frac{\partial}{\partial y_j}(S\bigcirc T) = \frac{\partial S}{\partial y_j}\bigcirc T + S\bigcirc \frac{\partial T}{\partial y_j},$$

completing the proof.

Remark 7. By making use of Remark 4 we can show that if $\Delta S\bigcirc T$ exists, then $S\bigcirc T$, $\frac{\partial S}{\partial x_j}\bigcirc T$, $S\bigcirc \frac{\partial T}{\partial x_j}$ and $S\bigcirc \Delta T$ exist for $j=1, 2, \cdots, N$ and

$$\frac{\partial}{\partial x_j}(S\bigcirc T) = \frac{\partial S}{\partial x_j}\bigcirc T + S\bigcirc \frac{\partial T}{\partial x_j},$$

$$\Delta(S\bigcirc T) = -\Delta S\bigcirc T + 2\sum_j \frac{\partial}{\partial x_j}\left(\frac{\partial S}{\partial x_j}\bigcirc T\right) + S\bigcirc \Delta T$$

$$= \Delta S\bigcirc T + 2\sum_j \frac{\partial}{\partial x_j}\left(S\bigcirc \frac{\partial T}{\partial x_j}\right) - S\bigcirc \Delta T.$$

Remark 8. Let $S = D^p S_1 \in \mathcal{D}'(\Omega)$ and $T = D^q T_1 \in \mathcal{D}'(\Omega)$ be such that $D^{q'} S_1 \bigcirc D^{p'} T_1$ exists for every $q' \leqq q$, $p' \leqq p$. Then $S\bigcirc T$ exists and

$$S\bigcirc T = \sum_{p'\leqq p}\sum_{q'\leqq q}(-1)^{|p'+q'|}\binom{p}{p'}\binom{q}{q'}D^{p+q-p'-q'}(D^{q'}S_1\bigcirc D^{p'}T_1).$$

In fact, we can write for any $\alpha \in \mathcal{D}(\Omega)$

$$\alpha S * \check{T} = \alpha D^p S_1 * (D^q T_1)^{\vee}$$

$$= \sum_{p'\leqq p}(-1)^{|p-p'|}\binom{p}{p'}D^{p'}((D^{p-p'}\alpha)S_1 * (D^q T_1)^{\vee})$$

$$= \sum_{p'\leqq p}(-1)^{|p+q|}\binom{p}{p'}(D^q((D^{p-p'}\alpha)S_1) * (D^{p'}T_1)^{\vee})$$

$$= \sum_{p'\leqq p}\sum_{q'\leqq q}(-1)^{|p+q|}\binom{p}{p'}\binom{q}{q'}((D^{p-p'+q-q'}\alpha)(D^{q'}S_1) * (D^{p'}T_1)^{\vee}).$$

Consequently our assertion will be clear. We can also show that if $D^{q'}S_1\bigcirc D^{p'}T_1$ exists for every $q'\leqq q+(1, 0, \cdots, 0)$, $p'\leqq p$, then $S\bigcirc T$, $S\bigcirc\frac{\partial T}{\partial x_1}$ and $\frac{\partial S}{\partial x_1}\bigcirc T$ exist and $\frac{\partial}{\partial x_1}(S\bigcirc T)=\frac{\partial S}{\partial x_1}\bigcirc T+S\bigcirc\frac{\partial T}{\partial x_1}$. The results also hold true of the multiplication discussed in [18]. This gives an extended version of Mikusiński's theorems ([13], pp. 257–258).

We shall introduce the notion of partial multiplication as follows:

DEFINITION IV. *Let $S(x)\in\mathcal{D}'(R_x^m)$ and $T(x, y)\in\mathcal{D}'(R^N)$. If $(S(x)\otimes 1_y)\bigcirc T(x, y)$ exists in $\mathcal{D}'(R^N)$, then it is called the multiplicative product of S and T and denoted by $S\bigcirc T$.*

PROPOSITION 4. *Let $S(x)\in\mathcal{D}'(R_x^m)$ and $T(x, y)\in\mathcal{D}'(R^N)$. A necessary and sufficient condition for the existence of $S\bigcirc T$ is that $S(x)\bigcirc<T(x, y), \psi(y)>_y$ exists in $\mathcal{D}'(R_x^m)$ for every $\psi(y)\in\mathcal{D}(R_y^n)$. In this case $<S\bigcirc T, \psi>_y=S(x)\bigcirc<T(x, y), \psi(y)>_y$.*

PROOF. Let $\phi\in\mathcal{D}(R^N)$ be such that $\phi(x, y)\geqq 0$ and $\iint\phi(x, y)dxdy=1$. If we put $\psi(x)=\int\phi(x, y)dy$, then $\psi\in\mathcal{D}(R_x^m)$, $\psi\geqq 0$ and $\int\psi(x)\,dx=1$. Clearly $\psi_\lambda(x)=\int\phi_\lambda(x, y)dy$, λ being positive. Conversely, given $\psi(x)\in\mathcal{D}(R_x^m)$ such that $\psi(x)\geqq 0$ and $\int\psi(x)dx=1$, we can also choose $\phi(x, y)\in\mathcal{D}(R^N)$ satisfying the above conditions. Then we can assert the proposition from the relation

$$T\big((S(x)\otimes 1_y)*\phi_\lambda\big)=T(S*\psi_\lambda).$$

Thus the proof is completed.

From these consideration it is clear that Proposition 4 is also true of the partial multiplication ST which is defined as $(S(x)\otimes 1_y)T(x, y)$.

Now we shall turn to the consideration of simultaneous multiplication of more than two distributions. First we show

PROPOSITION 5. *Let $S, T\in\mathcal{D}'(\Omega)$. In order that the multiplicative product $S\bigcirc T$ exists, it is necessary and sufficient that for any $\phi, \psi\in\mathcal{D}(R^N)$ such that $\phi\geqq 0$, $\psi\geqq 0$, $\int\phi(x)dx=1$ and $\int\psi(x)\,dx=1$, the distributional limit $\lim\limits_{\lambda,\lambda'\to+0}(S*\phi_\lambda)(T*\psi_{\lambda'})$ exists and does not depend on the choice of ϕ, ψ. If this is the case,*

$$S\bigcirc T=\lim_{\lambda,\lambda'\to+0}(S*\phi_\lambda)(T*\psi_{\lambda'}).$$

PROOF. The sufficiency is obvious in view of Definition II. To prove the necessity, it suffices to prove that $\lim\limits_{\lambda,\lambda'\to+0}(\alpha S*\phi_\lambda)(\beta T*\psi_{\lambda'})$ exists for every $\alpha, \beta\in\mathcal{D}(\Omega)$, that is, $<(\alpha S*\phi_\lambda)(\beta T*\psi_{\lambda'}), \chi>$ converges for any $\chi\in\mathcal{D}(R^N)$ as λ,

$\lambda' \to +0$. In order prove this, we may assume χ to be extended to a periodic function with period $2l$ relative to each coordinate. Consider the Fourier expansion of χ:

$$\chi = \sum_m c_m e^{i\frac{\pi}{l}<m,x>} = \sum_m c_m e(m),$$

where $\sum_m |c_m|(1+|m|)^k < \infty$ for every positive integer k. Now, we can write for small λ, $\lambda' > 0$

$$<(\alpha S * \phi_\lambda)(\beta T * \psi_{\lambda'}), \chi> = \sum_m c_m <(\alpha S * \phi_\lambda)(\beta T * \psi_{\lambda'}), e(m)>$$
$$= \sum_m c_m <(e(m)\alpha S) * (\beta T)^\vee, e(-m)\check{\phi}_\lambda * \psi_{\lambda'}>.$$

For the sake of simplicity, we assume that $\operatorname{supp}\phi$, $\operatorname{supp}\psi \subset P_1$. We may also assume that $\lambda \leqq \lambda'$. Since $\operatorname{supp}(e(-m)\check{\phi}_\lambda * \psi_{\lambda'}) \subset P_{2\lambda'}$ and $\lambda'^{N+|p|} D_x^p(e(-m)\check{\phi}_\lambda * \psi_{\lambda'})$ is bounded because of the equality

$$D_x^p(e(-m)\check{\phi}_\lambda * \psi_{\lambda'}) = \int e^{-i\frac{\pi}{l}<m,y>} \check{\phi}_\lambda(y) \frac{1}{\lambda'^{N+|p|}} \psi^{(p)}\left(\frac{x-y}{\lambda'}\right) dy,$$

it follows from Lemma 2 that $<(e(m)\alpha S) * (\beta T)^\vee, e(-m)\check{\phi}_\lambda * \psi_{\lambda'}>$ converges to $((e(m)\alpha S) * (\beta T)^\vee)(0)$ as $\lambda' \to +0$.

On the other hand, if we consider the mapping

$$\mathcal{D}_{\operatorname{supp}\alpha} \times \mathcal{D}_{\overline{P_{2\lambda'}}} \ni (f, g) \to <(fS) * (\beta T)^\vee, g>,$$

then we can find a multi-index p and a constant M independent of λ' such that

$$|<(fS) * (\beta T)^\vee, g>| \leqq M \sup|D^p f| \lambda'^{N+|p|} \sup|D^p g|$$

remains true. Then we obtain with a new constant M' and a multi-index p'

$$|<(e(m)\alpha S) * (\beta T)^\vee, e(-m)\check{\phi}_\lambda * \psi_{\lambda'}>| \leqq M'(1+|m|)^{|p'|}.$$

Consequently

$$\lim_{\lambda,\lambda' \to +0} <(\alpha S * \phi_\lambda)(\beta T * \psi_{\lambda'}), \chi> = \sum_m c_m \lim_{\lambda,\lambda' \to +0} <(e(m)\alpha S) * (\beta T)^\vee, e(-m)\check{\phi}_\lambda * \psi_{\lambda'}>$$
$$= \sum_m c_m((e(m)\alpha S) * (\beta T)^\vee)(0)$$
$$= ((\chi\alpha S) * (\beta T)^\vee)(0)$$
$$= <(\alpha S) \bigcirc (\beta T), \chi>,$$

where we are justified to interchange the order of $\sum_m$ and $\lim_{\lambda,\lambda' \to +0}$ since $\sum_m |c_m|(1+|m|)^{|p'|} < \infty$ as already remarked.

Thus the proof is completed.

Definition V. *Let S, T, $W \in \mathcal{D}'(\Omega)$. If, for any ϕ, ψ and $\chi \in \mathcal{D}(R^N)$ such* *that* $\phi \geqq 0$, $\psi \geqq 0$, $\chi \geqq 0$ *and* $\int \phi(x)dx = \int \psi(x)dx = \int \chi(x)dx = 1$, *the distributional limit*

$$\lim_{\lambda, \lambda', \lambda'' \to +0} (S*\phi_\lambda)(T*\psi_{\lambda'})(W*\chi_{\lambda''})$$

exists and does not depend on the choice of ϕ, ψ *and* χ, *then the limit will be called as the multiplicative product of* S, T *and* W, *and denoted by* $S \bigcirc T \bigcirc W$.

Now we can show

PROPOSITION 6. *If* $T \bigcirc W$ *and* $S \bigcirc T \bigcirc W$ *exist, then* $S \bigcirc (T \bigcirc W)$ *exists and is equal to* $S \bigcirc T \bigcirc W$.

PROOF. By Definition II, Proposition 5 and Definition V we have

$$\begin{aligned} S \bigcirc T \bigcirc W &= \lim_{\lambda, \lambda' \lambda'' \to +0} (S*\phi_\lambda)(T*\psi_{\lambda'})(W*\chi_{\lambda''}) \\ &= \lim_{\lambda \to +0} (S*\phi_\lambda)(T \bigcirc W) \\ &= S \bigcirc (T \bigcirc W), \end{aligned}$$

which completes the proof.

A locally convex space $\mathcal{H} \subset \mathcal{D}'(R^N)$ with topology finer than $\mathcal{D}'(R^N)$ is called a space of distributions. In addition, if $\mathcal{D}(R^N)$ is contained in $\mathcal{H}$ with a finer topology and dense in $\mathcal{H}$, $\mathcal{H}$ is called to be normal. Let $\mathcal{H}$, $\mathcal{L}$ be spaces of distributions. We assume that $\mathcal{H}$ is normal. $S \in \mathcal{D}'(R^N)$ is called a multiplicator of $\mathcal{H}$ into $\mathcal{L}$ if there exists a continuous linear mapping $<S>$ of $\mathcal{H}$ into $\mathcal{L}$ such that $<S>\alpha = \alpha S$ for every $\alpha \in \mathcal{D}(R^N)$. When $\mathcal{H} = \mathcal{L}$, we shall say that S is a multiplicator of $\mathcal{H}$.

PROPOSITION 7. *Let* $\mathcal{H}$ *be a normal barrelled space of distribution. Given* S, *if* $S \bigcirc T$ *exists for every* $T \in \mathcal{H}$ *then* S *is a multiplicator of* $\mathcal{H}$ *into* $\mathcal{D}'(R^N)$ *and* $<S>T = S \bigcirc T$ *for every* $T \in \mathcal{H}$ *and* $\alpha S \in \mathcal{H}'$ *for every* $\alpha \in \mathcal{D}(R^N)$.

In addition, if $S \bigcirc \mathcal{H} \subset \mathcal{L}$ *and* $\mathcal{D}(R^N)$ *is strictly dense in* $\mathcal{L}'_c$, *where* $\mathcal{L}$ *is a space of distributions, then* S *is a multiplicator of* $\mathcal{H}$ *into* $\mathcal{L}$.

PROOF. Let $\phi \in \mathcal{D}(R^N)$ be such that $\phi \geqq 0$ and $\int \phi(x)dx = 1$. Then by Definition II

$$S \bigcirc T = \lim_{\lambda \to +0} S(T*\phi_\lambda).$$

Since the mapping $\mathcal{H} \ni T \to S(T*\phi_\lambda) \in \mathcal{D}'(R^N)$ is continuous and $\mathcal{H}$ is barrelled, it follows from the Banach-Steinhaus theorem that $<S>$: $\mathcal{H} \ni T \to S \bigcirc T \in \mathcal{D}'(R^N)$ is continuous. Then $<S>\alpha = \alpha S$ for every $\alpha \in \mathcal{D}(R^N)$. This shows that S is a multiplicator of $\mathcal{H}$ into $\mathcal{D}'(R^N)$. Therefore there exists for any $\alpha \in \mathcal{D}(R^N)$ a unique $W_\alpha \in \mathcal{H}'$ such that $<S \bigcirc T, \alpha> = <T, W_\alpha>_{\mathcal{H}, \mathcal{H}'}$. Let

T be taken arbitrarily from $\mathcal{D}(R^N)$. Then we can conclude that $\alpha S = W_\alpha \in \mathcal{H}'$.

Now assume that $\mathcal{L}$ has the properties stated in the last part of the proposition. If u is a continuous linear mapping from a barrelled space E into $\mathcal{D}'(R^N)$ with range in $\mathcal{L}$, then u must be continuous from E into $\mathcal{L}$ ([17], p. 176). Accordingly S will be a multiplicator of $\mathcal{H}$ into $\mathcal{L}$. Thus the proof is completed.

Remark 9. Let $\mathcal{H}$ be a normal space of distributions. Assume that $\mathcal{H}$ has the approximation properties by regularization and truncation ([16], p. 7). It was shown ([18], p. 232) that if S is a multiplicator of $\mathcal{H}$ into $\mathcal{D}'(R^N)$, then ST exists for every $T \in \mathcal{H}$, and $\langle S \rangle T = ST$. Furthermore we assume that $\mathcal{H}$ is barrelled. Let $S \bigcirc T$ exists for every $T \in \mathcal{H}$. Then by Proposition 7 $T \to S \bigcirc T$ is a multiplicator of $\mathcal{H}$ into $\mathcal{D}'(R^N)$ so that ST exists for every T and $S \bigcirc T = ST$. The result is not true if the approximation properties are not satisfied. Consider the example given in Remark 3 in [18], where $\mathcal{H}$, $\mathcal{K}$ were defined by

$$\mathcal{H} = \left\{ f\,;\ \|f\|_{\mathcal{H}}^2 = \int \frac{|f(x)|^2}{|x|}\, dx < \infty \right\};$$

$$\mathcal{K} = \left\{ g;\ \|g\|_{\mathcal{K}}^2 = \int |g(x)|^2 |x|\, dx < \infty \right\}; \qquad (N \geqq 2).$$

It was shown there that the ordinary product fg, $f \in \mathcal{H}$, $g \in \mathcal{K}$, is always summable while for some f, g their multiplicative product in the sense of [4] does not exist. $\mathcal{H}$ and $\mathcal{K}$ are normal barrelled spaces with the approximation property by truncation, not by reguralization. Now we show that $f \bigcirc g$ always exists and is equal to fg. Let V_ε be the volume of the ball with center 0 and radius $\varepsilon > 0$. Put $M_\varepsilon(g) = \frac{1}{V_\varepsilon}\int_{|t| \leqq \varepsilon} g(x-t)\,dt$ for any $g \in \mathcal{K}$. Noting that $\frac{|x|}{V_\varepsilon}\int_{|t| \leqq \varepsilon} \frac{dt}{|x-t|}$ is bounded, we obtain with a constant $C > 0$

$$\int |M_\varepsilon(g)|^2 |x|\, dx \leqq \frac{1}{V_\varepsilon^2}\int |x|\, dx \Big(\int_{|t| \leqq \varepsilon} |x-t|\, |g(x-t)|^2 dt\Big)\Big(\int_{|t| \leqq \varepsilon} \frac{1}{|x-t|} dt\Big)$$

$$\leqq C^2 \|g\|_{\mathcal{K}}^2.$$

Consequently we have for any $f \in \mathcal{H}$ and $g \in \mathcal{K}$

$$\frac{1}{V_\varepsilon}\int_{|t| \leqq \varepsilon} |(f * \check{g})(t)|\, dt \leqq \int |f(x)|\, M_\varepsilon(|g|)(x)\, dx \leqq C \|f\|_{\mathcal{H}} \|g\|_{\mathcal{K}}.$$

This implies that

$$\frac{1}{V_\varepsilon}\int_{|t| \leqq \varepsilon} \Big|(f * \check{g})(t) - \int f(x) g(x)\, dx\Big|\, dt \to 0$$

as $\varepsilon \to 0$, that is, $f * \check{g}$ has the value $\int f(x) g(x)\, dx$ at 0 in the sense of Łojasiewicz

[12]. Since every $\alpha \in \mathcal{D}(R^N)$ is a multiplicator of $\mathcal{H}$, it follows therefore that $f \bigcirc g$ exists for every $f \in \mathcal{H}$, $g \in \mathcal{K}$ and is equal to fg.

§ 5. An extension of the multiplicative product $S \bigcirc T$ in the case $N=1$

Hereafter we shall assume that $N=1$. The foregoing discussions about the multiplication between distributions can be extended preserving normality through an extension of the notion of the value of a distribution at a point. For this end, recall the notion of the right and left hand limits of a distribution at a point.

Let S be a distribution defined in a 0-neighbourhood. S has a right hand limit c_+ at 0 if the distributional limit $\lim_{\lambda \to +0} S(\lambda x)$ exists in the positive axis $x>0$ and is a constant function c_+([11], p. 3). We write $\lim_{x \to +0} S = c_+$. The condition may be written ([11], p. 5):

$$S = c_+ Y + D^p F^+,$$

where Y is the Heaviside function and F^+ is a continuous function in an open interval $(0, a)$ such that $F^+ = o(|x|^p)$ as $x \to 0$.

Similarly we can define the left hand limit.

We shall say that S has no mass at 0 if $\lim_{\lambda \to +0} \lambda S(\lambda x) = 0$ ([12], p. 23).

If $\lim_{x \to +0} S = c_+$, $\lim_{x \to -0} S = c_-$ and moreover S has no mass at $x=0$, then we write

$$S[0] = \frac{c_+ + c_-}{2},$$

which will be referred to as the extended value of S at 0.

A necessary and sufficient condition for the existence of $S[0]$ is that there exists a non-negative integer p, a 0-neighbourhood U and a continuous function $F(x)$ in U, for which

$$S = c_+ Y + c_- \check{Y} + D^p F$$

in U, where $F(x) = o(|x|^p)$ as $|x| \to 0$.

Let Ω be a non-empty open subset of R and $S, T \in \mathcal{D}'(\Omega)$. When $S * \check{T}$ is defined in a neighbourhood of 0 and has the extended value $(S * \check{T})[0]$, we shall define the extended scalar product $[S, T]$ of S and T by the formula

$$[S, T] = (S * \check{T})[0].$$

If the extended scalar product $[\alpha S, T]$ exists for any $\alpha \in \mathcal{D}(\Omega)$, then for any $\phi^+, \phi^- \in \mathcal{D}(R)$ such that $\phi^+ \geqq 0$, $\phi^- \geqq 0$, $\operatorname{supp} \phi^+ \subset (0, \infty)$, $\operatorname{supp} \phi^- \subset (-\infty, 0)$

and $\int \phi^+(x)dx = \int \phi^-(x)dx = 1$, $<\alpha S * \check{T}, \frac{1}{\lambda}\phi^+\left(\frac{x}{\lambda}\right)>$ and $<\alpha S * \check{T}, \frac{1}{\lambda}\phi^-\left(\frac{x}{\lambda}\right)>$ are defined for small $\lambda > 0$ and

$$(\alpha S * \check{T})[0] = \frac{1}{2}\left(\lim_{\lambda \to +0} <\alpha S * \check{T}, \frac{1}{\lambda}\phi^+\left(\frac{x}{\lambda}\right)> + \lim_{\lambda \to +0} <\alpha S * \check{T}, \frac{1}{\lambda}\phi^-\left(\frac{x}{\lambda}\right)>\right).$$

By virtue of the Banach-Steinhaus theorem we see that there exists a unique distribution $W \in \mathcal{D}'(\Omega)$ such that

$$< W, \alpha > = [\alpha S, T], \qquad \alpha \in \mathcal{D}(\Omega).$$

DEFINITION VI. *Let $S, T \in \mathcal{D}'(\Omega)$. If $[\alpha S, T]$ exists for every $\alpha \in \mathcal{D}(\Omega)$, then $W \in \mathcal{D}'(\Omega)$ defined by the formula*

$$< W, \alpha > = [\alpha S, T] = (\alpha S * \check{T})[0], \qquad \alpha \in \mathcal{D}(\Omega),$$

will be called the multiplicative product of S and T and denoted by $S \times_\circ T$.

From the definition we see that if $S \bigcirc T$ exists then $S \times_\circ T$ exists and coincides with $S \bigcirc T$. The converse is not true: $Y \times_\circ \delta = \frac{1}{2}\delta$ but $Y \bigcirc \delta$ does not exist.

In what follows we shall show that the multiplication just defined is normal.

I_1 and I_3 are obvious. I_4, III_1 and III_2 may be verified by the same way as in Section 3.

As for I_2, assume that $S \times_\circ T = W$ exists. By the definition of multiplication, given a non-empty open subset $\varDelta \subset\subset \Omega$, there exists a 0-neighbourhood $U \subset R_y$, for which we can write in $\varDelta \times U$

$$S(x) \otimes T(x-y) = W_+(x) \otimes Y(y) + W_-(x) \otimes \check{Y}(y) + V(x, y),$$

where V is a distribution in $\varDelta \times U$ with 0 as the section for $y=0$ and $W = \frac{W_+ + W_-}{2}$. After the change of variables, $x = x' - y'$ and $y = -y'$, we can write

$$S(x'-y') \otimes T(x') = W_+(x'-y') \otimes \check{Y}(y') + W_-(x'-y') \otimes Y(y') + V(x'-y', -y').$$

It is easy to verify that $V(x'-y', -y')$ has the section 0 for $y'=0$. Consequently we only have to show that both

$$W_+(x'-y') \otimes \check{Y}(y') - W_+(x') \otimes \check{Y}(y'),$$

and

$$W_-(x'-y') \otimes Y(y') - W_-(x') \otimes Y(y')$$

have the section 0 for $y'=0$.

Let $\phi \in \mathcal{D}(\varDelta \times U)$. Then we have with small $\lambda > 0$

$$< W_+(x-\lambda y)\otimes \check{Y}(\lambda y)-W_+(x)\otimes \check{Y}(\lambda y),\ \phi(x,\ y)>$$

$$=\iint W_+(x-\lambda y)\check{Y}(y)\phi(x,\ y)dxdy-\iint W_+(x)\check{Y}(y)\phi(x,\ y)dxdy$$

$$=\iint W_+(x)\check{Y}(y)(\phi(x+\lambda y,\ y)-\phi(x,\ y))dxdy,$$

which yields

$$\lim_{\lambda\to+0} < W_+(x-\lambda y)\otimes \check{Y}(\lambda y)-W_+(x)\otimes \check{Y}(\lambda y),\ \phi(x,\ y)> =0.$$

Thus $W_+(x-y)\otimes \check{Y}(y)-W_+(x)\otimes \check{Y}(y)$ has the section 0 for $y=0$. Similarly the same is true of $W_-(x-y)\otimes Y(y)-W_-(x)\otimes Y(y)$. Thus the proof is completed.

As for II, from the proof of the corresponding case of Section 3 it is sufficient to note that if $\frac{dS}{dx}[0]$ exists, so does $S[0]$. If we put $T=S-\frac{c_+-c_-}{2}(x_++x_-)$ with $c_+=\lim_{x\to+0}\frac{dS}{dx}$ and $c_-=\lim_{x\to-0}\frac{dS}{dx}$, then $\frac{dT}{dx}=\frac{dS}{dx}-\frac{c_+-c_-}{2}(Y-\check{Y})$ and therefore $\frac{dT}{dx}(0)$ exists. It follows from Lemma 3 that the value $T(0)$ exists and a fortiori $S(0)$.

Finally we shall show that IV is fulfilled. We shall continue to use the same notation as in the proof of I_2. Assume that $S\times_0 T=W$ exists. Then we can write as before

$$S(x)\otimes T(x-y)=W_+(x)\otimes Y(y)+W_-(x)\otimes \check{Y}(y)+V(x,\ y).$$

The distribution $\tilde{S}(x')\otimes\tilde{T}(x'-y')=S(\varPhi(x'))\otimes T(\varPhi(x'-y'))$ is obtained from $S(x)\otimes T(x-y)$ after the change of variables:

$$x=\varPhi(x'),\quad y=\varPhi(x')-\varPhi(x'-y').$$

Consequently we have

$$\begin{aligned}S(x')\otimes T(x'-y')=&W_+(\varPhi(x'))\otimes Y(\varPhi(x')-\varPhi(x'-y'))\\&+W_-(\varPhi(x'))\otimes \check{Y}(\varPhi(x')-\varPhi(x'-y'))+\tilde{V}(x',\ y'),\end{aligned}$$

where $\tilde{V}(x',0)=0$ as seen from the proof of IV given in Section 3.

On the other hand, we can write

$$W_+(\varPhi(x'))\otimes Y(\varPhi(x')-\varPhi(x'-y'))+W_-(\varPhi(x'))\otimes \check{Y}(\varPhi(x')-\varPhi(x'-y'))$$

$$=\begin{cases}W_+(\varPhi(x'))\otimes Y(y')+W_-(\varPhi(x'))\otimes \check{Y}(y') & \text{for } \varPhi'>0,\\ W_+(\varPhi(x'))\otimes \check{Y}(y')+W_-(\varPhi(x'))\otimes Y(y') & \text{for } \varPhi'<0.\end{cases}$$

This shows that $\tilde{S}\times_0\tilde{T}$ exists and $\tilde{S}\times_0\tilde{T}=\tilde{W}$, completing the proof.

Thus we have shown

THEOREM 2. *The multiplication given by Definition* VI *is normal.*

We shall consider the multiplicative product in the sense of Tillmann [19]. Let $S, T \in \mathcal{D}'(R)$. We have shown in [6] (p. 71) that if $(\alpha S*\check{T})\bigcirc\delta = c_\alpha\delta$ exists for each $\alpha \in \mathcal{D}(R)$, where c_α is a constant depending on α, then $S\bigcirc T$ exists and $<S\bigcirc T, \alpha> = c_\alpha$.

LEMMA 8. *If $S\times_0\delta$ exists, so does $S[0]$. Conversely, if $S[0]$ exists, then both $S\times_0\delta$ and $S\bigcirc\delta$ exist and are equal to $S[0]\delta$.*

PROOF. From the identity $S*(\alpha\delta)^\vee = \alpha(0)S$, $\alpha \in \mathcal{D}(R)$, we see that the first part of the lemma is clear, and that if $S[0]$ exists, $S\times_0\delta$ exists and is equal to $S[0]\delta$. On the other hand, we can write in a 0-neighbourhood

$$S = c_+Y + c_-\check{Y} + D^pF, \quad F(x) = o(|x|^p),$$

where $(D^pF)\bigcirc\delta = 0$ since $(D^pF)(0) = 0$ and $Y\bigcirc\delta = \check{Y}\bigcirc\delta = \frac{1}{2}\delta$ ([6], p. 66, p. 69). Consequently $S\bigcirc\delta = \frac{c_+ + c_-}{2}\delta = S[0]\delta$, completing the proof.

By aid of this lemma we shall show

THEOREM 3. *Let $S, T \in \mathcal{D}'(R)$. If $S\times_0 T$ exists, then $S\bigcirc T$ exists and is equal to $S\times_0 T$.*

PROOF. Let $S\times_0 T$ exist. That is, $(\alpha S*\check{T})[0]$ exists for every $\alpha \in \mathcal{D}(R)$. Hence it follows that $(\alpha S*\check{T})\bigcirc\delta$ exists and is equal to $<S\times_0 T, \alpha>\delta$. Consequently $S\bigcirc T$ exists and is equal to $S\times_0 T$.

Note that the converse of the theorem is not true: $\mathrm{Pf}\frac{1}{x}\bigcirc\delta = -\frac{1}{2}\delta'$ ([2], p. 251) but $\mathrm{Pf}\frac{1}{x}\times_0\delta$ does not exist.

§ 6. Further extension of the multiplicative product in the case $N=1$

This section is devoted to a further extension of the preceding discussion so that the multiplicative product of δ by δ makes sense.

In the definition of $S[0]$, we drop the condition that S has no mass at 0. We shall denote the generalized value thus defined by $S\{0\}$ instead of $S[0]$. For example $\delta^{(j)}\{0\} = 0$ for $j = 0, 1, 2, \cdots$. $S\{0\}$ exists if and only if we can write S in the form

$$S = c_+Y + c_-\check{Y} + D^pF + a_0\delta + a_1\delta' + \cdots + a_m\delta^{(m)}$$

with constants $c_+, c_-, a_0, \cdots, a_m$ in a 0-neighbourhood U, where F is a continuous function in U such that $F=o(|x|^p)$ as $|x|\to 0$.

Let Ω be any non-empty open subset of R and $S,\ T\in\mathcal{D}'(\Omega)$. When $S*\check{T}$ is defined in a 0-neighbourhood and $(S*\check{T})\{0\}$ exists, we shall define the generalized scalar product $\{S,\ T\}$ of S and T by the formula

$$\{S,\ T\}=(S*\check{T})\{0\}.$$

If $\{\alpha S,\ T\}$ exists for every $\alpha\in\mathcal{D}(\Omega)$, then we can prove, as in Section 5, that the linear form $\alpha\to\{\alpha S,\ T\}$ is continuous on $\mathcal{D}(\Omega)$. Thus there exists a unique distribution $W\in\mathcal{D}'(\Omega)$ such that

$$<W,\alpha>=\{\alpha S,\ T\},\qquad \alpha\in\mathcal{D}(\Omega).$$

DEFINITION VII. *Let $S,\ T\in\mathcal{D}'(\Omega)$. If $\{\alpha S,\ T\}$ exists for every $\alpha\in\mathcal{D}(\Omega)$, then $W\in\mathcal{D}'(\Omega)$ defined by the formula*

$$<W,\alpha>=\{\alpha S,\ T\}=(\alpha S*\check{T})\{0\},\qquad \alpha\in\mathcal{D}(\Omega)$$

will be called the multiplicative product of S and T and denoted by $S\times_1 T$.

From the definition we see that if $S\times_0 T$ exists, then $S\times_1 T$ also exists and is equal to $S\times_0 T$. The converse is not true: $\delta^{(j)}\times_1\delta^{(k)}=0$ for any non-negative integers j, k but $\delta^{(j)}\times_0\delta^{(k)}$ does not exist.

THEOREM 4. *The multiplication given by Definition VII is normal.*

The proof is omitted since it may be carried out with necessary modifications along the same line as in the proof of Theorem 3.

We shall denote by $S\cdot T$ the multiplicative product of the general sense of Tillmann ([6], p. 56, [19], p. 108).

THEOREM 5. *Let $S,\ T\in\mathcal{D}'(R)$. If $S\times_1 T$ exists, then $S\cdot T$ exists and is equal to $S\times_1 T$.*

PROOF. Let K be a compact subset of R and $\alpha\in\mathcal{D}(R)$ be chosen equal to 1 in a neighbourhood of K so that $\alpha S*(\phi T)^{\vee}$ may coincide with $S*(\phi T)^{\vee}$ in a 0-neighbourhood for every $\phi\in\mathcal{D}_K$. We use the notations and the results of [6]. Putting $S_1=\alpha S$, $S_2=(1-\alpha)S$, $T_1=\alpha T$ and $T_2=(1-\alpha)T$, we can write $\hat{S}_\varepsilon=(\hat{S}_1)_\varepsilon+(\hat{S}_2)_\varepsilon$, $\hat{T}_\varepsilon=(\hat{T}_1)_\varepsilon+(\hat{T}_2)_\varepsilon$ ([6], p. 61). Both $\hat{S}_2(z)$ and $\hat{T}_2(z)$ are analytic in $C\setminus(R\setminus K)$, where C is a complex plane and each of $(\hat{S}_1)_\varepsilon(\hat{T}_2)_\varepsilon$, $(\hat{S}_2)_\varepsilon(\hat{T}_1)_\varepsilon$ and $(\hat{S}_2)_\varepsilon(\hat{T}_2)_\varepsilon$ tends to 0 in $\mathcal{D}'_K$ as $\varepsilon\downarrow 0$. Put $h_\varepsilon=\dfrac{1}{\pi}\dfrac{\varepsilon}{x^2+\varepsilon^2}$. $S\cdot T$ was defined as $\mathrm{Pf}\,\hat{S}_\varepsilon\hat{T}_\varepsilon$. We note that

$$\mathrm{Pf}<\hat{S}_\varepsilon\hat{T}_\varepsilon,\ \phi>=\mathrm{Pf}<(\alpha S*h_\varepsilon)(\alpha T*h_\varepsilon),\ \phi>,\qquad \phi\in\mathcal{D}_K$$

so far as either side of the equation makes sense. Here we have used the symbol Pf to denote the finite part of the limit in the sense of Hadamard. Let $\beta\in\mathcal{D}(R)$ be chosen equal to 1 in a 0-neighbourhood. We assume that $\beta=\check{\beta}$. Then we can write the above equation as follows:

$$\mathrm{Pf}<\hat{S}_\varepsilon \hat{T}_\varepsilon, \phi> = \mathrm{Pf}<(\alpha S * h_\varepsilon)(\alpha T * \beta h_\varepsilon), \phi>.$$

Since $\alpha T * \beta h_\varepsilon$ is of compact support we may assume that ϕ is a periodic function with period $2l$ relative to each coordinate with a sufficiently large l. Consider the Fourier expansion of ϕ:

$$\phi(x) = \sum_m c_m e^{i\frac{\pi}{l}<m, x>} = \sum_m c_m e(m),$$

where $\sum_m |c_m|(1+|m|)^k < \infty$ for every positive integer k. Then we have

$$\begin{aligned} <(\alpha S * h_\varepsilon)(\alpha T * \beta h_\varepsilon), \phi> &= \sum_m c_m <(\alpha S * h_\varepsilon)(\alpha T * \beta h_\varepsilon), e(m)> \\ &= \sum_m c_m <\alpha S * h_\varepsilon, e(m)\alpha T * e(m)\beta h_\varepsilon> \\ &= \sum_m c_m <(\alpha S * (e(m)\alpha T)^\vee * h_\varepsilon) h_\varepsilon, e(m)\beta>. \end{aligned}$$

By the hypothesis $S \times_1 T$ exists, we can therefore write for any $\gamma \in \mathcal{B}$

$$\begin{aligned} \alpha S * (\gamma \alpha T)^\vee &= c_+ Y + c_- \check{Y} + V + a_0 \delta + a_1 \delta' + \cdots + a_n \delta^{(n)} \\ &= H(\gamma) + \Delta(\gamma). \end{aligned}$$

Here V is a distribution such that $V(0)=0$ and c_+, c_-, a_0, a_1, $\cdots$, a_n are continuous linear forms of $\gamma \in \mathcal{B}$. $H(\gamma)$ and $\Delta(\gamma)$ denote $c_+ Y + c_- \check{Y} + V$ and $a_0 \delta + a_1 \delta' + \cdots + a_n \delta^{(n)}$ respectively. Using these symbols we have

$$\begin{aligned} <(\alpha S * h_\varepsilon)(\alpha T * \beta h_\varepsilon), \phi> &= \sum_m c_m <(H(e(m)) * h_\varepsilon) h_\varepsilon, e(m)\beta> \\ &\quad + \sum_m c_m <(\Delta(e(m)) * h_\varepsilon) h_\varepsilon, e(m)\beta>. \end{aligned}$$

From the definition of $\alpha S \times_1 T$ together with the fact that $\Delta(\gamma)\cdot\delta = 0$, we obtain

$$<\alpha S \times_1 T, \gamma\alpha> \delta = (\alpha S * (\gamma \alpha T)^\vee)\cdot\delta = H(\gamma)\cdot\delta.$$

Consequently we obtain for any $\chi \in \mathcal{B}$

$$<(\alpha S * (\gamma \alpha T)^\vee)\cdot\delta, \chi> = \lim_{\varepsilon \to +0} <H(\gamma), h_\varepsilon * \beta h_\varepsilon \chi>.$$

Now, for each $\varepsilon > 0$ the bilinear form u_ε: $(\gamma, \chi) \to <H(\gamma), h_\varepsilon * \beta h_\varepsilon \chi>$ on $\mathcal{B} \times \mathcal{B}$ is continuous. As $\mathcal{B}$ is of type (**F**), we can find an integer $k_0 \geqq 0$ and a positive constant M independent of ε such that

$$|<H(\gamma), h_\varepsilon * \beta h_\varepsilon \chi>| \leqq M \sup_{j \leqq k_0} |D^j \gamma| \sup_{j \leqq k_0} |D^j \chi|.$$

Hence we have with new constants k', M_1

$$|<(H(e(m)) * h_\varepsilon) h_\varepsilon, e(m)\beta>| \leqq M_1 (1+|m|)^{2k'}.$$

Because of the fact that $\sum_m |c_m|(1+|m|)^{2k'} < \infty$ as already remarked, the series $\sum_m |c_m| \sup_{0<\varepsilon<1} |<(H(e(m))*h_\varepsilon)h_\varepsilon,\ e(m)\beta>|$ converges, which we shall for simplicity say that $\sum_m c_m<(H(e(m))*h_\varepsilon)h_\varepsilon,\ e(m)\beta>$ converges normally. Since

$$\lim_{\varepsilon\to+0}<(H(e(m))*h_\varepsilon)h_\varepsilon,\ e(m)\beta> = <<\alpha S\times_1 T,\ e(m)\alpha>\delta,\ e(m)\beta>$$
$$= <\alpha S\times_1 T,\ e(m)\alpha>,$$

it follows therefore that

$$\lim_{\varepsilon\to+0}\sum_m c_m<(H(e(m))*h_\varepsilon)h_\varepsilon,\ e(m)\beta> = \sum_m c_m<\alpha S\times_1 T,\ e(m)\alpha>$$
$$= <\alpha S\times_1 T,\ \phi\alpha>$$
$$= <S\times_1 T,\ \phi>.$$

On the other hand, we have

$$\mathrm{Pf}\sum_m c_m<(\Delta(e(m))*h_\varepsilon)h_\varepsilon,\ e(m)\beta>$$
$$= \mathrm{Pf}\sum_m c_m<(a_0(e(m))h_\varepsilon + \cdots + a_n(e(m))h_\varepsilon^{(n)})h_\varepsilon,\ e(m)\beta>.$$

With the aid of the formulas established in [6] (p. 69):

$$<h_\varepsilon^{(2p)}h_\varepsilon,\ \phi> = (-1)^p\frac{\phi(0)}{\varepsilon^{2p+1}}\int_{-\infty}^{\infty}(h^{(p)})^2dx + \cdots + \frac{\phi^{(2p)}(0)}{(2p)!\,\varepsilon}\int_{-\infty}^{\infty}h^{(2p)}hx^{2p}dx + o(1)$$

and

$$<h_\varepsilon^{(2p-1)}h_\varepsilon,\ \phi> = (-1)^p\frac{(2p-1)\phi'(0)}{2\varepsilon^{2p-1}}\int_{-\infty}^{\infty}(h^{(p-1)})^2dx + \cdots$$
$$+\frac{\phi^{(2p-1)}(0)}{(2p-1)!\,\varepsilon}\int_{-\infty}^{\infty}h^{(2p-1)}hx^{2p-1}dx + o(1)$$

as $\varepsilon\to+0$, where $h=\dfrac{1}{\pi}\dfrac{1}{x^2+1}$,

we can show that

$$\mathrm{Pf}\sum_m c_m a_j(e(m))<h_\varepsilon^{(j)}h_\varepsilon,\ e(m)\beta> = 0.$$

First consider the case $j=2p$. $\sum_m c_m a_{2p}(e(m))<h_\varepsilon^{(2p)}h_\varepsilon,\ e(m)\beta>$ is a linear combination of $\dfrac{\beta^{(k)}(0)}{\varepsilon^j}$, $j=1, 2, \dots, 2p+1$, $k=0, 1, \dots, 2p$. Since $|a_{2p}(e(m))| \leqq M_2|m|^r$ with an integer $r\geqq 0$ and a positive constant M_2 independent of ε, we can easily verify that the coefficient of $\dfrac{\beta^{(k)}(0)}{\varepsilon^j}$ converges normally. Thus we can write

$$\sum_m c_m a_{2p}(e(m))<h_\varepsilon^{(2p)}h_\varepsilon,\ e(m)\beta>$$
$$= \frac{<W_1,\ \beta>}{\varepsilon} + \frac{<W_2,\ \beta>}{\varepsilon^2} + \cdots + \frac{<W_{2p+1},\ \beta>}{\varepsilon^{2p+1}} + o(1)$$

with distributions $W_1, W_2, \cdots, W_{2p+1} \in \mathcal{D}'(R)$. Similarly for the case $j=2p-1$.

Thus we have shown that $S\cdot T$ exists and is equal to $S\times_1 T$, completing the proof.

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