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Asymptotic Expansion of the Distribution of the Generalized Variance in the Non-central Case

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1. Introduction and Summary

The generalized variance (the determinant of the sample variance and covariance matrix) was defined by Wilks [8] as a measure of the spread of observations. In this paper we study asymptotic expansion of the distribution of the generalized variance in the non-central case. In general, if the rows of a $n\times p$ matrix $\boldsymbol{X}$ are independently normally distributed with common covariance matrix $\boldsymbol{\Sigma}$ and mean $\mathbf{E}[\boldsymbol{X}]=\boldsymbol{M}$, then the generalized variance is defined as the determinant of a matrix $\boldsymbol{S}=(1/n)\boldsymbol{X}'\boldsymbol{X}$. Asymptotic expansion of the distribution of $|\boldsymbol{S}|$ depends on the order of the non-centrality matrix $\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{M}'\boldsymbol{M}\boldsymbol{\Sigma}^{-\frac{1}{2}}=\boldsymbol{\Omega}$ with respect to n. It is in general true that $\boldsymbol{\Omega}=\mathbf{0}(1)$ or $\boldsymbol{\Omega}=\mathbf{0}(n)$, which means that all elements of $\boldsymbol{\Omega}$ are $0(1)$ or $0(n)$ as $n\to\infty$.

In section 2 we derive the limiting distribution of $|\boldsymbol{S}|$ under the assumption that $\boldsymbol{\Omega}=n\boldsymbol{\theta}_n=\mathbf{0}(n)$ and $\lim_{n\to\infty}\sqrt{n}(\boldsymbol{\theta}_n-\boldsymbol{\theta})=\mathbf{0}$. If $\boldsymbol{\Omega}$ may be regarded as a constant matrix, asymptotic expansion of the distribution of $|\boldsymbol{S}|$ is obtained up to the order $n^{-\frac{3}{2}}$ by inverting the characteristic function expressed in terms of hypergeometric function with matrix argument (see section 3).

2. Limiting distribution of $|\boldsymbol{S}|$ when $\boldsymbol{\Omega}=\mathbf{0}(n)$

In this section we assume that $\boldsymbol{\Omega}=n\boldsymbol{\theta}_n=\mathbf{0}(n)$ and $\lim_{n\to\infty}\sqrt{n}(\boldsymbol{\theta}_n-\boldsymbol{\theta})=\mathbf{0}$. At first we shall consider limiting distribution of a function of the non-central Wishart matrix $\boldsymbol{X}'\boldsymbol{X}$. Let $C_{\boldsymbol{X}'\boldsymbol{X}}(\boldsymbol{T})$ be the characteristic function of $\boldsymbol{X}'\boldsymbol{X}$, where $\boldsymbol{T}$ is the $p\times p$ symmetric matrix having $\{(1+\delta_{ij})/2\}t_{ij}$ as its (i,j) element with Kronecker delta δ_{ij}. From the result of Anderson [1] $C_{\boldsymbol{X}'\boldsymbol{X}}(\boldsymbol{T})$ can be expressed by our notation as

$$(2.1)\quad C_{\boldsymbol{X}'\boldsymbol{X}}(\boldsymbol{T})=|\boldsymbol{I}-2i\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{T}\boldsymbol{\Sigma}^{\frac{1}{2}}|^{-(n/2)}\operatorname{etr}\left\{-\frac{1}{2}\boldsymbol{\Omega}+\frac{1}{2}\boldsymbol{\Omega}^{\frac{1}{2}}(\boldsymbol{I}-2i\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{T}\boldsymbol{\Sigma}^{\frac{1}{2}})^{-1}\boldsymbol{\Omega}^{\frac{1}{2}}\right\}.$$

Put $\boldsymbol{S}^*=\sqrt{n}\{\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{S}\boldsymbol{\Sigma}^{-\frac{1}{2}}-(\boldsymbol{I}+\boldsymbol{\theta})\}$. Then we can express the characteristic function of $\boldsymbol{S}^*$ as

$$(2.2)\quad C_S(\boldsymbol{T})=\left|\boldsymbol{I}-\frac{2i}{\sqrt{n}}\boldsymbol{T}\right|^{-(n/2)}\operatorname{etr}\left\{-\frac{n}{2}\boldsymbol{\theta}_n+\frac{n}{2}\left(\boldsymbol{I}-\frac{2i}{\sqrt{n}}\boldsymbol{T}\right)^{-1}\boldsymbol{\theta}_n-\sqrt{n}\,i\boldsymbol{T}(\boldsymbol{I}+\boldsymbol{\theta})\right\}.$$

$C_{S^*}(\boldsymbol{T})$ can be expanded by using the well known asymptotic formulas

$$(2.3)\quad \left|\boldsymbol{I}-\frac{2i}{\sqrt{n}}\boldsymbol{T}\right|^{-(n/2)}=\exp\left\{-\frac{n}{2}\log\left|\boldsymbol{I}-\frac{2i}{\sqrt{n}}\boldsymbol{T}\right|\right\}$$

$$=\operatorname{etr}\{\sqrt{n}\,i\boldsymbol{T}-\boldsymbol{T}^2\}\{1+0(n^{-\frac{1}{2}})\}$$

$$(2.4)\quad \left(\boldsymbol{I}-\frac{2i}{\sqrt{n}}\boldsymbol{T}\right)^{-1}=\boldsymbol{I}+\frac{2i}{\sqrt{n}}\boldsymbol{T}+\left(\frac{2i}{\sqrt{n}}\boldsymbol{T}\right)^2+\boldsymbol{0}(n^{-\frac{3}{2}}),$$

which hold for large n such that the maximum of the absolute values of the characteristic roots of $(2i/\sqrt{n})\boldsymbol{T}$ is less than unity. Applying the formulas (2.3) and (2.4) to the expression of $C_{S^*}(\boldsymbol{T})$ in (2.2), we get

$$(2.5)\quad C_{S^*}(\boldsymbol{T})=\operatorname{etr}\{\sqrt{n}\,i\boldsymbol{T}(\boldsymbol{\theta}_n-\boldsymbol{\theta})\}\operatorname{etr}\{-\boldsymbol{T}^2(\boldsymbol{I}+2\boldsymbol{\theta}_n)\}\{1+0(n^{-\frac{1}{2}})\}.$$

Therefore we have

$$(2.6)\quad \lim_{n\to\infty}C_{S^*}(\boldsymbol{T})=\operatorname{etr}\{-\boldsymbol{T}^2(\boldsymbol{I}+2\boldsymbol{\theta})\}.$$

(2.6) shows that the limiting distribution of $\boldsymbol{S}^*=(s^*_{ij})$ is the multivariate normal distribution with mean zero and covariances $\mathbf{E}[s^*_{ij}s^*_{kl}]=q_{ijkl}$, where q_{ijkl} is difined by

$$(2.7)\quad 2tr\boldsymbol{T}^2(\boldsymbol{I}+2\boldsymbol{\theta})=\sum_{i\leq j}\sum_{k\leq l}q_{ijkl}t_{ij}t_{kl}.$$

Now we will generalize the well known result for obtaining limiting distributions of statistics (for example, Theorem 4.2.5 in Anderson [2] and Siotani and Hayakawa [6]) to the non-central case.

Lemma. *Let $n\boldsymbol{S}$ have the non-central Wishart distribution with n degrees of freedom and the non-centrality matrix $\boldsymbol{\Omega}$ such that $\boldsymbol{\Omega}=n\boldsymbol{\theta}_n=\mathbf{0}(n)$ and $\lim_{n\to\infty}\sqrt{n}(\boldsymbol{\theta}_n-\boldsymbol{\theta})=\mathbf{0}$. Suppose $f(\boldsymbol{W})$ is a real valued function of a $p\times p$ symmetric matrix $\boldsymbol{W}$ with first and second derivatives existing in a neighborhood of $\boldsymbol{W}=\boldsymbol{I}+\boldsymbol{\theta}$. Then the statistic $\sqrt{n}\{f(\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{S}\boldsymbol{\Sigma}^{-\frac{1}{2}})-f(\boldsymbol{I}+\boldsymbol{\theta})\}$ has asymptotically the normal distribution with mean zero and variance $2tr\boldsymbol{F}^2(\boldsymbol{I}+2\boldsymbol{\theta})$, where $\boldsymbol{F}=(\{(1+\delta_{ij})/2\}f_{ij})$ and $f_{ij}=\partial f(\boldsymbol{W})/\partial w_{ij}|_{\boldsymbol{W}=\boldsymbol{I}+\boldsymbol{\theta}}$.*

Proof. From (2.6) and Theorem 4.2.5 in Anderson [2] we can see that the asymptotic distribution of $\sqrt{n}\{f(\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{S}\boldsymbol{\Sigma}^{-\frac{1}{2}})-f(\boldsymbol{I}+\boldsymbol{\theta})\}$ is normal with mean zero. By using (2.7) its asymptotic variance can be expressed as

$$\sum_{i\leq j}\sum_{k\leq l}\frac{\partial f(\boldsymbol{W})}{\partial w_{ij}}\bigg|_{\boldsymbol{W}=\boldsymbol{I}+\boldsymbol{\theta}}\frac{\partial f(\boldsymbol{W})}{\partial w_{kl}}\bigg|_{\boldsymbol{W}=\boldsymbol{I}+\boldsymbol{\theta}}q_{ijkl}$$
$$=2tr\boldsymbol{F}^2(\boldsymbol{I}+2\boldsymbol{\theta}).$$

Putting $f(\boldsymbol{W})=|\boldsymbol{W}|$ in the above lemma and noting that the equality $(\{(1+\delta_{ij})/2\}\partial|\boldsymbol{W}|/\partial w_{ij})=|\boldsymbol{W}|\boldsymbol{W}^{-1}$ holds for any symmetric matrix $\boldsymbol{W}$, we have the following theorem.

THEOREM 1. *Let nS have the non-central Wishart distribution with n degrees of freedom and the non-centrality matrix $\boldsymbol{\Omega}$ such that $\boldsymbol{\Omega}=n\boldsymbol{\theta}_n=\mathbf{O}(n)$ and $\lim_{n\to\infty}\sqrt{n}(\boldsymbol{\theta}_n-\boldsymbol{\theta})=\mathbf{0}$. Then the distribution of $\sqrt{n}\{(|\boldsymbol{S}|/|\boldsymbol{\Sigma}|)-|\boldsymbol{I}+\boldsymbol{\theta}|\}$ is asymptotically normal with mean zero and variance $2|\boldsymbol{I}+\boldsymbol{\theta}|^2tr(\boldsymbol{I}+2\boldsymbol{\theta})(\boldsymbol{I}+\boldsymbol{\theta})^{-2}$.*

3. Asymptotic expansion of the distribution of $|S|$ when $\boldsymbol{\Omega}=\mathbf{O}(1)$

In this section we shall obtain asymptotic expansion of the distribution of $|\boldsymbol{S}|$ under the assumption that the non-centrality matrix $\boldsymbol{\Omega}$ is a constant matrix. Constantine [3] showed that the hth moment of $|\boldsymbol{S}|$ in the non-central case could be expressed by the hypergeometric function of matrix argument. His result can be expressed by our notation as

$$\text{(3.1)}\qquad \mathbf{E}[|\boldsymbol{S}|^h]=|\boldsymbol{\Sigma}|^h\left(\frac{2}{n}\right)^{ph}\frac{\Gamma_p\left(\frac{n}{2}+h\right)}{\Gamma_p\left(\frac{n}{2}\right)}{}_1F_1\left(-h;\frac{n}{2};-\frac{1}{2}\boldsymbol{\Omega}\right),$$

where $\Gamma_p(a)$ and the hypergeometric function ${}_1F_1$ are defined by

$$\Gamma_p(a)=\pi^{p(p-1)/4}\prod_{\alpha=1}^p\Gamma(a-(\alpha-1)/2)$$

$$\text{(3.2)}\qquad {}_1F_1(a;b;Z)=\sum_{k=0}^{\infty}\sum_{(\kappa)}\frac{(a)_\kappa}{(b)_\kappa}\frac{C_\kappa(Z)}{k!}$$

$$(a)_\kappa=\prod_{\alpha=1}^p(a-(\alpha-1)/2)(a+1-(\alpha-1)/2)\cdots(a+k_\alpha-1-(\alpha-1)/2).$$

The function $C_\kappa(Z)$ is a zonal polynomial of the $p\times p$ symmetric matrix Z corresponding to the partition $\kappa=(k_1, k_2, \cdots, k_p)$, with $k_1+\cdots+k_p=k$, $k_1\geq\cdots\geq k_p\geq 0$. The symbol $\sum_{(\kappa)}$ means the sum of all such partitions.

Put $\hat{\lambda}=\sqrt{n}\log(|\boldsymbol{S}|/|\boldsymbol{\Sigma}|)$. We can express the characteristic function of $\hat{\lambda}$ as

$$\text{(3.3)}\qquad C(t)=\mathbf{E}[e^{it\hat{\lambda}}]$$
$$=\mathbf{E}[(|\boldsymbol{S}|/|\boldsymbol{\Sigma}|)^{it\sqrt{n}}]$$

$$=\left(\frac{2}{n}\right)^{itp\sqrt{n}}\frac{\Gamma_p\left(\frac{n}{2}+it\sqrt{n}\right)}{\Gamma_p\left(\frac{n}{2}\right)}{}_1F_1\left(-it\sqrt{n}\ ;\ \frac{n}{2}\ ;\ -\frac{1}{2}\boldsymbol{\Omega}\right).$$

Applying the Kummer transformation formula ${}_1F_1(a;\, b;\, Z)=(\operatorname{etr} Z)\cdot{}_1F_1(b-a;\, b;\, -Z)$ (see Herz [4]) to this last expression, we can write $C(t)$ as

$$\text{(3.4)}\qquad \left(\frac{2}{n}\right)^{itp\sqrt{n}}\frac{\Gamma_p\left(\frac{n}{2}+it\sqrt{n}\right)}{\Gamma_p\left(\frac{n}{2}\right)}\operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}\right){}_1F_1\left(\frac{n}{2}+it\sqrt{n}\ ;\ \frac{n}{2}\ ;\ \frac{1}{2}\boldsymbol{\Omega}\right)$$

$$=C_1(t)C_2(t).$$

In the case that the non-centrality matrix $\boldsymbol{\Omega}$ is equal to zero, $\operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}\right)\cdot{}_1F_1\left(\frac{n}{2}+it\sqrt{n}\ ;\ \frac{n}{2}\ ;\ \frac{1}{2}\boldsymbol{\Omega}\right)$ which we shall denote by $C_2(t)$ is equal to unity. So $(2/n)^{itp\sqrt{n}}\Gamma_p\left(\frac{n}{2}+it\sqrt{n}\right)/\Gamma_p\left(\frac{n}{2}\right)$ gives us the characteristic function of $\hat{\lambda}$ in the central case, which we shall denote by $C_1(t)$. We shall use the following asymptotic formula for the gamma function as in Anderson ([2], p. 204).

$$\text{(3.5)}\qquad \log\Gamma(x+h)=\log\sqrt{2\pi}+\left(x+h-\frac{1}{2}\right)\log x-x-\sum_{r=1}^{m}\frac{(-1)^r B_{r+1}(h)}{r(r+1)x^r}$$
$$+0(|x|^{-m-1})$$

which holds for large $|x|$ and fixed h with the Bernoulli polynomial $B_r(h)$ of degree r, $B_2(h)=h^2-h+(1/6)$, $B_3(h)=h^3-(3/2)h^2+(1/2)h$, etc.. Applying the formula (3.5) to each gamma function in $C_1(t)$, we get

$$\text{(3.6)}\qquad \log C_1(t)=-pt^2-\frac{1}{\sqrt{n}}\left\{qit+\frac{2}{3}p(it)^3\right\}+\frac{1}{n}\left\{q(it)^2+\frac{2}{3}p(it)^4\right\}$$
$$-\frac{1}{n\sqrt{n}}\left\{\frac{1}{12}p(2p^2+3p-1)(it)+\frac{4}{3}q(it)^3+\frac{4}{5}p(it)^5\right\}+0(n^{-2}),$$

where $q=p(p+1)/2$. This formula implies the asymptotic expansion of $C_1(t)$.

$$\text{(3.7)}\qquad C_1(t)=e^{-pt^2}\Big[1-\frac{1}{\sqrt{n}}\left\{qit+\frac{2}{3}p(it)^3\right\}+\frac{1}{n}\left\{\frac{1}{2}q(q+2)(it)^2\right.$$
$$\left.+\frac{2}{3}p(q+1)(it)^4+\frac{2}{9}p^2(it)^6\right\}-\frac{1}{n\sqrt{n}}\left\{\frac{1}{12}p(2p^2+3p-1)(it)\right.$$
$$+\frac{1}{6}q(q+2)(q+4)(it)^3+\frac{1}{15}p(5q^2+20q+12)(it)^5$$

$$+\frac{2}{9}p^2(q+2)(it)^7+\frac{4}{81}p^3(it)^9\Big\}+0(n^{-2})\Big].$$

Now we shall consider the second term $C_2(t)$ of (3.4). From definition (3.2) we have

$$C_2(t)=\operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}\right)\sum_{k=0}^{\infty}\sum_{(\kappa)}\frac{\left(\frac{n}{2}+it\sqrt{n}\right)_{\kappa}}{\left(\frac{n}{2}\right)_{\kappa}}\frac{C_{\kappa}\left(\frac{1}{2}\boldsymbol{\Omega}\right)}{k!}. \tag{3.8}$$

The coefficient of each term can be arranged according to the descending order of powers of n as

$$\left(\frac{n}{2}+it\sqrt{n}\right)_{\kappa}=\left(\frac{n}{2}\right)^k\Big[1+\frac{2}{\sqrt{n}}itk+\frac{1}{n}\Big\{\sum_{\alpha=1}^{p}k_{\alpha}(k_{\alpha}-\alpha)+2(it)^2k(k-1)\Big\}$$
$$+\frac{2}{n\sqrt{n}}\Big\{it(k-1)\sum_{\alpha=1}^{p}k_{\alpha}(k_{\alpha}-\alpha)+\frac{2}{3}(it)^3k(k-1)(k-2)\Big\}+0(n^{-2})\Big] \tag{3.9}$$

$$\left(\frac{n}{2}\right)_{\kappa}=\left(\frac{n}{2}\right)^k\Big\{1+\frac{1}{n}\sum_{\alpha=1}^{p}k_{\alpha}(k_{\alpha}-\alpha)+0(n^{-2})\Big\}. \tag{3.10}$$

Hence we can write $C_2(t)$ as

$$C_2(t)=\operatorname{etr}\left(-\frac{1}{2}\boldsymbol{\Omega}\right)\sum_{k=0}^{\infty}\sum_{(\kappa)}\frac{C_{\kappa}\left(\frac{1}{2}\boldsymbol{\Omega}\right)}{k!}\Big[1+\frac{2}{\sqrt{n}}itk+\frac{2}{n}(it)^2k(k-1)$$
$$-\frac{2}{n\sqrt{n}}\Big\{it\sum_{\alpha=1}^{p}k_{\alpha}(k_{\alpha}-\alpha)-\frac{2}{3}(it)^3k(k-1)(k-2)\Big\}+0(n^{-2})\Big]. \tag{3.11}$$

Now we shall evaluate each term of the above infinite series. Since the identity $(trZ)^k=\sum_{(\kappa)}C_{\kappa}(Z)$ (see James [5]) holds for any symmetric matrix Z, We have

$$\sum_{k=0}^{\infty}\sum_{(\kappa)}\frac{C_{\kappa}(Z)}{k!}k(k-1)\cdots(k-r+1) \tag{3.12}$$
$$=\sum_{k=r}^{\infty}\sum_{(k)}\frac{C_{\kappa}(Z)}{(k-r)!}$$
$$=\sum_{k=r}^{\infty}\frac{(trZ)^k}{(k-r)!}=(trZ)^r etrZ,$$

which holds for any non-negative integer r. Sugiura [7] proved the following formula.

$$\sum_{k=0}^{\infty} \sum_{(\kappa)} \frac{C_\kappa(Z)}{k!} \left\{ \sum_{\alpha=1}^{p} k_\alpha(k_\alpha - \alpha) \right\} = trZ^2\, etrZ. \tag{3.13}$$

Applying the formula (3.12) and (3.13) to the expression of $C_2(t)$ in (3.11), we can simplify the expression (3.11) for the function $C_2(t)$ as

$$C_2(t) = 1 + \frac{it}{\sqrt{n}} tr\boldsymbol{\Omega} + \frac{(it)^2}{2n}(tr\boldsymbol{\Omega})^2 - \frac{1}{6n\sqrt{n}}\{3(it)tr\boldsymbol{\Omega}^2 - (it)^3(tr\boldsymbol{\Omega})^3\} + 0(n^{-2}). \tag{3.14}$$

Combining this result with the expression for $C_1(t)$ in (3.7), we obtain the following asymptotic expansion of the characteristic function $C(t)$.

$$C\left(\frac{t}{\sqrt{2p}}\right) = e^{-\frac{t^2}{2}} \left\{1 - n^{-\frac{1}{2}} A_1 + n^{-1} A_2 - n^{-\frac{3}{2}} A_3 + 0(n^{-2})\right\}, \tag{3.15}$$

where the coefficients A_1, A_2 and A_3 are given by

$$A_1 = \frac{1}{3\sqrt{2p}}\{3it(q - tr\boldsymbol{\Omega}) + (it)^3\}$$

$$A_2 = \frac{1}{36p}\{9(it)^2[\mathrm{q}(\mathrm{q}+2) - 2q\,tr\boldsymbol{\Omega} + (tr\boldsymbol{\Omega})^2] + 6(it)^4[q + 1 - \mathrm{tr}\boldsymbol{\Omega}] + (it)^6\}$$

$$\begin{aligned} A_3 = \frac{1}{180\sqrt{2p}p}\{&15it[p^2(2p^2 + 3p - 1) + 6p\,tr\boldsymbol{\Omega}^2] + 15(it)^3[q(q+2)(q+4) \\ &- 3q(q+2)tr\boldsymbol{\Omega} + 3q(tr\boldsymbol{\Omega})^2 - (tr\boldsymbol{\Omega})^3] + 3(it)^5[5q^2 + 20q + 12 \\ &- 10(q+1)tr\boldsymbol{\Omega} + 5(tr\boldsymbol{\Omega})^2] + 5(it)^7[q + 2 - tr\boldsymbol{\Omega}] + (5/9)(it)^9\}. \end{aligned}$$

By inverting this characteristic function, we can finally obtain the following theorem.

THEOREM 2. *Let $n\boldsymbol{S}$ have the non-central Wishart distribution with n degrees of freedom and the non-centrality matrix $\boldsymbol{\Omega} = \boldsymbol{\Sigma}^{-\frac{1}{2}} \boldsymbol{M}'\boldsymbol{M} \boldsymbol{\Sigma}^{-\frac{1}{2}}$. Assume that the non-centrality matrix $\boldsymbol{\Omega}$ may be regarded as a constant matrix with respect to n. Then the asymptotic expansion of the distribution of $|\boldsymbol{S}|$ can be obtained up to the order $n^{-\frac{3}{2}}$ in the following way. Let $\lambda = (\sqrt{n}/\sqrt{2p})\log(|\boldsymbol{S}|/|\boldsymbol{\Sigma}|)$. Then we have*

$$\begin{aligned} P(\lambda \leq x) = {}& \Phi(x) + \frac{1}{3\sqrt{2pn}}\{3\Phi'(x)(q - tr\boldsymbol{\Omega}) + \Phi^{(3)}(x)\} \\ &+ \frac{1}{36pn}\{9\Phi^{(2)}(x)[q(q+2) - 2q\,tr\boldsymbol{\Omega} + (tr\boldsymbol{\Omega})^2] + 6\Phi^{(4)}(x)[q + 1 - tr\boldsymbol{\Omega}] + \Phi^{(6)}(x)\} \\ &+ \frac{1}{180\sqrt{2p}p\sqrt{n}n}\{15\Phi'(x)[p^2(2p^2 + 3p - 1) + 6p\,tr\boldsymbol{\Omega}^2] + 15\Phi^{(3)}(x)[q(q+2)(q+4) \\ &- 3q(q+2)tr\boldsymbol{\Omega} + 3q(tr\boldsymbol{\Omega})^2 - (tr\boldsymbol{\Omega})^3] + 3\Phi^{(5)}(x)[5q^2 + 20q + 12 - 10(q+1)tr\boldsymbol{\Omega} \\ &+ 5(tr\boldsymbol{\Omega})^2] + 5\Phi^{(7)}(x)[q + 2 - tr\boldsymbol{\Omega}] + (5/9)\Phi^{(9)}(x)\} + 0(n^{-2}), \end{aligned} \tag{3.16}$$

where $q=p(p+1)/2$ *and* $\Phi^{(r)}(x)$ *denotes the rth derivative of the standard normal distribution function* $\Phi(x)$.

COROLLARY 1. *If the non-centrality matrix* $\boldsymbol{\Omega}$ *is the null matrix,* λ *can be expanded asymptotically as*

$$
\begin{aligned}
(3.17)\quad P(\lambda\leq x)=&\Phi(x)+\frac{1}{3\sqrt{2pn}}\{3q\Phi'(x)+\Phi^{(3)}(x)\}+\frac{1}{36pn}\{9\Phi^{(2)}(x)q(q+2)\\
&+6\Phi^{(4)}(x)(q+1)+\Phi^{(6)}(x)\}+\frac{1}{180\sqrt{2p}p\sqrt{n}n}\{15\Phi'(x)p^2(2p^2+3p-1)\\
&+15\Phi^{(3)}(x)q(q+2)(q+4)+3\Phi^{(5)}(x)(5q^2+20q+12)+5\Phi^{(7)}(x)(q+2)\\
&+(5/9)\Phi^{(9)}(x)\}+0(n^{-2}).
\end{aligned}
$$

This corollary will be obtained at once by putting $\boldsymbol{\Omega}=\mathbf{0}$ in (3.16). The asymptotic expansion (3.16) may be useful not only in the case of $\boldsymbol{\Omega}=\mathbf{0}(1)$, but also in the case of $\boldsymbol{\Omega}=\mathbf{0}(n)$. However, we could not succeed in deriving it.

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