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Non-Immersion Theorems for Lens Spaces. II

Dedicated to Professor Atuo Komatu on his 60th birthday

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§1. Introduction

Throughout this note we assume that p is an odd prime. Let Z_p be the cyclic group of order p with generator γ. Let S^{2n+1} be the unit sphere in complex $(n+1)$-space. Define an action of Z_p on S^{2n+1} by the formula:

$$\gamma(z_0, z_1, \cdots, z_n) = (\lambda z_0, \lambda z_1, \cdots, \lambda z_n), \quad \text{where} \quad \lambda = e^{2\pi i/p},$$

for $(z_0, z_1, \cdots, z_n) \in S^{2n+1}$. The orbit space S^{2n+1}/Z_p is the lens space mod p and is written by $L^n(p)$. It is a compact, connected, orientable C^∞-manifold of dimension $2n+1$ and has the structure of a CW-complex with one cell in each dimension $0, 1, \cdots, 2n+1$. Let $L_0^n(p)$ be the $2n$-skeleton of $L^n(p)$.

The purpose of this paper is to prove some results on the stable homotopy type of the stunted space $L_0^n(p)/L_0^m(p)$ $(n>m)$ and on the non-immersibility of the lens space $L^n(p)$ in the Euclidean space.

After some preparations in §2, we determine the structure of the reduced Grothendieck ring $\tilde{K}(L_0^n(p)/L_0^m(p))$ of complex vector bundles in §3. Using the Adams operation we shall prove the following result in §4.

Theorem A. *Let $n>m$. If $L_0^n(p)/L_0^m(p)$ is stably homotopy equivalent to $L_0^{n+t}(p)/L_0^{m+t}(p)$, then $t \equiv 0 \pmod{p^{[(n-m-1)/(p-1)]}}$.*

We notice that the following result is known by Theorem 3 of [4]: $L_0^n(p)/L_0^m(p)$ is stably homotopy equivalent to $L_0^{n+t}(p)/L_0^{m+t}(p)$, if $t \equiv 0$ $(\bmod\, p^{[(n-m)/(p-1)]})$.

Together with Theorem 3 of [5], Theorem A can be used to give a condition for the immersibility of $L^n(p)$ in the Euclidean space $R^{2n+2m+1}$.

Theorem B. *Let n and m be integers with $n>m>0$. Assume that $n+m+1 \not\equiv 0 \pmod{p^{[(n-m-1)/(p-1)]}}$. If there is an immersion of $L^n(p)$ in $R^{2n+2m+1}$, then the Euler class of its normal bundle is zero.*

This will be proved in §5. From Theorem B we have the following result.

Theorem C. *Let n and m be integers with $n>m>0$. Assume that the following two conditions are satisfied:*

(i) $\binom{n+m}{m} \not\equiv 0 \pmod{p}$

(ii) $n+m+1 \not\equiv 0 \pmod{p^{[(n-m-1)/(p-1)]}}$.

Then $L^n(p)$ is not immersible in $R^{2n+2m+1}$.

This is a generalization of Theorems 4, 4′, 5, 5′ of [5]. By [8], $L^n(p)$ is immersible in $R^{2n+2[n/2]+2}$. Theorem C shows that this is best possible for some n. In fact we have the following two results which follow directly from the above fact and Theorem C.

COROLLARY D. *Let $n=2m$ and assume that the following conditions are satisfied:*

(i) $\binom{3m}{m} \not\equiv 0 \pmod{p}$

(ii) $m \geqq p$; $3m \neq 4p-1$; $3m \neq 5p-1$; $m \neq 16$ *if* $p=7$.

Then $L^n(p)$ is immersible in R^{3n+2} and not in R^{3n+1}.

COROLLARY E. *Let $n=2m+1$ and assume that the following conditions are satisfied:*

(i) $\binom{3m+1}{m} \not\equiv 0 \pmod{p}$

(ii) $m \geqq p-1$; $3m \neq 4p-2$; $3m \neq 5p-2$.

Then $L^n(p)$ is immersible in R^{3n+1} and not in R^{3n}.

Corollary D (resp. E) is an improvement of Theorems 7 and 7′ (resp. 8 and 8′) of [6].

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§2. Preliminaries

Let CP^n be the complex projective space of complex dimension n. Let $\xi \in K(CP^n)$ be (the equivalence class of) the canonical line bundle over CP^n and $1 \in K(CP^n)$ be (the equivalence class of) the complex 1-dimensional trivial bundle over CP^n. Put $\mu=\xi-1 \in \tilde{K}(CP^n)$. Let $\pi: L^n(p) \to CP^n$ be the map defined by $\pi q(z_0, z_1, \cdots, z_n)=[z_0, z_1, \cdots, z_n]$ for $(z_0, z_1, \cdots, z_n) \in S^{2n+1}$, where $q: S^{2n+1} \to L^n(p)$ is the natural projection. Let $\pi_0: L_0^n(p) \to CP^n$ be the restriction of π to the $2n$-skeleton $L_0^n(p)$. Then $\pi_0^*: \tilde{K}(CP^n) \to \tilde{K}(L_0^n(p))$ is an epimorphism [3, (2.7)]. Define

$$\sigma = \pi_0^* \mu \in \tilde{K}(L_0^n(p)).$$

The following result is proved in [3, Theorem 1].

(2.1) *Let* $n=s(p-1)+r$ $(0 \leqq r < p-1)$. *Then*

$$\tilde{K}(L^n(p)) \cong \tilde{K}(L_0^n(p)) \cong (Z_{p^{s+1}})^r + (Z_{p^s})^{p-r-1}$$

and $\sigma, \cdots, \sigma^r$ *generate additively the first* r *factors and* $\sigma^{r+1}, \cdots, \sigma^{p-1}$ *the last* $p-r-1$ *factors. Moreover, its ring structure is given by*

$$\sigma^p = -\sum_{i=1}^{p-1} \binom{p}{i} \sigma^i, \qquad \sigma^{n+1} = 0.$$

In the above statement, $(Z_m)^t$ denotes the direct sum of t-copies of the cyclic group Z_m of order m.

Suppose that $n > m$. Let $i: L_0^m(p) \to L_0^n(p)$ be the inclusion and $j: L_0^n(p) \to L_0^n(p)/L_0^m(p)$ be the projection.

(2.2) *We have the exact sequence*:

$$0 \to \tilde{K}(L_0^n(p)/L_0^m(p)) \xrightarrow{j^*} \tilde{K}(L_0^n(p)) \xrightarrow{i^*} \tilde{K}(L_0^m(p)) \to 0.$$

Proof. Let μ' and σ' be the generators of $\tilde{K}(CP^m)$ and $\tilde{K}(L_0^m(p))$ respectively. As is well-known, $k^*\mu=\mu'$, where k^* is induced by the inclusion $k: CP^m \to CP^n$. So we have $i^*\sigma = \sigma'$. Thus i^* is an epimorphism. Since $\tilde{K}^{-1}(L_0^m(p))=0$ [3, (2.4)], the result follows from the Puppe exact sequence. q.e.d.

Let $\# A$ denote the number of the elements of a finite set A.

(2.3) $\# \tilde{K}(L_0^n(p)/L_0^m(p)) = p^{n-m}$.

Proof. By (2.1), $\# \tilde{K}(L_0^n(p)) = p^n$ and $\# \tilde{K}(L_0^m(p)) = p^m$. Thus we have the desired result from the exact sequence in (2.2). q.e.d.

§3. The structure of $\tilde{K}(L_0^n(p)/L_0^m(p))$

If $n > m$, $\sigma^{m+1}, \cdots, \sigma^n$ belong to the kernel of i^* (= the image of j^*, by (2.2)), because $\sigma'^{m+1}=0$ and $i^*\sigma=\sigma'$. We define, for $t > m$,

$$\sigma^{(t)} = j^{*-1}\sigma^t \in \tilde{K}(L_0^n(p)/L_0^m(p)).$$

We are ready to determine the structure of $\tilde{K}(L_0^n(p)/L_0^m(p))$.

Theorem (3.1) *Let* p *be an odd prime, and assume that* $n > m$. *Then*

$$\tilde{K}(L_0^n(p)/L_0^m(p)) \cong \sum_{i=1}^{p-1} G_i \qquad (\textit{direct sum}),$$

where each G_i *is the cyclic group of order* $p^{1+[(n-m-i)/(p-1)]}$ *generated by* $\sigma^{(m+i)}$ $(1 \leqq i < p)$.

PROOF. First, we show that $\tilde{K}(L_0^n(p)/L_0^m(p))$ is generated by $\sigma^{(m+1)}, \cdots, \sigma^{(m+p-1)}$. Let $n = s(p-1)+r$ and $m = s'(p-1)+r'$ $(0 \leqq r, r' < p-1)$. Since $i^*\sigma = \sigma'$, the kernel of i^* in (2.2) is additively generated by $p^{s'+1}\sigma, \cdots, p^{s'+1}\sigma^{r'}, p^{s'}\sigma^{r'+1}, \cdots, p^{s'}\sigma^{p-1}$. On the other hand, the first relation in (2.1) implies that

$$\begin{aligned} p\sigma &= -\frac{p}{2}\binom{p-1}{1}\sigma^2 - \cdots - \frac{p}{r}\binom{p-1}{r-1}\sigma^r - \cdots - p\sigma^{p-1} - \sigma^p \\ &= pa_1\sigma^2 + \cdots + pa_{p-2}\sigma^{p-1} + a_{p-1}\sigma^p, \end{aligned}$$

where $(a_i, p) = 1$ for $i = 1, 2, \cdots, p-1$. Repeated application of this equality shows that $p\sigma$ is expressed as a linear combination $\sigma^p, \sigma^{p+1}, \cdots, \sigma^{2p-2}$. By induction, $p^t\sigma^i$ can be expressed as a linear combination of $\sigma^{t(p-1)+i}, \sigma^{t(p-1)+i+1}, \cdots, \sigma^{t(p-1)+i+p-2}$. Since the minimum of the set of integers $\{(s'+1)(p-1)+1, \cdots, (s'+1)(p-1)+r', s'(p-1)+r'+1, \cdots, s'(p-1)+p-1\}$ is $s'(p-1)+r'+1 = m+1$, the kernel of i^* is generated by $\sigma^{m+1}, \cdots, \sigma^{m+p-1}$. Therefore, by (2.2), $\tilde{K}(L_0^n(p)/L_0^m(p))$ is generated by $\sigma^{(m+1)}, \cdots, \sigma^{(m+p-1)}$.

Since σ^i is of order $p^{1+[(n-i)/(p-1)]}$ [3, (2.10)], $\sigma^{(i)}$ is also of order $p^{1+[(n-i)/(p-1)]}$ by (2.2). We see easily that

$$\prod_{i=1}^{p-1} p^{1+[(n-m-i)/(p-1)]} = p^{n-m}.$$

Combining this with (2.3), we have the desired result. q.e.d.

REMARK. In the similar way to the proof of (3.1) we can determine the structure of $\widetilde{KO}(L_0^n(p)/L_0^m(p))$ by making use of [3, Theorem 2]. Let $r: \tilde{K}(X) \to \widetilde{KO}(X)$ be a group-homomorphism induced by the standard injection $r_n: GL(n, C) \to GL(2n, R)$. Define

$$\begin{aligned} &\bar{\sigma} = r\sigma \in \widetilde{KO}(L_0^n(p)), \\ &\bar{\sigma}^{(t)} = j^{*-1}\bar{\sigma}^t \in \widetilde{KO}(L_0^n(p)/L_0^m(p)), \qquad \text{for} \quad t > [m/2], \end{aligned}$$

where $j^*: \widetilde{KO}(L_0^n(p)/L_0^m(p)) \to \widetilde{KO}(L_0^n(p))$ is induced by the projection. Then we have the following result.

(3.2) *Let* p *be an odd prime,* $q = (p-1)/2$, *and assume that* $n > m$. *Then*

$$\widetilde{KO}(L_0^n(p)/L_0^m(p)) \cong \sum_{i=1}^{q} G_i \qquad (\textit{direct sum}),$$

where each G_i *is the cyclic group of order* $p^{1+[(n-2[m/2]-2i)/(p-1)]}$ *generated by*

$\bar{\sigma}^{([m/2]+i)}$ $(1 \leqq i \leqq q)$.

§4. Proof of Theorem A

Let p be an odd prime, and m be a positive integer. Let $v(m)$ denote the maximum power of p which divides m, that is, $m = up^{v(m)}$ for some integer u such that $(u, p) = 1$.

(4.1) *Let t be a positive integer. Then*

$$v\left((p \pm 1)^t - (\pm 1)^t\right) = v(t) + 1.$$

PROOF. Let f be a positive integer. If x and y are integers such that $x - y \equiv p^f (\bmod\, p^{f+1})$, then obviously

$$(1) \qquad x^p - y^p \equiv y^{p-1} p^{f+1} \qquad (\bmod\, p^{f+2}),$$

$$(2) \qquad x^n - y^n \equiv n y^{n-1} p^f \qquad (\bmod\, p^{f+1}), \quad \text{for any integer} \quad n > 0.$$

Since $(p \pm 1) - (\pm 1) = p$, repeated application of (1) shows that

$$(p \pm 1)^{p^f} - (\pm 1)^{p^f} \equiv p^{f+1} \qquad (\bmod\, p^{f+2}).$$

Then, by (2), for any integer $u > 0$ we have

$$(p \pm 1)^{up^f} - (\pm 1)^{up^f} \equiv (-1)^{u-1} u p^{f+1} \qquad (\bmod\, p^{f+2}).$$

The result follows if we suppose that $(u, p) = 1$. q.e.d.

PROOF OF THEOREM A. Suppose that there is a homotopy equivalence g: $S^{2t+r}(L_0^n(p)/L_0^m(p)) \to S^r(L_0^{n+t}(p)/L_0^{m+t}(p))$ for some integers r and t. g induces an isomorphism of $\tilde{K}$-groups. We may assume that r is even.

Let Ψ^k: $\tilde{K}(L_0^n(p)) \to \tilde{K}(L_0^n(p))$ be the Adams operation. Since $1 + \sigma (= 1 + \pi_0^* \mu = \pi_0^* \xi)$ is a complex line bundle over $L_0^n(p)$, we have $\Psi^k(1+\sigma) = (1+\sigma)^k$ [1, Theorem 5.1]. Therefore $\Psi^k \sigma = (1+\sigma)^k - 1$. The relation $(1+\sigma)^p = 1$ [3, (2.8)] shows that Ψ^{p+1} is the identity. By (2.2) we see that Ψ^{p+1}: $\tilde{K}(L_0^n(p)/L_0^m(p)) \to \tilde{K}(L_0^n(p)/L_0^m(p))$ is also the identity.

Consider the following diagram:

$$\begin{array}{ccc}
\tilde{K}(L_0^n(p)/L_0^m(p)) & \xrightarrow{I^{(2t+r)/2}} & \tilde{K}(S^{2t+r}(L_0^n(p)/L_0^m(p))) \\
\Psi^{p+1} \downarrow & & \Psi^{p+1} \downarrow \\
\tilde{K}(L_0^n(p)/L_0^m(p)) & \xrightarrow{I^{(2t+r)/2}} & \tilde{K}(S^{2t+r}(L_0^n(p)/L_0^m(p)))
\end{array}$$

where I denotes the isomorphism defined by the Bott periodicity [2, Theorem 1]. By [1, Corollary 5.3], we have

$$\Psi^{p+1} I^{(2t+r)/2} = (p+1)^{(2t+r)/2} I^{(2t+r)/2} \Psi^{p+1} = (p+1)^{(2t+r)/2} I^{(2t+r)/2},$$

and so the right-hand operation Ψ^{p+1} in the diagram is given by $\Psi^{p+1}=(p+1)^{(2t+r)/2}$. Similarly, the operation

$$\Psi^{p+1}\colon \tilde{K}(S^r(L_0^{n+t}(p)/L_0^{m+t}(p)))\to\tilde{K}(S^r(L_0^{n+t}(p)/L_0^{m+t}(p)))$$

is given by $\Psi^{p+1}=(p+1)^{r/2}$.

Now, from the commutative diagram

$$\begin{array}{ccc} \tilde{K}(S^r(L_0^{n+t}(p)/L_0^{m+t}(p))) & \xrightarrow{g^*} & \tilde{K}(S^{2t+r}(L_0^n(p)/L_0^m(p))) \\ \Psi^{p+1}\downarrow & & \Psi^{p+1}\downarrow \\ \tilde{K}(S^r(L_0^{n+t}(p)/L_0^{m+t}(p))) & \xrightarrow{g^*} & \tilde{K}(S^{2t+r}(L_0^n(p)/L_0^m(p))) \end{array}$$

we have $(p+1)^{(2t+r)/2}g^*=g^*(p+1)^{r/2}=(p+1)^{r/2}g^*$. The Bott periodicity and Theorem (3.1) imply that

$$\tilde{K}(S^{2t+r}(L_0^n(p)/L_0^m(p)))\simeq\tilde{K}(L_0^n(p)/L_0^m(p))\simeq Z_{p^{1+[(n-m-1)/(p-1)]}}+\cdots$$

Therefore $(p+1)^{(2t+r)/2}\equiv(p+1)^{r/2}$ (mod $p^{1+[(n-m-1)/(p-1)]}$), and so $(p+1)^t-1\equiv 0$ (mod $p^{1+[(n-m-1)/(p-1)]}$). Thus (4.1) shows that $t\equiv 0$ (mod $p^{[(n-m-1)/(p-1)]}$). q.e.d.

REMARK. The Adams operation $\Psi_R^k\colon \widetilde{KO}(L_0^n(p))\to\widetilde{KO}(L_0^n(p))$ is determined by the equation:

$$c\Psi_R^k(\bar\sigma)=(1+\sigma)^k+(1+\sigma)^{-k}-2$$

where $c\colon \widetilde{KO}(L_0^n(p))\to\tilde{K}(L_0^n(p))$ is the complexification. Using this, we can prove the following.

(4.2) *Let p be an odd prime, $q=(p-1)/2$, and k be any integer. The Adams operation $\Psi_R^k\colon \widetilde{KO}(L_0^n(p))\to\widetilde{KO}(L_0^n(p))$ is given by*

i) $\Psi_R^{p\pm k}=\Psi_R^k$

ii) $\Psi_R^k(\bar\sigma)=\sum_{i=1}^{k}\frac{k}{i}\binom{k+i-1}{2i-1}\bar\sigma^i$ *for* $1\leqq k\leqq q$,

where $\bar\sigma\in\widetilde{KO}(L_0^n(p))$ is the generator.

§5. Proof of Theorem B

The following result is Theorem 3 of [5].

(5.1) *Let n and m be integers with $n>m$ and let $n=s(p-1)+r$ $(0\leqq r<p-1)$. Assume that a is a positive integer such that $2ap^v>4n+3$ where $v=s$ or $s+1$ according as $[r/2]=0$ or $[r/2]\geqq 1$. Put $t=ap^v-n-m-1$. If $L^n(p)$ is im-*

mersed in $R^{2n+2m+1}$ *with a normal bundle whose Euler class is non-zero, then there is a map*

$$g\colon S^{2t}(L^n(p)/L^{m-1}(p)) \to L^{n+t}(p)/L^{m+t-1}(p)$$

which induces isomorphisms of all cohomology groups with Z_p *coefficients.*

Proof of Theorem B. Suppose that $L^n(p)$ is immersed in $R^{2n+2m+1}$ with a normal bundle whose Euler class is non-zero. Let a be an integer such that $2ap^v > 4n+3$, and put $t = ap^v - n - m - 1$. Then by (5.1) there exists a map

$$g\colon S^{2t}(L^n(p)/L^{m-1}(p)) \to L^{n+t}(p)/L^{m+t-1}(p)$$

which induces isomorphisms of all cohomology groups with Z_p coefficients. We may assume that g is a cellular map. Then clearly g defines a map

$$g_0\colon S^{2t}(L_0^n(p)/L_0^m(p)) \to L_0^{n+t}(p)/L_0^{m+t}(p)$$

which induces isomorphisms of all cohomology groups with Z_p coefficients. Since the mod p reduction $Z \to Z_p$ induces an isomorphism $H^i(L_0^n(p)/L_0^m(p); Z) \cong H^i(L_0^n(p)/L_0^m(p); Z_p)$ ($\cong Z_p$ for $2m < i \leqq 2n$, i even; $\cong 0$ for other $i > 0$), and since the spaces are simply connected, g_0 is a homotopy equivalence. Therefore, by Theorem A, we have $t \equiv 0 \pmod{p^{[(n-m-1)/(p-1)]}}$. As we may take a such that $ap^v \equiv 0 \pmod{p^{[(n-m-1)/(p-1)]}}$, we see that $n+m+1 \equiv 0 \pmod{p^{[(n-m-1)/(p-1)]}}$. But this is inconsistent with the assumption. Consequently, if there is an immersion of $L^n(p)$ in $R^{2n+2m+1}$, then the Euler class of its normal bundle is zero. q.e.d.

Remark. If there is an embedding of $L^n(p)$ in $R^{2n+2m+1}$, the Euler class of its normal bundle is zero (cf. [7, Theorem 14]).

§6. Proof of Theorem C

We shall apply the previous results to the problem of finding the least integer k such that $L^n(p)$ can be immersed in R^{2n+1+k} (cf. [5], [6] and [8]). First, we recall the Pontrjagin class of $L^n(p)$ (cf. [9, Corollary 3.2]).

(6.1) *The* mod p *Pontrjagen class* p_i *and the* mod p *dual Pontrjagin class* $\bar{p}_i$ *are given by the equations:*

$$p = p(L^n(p)) = (1+x^2)^{n+1}$$

$$\bar{p} = \bar{p}(L^n(p)) = (1+x^2)^{-n-1} = \sum_{i=1}^{[n/2]} (-1)^i \binom{n+i}{i} x^{2i}$$

where x *is a generator of* $H^2(L^n(p); Z_p)$ ($\cong Z_p$).

PROOF OF THEOREM C. Suppose that $n+m+1 \not\equiv 0 \pmod{p^{[(n-m-1)/(p-1)]}}$. According to Theorem B, if there is an immersion $L^n(p)$ in $R^{2n+2m+1}$, then the $\bmod p$ Euler class χ of its $2m$-dimensional normal bundle is zero. Since $\chi^2=\bar{p}_m$ (cf. [7, Theorem 31]), $\bar{p}_m=0$. Thus, by (6.1), we have $\binom{n+m}{m} \equiv 0 \pmod p$. This is inconsistent with the assumption (i). Therefore, $L^n(p)$ is not immersible in $R^{2n+2m+1}$. q.e.d.

REMARK. As is well-known, if an m-dimensional manifold M is immersible in $R^{m+k}(k>0)$, then $\bar{p}_i(M)=0$ except 2-torsions for any $i>[k/2]$. Hence we have the following.

(6.2) *Let n and m be integers with $n>m>0$. If*

$$\binom{n+m}{m} \not\equiv 0 \qquad (\bmod p),$$

then $L^n(p)$ is not immersible in R^{2n+2m}.

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