Hiroshima Math. J.
4 (1974), 279–284

Nonoscillation of Elliptic Differential Equations of Second Order

Norio Yoshida

(Received November 22, 1973)

Nonoscillation criteria for second order elliptic equations have been obtained by Headley [4], Headley and Swanson [5], Kreith [6], Kuks [7] and Swanson [10]. The purpose of this paper is to establish nonoscillation criteria for the non-self-adjoint elliptic equation

$$Lu=\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_ju)+2\sum_{i=1}^{n} b_i(x)D_iu+c(x)u=0\,. \tag{1}$$

Nonoscillation criteria for (1) due to Swanson [10] will be derived from our main theorem.

Let R be an unbounded domain in n-dimensional Euclidean space E^n with piecewise smooth boundary ∂R. A generic point of E^n is denoted by $x=(x_1,\dots, x_n)$. Partial differentiation with respect to x_i is denoted by D_i, $i=1,\dots, n$. It is assumed that the coefficients a_{ij}, b_i and c of L are real-valued and continuous on $\bar{R}$, that the b_i are differentiable in R and that the matrix (a_{ij}) is symmetric and positive definite in R. The domain of L relative to R, $\mathfrak{D}(L; R)$, is the set of continuous functions on $\bar{R}$ which have uniformly continuous first derivatives in R and for which all derivatives involved in L exist and are continuous in R. A solution of equation (1) is a function $u\in\mathfrak{D}(L; R)$ which satisfies (1) at every point of R.

Definition 1. A bounded domain G with $\bar{G}\subset R$ is a *nodal domain* of a nontrivial solution u of (1) iff $u=0$ on ∂G. The partial differential equation (1) is said to be *strongly oscillatory* in R iff for arbitrary $r>0$ there exists a nontrivial solution u_r of (1) with a nodal domain contained in R_r, where

$$R_r=R\cap\{x\in E^n\colon |x|>r\}\,.$$

Equation (1) is said to be *nonoscillatory* in R iff it is not strongly oscillatory in R, i.e. iff there exists a number $s>0$ such that no nontrivial solution of (1) has a nodal domain contained in R_s.

Definition 2. Consider the two self-adjoint operators

$$L_0u=\sum_{i,j=1}^{n} D_i(\alpha_{ij}(x)D_ju)+\gamma(x)u, \tag{2}$$

$$L_1 v = \sum_{i,j=1}^{n} D_i(A_{ij}(x)D_j v) + C(x)v, \tag{3}$$

where α_{ij}, γ, A_{ij}, C are real-valued and continuous on $\bar{R}$ and the matrices (α_{ij}), (A_{ij}) are symmetric and positive definite in R. We say that L_1 belongs to $\mathfrak{M}[L_0; R_s]$ for some $s>0$ iff for every bounded domain G with $\bar{G} \subset R_s$ the functional

$$V[u; G] = \int_G \Big[\sum_{i,j=1}^{n} (\alpha_{ij} - A_{ij}) D_i u D_j u + (C-\gamma)u^2\Big] dx \tag{4}$$

is nonnegative for all real-valued piecewise C^1 functions u on $\bar{G}$ vanishing on ∂G. The functional $V[u; G]$ in (4) is called the *variation*, relative to G, of L_1 from L_0. For example, $L_1 \in \mathfrak{M}[L_0; R_s]$ if the matrix $(\alpha_{ij} - A_{ij})$ is positive semi-definite in R_s and $C-\gamma$ is nonnegative in R_s.

Our main result is stated in the following

THEOREM. *Equation* (1) *is nonoscillatory in R if for some number $s>0$ there exist a self-adjoint elliptic operator $L_1 \in \mathfrak{M}\left[\frac{1}{2}(L+L^*); R_s\right]$, L^* being the formal adjoint of L, and a function $w \in \mathfrak{D}(L_1; R_s)$ with the property that*

(i) $w>0$ in $\bar{R}_s$;

(ii) $L_1 w \leqq 0$ in R_s.

To prove the theorem we require the following three lemmas that provide useful information regarding bounds for eigenvalues of self-adjoint and non-self-adjoint elliptic operators.

Let G be a bounded domain with piecewise smooth boundary ∂G and such that $\bar{G} \subset R$. By an *eigenvalue* λ of L relative to G we mean a number λ with the property that there exists a nontrivial solution $u \in \mathfrak{D}(L; G)$ of the problem

$$Lu + \lambda u = 0 \quad \text{in} \quad G, \qquad u = 0 \quad \text{on} \quad \partial G.$$

The solution u is called an *eigenfunction* associated with the eigenvalue λ.

LEMMA 1. (Allegretto [1]) *Let L be the elliptic operator defined by* (1). *Then, no eigenvalue of L relative to G can be less than the smallest eigenvalue of $\frac{1}{2}(L+L^*)$ relative to G.*

LEMMA 2. (Swanson [8]) *Let L_0 and L_1 be the elliptic operators defined by* (2) *and* (3), *respectively. If there exists an eigenvalue λ of L_1 relative to G with an associated eigenfunction u satisfying $V[u; G] \geqq 0$, then λ cannot be less than the smallest eigenvalue of L_0 relative to G.*

LEMMA 3. *Let μ_0 be the smallest eigenvalue of the self-adjoint elliptic operator L_1 relative to G. Then, for any $v \in \mathfrak{D}(L_1; G)$ such that $v>0$ in $\bar{G}$,*

$$\mu_0 > \inf_{x\in G} \frac{-L_1 v}{v}.$$

PROOF. Our method is essentially that used by Swanson [9]. If u is an eigenfunction of L_1 associated with μ_0, then the following identity holds (see [2]):

$$\sum_{i,j=1}^{n} A_{ij} D_i u D_j u - C u^2 = \sum_{i,j=1}^{n} A_{ij} X^i X^j + \sum_{i=1}^{n} D_i(u^2 Y^i) - \frac{u^2}{v} L_1 v, \tag{5}$$

where

$$X^i = v D_i\left(\frac{u}{v}\right), \qquad Y^i = \frac{1}{v} \sum_{j=1}^{n} A_{ij} D_j v,\ i = 1, \dots, n.$$

Since $u=0$ on ∂G and $v>0$ on $\bar{G}$, u/v is nonconstant in G and hence we have $\int_G \sum_{i,j=1}^{n} A_{ij} X^i X^j dx > 0$. By integrating (5) over G and applying Green's formula, we obtain

$$\begin{aligned}\int_G \left[\sum_{i,j=1}^{n} A_{ij} D_i u D_j u - C u^2\right] dx &= \int_G \left[\sum_{i,j=1}^{n} A_{ij} X^i X^j - \frac{u^2}{v} L_1 v\right] dx \\ &> -\int_G \frac{u^2}{v} L_1 v \, dx \\ &\geqq \inf_{x\in G} \frac{-L_1 v}{v} \int_G u^2 dx,\end{aligned}$$

from which the desired conclusion immediately follows with the use of Courant's Minimum Principle [3, p. 399].

PROOF OF THEOREM. Suppose to the contrary that equation (1) is strongly oscillatory. Then, there exists a nontrivial solution u of (1) with a nodal domain G contained in R_s. By Lemma 1, the smallest eigenvalue λ_0 of $\frac{1}{2}(L+L^*)$ relative to G is nonpositive.

By hypothesis, there exists a self-adjoint elliptic operator $L_1 \in \mathfrak{M}\left[\frac{1}{2}(L+L^*); R_s\right]$. Since the variation, relative to G, of L_1 from $\frac{1}{2}(L+L^*)$ is nonnegative, we can apply Lemma 2 to conclude that the smallest eigenvalue μ_0 of L_1 relative to G does not exceed λ_0, i.e. $\mu_0 \leqq 0$. On the other hand, it follows from Lemma 3 that μ_0 is greater than $\inf_{x\in G} [-L_1 w/w]$ which is nonnegative on account of (i) and (ii), i.e. $\mu_0 > 0$. The contradiction proves our theorem.

REMARK 1. The above theorem is an extension of the sufficiency part of Kuks' nonoscillation theorem [7, Theorem 3] for self-adjoint elliptic equations.

Our method is different from the one used by Kuks.

COROLLARY 1. *Equation* (1) *is nonoscillatory in R if for some $s>0$ there exists a function $h \in \mathfrak{D}(L; R_s)$ such that*

$$\sum_{i,j=1}^{n} [D_i(a_{ij}(x)D_jh)+a_{ij}(x)D_ihD_jh]+c(x)-\operatorname{div} b(x) \leqq 0$$

in R_s, where $b(x)=(b_1(x), \ldots, b_n(x))$.

PROOF. This corollary follows from the observation that the function $w=\exp[h(x)]$ satisfies

$$\frac{1}{w}\frac{Lw+L^*w}{2}=\sum_{i,j=1}^{n}[D_i(a_{ij}D_jh)+a_{ij}D_ihD_jh]+c-\operatorname{div} b .$$

Let $\lambda(x)$ be the smallest eigenvalue of the matrix $(a_{ij}(x))$, $x \in R$, and let f be an arbitrary positive-valued function of class $C^1(0, \infty)$ such that

$$f(r) \leqq \min_{x \in S_r} \lambda(x), \qquad 0<r<\infty ,$$

where $S_r=\{x \in \bar{R}: |x|=r\}$. We define the function g by

$$g(r)=\max_{x \in S_r}[c(x)-\operatorname{div} b(x)], \qquad 0<r<\infty .$$

Let us consider the self-adjoint elliptic operator

$$L_2v=\sum_{i=1}^{n} D_i(f(|x|)D_iv)+g(|x|)v . \tag{6}$$

Since, for all $x \in R$ and all $\xi \in E^n$,

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqq \lambda(x)|\xi|^2 \geqq f(|x|)|\xi|^2 ,$$

the operator $L_2 \in \mathfrak{M}\left[\frac{1}{2}(L+L^*); R\right]$, moreover, if $v \in \mathfrak{D}(L_2; R)$ depends only on $r=|x|$, then

$$L_2v=r^{1-n}\frac{d}{dr}\left(r^{n-1}f(r)\frac{dv}{dr}\right)+g(r)v .$$

COROLLARY 2. *Equation* (1) *is nonoscillatory in R if*

$$\limsup_{r \to \infty}\left[r^2g(r)-\frac{(n-2)^2}{4}f(r)-\frac{n-2}{2}rf'(r)\right]<0 . \tag{7}$$

PROOF. Observe that the function $w=r^{(2-n)/2}$, $r=|x|$, satisfies

$$r^{n-1}L_2w=\frac{d}{dr}\left(r^{n-1}f(r)\frac{dw}{dr}\right)+r^{n-1}g(r)w$$
$$=r^{(n-4)/2}\left[r^2g(r)-\frac{(n-2)^2}{4}f(r)-\frac{n-2}{2}rf'(r)\right].$$

REMARK 2. If L is uniformly elliptic in R with ellipticity constant κ, then we can take $f(r)\equiv\kappa$, and in this case condition (6) reduces to

$$\limsup_{r\to\infty} r^2g(r)<\frac{(n-2)^2}{4}\kappa,$$

which is a nonoscillation criterion of Swanson [10, Theorem 2].

The following corollary was first obtained by Swanson [10, Theorem 1].

COROLLARY 3. *Equation* (1) *is nonoscillatory in* R *if the ordinary differential equation*

$$\frac{d}{dr}\left(r^{n-1}f(r)\frac{dy}{dr}\right)+r^{n-1}g(r)y=0 \tag{8}$$

is nonoscillatory at $r=+\infty$.

PROOF. Since (8) is nonoscillatory at $r=+\infty$, there exists a solution $y(r)$ which does not vanish on some half-line $[s, +\infty)$. Without loss of generality we may assume that $y(r)>0$ on $[s, +\infty)$. Define the function w in R by $w(x)=y(r)$, $r=|x|$. Then w satisfies the elliptic equation $L_2w=0$ in R_s, where L_2 is the operator defined in (6). Now the conclusion follows from the main theorem.

Our final result is an extension of that of Kuks [7, Corollary 1].

COROLLARY 4. *Let the operator* L *be defined in* $R=\prod_{i=1}^{n} I_i$, *where* $I_i=[s_i, +\infty)$, $i=1,\dots, n$, *and uniformly elliptic in* R *with ellipticity constant* κ. *Assume that each of the ordinary differential equations*

$$\kappa\frac{d^2y}{dx_i^2}+c_i(x_i)y=0,\ i=1,\dots, n, \tag{9}$$

is nonoscillatory in I_i, $i=1,\dots, n$. *If*

$$\sum_{i=1}^{n} c_i(x_i)\geqq c(x)-\operatorname{div} b(x) \qquad in \quad R,$$

then equation (1) *is nonoscillatory in* R.

PROOF. By hypothesis there exist solutions $y_i=y_i(x_i)$ of (9) such that $y_i(x_i)>0$ in $I_i'=[s_i', +\infty)\subset I_i$, $i=1,\dots, n$. Now the conclusion follows from the main theorem by taking

$$L_1 = \kappa\Delta + \sum_{i=1}^{n} c_i(x_i) \quad \text{and} \quad w = \prod_{i=1}^{n} y_i(x_i).$$

ACKNOWLEDGEMENT. The author wishes to thank Professor T. Kusano for his helpful suggestions concerning this paper.

References

[1] W. Allegretto, Eigenvalue comparison and oscillation criteria for elliptic operators, J. London Math. Soc. **3** (1971), 571–575.

[2] C. Clark and C. A. Swanson, Comparison theorems for elliptic differential equations, Proc. Amer. Math. Soc. **16** (1965), 886–890.

[3] R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. I, Wiley (Interscience), New York, 1953.

[4] V. B. Headley, Some oscillation properties of selfadjoint elliptic equations, Proc. Amer. Math. Soc. **25** (1970), 824–829.

[5] V. B. Headley and C. A. Swanson, Oscillation criteria for elliptic equations, Pacific J. Math. **27** (1968), 501–506.

[6] K. Kreith, A Picone identity for strongly elliptic systems, Duke Math. J. **38** (1971), 473–481.

[7] L. M. Kuks, Sturm's theorem and oscillation of solutions of strongly elliptic systems, Soviet Math. Dokl. **3** (1962), 24–27.

[8] C. A. Swanson, A generalization of Sturm's comparison theorem, J. Math. Anal. Appl. **15** (1966), 512–519.

[9] C. A. Swanson, An identity for elliptic equations with applications, Trans. Amer. Math. Soc. **134** (1968), 325–333.

[10] C. A. Swanson, Nonoscillation criteria for elliptic equations, Canad. Math. Bull. **12** (1969), 275–280.

Department of Mathematics,
Faculty of Science,
Hiroshima University