HIROSHIMA MATH. J.
4 (1974), 265-277

Principal Oriented Bordism Algebra $\Omega_*(Z_{2^k})$

Yutaka KATSUBE

(Received October 4, 1973)

Introduction

The principal oriented bordism module $\Omega_*(G)$ for a finite group G is defined to be the module of equivariant bordism classes of closed oriented principal G-manifolds, and is a module over the oriented bordism ring Ω_* of R. Thom (cf. [1]).

This module $\Omega_*(G)$ and the unoriented one $\mathfrak{N}_*(G)$ are studied by several authors. If G is a finite cyclic group, the Ω_*-module structure of $\Omega_*(G)$ is determined by P. E. Conner and E. E. Floyd [1, Ch. VII] for $G=Z_{p^k}$ (p: odd prime), and by K. Shibata [3, §§1–4] for $G=Z_2$. Also it is proved by N. Hassani [2] that there is an isomorphism $\Omega_*(Z_{qr})\cong\Omega_*(Z_q)\otimes_{\Omega_*}\Omega_*(Z_r)$ of Ω_*-modules if q and r are relatively prime.

The main purpose of this note is to study the Ω_*-module structure of $\Omega_*(Z_{2^k})$ for $k>1$. Also, we study the Pontrjagin products in $\Omega_*(Z_{2^k})$ and $\mathfrak{N}_*(Z_{2^k})$ for $k>1$.

In § 1, we are concerned with the unoriented bordism module

$$\mathfrak{N}_*(Z_{2^k})\cong\mathfrak{N}_*\otimes H_*(Z_{2^k};Z_2)\qquad\text{(cf. [1, (19.3)])}.$$

It is easy to see that this is a free $\mathfrak{N}_*$-module with basis $\{[T,S^{2n+1}], i[a,S^{2n}]|n\geqq 0\}$ (Proposition 1.7), where (T, S^{2n+1}) is the Z_{2^k}-manifold with the diagonal action T of $\exp(\pi\sqrt{-1}/2^{k-1})$ and $i(a, S^{2n})$ is the extension of the Z_2-manifold (a, S^{2n}) with the antipodal action a. Also we study in Theorem 1.22 the product formulae in $\mathfrak{N}_*(Z_{2^k})$ using the results for $\mathfrak{N}_*(Z_2)$ of F. Uchida [6].

In § 2, we are concerned with

$$\tilde{\Omega}_n(Z_{2^k})\cong\textstyle\sum_{p+q=n}\tilde{H}_p(Z_{2^k};\Omega_q)\qquad\text{(cf. [1, Th. 14.2])}.$$

Using the homomorphism $r:\Omega_*(Z_{2^k})\to\mathfrak{N}_*(Z_{2^k})$ obtained by ignoring orientations, and the results for $\Omega_*(Z_2)$ in [3], we prove in Theorem 2.18 that the Ω_*-module $\tilde{\Omega}_*(Z_{2^k})$ $(k>1)$ is a quotient module of the free Ω_*-module

$$\Omega_*\{\{[T,S^{2n+1}], iE^{2n+1}W(\omega)|n\geqq 0,\ \omega\in\pi\}\},$$

where $E^{2n+1}W(\omega)\in\tilde{\Omega}_*(Z_2)$. Finally, we study in Theorem 2.22 the Pontrjagin product in $\tilde{\Omega}_*(Z_{2^k})$.

Recently, E. R. Wheeler [8] has discussed the bordism module of closed oriented (not necessarily principal) G-manifolds for a finite cyclic group G.

The author wishes to express his thanks to Professor M. Sugawara for making useful suggestions, and also to Dr. K. Shibata and Dr. K. Komiya for their criticisms.

§1. The unoriented bordism algebra $\mathfrak{N}_*(Z_{2^k})$

For a given finite group G, an n-dimensional principal G-manifold (G, B^n) is a pair of a compact n-manifold B^n and a free action of G on B^n as a group of diffeomorphisms, and two closed principal G-manifolds (G, M^n) and (G, N^n) are equivariantly bordant (G-bordant), if there is a principal G-manifold (G, B^{n+1}) with $(G, \dot{B}^{n+1})=(G, M^n \cup N^n)$. Denote the G-bordism class of (G, M^n) by $[G, M^n]$, and the collection of all such classes by $\mathfrak{N}_n(G)$. $\mathfrak{N}_n(G)$ is a module with respect to the disjoint union, and the direct sum

$$(1.1) \qquad \mathfrak{N}_*(G) = \sum_{n=0}^{\infty} \mathfrak{N}_n(G)$$

is the principal G-bordism module. For the unit group e,

$$\mathfrak{N}_* = \sum_{n=0}^{\infty} \mathfrak{N}_n, \qquad \mathfrak{N}_n = \mathfrak{N}_n(e),$$

is the usual bordism ring with respect to the multiplication induced by the cartesian product $M \times N$, and $\mathfrak{N}_*(G)$ of (1.1) can be given a structure of (left) $\mathfrak{N}_*$-module by

$$[N][G, M] = [G, N \times M],$$

where G acts on $N \times M$ by $g(x, y)=(x, gy)$ (cf. [1, §§2, 19]).

For an element $[G, M] \in \mathfrak{N}_n(G)$, let $f\colon M/G \to BG$ be the classifying map of the principal G-bundle $M \to M/G$. Then the element

$$\mu[G, M] = f_*(M/G) \in H_n(BG; Z_2) = H_n(G; Z_2)$$

is defined, where $M/G \in H_n(M/G; Z_2)$ means the fundamental class, and

(1.2) [1, (8.1)] $\mu\colon \mathfrak{N}_n(G) \longrightarrow H_n(G; Z_2)$ *is epimorphic.*

Let Z_{2^k} be the cyclic group of order 2^k. For the non zero element $c_n \in H_n(Z_{2^k}; Z_2)=Z_2$, we can take $C_n \in \mathfrak{N}_n(Z_{2^k})$ with $\mu C_n = c_n$ by (1.2). Then, a homomorphism of $\mathfrak{N}_*$-modules

$$h\colon \mathfrak{N}_* \otimes H_*(Z_{2^k}; Z_2) \longrightarrow \mathfrak{N}_*(Z_{2^k}) \qquad (k \geqq 1)$$

is obtained by $h(1 \otimes c_n) = C_n$, and

(1.3) [1, (19.3)] *h is an isomorphism of $\mathfrak{N}_*$-modules.*

We denote a principal Z_{2^k}-manifold (Z_{2^k}, M) and its orbit manifold M/Z_{2^k} also by (T, M) and M/T, respectively, using the action $T: M \to M$ of the generator of Z_{2^k}. Consider the extension

$$(1.4) \qquad i: \mathfrak{N}_n(Z_2) \longrightarrow \mathfrak{N}_n(Z_{2^k})$$

defined by $i(A, M) = (Z_{2^k}, (Z_{2^k} \times M)/(iA \times A))$ where $i: Z_2 \subset Z_{2^k}$ is the inclusion and the action of Z_{2^k} is $g'[g, x] = [g'g, x]$ (cf. [1, §20]).

We consider the principal Z_2-manifold

$$(1.5) \qquad (a, S^n) \quad (a \text{ is the antipodal action}),$$

and the principal Z_{2^k}-manifolds

$$(1.6) \qquad \begin{aligned} &(T, S^{2n+1}), \quad T(z_0, \dots, z_n) = (Tz_0, \dots, Tz_n), \\ &i(a, S^{2n}) = (T \times 1, (Z_{2^k} \times S^{2n})/(ia \times a)), \end{aligned}$$

where $T = \exp(\pi\sqrt{-1}/2^{k-1})$ is the generator of Z_{2^k}.

PROPOSITION 1.7. (i) *$\mathfrak{N}_*(Z_{2^k})$ $(k \geqq 1)$ is a free $\mathfrak{N}_*$-module with basis* $\{[T, S^{2n+1}], i[a, S^{2n}] | n \geqq 0\}$.

(ii) $i[a, S^{2n+1}] = 0$ *if* $k > 1$.

PROOF. (i) is an immediate consequence of (1.3), since $\mu[T, S^{2n+1}] = c_{2n+1}$, $i_*\mu[a, S^{2n}] = c_{2n}$.

(ii) The classifying map of the Z_{2^k}-bundle

$$i(a, S^{2n+1}) \longrightarrow i(a, S^{2n+1})/(T \times 1) = S^{2n+1}/a$$

is given by the projection $i: S^{2n+1}/a \to S^{2n+1}/T$ induced by $i: Z_2 \subset Z_{2^k}$, and $i_*: H_{2n+1}(S^{2n+1}/a; Z_2) \to H_{2n+1}(S^{2n+1}/T; Z_2)$ $(k > 1)$ is zero. Therefore, the Stiefel-Whitney numbers of the above bundle are zero and we have $i[a, S^{2n+1}] = 0$ by[1, Th. 17.2]. q.e.d.

Now, $\mathfrak{N}_*(Z_{2^k})$ is an algebra over $\mathfrak{N}_*$ with respect to the Pontrjagin product induced by the tensor product of principal Z_{2^k}-bundles. Explicitly, for principal Z_{2^k}-manifolds (T_1, M_1) and (T_2, M_2), the product is defined by

$$(1.8) \qquad (T_1, M_1)(T_2, M_2) = (T, (M_1 \times M_2)/T_1 \times T_2^{-1}),$$

where both $T_1 \times 1$ and $1 \times T_2$ induce the same action T.

It is clear that the extension i of (1.4) and the augmentation

$$(1.9) \qquad \varepsilon_*: \mathfrak{N}_*(Z_{2^k}) \longrightarrow \mathfrak{N}_*, \qquad \varepsilon_*[T, M] = [M/T],$$

are homomorphisms of $\mathfrak{N}_*$-algebras.

The product formulae in $\mathfrak{N}_*(Z_2)$ are given by [6], [4], and we study those in $\mathfrak{N}_*(Z_{2^k})$ $(k>1)$.

LEMMA 1.10. *For the manifolds of* (1.6), *we have*

(i) $$[T, S^1][T, S^{2n+1}]=0\,,$$

(ii) $$(T, S^1)(i(a, S^{2n}))=(T\times 1, (S^1\times S^{2n})/(a\times a))\,.$$

PROOF. (i) Consider the multiplication

$$m\colon S^1\times S^{2n+1}\longrightarrow S^{2n+1},\quad m(z,(z_0,\dots,z_n))=(zz_0,\dots,zz_n)$$

and the map

$$f\colon S^1\times S^{2n+1}\longrightarrow S^1\times S^{2n+1},\quad f(z,x)=(z, m(z,x))\,.$$

Then, $f(T\times T^{-1})=(T\times 1)f$ and $f(1\times T)=(1\times T)f$. Therefore, f induces an equivariant diffeomorphism

$$f\colon (T, S^1)(T, S^{2n+1})=(1\times T, (S^1\times S^{2n+1})/(T\times T^{-1}))\longrightarrow (1\times T, S^1\times S^{2n+1})\,.$$

This shows (i) since $[1\times T, S^1\times S^{2n+1}]=0$.

(ii) The desired result follows immediately from

$$((S^1\times Z_{2^k}\times S^{2n})/(T\times T^{-1}\times 1))/(1\times ia\times a)=(S^1\times S^{2n})/(a\times a)\,. \qquad \text{q.e.d.}$$

Let

$$\Delta\colon \mathfrak{N}_n(Z_{2^k})\longrightarrow \mathfrak{N}_{n-2}(Z_{2^k}) \tag{1.11}$$

be the Smith homomorphism defined as follows (cf. [1, §26 and (34.7)]): For a principal Z_{2^k}-manifold (Z_{2^k}, M^n), we can take a differentiable equivariant map $\varphi\colon (Z_{2^k}, M^n)\to(T, S^{2N+1})$ which is transverse regular on S^{2N-1}, since S^{2N+1}/T is the $(2N+1)$-skeleton of BZ_{2^k}, where (T, S^{2N+1}) is the one in (1.6) and $2N+1>n$. Then,

$$\Delta[Z_{2^k}, M^n]=[Z_{2^k}, \varphi^{-1}(S^{2N-1})]\,.$$

It is easy to see that Δ is a homomorphism of $\mathfrak{N}_*$-modules, and

LEMMA 1.12. *For the generators of Proposition* 1.7, *we have*

$$\Delta[T, S^{2n+1}]=[T, S^{2n-1}],\quad \Delta i[a, S^{2n}]=i[a, S^{2n-2}]\,.$$

LEMMA 1.13. $$\Delta([T, S^1]i[a, S^{2n}])=[T, S^1]i[a, S^{2n-2}]\,.$$

PROOF. Consider the differentiable map

$$S^1 \times S^{2n} \xrightarrow{1\times i_0} S^1 \times S^{2n+1} \xrightarrow{m} S^{2n+1},$$

where $i_0: S^{2n}=0\times S^{2n}\subset S^{2n+1}$ and m is the multiplication in the proof of Lemma 1.10. Since $m(a\times a)=m$ and $m(T\times 1)=Tm$, the composition $m(1\times i_0)$ induces the differentiable equivariant map

$$(T, S^1)(i(a, S^{2n}))=(T\times 1, (S^1\times S^{2n})/(a\times a)) \longrightarrow (T, S^{2n+1}),$$

by Lemma 1.10 (ii). It is clear that this map is transverse regular on S^{2n-1} and the inverse image of S^{2n-1} is $(S^1\times S^{2n-2})/(a\times a)$, and so we have the lemma. q.e.d.

For $\mathfrak{N}_*(Z_2)$, we can also define the Smith homomorphism

(1.14) $$\Delta_1: \mathfrak{N}_n(Z_2) \longrightarrow \mathfrak{N}_{n-1}(Z_2)$$

in the same way as (1.11) using the classifying space S^N/a of Z_2 (cf. [1, (26.1)]). It is clear that

(1.15) $$\Delta_1[a, S^m]=[a, S^{m-1}],$$

and the following is proved in [6, Lemma 2.2 (b)]:

(1.16) $\varepsilon_*\Delta_1([a, S^1][a, S^m])=0$ *for* $m\geqq 1$.

Now we prove the following theorem, which is an analogy of [6, Th 2.4].

THEOREM 1.17. *For the elements of* $\mathfrak{N}_*(Z_{2^k})$ *in Proposition* 1.7,

$$[T, S^{2n+1}]=[T, S^1](\textstyle\sum_{j=0}^{n}[P^{2j}]i[a, S^{2n-2j}])$$

where $[P^{2j}]\in\mathfrak{N}_{2j}$ *is the bordism class of the real projective space* $P^{2j}=S^{2j}/a$.

PROOF. We notice that $\varepsilon_*[T, S^{2n+1}]=[S^{2n+1}/T]=0$ since S^{2n+1}/T is the boundary of the associated disk bundle of the canonical S^1-bundle $S^{2n+1}/T \to S^{2n+1}/S^1$. Consider the element

$$y_n=-[T, S^{2n+1}]+[T, S^1](\textstyle\sum_{j=0}^{n}[P^{2j}]i[a, S^{2n-2j}])$$

of $\tilde{\mathfrak{N}}_*(Z_{2^k})=\operatorname{Ker}\varepsilon_*$. Then $\Delta(y_n)=y_{n-1}$ by Lemmas 1.12–13.

It is clear that $y_0=0$. If $y_{n-1}=0$, then $\Delta(y_n)=0$ and we have

$$y_n=x[T, S^1] \qquad \text{for some } x\in\mathfrak{N}_*$$

by Proposition 1.7 and Lemma 1.12, since $y_n\in\tilde{\mathfrak{N}}_*(Z_{2^k})$. Mapping this equality by the transfer

$$t:\ \mathfrak{N}_*(Z_{2^k}) \longrightarrow \mathfrak{N}_*(Z_2), \qquad t[T, M]=[T^{2^{k-1}}, M]$$

(cf. [1, § 20]), we have

$$x[a, S^1] = ty_n = -[a, S^{2n+1}] + \sum_{j=0}^{n}[P^{2j}][a, S^1][a, S^{2n-2j}]$$

by Lemma 1.10 (ii) and (1.8). Applying Δ_1 of (1.14) and the augmentation ε_* of (1.9) for $k=1$, we have $x=\varepsilon_*\Delta_1(ty_n)=-[P^{2n}]+[P^{2n}]=0$ by (1.15) and (1.16), and so $y_n=0$ as desired. q.e.d.

COROLLARY 1.18. $[T, S^1]i[a, S^{2n}]=\sum_{j=0}^{n}a_{2j}[T, S^{2n-2j+1}]$, *where the elements* $a_{2j}\in\mathfrak{N}_{2j}$ $(j\geqq 0)$ *are defined by*

$$(1.19) \qquad a_0 = 1, \quad \sum_{j=0}^{m} a_{2j}[P^{2m-2j}] = 0 \qquad \textit{for any } m\geqq 1 .$$

PROOF. The right hand side of the desired equality is equal to

$$\sum_{j=0}^{n}a_{2j}[T, S^1](\sum_{l=0}^{n-j}[P^{2l}]i[a, S^{2n-2j-2l}])$$

by the above theorem, and so to $[T, S^1]\,i[a, S^{2n}]$ by (1.19). q.e.d.

Let $\alpha_j(m, n)\in\mathfrak{N}_{m+n-j}$ be the elements defined by

$$(1.20) \qquad [a, S^m][a, S^n] = \sum_{j\geqq 0}\alpha_j(m, n)[a, S^j] \quad \textit{in } \mathfrak{N}_{m+n}(Z_2) .$$

(1.21) (cf. [6, Lemma 3.1], [4, Th. 4.1]) *The above elements* $\alpha_i(m, n)$ *are determined by the following relations*:

(a) $\sum_{j\geqq 1}z_{j-1}\alpha_{i+j}(m, n) = \sum_{j\geqq 1}z_{j-1}(\alpha_i(m-j, n)+\alpha_i(m, n-j))$,

where $z_j\in\mathfrak{N}_j$, $z_0=1$ *and* $z_j=0$ *if* $i+1=2^s$.

(b) $\alpha_0(m, n)=[P^m][P^n]+\sum_{j\geqq 1}\alpha_j(m, n)[P^j]$.

(c) $[H_{m,n}]=\sum_{j\geqq 1}\alpha_j(m, n)[P^{j-1}]$,

where $H_{m,n}$ *is Milnor's hypersurface in* $P^m\times P^n$.

The commutative algebra $\mathfrak{N}_*(Z_{2^k})$ over $\mathfrak{N}_*$ with the Pontrjagin product defined by (1.8) is given by the following theorem.

THEOREM 1.22. $\mathfrak{N}_*(Z_{2^k})(k>1)$ *is a free* $\mathfrak{N}_*$*-module with basis* $\{[T, S^{2n+1}], i[a, S^{2n}]|n\geqq 0\}$ *of* (1.6), *and*

$$[T, S^{2m+1}][T, S^{2n+1}]=0, \quad i[a, S^{2m}]\,i[a, S^{2n}]=\sum_j\alpha_{2j}(2m, 2n)i[a, S^{2j}] ,$$

$$[T, S^{2m+1}]i[a, S^{2n}]=\sum_j(\sum_{s,t}[P^{2s}]\alpha_{2t}(2m-2s, 2n)a_{2t-2j})[T, S^{2j+1}] ,$$

where a_{2t-2j} *and* $\alpha_{2j}(2m, 2n)$ *are the elements of* (1.19–20).

PROOF. The first half is Proposition 1.7 (i). The equalities are seen by routine calculations, by Theorem 1.17, Lemma 1.10 (i), (1.20), Proposition 1.7 (ii), Corollary 1.18 and the fact that i is a homomorphism of $\mathfrak{N}_*$-algebras.

q.e.d.

§2. The oriented bordism algebra $\Omega_*(Z_{2^k})$ over Ω_*

The principal oriented G-bordism module and the oriented bordism ring

$$\Omega_*(G) = \sum_{n=0}^{\infty}\Omega_n(G) \quad \text{and} \quad \Omega_* = \sum_{n=0}^{\infty}\Omega_n$$

are defined in the same way as $\mathfrak{N}_*(G)$ and $\mathfrak{N}_*$ in § 1, provided that manifolds are oriented and G-actions preserve the orientations (cf. [1, §§2, 19]). $\Omega_*(G)$ is a module over Ω_*, and there are homomorphisms

(2.1) $$r\colon \Omega_*(G) \longrightarrow \mathfrak{N}_*(G), \qquad r\colon \Omega_* \longrightarrow \mathfrak{N}_*,$$

obtained by ignoring the orientations. Also, the augmentation homomorphism

(2.2) $$\varepsilon_*\colon \Omega_*(G) \longrightarrow \Omega_*, \qquad \varepsilon_*[G, M] = [M/G],$$

defines the direct sum decomposition of Ω_*-modules:

$$\Omega_*(G) = \tilde{\Omega}_*(G)\oplus\Omega_*, \qquad \tilde{\Omega}_*(G) = \operatorname{Ker}\varepsilon_*.$$

Wall's results on Ω_* can be stated as follows:

Let π denote the set of partitions $\omega=(a_1, .., a_r)$ with unequal parts a_j, none of which is a power of 2, and set $|\omega|=r$. Let $\omega\cap\omega'$, $\omega\ominus\omega'$, $\omega_j\in\pi$ for $\omega, \omega'\in\pi$ be the intersection, the symmetric difference and the partition obtained from $\omega=(a_1,\dots, a_r)$ by omitting a_j, respectively. Then,

THEOREM 2.3. (C. T. C. Wall [7]) *The oriented bordism ring Ω_* is the quotient ring of the integral polynomial ring*

$$Z[h_{4k}, g(\omega)|k\geqq 0, \omega\in\pi]$$

by the ideal generated by the elements

$$2g(\omega), \quad \sum_j g(a_j)g(\omega_j) \quad (|\omega|\geqq 3),$$

$$g(\omega)g(\omega') - \sum_j h(\omega_j\cap\omega')g(a_j)g(\omega_j\ominus\omega'),$$

where $h((a_1,\dots, a_r)) = h_{4a_1}\cdots h_{4a_r}$.

K. Shibata [3] studies the principal oriented Z_2-bordism algebra $\Omega_*(Z_2)$ $(=\hat{\Omega}^+_*(Z_2))$, together with the bordism algebra $\Omega^-_*(Z_2)$ $(=\hat{\Omega}^-_*(Z_2))$ of orientation-reversing principal Z_2-manifolds. Let

$$(2.4)\qquad \begin{aligned}&\Delta_1: \Omega_m^-(Z_2) \longrightarrow \Omega_{m-1}(Z_2), \quad E^{2n+2}: \Omega_m^-(Z_2) \longrightarrow \Omega_{m+2n+2}^-(Z_2),\\ &E^{2n+1} = \Delta_1 E^{2n+2}: \Omega_m^-(Z_2) \longrightarrow \Omega_{m+2n+1}(Z_2)\end{aligned}$$

be the Smith homomorphism defined in the same way as (1.14), and the homomorphism of Ω_*-modules defined by

$$E^{2n+2}[A, M] = [a, S^{2n+2}][A, M] \qquad \text{for} \quad [A, M] \in \Omega_m^-(Z_2),$$

where (a, S^{2n+2}) is the one in (1.5) and the product is defined by (1.8). Then

(2.5) [3, p. 205] $E^m[a, S^0] = [a, S^m]$ for $m>0$.

For a partition $\omega=(a_1,\dots, a_r)\in\pi$, let

$$(2.6)\qquad X_\omega = X_{2a_1}\cdots X_{2a_r} \in \mathfrak{M}_* \subset \mathfrak{N}_*, \quad W(\omega) \in \Omega_*^-(Z_2)$$

be the bordism classes of the unoriented manifold $M_\omega = M_{2a_1}\cdots M_{2a_r}$ in [7, §4] and of the orientation bundle over M_ω with the orientation-reversing transformation as a Z_2-action. Consider the following elements of $\Omega_*^-(Z_2)$:

$$(2.7)\qquad \begin{aligned}&A(\omega) = \textstyle\sum_j g(a_j)W(\omega_j) - g(\omega)[a, S^0],\\ &B(\omega, \omega') = \textstyle\sum_j h(\omega_j \cap \omega')g(a_j)W(\omega_j \ominus \omega') - g(\omega)W(\omega'),\end{aligned}$$

for $\omega, \omega' \in \pi$, where $h(\omega), g(\omega) \in \Omega_*$ are the elements in Theorem 2.3.

THEOREM 2.8. (K. Shibata [3, Th. 4.5, Cor. 3.3 (6)]) *The principal oriented Z_2-bordism module $\tilde{\Omega}_*(Z_2)$ is the quotient module of the free Ω_*-module*

$$\Omega_*\{\{[a, S^{2n+1}], E^{2n+1}W(\omega) | n \geqq 0, \omega \in \pi\}\}$$

by the submodule generated by the elements $2[a, S^{2n+1}]$, $2E^{2n+1}W(\omega)$, $E^{2n+1}A(\omega)$ $(|\omega| \geqq 2)$, $E^{2n+1}B(\omega, \omega')$, *where* E^{2n+1} *is the homomorphism of* (2.4).

For our purpose, we use also the following

THEOREM 2.9. (cf. [1, Th. 14.2]) *There is an isomorphism*

$$\theta: \tilde{\Omega}_n(Z_{2^l}) \xrightarrow[\cong]{} \textstyle\sum_{p+q=n} \tilde{H}_p(Z_{2^l}; \Omega_q) \qquad (l \geqq 1).$$

We see easily that this isomorphism θ is natural by the proof of [1, pp. 39–41], and so we have the commutative diagram

$$(2.10)\qquad \begin{array}{ccccc} \tilde{\mathfrak{N}}_n(Z_2) & \xleftarrow{r} & \tilde{\Omega}_n(Z_2) & \xrightarrow[\cong]{\theta} & H_{n,1}\oplus G_{n,1}\\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i_*\oplus i_*}\\ \tilde{\mathfrak{N}}_n(Z_{2^k}) & \xleftarrow{r} & \tilde{\Omega}_n(Z_{2^k}) & \xrightarrow[\cong]{\theta} & H_{n,k}\oplus G_{n,k}, \end{array}$$

where r's are the homomorphisms of (2.1), the left i is the extension of (1.4), the middle i is the one defined in the same way,

$$H_{n,l}=\sum_m H_{2m+1}(Z_{2^l};\,\Omega_{n-2m-1}),\qquad G_{n,l}=\sum_m \tilde{H}_{2m}(Z_{2^l};\,\Omega_{n-2m}),$$

and i_*'s are the induced homomorphisms of the inclusion $i\colon Z_2\subset Z_{2^k}$. We notice that

$$\operatorname{Ker} r=2\tilde{\Omega}_n(Z_{2^l})\qquad (l\geqq 1) \tag{2.11}$$

by Rohlin's theorem (cf. [1, Th. 16.2]).

LEMMA 2.12. (i) $i_*\colon G_{n,1}\longrightarrow G_{n,k}$ *is isomorphic and* $i_*H_{n,1}\subset 2^{k-1}H_{n,k}$.
(ii) $r\theta^{-1}i_*$ *is monomorphic on* $G_{n,1}$ *and* $r\theta^{-1}i_*(H_{n,1})=0$ *if* $k>1$.

PROOF. Since Ω_* is a direct sum of some copies of Z and Z_2, we have the lemma by the well known facts for $H_*(Z_{2^l};Z)$ and $H_*(Z_{2^l};Z_2)$ and by $2\mathfrak{N}_n(Z_{2^l})=0$. q.e.d.

LEMMA 2.13. (i) $[T,S^{2n+1}]\in\tilde{\Omega}_{2n+1}(Z_{2^k})$ $(k\geqq 1)$ *is of order* 2^k.
(ii) $x[T,S^{2n+1}]=0$ *if and only if* $x\in 2^k\Omega_*$, *for* $x\in\Omega_*$.

PROOF. (i) It is clear that $\mu[T,S^{2n+1}]$ is a generator, where $\mu\colon\Omega_{2n+1}(Z_{2^k})\to H_{2n+1}(Z_{2^k};Z)=Z_{2^k}$ is the natural homomorphism defined in the same way as (1.2) (cf. [1, §6]). Therefore we have the desired result by Theorem 2.9.

(ii) It is sufficient to prove $x\in 2^k\Omega_*$ if $x[T,S^{2n+1}]=0$. By [1, §7], there is a commutative diagram

$$\begin{array}{ccc}
\Omega_m\otimes\Omega_{2n+1}(Z_{2^k}) & \xrightarrow{\kappa} & J_{2n+1,m}\subset\Omega_{m+2n+1}(Z_{2^k})\\
\Big\downarrow{\scriptstyle 1\otimes\mu} & & \Big\downarrow\\
\Omega_m\otimes H_{2n+1}(Z_{2^k};Z) & \xrightarrow{\kappa} & H_{2n+1}(Z_{2^k};\Omega_m)
\end{array}$$

where κ's are the homomorphisms defined by the multiplications, and the lower κ is monomorphic. Therefore we have the desired result. q.e.d.

PROPOSITION 2.14. (i) *The* Ω_*-*submodule* $\mathfrak{H}_k$ *of* $\tilde{\Omega}_*(Z_{2^k})$ $(k\geqq 1)$, *generated by the elements* $[T,S^{2n+1}]$ $(n\geqq 0)$, *is the quotient module of the free* Ω_*-*module*

$$\Omega_*\{\{[T,S^{2n+1}]\,|\,n\geqq 0\}\}$$

by the submodule generated by the elements $2^k[T,S^{2n+1}]$ $(n\geqq 0)$.

(ii) *By the isomorphism* θ *in* (2.10), $\mathfrak{H}_{n,l}=\mathfrak{H}_l\cap\tilde{\Omega}_n(Z_{2^l})$ *is mapped isomorphically onto* $H_{n,l}$.

(iii) $i\mathfrak{H}_1\subset 2^{k-1}\mathfrak{H}_k$ *for the extension* i *in* (2.10).

PROOF. (i) Consider the Smith homomorphism

$$\Delta\colon \Omega_n(Z_{2^k}) \longrightarrow \Omega_{n-2}(Z_{2^k})$$

defined in the same way as (1.11). Then we have $\Delta[T, S^{2j+1}]=[T, S^{2j-1}]$ in the same way as Lemma 1.12.

Assume that

$$\textstyle\sum_{j=0}^{n} x_j[T, S^{2j+1}] = 0 \qquad (x_j \in \Omega_*)\,.$$

Then the image of the left hand side of this equality by Δ^n is equal to $x_n[T, S^1]$, and so we have $x_n \in 2^k\Omega_*$ by Lemma 2.13 (ii). Therefore we have (i).

(ii) Consider a commutative diagram similar to (2.10) for the inclusion $j\colon Z_{2^l} \subset Z_{2^{l+1}}$. Then, we see that $r\theta^{-1}j_*$ is monomorphic on $G_{n,l}$ and $r\theta^{-1}j_*H_{n,l}=0$ in the same way as Lemma 2.12 (ii). Then, we obtain

$$\theta(\mathfrak{H}_{n,l}) \subset H_{n,l}$$

since we have $jr[T, S^{2n+1}]=0$ in the same way as Proposition 1.7 (ii).

On the other hand, there is a group homomorphism

$$\varphi\colon H_{n,l} = \textstyle\sum_m H_{2m+1}(Z_{2^l}; Z) \otimes \Omega_{n-2m-1} \longrightarrow \mathfrak{H}_{n,l}\,,$$

defined by $\varphi(d_{2m+1} \otimes x) = x[T, S^{2m+1}]$ $(x \in \Omega_{n-2m-1})$, where $d_{2m+1} \in H_{2m+1}(Z_{2^l}; Z) = Z_{2^l}$ is the generator. It is clear by (i) that φ is isomorphic. These show that $\theta(\mathfrak{H}_{n,l}) = H_{n,l}$ as desired.

(iii) The desired result follows immediately from (ii) and Lemma 2.12 (i). q.e.d.

Consider the Ω_*-submodule

$$\mathfrak{G}_k \subset \tilde{\Omega}_*(Z_{2^k}) \qquad (k \geqq 1) \tag{2.15}$$

generated by the elements $iE^{2n+1}W(\omega)$ $(n \geqq 0, \omega \in \pi)$, and the elements

$$\begin{aligned} A_{n,k}(\omega) &= \textstyle\sum_j g(a_j) iE^{2n+1}W(\omega_j)\,, \\ B_{n,k}(\omega, \omega') &= \textstyle\sum_j h(\omega_j \cap \omega') g(a_j) iE^{2n+1} W(\omega_j \ominus \omega') \\ &\quad - g(\omega) iE^{2n+1} W(\omega')\,, \end{aligned} \tag{2.16}$$

of $\mathfrak{G}_k$ $(k>1)$, where $i\colon \Omega_*(Z_2) \to \Omega_*(Z_{2^k})$ is the extension in (2.10) and the elements are the ones in Theorem 2.8.

LEMMA 2.17. (i) $\mathfrak{G}_k = i\mathfrak{G}_1, \quad i(\mathfrak{G}_1 \cap \mathfrak{H}_1) = 0\,,$

(ii) $A_{n,k}(\omega) = iE^{2n+1}A(\omega),\ \ B_{n,k}(\omega, \omega') = iE^{2n+1}B(\omega, \omega'),$ *for* $k>1\,,$

where $\mathfrak{H}_1$ *is the one in Proposition* 2.14 *and* $A(\omega)$ *and* $B(\omega, \omega')$ *are the elements of* (2.7).

PROOF. Take $g \in \mathfrak{G}_1$ and $h \in \mathfrak{H}_1$ such that $g = h$. Then $g - h$ is a linear combination of the elements $E^{2n+1}A(\omega)$ in Theorem 2.8, and so

$$h = \Sigma x_{n,\omega} g(\omega)[a, S^{2n+1}]$$

by (2.7) and (2.5). Therefore $ih = 0$ and $i(\mathfrak{G}_1 \cap \mathfrak{H}_1) = 0$ for $k > 1$, by Proposition 2.14 (iii) and Theorem 2.3. The first equality of (ii) follows in the same way. q.e.d.

Now, we are ready to prove our main theorem.

THEOREM 2.18. *The principal oriented* Z_{2^k}*-bordism module* $\tilde{\Omega}_*(Z_{2^k})$ $(k > 1)$ *is the direct sum*

$$\tilde{\Omega}_*(Z_{2^k}) = \mathfrak{H}_k \oplus \mathfrak{G}_k,$$

where the submodule $\mathfrak{H}_k$ *is given by Proposition* 2.14 (i) *and* $\mathfrak{G}_k$ $(k > 1)$ *of* (2.15) *is the quotient module of the free* Ω_**-module*

$$\Omega_*\{\{iE^{2n+1}W(\omega) | n \geqq 0, \omega \in \pi\}\}$$

by the submodule generated by the elements $2iE^{2n+1}W(\omega)$ *and* $A_{n,k}(\omega)$ $(|\omega| \geqq 2)$, $B_{n,k}(\omega, \omega')$ *of* (2.16).

PROOF. Since $\tilde{\Omega}_*(Z_2) = \mathfrak{H}_1 + \mathfrak{G}_1$ by Theorem 2.8, we see immediately that

$$\tilde{\Omega}_*(Z_{2^k}) = \mathfrak{H}_k + \mathfrak{G}_k = \mathfrak{H}_k + i\mathfrak{G}_1$$

by the right commutative square of (2.10), Lemma 2.12 (i) and Proposition 2.14 (ii).

Assume that

$$h + ig_1 = 0 \qquad \text{for } h \in \mathfrak{H}_k \text{ and } g_1 \in \mathfrak{G}_1.$$

Then, since $i_* p_G \theta g_1 = p_G \theta(h + ig_1) = 0$ by Proposition 2.14 (ii), we have $p_G \theta g_1 = 0$ by Lemma 2.12 (i), where $p_G \colon H_{n,l} \oplus G_{n,l} \to G_{n,l}$ is the projection. Therefore, by Proposition 2.14 (ii), there is an element $h_1 \in \mathfrak{H}_1$ such that $g_1 = h_1$ in $\tilde{\Omega}_*(Z_2)$. Therefore, we have

$$ig_1 = 0 \qquad \text{and} \qquad h = 0$$

by Lemma 2.17 (i). Also, by Theorem 2.8, $g_1 - h_1$ is a linear combination of the elements

$$2E^{2n+1}W(\omega), \quad E^{2n+1}A(\omega) \ (|\omega| \geqq 2), \quad E^{2n+1}B(\omega, \omega'),$$

and so $ig_1 = i(g_1 - h_1)$ is a linear combination of their i-images. Therefore, we have the theorem by Lemma 2.17 (ii). q.e.d.

In the rest of this note, we study the Pontrjagin product in $\Omega_*(Z_{2^k})$, which is defined in the same way as (1.8) for $\mathfrak{N}_*(Z_{2^k})$.

We consider the commutative diagram

$$\begin{array}{ccc} \Omega_m^-(Z_2) & \xrightarrow{E^{2n+1}} & \Omega_{m+2n+1}(Z_2) \\ \downarrow r & & \downarrow r \\ \mathfrak{N}_m(Z_2) & \xrightarrow{E^{2n+1}} & \mathfrak{N}_{m+2n+1}(Z_2) \end{array} \tag{2.19}$$

where r's are the homomorphisms of algebras obtained by ignoring the orientations (cf. (2.1)), the upper E^{2n+1} is the one of (2.4) and the lower E^{2n+1} is defined in the same way.

PROPOSITION 2.20. (i) $rW(\omega) = X_\omega[a, S^0] + rg(\omega)[a, S^1]$,

(ii) $$rE^{2n+1}W(\omega) = X_\omega[a, S^{2n+1}] + \sum_{j=0}^{n+1} a_{2j} rg(\omega)[a, S^{2n-2j+2}],$$

where $W(\omega) \in \Omega_*^-(Z_2)$, X_ω, $a_{2j} \in \mathfrak{N}_*$ *and* $g(\omega) \in \Omega_*$ *are the elements of* (2.6), (1.20) *and Theorem* 2.3.

PROOF. (i) Since $\varepsilon_* W(\omega) = X_\omega$ (ε_* is the augmentation) by the definition of $W(\omega)$, we have

$$rW(\omega) = X_\omega[a, S^0] + \sum_{j>0} x_j([a, S^j] - [P^j][a, S^0]) \qquad (x_j \in \mathfrak{N}_*)$$

by Proposition 1.7. On the other hand, the orientation bundle $W(\omega) \to M_\omega$ can be clssified by $S^1 \to S^1/Z_2$ (cf. [7, p. 299]), and so $\Delta_1^m rW(\omega) = 0$ for $m \geqq 2$, by the definition of Δ_1 of (1.14). Also, $\varepsilon_* \Delta_1 rW(\omega) = rg(\omega)$ by the definition of $g(\omega)$ in [7, p. 309]. These facts, (1.15) and Proposition 1.16 show that $x_j = 0$ ($j \geqq 2$) and $x_1 = rg(\omega)$.

(ii) The equality follows immediately from (i), (2.19), (2.5), Corollary 1.18 and (1.15). q.e.d.

LEMMA 2.21. *The homomorphism* $r\colon \tilde{\Omega}_{2l}(Z_{2^k}) \longrightarrow \tilde{\mathfrak{N}}_{2l}(Z_{2^k})$ *of algebras in* (2.10) *is monomorphic.*

PROOF. In the commutative diagram (2.10), $r\theta^{-1}|G_{2l,k}$ is monomorphic by Lemma 2.12. Any element of $H_{2l,k} = \Sigma H_{2m+1}(Z_{2^k}; \Omega_{2l-2m-1})$ is of order 2, since $2\Omega_{2l-2m-1} = 0$ by Theorem 2.3. Therefore we have the lemma by (2.11). q.e.d.

THEOREM 2.22. *For the generators of the* Ω_**-module* $\tilde{\Omega}_*(Z_{2^k})$ *in Theorem* 2.18, *the Pontrjagin product is given as follows*:

(i) $[T, S^{2m+1}][T, S^{2n+1}]=0$.

(ii) *The images of the products*

$$[T, S^{2m+1}]iE^{2n+1}W(\omega) \quad \textit{and} \quad iE^{2m+1}W(\omega)\, iE^{2n+1}W(\omega')$$

by the monomorphism r *of the above lemma are determined by the equalities*

$$r[T, S^{2n+1}] = [T, S^{2n+1}],\ riE^{2n+1}W(\omega) = \textstyle\sum_{j=0}^{n+1} a_{2j} rg(\omega)i[a, S^{2n-2j+2}],$$

and the product formulae in $\mathfrak{N}_*(Z_{2^k})$ *of Theorem* 1.22. *In particular,*

$$iE^{2n+1}W(\omega)iE^{2n+1}W(\omega')=0.$$

Proof. The desired results follow immediately from Propositions 2.20 (ii), 1.7 (ii) and the fact that $\mathfrak{N}_*(Z_2)$ is the exterior algebra over $\mathfrak{N}_*$ (cf. [5]).

q.e.d.

References

[1] P. E. Conner and E. E. Floyd: *Differentiable Periodic Maps*, Erg. d. Math. Bd. **33**, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1964.

[2] N. Hassani: *Sur le bordisme des groupes cycliques*, C. R. Acad. Sci. Paris. **272** (1971), 776–778.

[3] K. Shibata: *Oriented and weakly complex bordism algebra of free periodic maps*, Trans. Amer. Math. Soc. **177** (1973), 199–220.

[4] ————: *A note on the formal group law of unoriented cobordism theory*, Osaka J. Math. **10** (1973), 33–42.

[5] J. C. Su: *A note on the bordism algebra of involutions*, Michigan Math. J. **12** (1965), 25–31.

[6] F. Uchida: *Bordism algebra of involutions*, Proc. Japan Acad. **46** (1970), 615–619.

[7] C. T. C. Wall: *Determination of the cobordism ring*, Ann. of Math. **72** (1960), 292–311.

[8] E. R. Wheeler: *The oriented bordism of cyclic groups*, to appear.

Department of Mathematics,
Faculty of Science,
Hiroshima University