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A Note on Coreflexive Coalgebras

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Introduction

E. J. Taft [6] has introduced the concept of coreflexive coalgebras. Finitedimensional coalgebras are coreflexive and the coalgebra of divided powers is coreflexive. The latter is a cocommutative coconnected coalgebra and its space of primitive elements is 1-dimensional. Taft has shown that if a cocommutative coconnected coalgebra is coreflexive, then the space of primitive elements is finite-dimensional. In this paper we show the converse of this result.

To this end, following D. E. Radford's idea in discussing coreflexivity in [3], we introduce a topology in the dual algebra of a coalgebra and give a necessary and sufficient condition for a coalgebra to be coreflexive.

Throughout this paper we employ the notations and terminology used in [4] and [6]. All vector spaces are over a fixed field k. For a vector space V and a subspace X of V

$$X^{\perp} = \{ v^* \in V^* \colon \langle v^*, X \rangle = 0 \}$$

and for a subspace Y of V^*

$$Y^{\perp} = \{ v \in V : \langle Y, v \rangle = 0 \}.$$

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1. The following lemma was indicated in [4], p. 240.

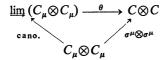
LEMMA 1. Let $\{C_{\mu}, \sigma_{\nu}^{\mu}\}$ be an inductive system with a directed set M. If every C_{μ} has a coalgebra structure and every σ_{ν}^{μ} is a coalgebra map, then $C = \lim_{\mu \to \infty} C_{\mu}$ has a coalgebra structure such that every canonical map $\sigma^{\mu}: C_{\mu} \to C$ is a coalgebra map.

Furthermore, the dual algebra C^* is isomorphic to $\lim_{\mu} C^*_{\mu}$ as algebras by the canonical map.

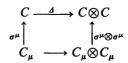
PROOF. We denote by Δ_{μ} and ε_{μ} the coalgebra structure of C_{μ} . Since σ_{ν}^{μ} is a coalgebra map the maps Δ_{μ} induce a map $\Delta': C \to \lim_{\lambda \to \infty} (C_{\mu} \otimes C_{\mu})$ such that

$$\begin{array}{ccc} C & \longrightarrow & \varinjlim \left(C_{\mu} \otimes C_{\mu} \right) \\ & \sigma^{\mu} & & \uparrow \\ \sigma^{\mu} & & \uparrow \\ C_{\mu} & \longrightarrow & C_{\mu} \otimes C_{\mu} \end{array}$$

is a commutative diagram. Further, we have a map θ induced by $\sigma^{\mu} \otimes \sigma^{\mu}$: $C_{\mu} \otimes C_{\mu} \rightarrow C \otimes C$ such that a diagram



commutes. Put $\Delta = \theta \Delta'$. Then the following diagram commutes:



Similarly, we have a map ε such that a diagram



commutes. It is then easily verified that (C, Δ, ε) is a coalgebra.

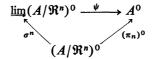
Let $\phi: C^* \to \lim_{\mu \to \infty} C^*_{\mu}$ be the canonical linear isomorphism. Then for c^* , $d^* \in C^*$, we have

$$\begin{split} \sigma_{\mu}\phi(c^{*}d^{*}) &= (\sigma^{\mu})^{*}(c^{*}d^{*}) = (\sigma^{\mu})^{*}(c^{*})(\sigma^{\mu})^{*}(d^{*}) \\ &= \sigma_{\mu}\phi(c^{*})\sigma_{\mu}\phi(d^{*}), \\ \sigma_{\mu}\phi\varepsilon &= (\sigma^{\mu})^{*}\varepsilon = \varepsilon\sigma^{\mu} = \varepsilon_{\mu}, \end{split}$$

where σ_{μ} denotes the projection of $\lim_{\mu} C_{\mu}^*$ to C_{μ}^* .

Let A be an algebra and let \mathfrak{R} be the Jacobson radical of A. If σ_m^n denotes the canonical map of A/\mathfrak{R}^m to A/\mathfrak{R}^n , then $\{A/\mathfrak{R}^n, \sigma_m^n\}$ and $\{(A/\mathfrak{R}^n)^{0*}, (\sigma_m^n)^{0*}\}$ are projective systems and $\{(A/\mathfrak{R}^n)^0, (\sigma_m^n)^0\}$ is an inductive system.

LEMMA 2. Let $\pi_n: A \to A/\Re^n$ be the canonical map, and let ψ be a map induced by $(\pi_n)^0: (A/\Re^n)^0 \to A^0$ such that a diagram



q.e.d.

commutes. Then ψ is a coalgebra isomorphism.

PROOF. We denote the coalgebra structures of $\lim_{\to} (A/\Re^n)^0$, A^0 and $(A/\Re^n)^0$ by $\{\Delta, \varepsilon\}$, $\{\overline{\Delta}, \overline{\varepsilon}\}$ and $\{\Delta_n, \varepsilon_n\}$ respectively. First we show that ψ is a coalgebra map. If $a^0 \in \lim_{\to} (A/\Re^n)^0$, then $\sigma^n(a_n^0) = a^0$ for some *n* and $a_n^0 \in (A/\Re^n)^0$. We have

$$\begin{split} \overline{\Delta}\psi(a^0) &= \overline{\Delta}\psi\sigma^n(a_n^0) = \overline{\Delta}(\pi_n)^0(a_n^0) \\ &= ((\pi_n)^0 \otimes (\pi_n)^0) \Delta_n(a_n^0) \\ &= (\psi \otimes \psi)(\sigma^n \otimes \sigma^n) \Delta_n(a_n^0) \\ &= (\psi \otimes \psi) \Delta \sigma^n(a_n^0) \\ &= (\psi \otimes \psi) \Delta (a^0), \\ \overline{\epsilon}\psi(a^0) &= \overline{\epsilon}\psi\sigma^n(a_n^0) = \overline{\epsilon}(\pi_n)^0(a_n^0) \\ &= \epsilon_n(a_n^0) = \epsilon\sigma^n(a_n^0) \\ &= \epsilon(a^0). \end{split}$$

Since $(\pi_n)^0$ is injective, ψ is injective. Finally, we show that ψ is surjective. Let $a^0 \in A^0$ and let a be a cofinite ideal contained in Ker a^0 . Then for some n > 0 $\Re^n \subseteq a$ and so $a^{\perp} \subseteq (\Re^n)^{\perp}$. We take an $a_n^0 \in (A/\Re^n)^0$ such that $(\pi_n)^0 (a_n^0) = a^0$. Then $a^0 = \psi \sigma^n (a_n^0)$. This completes the proof.

REMARK. By the last part of the above proof, it is clear that if A is a proper algebra then $\bigcap_{n\geq 0} \mathfrak{R}^n = \{0\}$. This is an extended result of Theorem 2.5 (b) in [1].

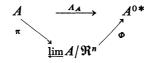
By Lemma 1, $\phi: (\lim_{n \to \infty} (A/\Re^n)^0)^* \to \lim_{n \to \infty} (A/\Re^n)^{0*}$ is an algebra isomorphism. Let Λ be a map induced by Λ_{A/\Re^n} such that a diagram

$$\underbrace{\lim_{\text{proj.}} A/\Re^n \xrightarrow{A} \lim_{\text{proj.}} (A/\Re^n)^{0*}}_{A/\Re^n \xrightarrow{A_A(\Re^n)} (A/\Re^n)^{0*}}$$

commutes. We denote by Φ the composite map

$$\underbrace{\lim A/\mathfrak{R}^n \longrightarrow \underline{\lim} (A/\mathfrak{R}^n)^{0*}}_{- \underbrace{\phi^{-1}} \to (\underline{\lim} (A/\mathfrak{R}^n)^0)^* \xrightarrow{(\psi^{-1})^*} A^{0*}.$$

LEMMA 3. The diagram



commutes, where π denotes the canonical map.

PROOF. For $a \in A$ and $a^0 \in A^0$ we have

$$<\Phi\pi(a), a^0> = <(\psi^{-1})^*\phi^{-1}\Lambda\pi(a), a^0>$$

= $<\phi^{-1}\Lambda\pi(a), \psi^{-1}(a^0)>.$

Since $\psi^{-1}(a^0) = \sigma^n(a_n^0)$ for some $a_n^0 \in (A/\Re^n)^0$,

$$\langle \Phi \pi(a), a^{0} \rangle = \langle (\sigma^{n})^{*} \phi^{-1} \Lambda \pi(a), a^{0}_{n} \rangle$$

$$= \langle (\Lambda_{A/\Re^{n}}) \pi_{n}(a), a^{0}_{n} \rangle$$

$$= \langle a^{0}_{n}, \pi_{n}(a) \rangle = \langle (\pi_{n})^{0}(a^{0}_{n}), a \rangle$$

$$= \langle \psi \sigma^{n}(a^{0}_{n}), a \rangle = \langle a^{0}, a \rangle$$

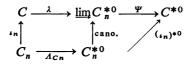
$$= \langle \Lambda_{A}(a), a^{0} \rangle .$$
q.e.d.

2. Let A be an algebra. Then $\mathfrak{T}[A] = \{\mathfrak{a} : \mathfrak{a} \text{ is a cofinite ideal of } A\} \neq \emptyset$ and is a filter base, and so we can define a uniform topology on A. With this topology A is a topological algebra. If A is proper then it is a Hausdorff space. We denote the closure of a subset X of A by \overline{X} . We can prove the following lemma by a way similar to Lemma 2.

LEMMA 4. Let A be a proper algebra. Then $\lim_{n \to \infty} (A/\overline{\Re^n})^0$ is isomorphic to A^0 as coalgebras.

Let C be a coalgebra and let R be its coradical. Then C^* is a proper algebra and R^{\perp} is the Jacobson radical of C^* ([1], Theorem 2.5). We introduce into C^* the linear weak topology determined by a dual pair $\langle C, C^* \rangle$. If X is a subspace of C^* , then the linear weak closure of X is $X^{\perp \perp}$ and clearly $\overline{X} \subseteq X^{\perp \perp}$. In particular, if C is a coreflexive coalgebra then $\overline{X} = X^{\perp \perp}$. With this topology C^* is again a topological algebra. We define $C_n = \wedge^{n+1}R, n \ge 0$, as usual.

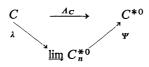
Let $\iota_n: C_n \to C$ be the inclusion map. Then we have the maps Ψ and λ induced respectively by $(\iota_n)^{*0}: (C_n)^{*0} \to C^{*0}$ and $\Lambda_{C_n}: C_n \to C_n^{*0}$ such that a diagram



commutes. Then the following lemma is easily verified.

LEMMA 5. A diagram

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commutes.

THEOREM 6. Let C be a coalgebra. Then C is coreflexive if and only if (i) all C_n are coreflexive and (ii) $C_n^{\perp} = \overline{\Re^{n+1}}$, where \Re denotes the Jacobson radical of C^{*}.

PROOF. Suppose that C is coreflexive. Then (i) is clear by Proposition 6.4 of [6]. We show (ii) by induction. If n=1, then by Proposition 9.0.0b) of [4]

$$C_1^{\perp} = (R \wedge R)^{\perp} = (R^{\perp}R^{\perp})^{\perp \perp} = (\mathfrak{R}^2)^{\perp \perp} = \overline{\mathfrak{R}^2} \,.$$

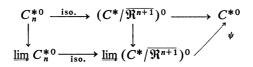
Assuming (ii) for n-1 we get

$$C_n^{\perp} = (R \wedge C_{n-1})^{\perp} = (R^{\perp}(C_{n-1})^{\perp})^{\perp \perp} = (\Re \overline{\Re^n})^{\perp \perp}$$
$$= \overline{\Re \overline{\Re^n}} = \overline{\Re^{n+1}}.$$

Conversely, we assume the conditions (i) and (ii). We prove the coreflexivity of C. First we show that Ψ is an isomorphism. The composite

$$C^* \xrightarrow{\text{cano.}} C^* / \overline{\mathfrak{R}^{n+1}} = C^* / C_n^{\perp} \xrightarrow{\text{cano.}} C_n^*$$

coincides with ι_n^* , and so by the universality of inductive systems and by a commutative diagram



the composite $\lim_{n \to \infty} C_n^{*0} \to \lim_{n \to \infty} (C^*/\overline{\Re^{n+1}})^0 \to C^{*0}$ and Ψ coincide with each other. By assumption Λ_{C_n} is an isomorphism, and so λ is also an isomorphism. Thus we see that Λ_C is an isomorphism, i.e., C is coreflexive. q.e.d.

LEMMA 7. Let C be a cocommutative coalgebra satisfying the minimum condition on subcoalgebras. Then every ideal of C^* is linearly closed and C^* is a Noetherian algebra.

PROOF. For $c^* \in C^*$ we denote by c_R^* the right translation by c^* . Then $c_R^*: C^* \to C^*$ is a continuous linear map with respect to the linear topology. Since C^* is linearly compact ([2], §10, 10), this implies that the ideal generated by c^* is linearly closed. Hence it is sufficient for us to show the last part.

Let a be an ideal of C^* . Then {b: b is a finitely generated ideal contained

in \mathfrak{a} is a non-empty family consisting of linearly closed ideals, and so it has a maximal element $\mathfrak{a}' \subseteq \mathfrak{a}$ by assumption. Then $\mathfrak{a}' = \mathfrak{a}$. In fact, if $\mathfrak{a}' \neq \mathfrak{a}$, then for any $c^* \in \mathfrak{a} - \mathfrak{a}'$, $\mathfrak{a}' + (c^*)$ is a finitely generated ideal contained in \mathfrak{a} and contains \mathfrak{a}' properly. Thus we see that C^* is Noetherian. This completes the proof.

THEOREM 8. Let C be a cocommutative coalgebra satisfying the minimum condition on subcoalgebras. Then C is coreflexive and C^* is \Re -adically complete, where \Re is the Jacobson radical of C^* .

PROOF. By Lemma 7, $(\Re^{n+1})^{\perp \perp} = \Re^{n+1}$, and so $C_n^{\perp} = (\Re^{n+1})^{\perp \perp} = \Re^{n+1} = \Re^{n+1}$. By assumption R is finite-dimensional, and so \Re is cofinite and all \Re^n are cofinite ideals. This implies that each C_n is finite-dimensional and therefore it is coreflexive. By Theorem 6, this implies that C is coreflexive.

Further since all \mathfrak{R}^n are cofinite, C^*/\mathfrak{R}^n are finite-dimensional and so they are reflexive algebras. Hence Φ in Lemma 3 is an isomorphism. Therefore π : $C^* \rightarrow \underline{\lim} C^*/\mathfrak{R}^n$ is also an isomorphism since so is Λ_{C^*} ([6], Proposition 6.1). This completes the proof.

COROLLARY 9. (i) Let C be a cocommutative coconnected coalgebra. Then C is coreflexive if and only if the space of primitive elements P(C) of C is finite-dimensional.

(ii) Let C_i , i = 1, 2, ..., n, be cocommutative coconnected coalgebras. Then $C_1 \otimes C_2 \otimes \cdots \otimes C_n$ is coreflexive if and only if every C_i is coreflexive.

PROOF. (i) By Heyneman's Theorem [5], if P(C) is finite-dimensional C satisfies the minimum condition on subcoalgebras. Hence C is coreflexive. The converse has been shown by Taft ([6], p. 1127).

(ii) By Corollary 11.0.7 of [4], $P(C_1 \otimes \cdots \otimes C_n) = P(C_1) \oplus \cdots \oplus P(C_n)$, and so this follows from (i). q.e.d.

References

- [1] L. A. Grünenfelder, Hopf-algebren und coradikal, Math. Z. 116 (1970), 166-182.
- [2] G. E. Köthe, Topological Vector Spaces I, Springer, Berlin-Heidelberg-New York, 1964.
- [3] D. E. Radford, Coreflexive coalgebras, J. Alg. 26 (1973), 512-535.
- [4] M. E. Sweedler, Hopf Algebras, Benjamin, New York, 1969.
- [5] _____, Weakening a theorem on divided powers, Trans. Amer. Math. Soc. 154 (1971), 427-428.
- [6] E. J. Taft, Reflexivity of algebras and coalgebras, Amer. J. Math. 94 (1972), 1111-1130.

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