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A theorem on splitting of algebraic vector bundles and its applications

Dedicated to Professor Yoshikazu Nakai on his 60th birthday

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0. Introduction

Let E be an algebraic vector bundle on a smooth projective algebraic scheme X defined over an algebraically closed field (arbitrary characteristic). Then it is known that after a suitable succession of blowing ups of $X, f: X' \to X, f^*(E)$ has a splitting of line bundles on X', i.e., there is a filtration of subbundles of $f^*(E)$ $F_0 \supset \cdots \supset F_r = 0$ ($r = \mathrm{rank}\, E$) such that every quotient F_i/F_{i+1} ($0 \leqq i \leqq r-1$) is a line bundle on X' (cf. [4]). In this paper, we shall prove another simple theorem on splitting of line bundles of algebraic vector bundles (cf. Theorem 2.1): Let E be an algebraic vector bundle on a smooth quasi-projective algebraic scheme defined over an algebraically closed field (arbitrary characteristic). Then there exists a finite and faithfully flat morphism $f: X' \to X$ such that $f^*(E)$ has a splitting of line bundles on X'. Hence we can prove the following (cf. Theorem 3.2) as a corollary: Let Z be an algebraic cycle of codim $= p$ on a smooth projective algebraic scheme X. Then there is a finite faithfully flat morphism $f: X' \to X$ such that $(p-1)!f^*(Z) = \sum \pm D_1 \cdots D_p$ (rat. equiv.), where D_k are divisors on X'. Hence in particular, $(p-1)!f^*(Z)$ is smoothable. Theorem 3.2 seems to be a useful fact to study algebraic cycles because it says that if a problem on algebraic cycles is not changed after multiplication of integers and pull back of finite faithfully flat morphisms, then we have only to consider the cycles Z of the forms $\sum \pm D_1 \cdots D_p$, where D_k are divisors on X. After introducing the notion of very ample vector bundles and studying their properties, we shall prove the above theorems.

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1. Very ample vector bundles

In [2], R. Hartshorne has introduced the notion of ampleness of algebraic vector bundles. Since then, we have obtained several useful algebro-geometric results using ample vector bundles. In this section, we shall define very ample vector bundles on algebraic schemes and study their properties.

Let k be an algebraically closed field with arbitrary characteristic, X an algebraic k-scheme and let E be an algebraic vector bundle on X, i.e., a locally free O_X-coherent sheaf with constant rank. We shall denote the associated projective bundle by $\pi: P(E)\to X$ and its tautological line bundle, i.e., an invertible sheaf on $P(E)$ by L_E.

DEFINITION 1.1. With the above notation, if L_E is a very ample line bundle on $P(E)$, then we define E to be *very ample*. Hence, a very ample vector buldle is ample in the sense of Hartshorne.

At first, we shall prove some formal properties of very ample vector bundles.

PROPOSITION 1.2. *Let E and E' be very ample vector bundles on a k-algebraic scheme X. Then we have the followings.*

(1) *Every quotient vector bundle of E is very ample.*

(2) *$E\oplus E'$ and $E\otimes E'$ are very ample.*

(3) *$E^{\otimes n}$, $S^n(E)$ $(n=1, 2,\dots)$ and $\wedge E^m$ $(1\leqq m\leqq \operatorname{rank} E)$ are very ample. Furthermore, let $T(E)$ be a positive tensor bundle of E (cf.[2]). If* char $k=0$, *then $T(E)$ is very ample.*

(4) *Let L be an ample line bundle and let F be a vector bundle on X. Then, there is a positive integer n_0 such that $L^{\otimes n}\otimes F$ is very ample for all $n\geqq n_0$.*

(5) *Let Y be a closed subscheme of X. Then, the restricted vector bundle $E|Y$ of E to Y is very ample.*

PROOF. (1). Let F be a quotient vector bundle of E. Then the projective bundle $P(F)$ is a closed subscheme of $P(E)$ and the tautological line bundle L_F of F is the restriction of L_E to $P(F)$. Thus, F is very ample. (2). Let $\varphi: P(E)\to P^{a-1}$ (resp. $\varphi': P(E')\to P^{b-1}$) be an embedding of $P(E)$ by L_E (resp. an embedding of $P(E')$ by $L_{E'}$). Suppose that $\{s^i | s^i\in H^0(P(E), L_E)=H^0(X, E), i=1,\dots, a\}$ and $\{\bar{s}^j | \bar{s}^j\in H^0(P(E'), L_{E'})=H^0(X. E'), j=1,\dots, b\}$ give those embeddings. Let $\{U_\alpha\}$ be an affine open covering of X such that $E|U_\alpha\cong \oplus^r O_{U_\alpha}$, $E'|U_\alpha\cong\oplus^{r'} O_{U_\alpha}$ and let $s^i|U_\alpha=(s_1^i,\dots, s_r^i)$ $(s_k^i\in\Gamma(U_\alpha, O_{U_\alpha}))$ and $\bar{s}^j|U_\alpha=(\bar{s}_1^k,\dots,\bar{s}_{r'}^j)$ $(\bar{s}_k^j\in\Gamma(U_\alpha, O_{U_\alpha}))$. Then, $\varphi|U_\alpha: P(E|U_\alpha)\cong U_\alpha\times P^{r-1}\ni (x, (\xi_1:\cdots:\xi_r))\to(\sum s_k^1(x)\xi_k:\cdots:\sum s_k^a(x)\xi_k)\in P^{a-1}$, where $\varphi|U_\alpha$ is the restricted morphism of φ to an open subscheme $P(E|U_\alpha)$. Similarly we have $\varphi'|U^\alpha: P(E'|U_\alpha)\cong U_\alpha\times P^{r'-1}\ni (x, (\eta_1:\cdots:\eta_{r'}))\to (\sum \bar{s}_k^1(x)\eta_k:\cdots:\sum\bar{s}_k^b(x)\eta_k)\in P^{b-1}$. Now we shall prove that the morphism $\varphi'': P(\mathrm{E}\oplus\mathrm{E}')\to P^{a+b-1}$ is an embedding, where φ'' is given by $\varphi''|U_\alpha: P(E\oplus E'|U_\alpha)\cong U_\alpha\times P^{r+r'-1}\ni (x, (\xi_1:\cdots:\xi_r:\eta_1:\cdots:\eta_{r'}))\to(\sum s_k^1(x)\xi_k:\cdots:\sum s_k^a(x)\,\xi_k:\sum\bar{s}_k^1(x)\eta_k:\cdots:\sum\bar{s}_k^b(x)\eta_k)\in P^{a+b-1}$ locally. In fact, since E and E' are very ample, φ'' is injective and the induced local ring homomorphism $\varphi''^*: O_{\varphi''(x)}\to O_x$ is surjective for all $x\in X$. Hence, we have only to prove that X is homeomorphic to a locally closed subscheme of P^{a+b-1} by φ''. Let ψ:

$P(E\oplus E')\to P(E)$ (resp. ψ': $P(E\oplus E')\to P(E')$) be the rational map obtained by the O_X-homomorphism: $E\ni e\to(e, 0)\in E\oplus E'$ (resp. $E'\ni e'\to(0, e')\in E\oplus E'$) and let $U=P(E\oplus E')-P(E')$ (resp. $U'=P(E\times E')-P(E)$). Then U (resp. U') is the domain of definition of ψ (resp. ψ') and ψ_U: $U\to P(E)$ (resp. $\psi'_{U'}$: $U'\to P(E')$) is an affine vector bundle over $P(E)$, i.e., $U=\mathrm{Spec}(S^{\cdot}(L_E^*\otimes\pi^*(E')))$, where π: $P(E)\to X$ is the structure morphism and $S^{\cdot}(L_E^*\times\pi^*(E'))$ is the symmetric O_X-Algebra of $L_E^*\otimes\pi^*(E')$ (L_E^* being the dual line bundle of L_E) (resp. $U'=\mathrm{Spec}(S^{\cdot}(L_{E'}^*\otimes\pi'^*(E)))$). Moreover, let $\{X_1,\dots, X_a, Y_1,\dots, Y_b\}$ be a homogeneous coordinate of P^{a+b-1}, $W=\bigcup_{i=1}^a P_{X_i}^{a+b-1}$ (resp. $W'=\bigcup_{j=1}^b P_{Y_j}^{a+b-1}$), where $P_{X_i}^{a+b-1}=\{\zeta=(\zeta_1:\cdots:\zeta_{a+b})|\zeta_i\neq 0,\ 1\leqq i\leqq a\}$ (resp. $P_{Y_j}^{a+b-1}=\{\zeta=(\zeta_1:\cdots:\zeta_{a+b})|\zeta_{a+j}\neq 0,\ 1\leqq j\leqq b\}$) and let $\bar{\psi}$: $W\in(x_1:\cdots: x_a:y_1:\cdots: y_b)\to(x_1:\cdots: x_a)\in P^{a-1}$ (resp. $\bar{\psi}'$: $W'\ni(x_1:\cdots: x_a: y_1:\cdots: y_b)\to(y_1:\cdots: y_b)\in P^{b-1}$) be the canonical projection. Then, P^{a+b-1} is covered by W and W' and $\bar{\psi}$: $W\to P^{a-1}$ (resp. $\bar{\psi}'$: $W'\to P^{b-1}$) is an affine bundle over P^{a-1}, i.e., $W=\mathrm{Spec}(S^{\cdot}(O_{P^{a-1}}(-1)^{\oplus b}))$ (resp. $W'=\mathrm{Spec}(S^{\cdot}(O_{P^{b-1}}(-1)^{\oplus a}))$). Since $L_E=\varphi^*(O_{P^{a-1}}(1))$ (resp. $L_{E'}=\varphi'^*(O_{P^{b-1}}(1))$), U (resp. U') is a closed subscheme of $\bar{\psi}^{-1}(\varphi(P(E)))$ (resp. $\bar{\psi}'^{-1}(\varphi'(P(E')))$). Therefore, $P(E\oplus E')$ is homemorphic to a locally closed subscheme of P^{a+b-1} because $P(E)$ (resp. $P(E')$) is homeomorphic to a locally closed subscheme of P^{a-1} through φ (resp. P^{b-1} through φ'). Hence, $E\oplus E'$ is very ample. We shall next prove that $E\otimes E'$ is very ample. Since E' is generated by global sections, $E\otimes E'$ is a quotient vector bundle of a direct sum of E'^s. Thus, $E\otimes E'$ is very ample by (1) and (2). (3), (4) and (5) are also easily proved by (1) and (2). q.e.d.

COROLLARY 1.3. *Let E be an ample vector bundle on X. Then there exists a positive integer n_0 such that $S^n(E)$ is very ample for all $n\geqq n_0$.*

PROOF. Let L be a very ample line bundle on X. Since E is ample, there is a positive integer n_0 such that $L^*\otimes S^n(E)$ is generated by global sections for all $n\geqq n_0$ (L^* being the dual line bundle of L). Hence $S^n(E)$ is very ample because $S^n(E)$ is a quotient vector bundle of $L^{\oplus N}$ for some positive integer N. q.e.d.

We shall next show some geometrical properties of very ample vector bundles.

Let E be a vector bundle (rank $E=r+1$) on a k-algebraic scheme X which is generated by global sections, say α: $O_X^{\oplus(n+1)}\to E$ a surjective homomorphism. Then α defines a morphism φ: $P(E)\to P^n$ and a morphism ψ: $X\to G(n, r)=$a parameter space of r-dimensional linear subsupaces of P^n as follows.

$$\psi\colon X\ni x\longrightarrow \mathrm{Im}\,\alpha(x)=(\alpha(x)\colon k(x)^{\oplus(n+1)}\longrightarrow E\otimes k(x))\in G(n, r)$$

where $k(x)$ is the residue field of x. For every $x\in X$, the r-dimensional linear subspace corresponding to $\psi(x)$ coicides with $\varphi(\pi^{-1}(x))$.

PROPOSITION 1.4. *If E is very ample, then the morphism ψ: $X\to G(n, r)$*

is an embedding for a sutiable choice of global sections of E.

PROOF. Let $\{s^i \mid s^i \in H^0(X, E), i=0, 1,\dots, n\}$ be a set of global sections of E which gives an embedding φ: $P(E) \to P^n$ and let $\{U_\alpha\}$ be an affine open covering of X such that $E \mid U_\alpha \cong \oplus^{r+1} O_{U_\alpha}$, $s^i \mid U_\alpha = (s^i_{0\alpha},\dots, s^i_{r\alpha})$ $(s^i_{j\alpha} \in \Gamma(U_\alpha, O_{U_\alpha}))$. Since φ is the following morphism on each open subscheme $\pi^{-1}(U_\alpha) \cong U_\alpha \times P^r$

$$\varphi \mid U_\alpha \colon U_\alpha \times P^r \ni (x, \xi_j) \longrightarrow (\textstyle\sum_j s^0_{j\alpha}(x)\xi_j \colon \cdots \colon \sum_j s^n_{j\alpha}(x)\xi_j) \in P^n,$$

the r-dimensional linear subsapce $\varphi(\pi^{-1}(x))$ in P^n for $x \in X$ is equal to the point $\psi(x) \in G(n, r)$. Therefore, ψ is injective because φ is an embedding. Hence, the problem is local and so we shall assume $X = U_\alpha$ for some α. For every $(i_0,\dots, i_r)$ $(0 \leqq i_0 < \cdots < i_r \leqq n\}$, let us put

$$s(i_0,\dots, i_r) = \begin{vmatrix} s_0^{i_0} \cdots\cdots\cdots s_0^{i_r} \\ \vdots \qquad\qquad \vdots \\ s_r^{i_0} \cdots\cdots\cdots s_r^{i_r} \end{vmatrix}.$$

Then, some $s(i_0,\dots, i_r)$ is an invertible element of $\Gamma(X, O_X)$. Suppose that $s(0,\dots, r)$ is invertible for simplicity. Taking a suitable base of $E \cong \oplus^{r+1} O_X$, we may assume that $s^i_j = \delta_{ij}$ for $0 \leqq i, j \leqq r$. Then $\psi(x)$ has following coordinate matrix in the open subset $U_{01\cdots r}$ of $G(n, r)$:

$$\begin{bmatrix} s_0^{r+1}(x) \cdots\cdots\cdots s_0^n(x) \\ \vdots \qquad\qquad\qquad \vdots \\ s_r^{r+1}(x) \cdots\cdots\cdots s_r^n(x) \end{bmatrix}.$$

Here, we shall denote by $U_{i_0\cdots i_r}$ the open subscheme of $G(n, r)$ defined for every pair $(i_0,\dots, i_r)$ $(0 \leqq i_0 < \cdots < i_r \leqq n)$ as follows. Let Ω be a universal domain over k and let $\{e_0,\dots, e_n\}$ be a basis of $(n+1)$-dimensional vector space $\Omega^{\oplus(n+1)}$. Then

$$U_{i_0\cdots i_r} = \{L \in \operatorname{Hom}(\Omega^{\oplus(n+1)}, \Omega^{\oplus(n+1)}) \mid L(e_{i_j}) \neq 0,\ 0 \leqq j \leqq r\}.$$

On the other hand, the following composite morphism of X to P^n for each $i (0 \leqq i \leqq r)$ is an embedding:

$$\begin{array}{ccccc} X & \longrightarrow & \pi^{-1}(X) \cong X \times P^r & \longrightarrow & P^n \\ \Cup & & \Cup & & \Cup \\ x & \longrightarrow & (x, (0\colon \cdots \colon 1 \colon \cdots \colon 0)) & \longrightarrow & (0\colon \cdots \colon 1 \colon \cdots \colon 0\colon s_i^{r+1}(x) \colon \cdots \colon s_i^n(X)). \end{array}$$

Hence the morphism ψ is an embedding. q.e.d.

COROLLARY 1.5. *Let E be an algebraic vector bundle on a quasi-projective k-aglebraic scheme X. Then, E is extendable to an algebraic vector bundle $\bar{E}$ on a projective algebraic k-scheme $\bar{X}$ containing X as an open subset.*

PROOF. Let L be a very ample line bundle on X such that $E'=E\otimes L$ is very ample. By Proposition 1.4, there is an embedding $\psi: X\to G(n, r)$ and $E'=\psi^*(Q)$, where Q is the universal quotient vector bundle of $G(n, r)$. Now, let $\bar{X}$ be the scheme-theoretic closure of X in $G(n, r)$ and let $\bar{E}'=Q\mid\bar{X}$. Then E' is extendable to $\bar{E}'$, i.e., $\bar{E}'|X=E'$. On the other hand, there is a projective algebraic scheme X' with a line bundle L' such that L is extendable to L' because L is very ample. Since X' and $\bar{X}$ are projective, there is a blowing up $f: \bar{X}'\to X'$ such that the canonical birational map $X'\to\bar{X}$ is resolved, i.e., there is a morphism $g: \bar{X}'\to\bar{X}$ and the diagram

$$\begin{tikzcd} & \bar{X}' \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X' \supset X & & \subset \bar{X} \end{tikzcd}$$

is commutative. q.e.d.

We shall show some results on chern classes of very ample vector bundles. Let X be a projective smooth algebraic scheme over k, E a vector bundle on X with rank $=r$ and let s be a global section of E. Let us denote the zero locus of s by $Z(s)$ and the tautological divisor associated to s by D. If $Z(s)$ is a subscheme of pure codimension r, then $Z(s)$ represents $c_r(E)$ (the r-th chern class of E). Let $\mathfrak{U}=\{U_\alpha\}$ be an affine open covering of X such that

$$E\mid U_\alpha\cong\oplus^r O_{U_\alpha},\qquad s\mid U_\alpha=(s_1^\alpha,\dots, s_r^\alpha)\quad (s_i^\alpha\in\Gamma(U_\alpha, O_X)).$$

Then $Z(s)$ is defined on U_α by the equations

$$s_1^\alpha=\cdots=s_r^\alpha=0$$

and D is defined on $\pi^{-1}(U_\alpha)\simeq U_\alpha\times P^{r-1}$ by the equation

$$s_1^\alpha X_1+\cdots+s_r^\alpha X_r=0.$$

Thus it is easy to see the following.

LEMMA 1.6. *D is a smooth divisor if and only if $Z(s)$ is either empty or a smooth subscheme of pure* codim $=r$.

COROLLARY 1.7. *Let X be a non-singular projective algebraic variety defined over an algebracially closed field k of* char $k=0$ *and let E be a vector bundle on X with* rank $=r(\geqq 2)$. *Assume that E is generated by global sections $\{s_1,\dots, s_t\}$, i.e., there is a surjective homomorphism $\alpha: O_X^{\oplus t}\to E$. Then there is a sufficiently general global section $s=\sum_{i=1}^t c_is_i$ $(c_i\in k)$ such that $Z(s)$ is either empty or a smooth subscheme of pure* codim $=r$.

PROOF. Let $\varphi: P(E)\to P^{t-1}$ be the morphism defined by global sections $\{s_1,\dots, s_t\}$. If $\dim \varphi(P(E))\geqq 2$, then there is a sufficiently general global section $s=\sum_{i=1}^{t} c_i s_i$ ($c_i\in k$) such that the tautological divisor D associated to s is irreducible and smooth by Bertini's theorem. By Lemma 1.6, $Z(s)$ is either empty or a smooth subscheme of pure codim $=r$. If $\dim \varphi(P(E))=1$, then $\varphi(P(E))$ is a line in P^{t-1} and $r=2$. Hence, there is a sufficiently general section $s=\Sigma c_i s_i$ with $Z(s)=\phi$. In this case, it is easy to see $E\simeq O_X\oplus O_X$. q.e.d.

More generally, let $s_1,\dots, s_i$ ($1\leqq i\leqq r$) be global sections of E with $s_i\,|\,U_\alpha=(s_{i1}^\alpha,\dots, s_{ir}^\alpha)$ ($s_{ij}^\alpha\in\Gamma(U_\alpha, O_X)$). For every α, let us put

I_α = the ideal generated by all i-minors of the matrix

$$\begin{bmatrix} s_{11}^\alpha & \cdots\cdots & s_{1r}^\alpha \\ \vdots & & \vdots \\ s_{i1}^\alpha & \cdots\cdots & s_{ir}^\alpha \end{bmatrix}.$$

Then the family of ideals $\{I_\alpha\}$ determines an ideal I of O_X such that $I\,|\,U_\alpha=I_\alpha$ for all α.

DEFINITION 1.8. With the above notation, we shall denote by $Z(s_1\wedge\cdots\wedge s_i)$ the closed subscheme of X defined by the ideal I.

Let $D_1,\dots, D_i$ be the tautological divisors associated to sections $s_1,\dots, s_i$ respecitvely. The intersection $D_1\cap\cdots\cap D_i$ is defined on each open subset $\pi^{-1}(U_\alpha)\cong U_\alpha\times P^{r-1}$ by

$$\begin{gathered} s_{11}^\alpha X_1+\cdots+s_{1r}^\alpha X_r = 0, \\ \vdots \\ s_{i1}^\alpha X_1+\cdots+s_{ir}^\alpha X_r = 0. \end{gathered}$$

Hence $Z(s_1\wedge\cdots\wedge s_i)$ is characterized set-theoretically as follows: $Z(s_1\wedge\cdots\wedge s_i)=\{x\in X\,|\dim \pi^{-1}(x)\cap D_1\cap\cdots\cap D_i\geqq r-i\}$. Now let us put $Z_k=Z(s_1\wedge\cdots\hat{s}_k\cdots\wedge s_i)$ for every k ($1\leqq k\leqq i$) (Z_k being a closed subscheme of $Z(s_1\wedge\cdots\wedge s_i)$) and $U=X-\bigcap_{k=1}^{i} Z_k$. Then we have the following as a generalization of Lemma 1.6.

LEMMA 1.9. 1) *$D_1\cap\cdots\cap D_i\cap\pi^{-1}(U)$ is a smooth subscheme of pure* codim $=i$ *if and only if either $Z(s_1\wedge\cdots\wedge s_i)\cap U$ is empty or a smooth subscheme of pure* codim $=r-i+1$ *and* $\mathrm{Sing}(Z(s_1\wedge\cdots\wedge s_i))=\bigcap_{k=1}^{i} Z_k$, *where* $\mathrm{Sing}(Z(s_1\wedge\cdots\wedge s_i))$ *denotes the singular locus of $Z(s_1\wedge\cdots\wedge s_i)$.*

2) *If $Z(s_1\wedge\cdots\wedge s_i)$ is of pure* codim $=r-i+1$, *then there is a rational map $f: Z(s_1\wedge\cdots\wedge s_i)\to P^{i-1}$ ($i\geqq 2$) such that the regular domain of f coincides with $Z(s_1\wedge\cdots\wedge s_i)-\bigcap_{k=1}^{i} Z_k$ and every $Z_k=f^{-1}(H_k)$, where H_k is a hyperplane of P^{i-1}.*

PROOF. When $Z(s_1\wedge\cdots\wedge s_i)\cap U=\phi$, our claim is obvious. Hence we assume $Z(s_1\wedge\cdots\wedge s_i)\cap U\neq\phi$. Since the problem is local, we may assume $X=U_\alpha\cap U$ and we omit the index α. Moreover, we may assume that $\det(s_{jk})$ $(1\leqq j,\ k\leqq i-1)$ is invertible on X. Then, taking a suitable basis of $E\simeq\oplus^r O_X$, we may assume that $D_1\cap\cdots\cap D_i$ is defined by

$$\begin{aligned} X_1+\cdots\cdots+s_{1i}X_i+\cdots\cdots+s_{1r}X_r&=0,\\ \ddots\qquad\qquad&\\ X_{i-1}+s_{i-1i}X_i+\cdots+s_{i-1r}X_r&=0,\\ s_{i1}X_1+\cdots\cdots+s_{ii}X_i+\cdots\cdots+s_{ir}X_r&=0.\end{aligned}$$

Here, let us put

$$f_j=\begin{vmatrix} 1 & & 0 & s_{1j}\\ & \ddots & & \vdots\\ 0 & & 1 & s_{i-1j}\\ s_{i1} & \cdots\cdots & s_{ii-1} & s_{ij}\end{vmatrix}=s_{ij}-\textstyle\sum_{k=1}^{i-1}s_{ik}s_{ki}\quad(i\leqq j\leqq r).$$

Then the ideal I is generated by the set $\{f_i,\dots,f_r\}$. Hence $\operatorname{codim}_X Z(s_1\wedge\cdots\wedge s_i)\leqq r-i+1$. Now let x be a point of $Z(s_1\wedge\cdots\wedge s_i)$, $\{z_1,\dots,z_n\}$ a regular system of parameters of X at x $(n=\dim X)$ and let us assume $\operatorname{rank}(\partial f_j/\partial z_k)_x=r-i+1-t$ $(i\leqq j\leqq r,\ 1\leqq k\leqq n)$. Consider the following Jacobian matrix:

$$\begin{bmatrix} \sum(\partial s_{1j}/\partial z_1)X_j,\dots\dots, & \sum(\partial s_{1j}/\partial z_n)X_j, & 1\ 0\cdots0, & s_{1i},\dots, & s_{1r}\\ \vdots & \vdots & \vdots\quad\vdots & \vdots & \vdots\\ \sum(\partial s_{i-1j}/\partial z_1)X_j,\dots, & \sum(\partial s_{i-1j}/\partial z_n)X_j, & 0\ 0\cdots1, & s_{i-1i},\dots, & s_{i-1r}\\ \sum(\partial s_{ik}/\partial z_1)X_k,\dots\dots, & \sum(\partial s_{ik}/\partial z_n)X_k, & s_{i1}\dots\dots & \dots\dots\dots, & s_{ir}\end{bmatrix}.$$

Since $x\in Z(s_1\wedge\cdots\wedge s_i)$, there are constants $c_1,\dots,c_{i-1}(\in k(x))$ such that $s_{ik}=\sum c_l s_{lk}$ $(1\leqq k\leqq r)$. Thus the following matrix is equivalent to the above matrix:

$$\begin{bmatrix} \sum(\partial s_{1j}/\partial z_1)X_j,\dots\dots, & \sum(\partial s_{1j}/\partial z_n)X_j, & 1\ 0\cdots0, & s_{1i},\dots, & s_{1r}\\ \vdots & \vdots & \vdots\quad\vdots & \vdots & \vdots\\ \sum(\partial s_{i-1k}/\partial z_1)X_j,\dots, & \sum(\partial s_{i-1j}/\partial z_n)X_j, & 0\ 0\cdots1, & s_{i-1i},\dots, & s_{i-1r}\\ g_1,\ \dots\dots\dots\dots & g_n, & 0\dots\dots & \dots\dots\dots, & 0\end{bmatrix},$$

where $g_m=\sum_{k=1}^{i-1}(\partial s_{ik}/\partial z_m)X_k+\sum_{j=i}^{r}(\partial s_{ij}/\partial z_m-\sum c_l\partial s_{lj}/\partial z_m)X_j(1\leqq m\leqq n)$. Hence the following linear equations have only a trivial solution if and only if $t=0$ because the rank of its coefficient matrix is equal to $r-t$ from our assumption:

$$\begin{aligned} X_1+\cdots\cdots+s_{1i}X_i+\cdots\cdots+s_{1r}X_r&=0,\\ \ddots\qquad\qquad&\\ X_{i-1}+s_{i-1i}X_i+\cdots+s_{i-1r}X_r&=0,\\ g_1=\cdots=g_n&=0.\end{aligned}$$

Therefore $D_1 \cap \cdots \cap D_i$ is a smooth subscheme of pure codim$=i$ if and only if $Z(s_1 \wedge \cdots \wedge s_i)$ is a smooth subscheme of pure codim $=r-i+1$. Since Sing$(Z(s_1 \wedge \cdots \wedge s_i))$ contains $\bigcap_{k=1}^{i} Z_k$ in general, Sing$(Z(s_1 \wedge \cdots \wedge s_i))=\bigcap_{k=1}^{i} Z_k$.

2). Let x be a general point of $Z(s_1 \wedge \cdots \wedge s_i)$. Then there is a unique $k(x)$-rational point $c(x)=(c_1(x): \cdots : c_i(x))$ of P^{i-1} such that $c_1(x)s_1(x)+\cdots+c_i(x)s_i(x)=0$. Hence we have a rational map $f\colon Z(s_1 \wedge \cdots \wedge s_i) \ni x \to c(x) \in P^{i-1}$ such that the regular domain of f coincides with $Z(s_1 \wedge \cdots \wedge s_i)-\bigcap_{k=1}^{i} Z_k$ and every $Z_k=f^{-1}(H_k)$, where H_k is a hyperplane defined by $X_k=0$. q.e.d.

Now let s_1 be a global section of E such that the associated tautological divisor D_1 is smooth and $Z(s_1) \neq \phi$, $f_1\colon X_1 \to X$ the blowing up of X with center $Z(s_1)$ and let F_1 be the exceptional divisor. Then we have the following exact sequence:

$$(*) \qquad 0 \longrightarrow O_{X_1}(F_1) \xrightarrow{\alpha} f_1^*(E) \xrightarrow{\beta} E_1 \longrightarrow 0,$$

where E_1 is a vector bundle on X_1 with rank $=r-1$. The exact sequence $(*)$ is expressed locally as follows:

$X=U=\mathrm{Spec}(A)$ such that $E\,|\,U \simeq \oplus^r O_X$.

$s_1\,|\,U=(x_1,\dots,x_r)$. $\{x_1,\dots,x_r\}$ is a part of regular system of parameters of X at the points of $Z(s_1)$.

$X_1=\bigcup_{i=1}^{r} U_i$, where $U_i=\mathrm{Spec}(A[x_1/x_i,\dots,x_r/x_i])$ $(1 \leqq i \leqq r)$.

On each affine open subset U_i,

$$\alpha_i\colon \quad 1 \longrightarrow (x_1/x_i,\dots,x_r/x_i),$$
$$\beta_i\colon (\xi_1,\dots,\xi_r) \longrightarrow (\xi_1-(x_1/x_i)\xi_i,\dots,\xi_r-(x_r/x_i)\xi_i).$$

From the exact sequence $(*)$, we have the following relation between chern classes of E and E_1: $c_i(E)=f_{1*}(c_i(E_1))$ $(1 \leqq i \leqq r-1)$. In fact, $c_i(f_1^*(E))=c_i(E_1)+F_1 \cdot c_{i-1}(E_1)$ $(1 \leqq i \leqq r-1)$ from the exact sequence $(*)$. Hence $c_i(E)=f_{1*}(c_i(E_1))$ because $f_{1*}(F_1 \cdot c_{i-1}(E_1))=0$. Let us consider the following commutative deagram:

$$\begin{tikzcd}
P(E_1) \arrow[r, "i"] \arrow[dr, "\pi_1"'] & P(f_1^*(E)) \arrow[r, "f_1'"] \arrow[d, "\pi'"] & P(E) \arrow[d, "\pi"] \\
& X_1 \arrow[r, "f_1"] & X.
\end{tikzcd}$$

Then an effective divisor $P(E_1)$ of $P(f_1^*(E))$ is defined on each open subset $\pi^{-1}(U_i)$ by the equation: $(x_1/x_i)X_1+\cdots+(x_r/x_i)X_r=0$. Therefore $h_1\colon P(E_1) \to D_1$ is the blowing up of D_1 with center $\pi^{-1}(Z(s_1))$, where $h_1=f_1' \circ i$ and the tautological line bundle L_{E_1} of E_1 is isomorphic to $h_1^*(L_E)$.

LEMMA 1.10. *With the above notation, let s_2 be a global section of E such that $Z(s_2)$ is a non-empty subscheme of pure* codim $=r$ *and let us put* $s_2'=\beta(f_1^*(s_2))$. *Then the following conditions are equivalent.*

(1) *$Z(s_2')$ is a non-empty smooth subscheme of pure* codim $=r-1$ *and $Z(s_2')\cap F_1$ is either empty or a smooth subscheme of pure* codim$=r$.

(2) *$Z(s_1\wedge s_2)$ is a subscheme of pure* codim$=r-1$ *with* $\mathrm{Sing}(Z(s_1\wedge s_2))=Z(s_1)\cap Z(s_2)$ *and $Z(s_1)\cap Z(s_2)$ is either empty or a smooth subscheme of pure* codim $=2r$. *In other words, D_2 intersects D_1 and $\pi^{-1}(Z(s_1))$ transversally.*

PROOF. (1)$\to$(2). Since the problem is local, we may assume $X=U$ as in the above argument. Let us put $s_2\,|\,U=(y_1,\dots,y_r)$. Then $Z(s_2')$ is defined on each affine open subset U_i by the equations:

$$(**)\qquad y_j-(x_j/x_i)y_i=0 \qquad (1\leqq j\leqq r, j\neq i).$$

Hence $Z(s_2')-F_1$ is isomorphic to $Z(s_1\wedge s_2)-Z(s_1)$ and $Z(s_2')$ is the proper transform of $Z(s_1\wedge s_2)$ by f_1. Thus $Z(s_1\wedge s_2)$ is a subscheme of pure codim $=r-1$ because every irreducible component of $Z(s_2')$ is not contained in F_1. The smoothness of $Z(s_2')$ implies that $Z(s_1\wedge s_2)-Z(s_1)$ is smooth. Now let x be a point of $Z(s_1)-Z(s_2)$, say $y_1(x)\neq 0$. Then $Z(s_1\wedge s_2)$ is defined in a neighbourhood of x by the equations: $x_i-(y_i/y_1)x_1=0$ $(2\leqq i\leqq r)$. Thus $Z(s_1\wedge s_2)$ is smooth at x because $\{x_1,\dots,x_r\}$ is a part of regular system of parameters of X at x. Therefore $Z(s_1\wedge s_2)-(Z(s_1)\cap Z(s_2))$ is smooth and so $\mathrm{Sing}(Z(s_1\wedge s_2))=Z(s_1)\cap Z(s_2)$. Assuming $Z(s_1)\cap Z(s_2)\neq\phi$, we shall prove that $Z(s_1)\cap Z(s_2)$ is a smooth subscheme of pure codim $=2r$. Let x be a point of $Z(s_1)\cap Z(s_2)$ and let $\{x_1,\dots,x_r,z_1,\dots,z_s\}$ $(r+s=\dim X)$ be a regular system of parameters of X at x. From our assumption, $Z(s_2')$ intersects transversally F_1 at the points lying over x. For simplicity, let us check this condition on U_1. Then $(x_1, x_2/x_1,\dots,x_r/x_1, z_1,\dots,z_s)$ is a regular system of parameters of U_1 at the point lying over x and F_1 is defined by $x_1=0$. Moreover $Z(s_2')$ is defined by the equations: $y_i-(x_i/x_1)y_1=0$ $(2\leqq i\leqq \mathrm{r})$. Hence we have $\mathrm{rank}(\partial y_j/\partial z_i-(x_j/x_1)\partial y_1/\partial z_i))=r-1$ $(1\leqq i\leqq s, 2\leqq j\leqq r)$ by direct calculation and so $\mathrm{rank}(\partial y_j/\partial z_i)_x=r$ $(1\leqq i\leqq s, 1\leqq j\leqq r)$. This implies that $Z(s_1)$ intersects $Z(s_2)$ transversally.

(2)$\to$(1). Since $Z(s_1\wedge s_2)$ is a subscheme of pure codim $=r-1$ with $\mathrm{Sing}(Z(s_1\wedge s_2))=Z(s_1)\cap Z(s_2)$, $D_1\cap D_2-\pi^{-1}(Z(s_1)\cap Z(s_2))$ is a smooth subscheme of pure codim $=2$ by Lemma 1.9. Let us assume that $Z(s_1)\cap Z(s_2)$ is a non-empty smooth subscheme of pure codim $=2r$. Then $(x_1,\dots,x_r,y_1,\dots,y_r)$ may be considered as a part of regular system of parameters of X at every point of $Z(s_1)\cap Z(s_2)$. Hence it is easily seen that D_1 meets D_2 transversally at the points lying over a point of $Z(s_1)\cap Z(s_2)$. Thus D_1 meets D_2 transversally. Moreover, D_2 intersects $\pi^{-1}(Z(s_1))$ transversally (including the case $Z(s_1)\cap Z(s_2)=\phi$) by Lemma 1.6. Now let D_2' be the tautological divisor associated to s_2'. Then,

$D_2' = h_1^*(D_1 \cap D_2)$. Since $h_1: P(E_1) \to D_1$ is the blowing up of D_1 with center $\pi^{-1}(Z(s_1))$, D_2' is smooth and meets $\pi_1^{-1}(F_1)$ transversally. Hence $Z(s_2')$ is a non-empty smooth subscheme of pure codim $= r-1$ and it intersects F_1 transversally (including the case $Z(s_2') \cap F_1 = \phi$). q.e.d.

LEMMA 1.11. *With the above notation, let $s_2, \dots, s_i$ be global sections of E satisfying the following conditions*: (i) $Z(s_2' \wedge \cdots \wedge s_i')$ *is a subscheme of pure* codim $= r-i+1$ *with no irreducible components contained in* F_1, *where* $s_k' = \beta(f_1^*(s_k))$ $(2 \leqq k \leqq i)$. (ii) $\operatorname{Sing}(Z(s_2' \wedge \cdots \wedge s_i')) = \bigcap_{k=2}^{i} Z_k'$, *where* $Z_k' = Z(s_2' \wedge \cdots \widehat{s_k'} \cdots \wedge s_i')$ *and* codim $(\operatorname{Sing}(Z(s_2' \wedge \cdots \wedge s_i'))) \geqq 2(r-i+2)$. (iii) codim $(Z(s_1) \cap Z(s_2 \wedge \cdots \wedge s_i)) \geqq 2r-i+2$. *Then we have the followings.*

(1) $Z(s_1 \wedge \cdots \wedge s_i)$ *is a subscheme of pure* codim $= r-i+1$.

(2) $\operatorname{Sing}(Z(s_1 \wedge \cdots \wedge s_i)) = \bigcap_{k=1}^{i} Z_k$, *where* $Z_k = Z(s_1 \wedge \cdots \widehat{s_k} \cdots \wedge s_i)$ $(1 \leqq k \leqq i)$ *and* codim $\bigcap_{k=1}^{i} Z_k \geqq 2(r-i+2)$.

PROOF. (1) is obvious. (2). In order to prove the first part, we have only to show that $Z(s_1 \wedge \cdots \wedge s_i) - \bigcap_{k=1}^{i} Z_k$ is smooth at every point x of $Z(s_1) - Z(s_2 \wedge \cdots \wedge s_i)$ from our assumption (ii). Since the problem is local, we may assume $X = U \ni x$. Let us put $s_j | U = (y_{j_1}, \dots, y_{jr})$ $(2 \leqq j \leqq r)$. For simplicity, we assume $\det(y_{jl}(x)) \neq 0$ $(2 \leqq j \leqq i,\ 1 \leqq l \leqq i-1)$. Then $Z(s_1 \wedge \cdots \wedge s_i)$ is defined in a neighbourhood of x by the equations:

$$0 = f_j = \begin{vmatrix} x_1 \cdots x_{i-1} & x_j \\ \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & \begin{matrix} y_{2j} \\ \vdots \\ y_{ij} \end{matrix} \end{vmatrix} = (-1)^{i+1}\left(x_j - \textstyle\sum_{l=1}^{i-1} x_l y_{l+1j}\right) \quad (i \leqq j \leqq r).$$

Hence $Z(s_1 \wedge \cdots \wedge s_i) - \bigcap_{k=1}^{i} Z_k$ is smooth at x because $\{x_1, \dots, x_r\}$ is a part of regular system of parameters of X at x. Since $\bigcap_{k=2}^{i} Z_k' - F_1$ is isomorphic to $\bigcap_{k=1}^{i} Z_k - Z(s_1)$, every irreducible component of $\bigcap_{k=1}^{i} Z_k$ not contained in $Z(s_1)$ has codim $\geqq 2(r-i+2)$ from (ii). Moreover since $2r+i+2 = 2(r-i+2) + (i-2) \geqq 2(r-i+2)$, every irreducible component of $\bigcap_{k=1}^{i} Z_k$ contained in $Z(s_1)$ has codim $\geqq 2(r-i+2)$ from (iii). Therefore codim $\bigcap_{k=1}^{i} Z_k \geqq 2(r-i+2)$.

q.e.d.

Let X be a non-singular projective algebraic variety ($\dim X = n \geqq 2$) defined over an algebraically closed field k of char $k=0$ and let E be an ample vector bundle on X generated by global sections with rank $= r$ $(2 \leqq r \leqq \dim X)$. If $t = \dim H^0(X, E)$, then there is a morphism $\varphi: P(E) \to P^{t-1}$ defined by the complete linear system $|L_E|$ which is finite because L_E is ample and hence $\dim \varphi(P(E)) = n+r-1$.

1) By Corollary 1.7, there is a global section s_1 of E such that $Z(s_1)$ is either

empty or a smooth subscheme of pure codim $=r$. Since E is ample, every chern class $c_i(E)$ $(1 \leqq i \leqq r)$ is not zero and hence $Z(s_1)=c_r(E)$ is not empty. Let D_1 be the irreducible smooth divisor associated to s_1 and let $\mathrm{tr}(L_E|D_1)$ be the trace of $|L_E|$ to D_1. Then the linear system $\mathrm{tr}(L_E|D_1)$ is free from base points and $\dim \varphi'(D_1)=n+r-2 \geqq 2$, where $\varphi'\colon D_1 \to P^{t-2}$ is the morphism defined by $\mathrm{tr}(L_E/D_1)$. Hence there is a sufficiently general global section s_2 of E such that D_2 is an irreducible smooth divisor and it intersects D_1 and $\pi^{-1}(Z(s_1))$ transversally (including the case $Z(s_1) \cap Z(s_2)=\phi$) by Bertini's theorem and Corollary 1.7. Then $Z(s_1 \wedge s_2)$ is a subscheme of pure codim $=r-1$ with $\mathrm{Sing}(Z(s_1 \wedge s_2))= Z(s_1) \cap Z(s_2)$ and $Z(s_1) \cap Z(s_2)$ is either empty or a smooth subscheme of pure codim$=2r$. Let $f_1\colon X \to X_0 = X$ be the blowing up of X_0 with center $Z(s_1)$, F_1 the exceptional divisor and let $h_1 = f_1' \circ i\colon P(E_1) \to D_1$, where

$$(*)_1 \qquad 0 \longrightarrow O_{X_1}(F_1) \xrightarrow{\alpha_1} f_1^*(E) \xrightarrow{\beta_1} E_1 \longrightarrow 0$$

and

$$\begin{tikzcd} P(E_1) \arrow[r, hook, "i"] \arrow[dr, "\pi_1"'] & P(f_1^*E)) \arrow[r, "f_1'"] \arrow[d, "\pi'"] & P(E) \arrow[d, "\pi"] \\ & X_1 \arrow[r, "f_1"] & X_0 . \end{tikzcd}$$

Then $Z(s_2^{(1)})$ is a smooth subscheme of pure codim $=r-1$ and it intersects F_1 transversally, where $s_2^{(1)}=\beta_1(f_1^*(s_2))$ by Lemma 1.10. In other words, if $D_2^{(1)}$ denotes the associated divisor to $s_2^{(1)}$, then $D_2^{(1)}=h_1^*(D_1 \cap D_2)$ is an irreducible smooth divisor and intersects $\pi_1^{-1}(F_1)$ transversally. Since $Z(s_2^{(1)})$ that is the proper transform of $Z(s_1 \wedge s_2)$ by f_1 represents $c_{r-1}(E_1)$, $Z(s_1 \wedge s_2)$ represents $c_{r-1}(E)$.

2) $L_{E_1} \simeq h_1^*(L_E)$ and every chern class $c_i(E_1)$ $(1 \leqq i \leqq r-1)$ is not zero. From $(*)_1$, we see that E_1 is generated by global sections which come from those of E. If we define L_1 to be the linear system of L_E generated by those sections, then $L_1 = h_1^*(\mathrm{tr}(L_E | D_1))$ and $\varphi_1 = \varphi' \circ h_1\colon P(E_1) \to P^{t-2}$ is the corresponding morphism. We shall assume $r \geqq 3$, i.e., $\mathrm{rank}\, E_1 = r-1 \geqq 2$. Since $\dim \varphi_1(D_2^{(1)}) = n+r-3 \geqq n \geqq 2$, there is a sufficiently general global section s_3 of E such that $D_3^{(1)}$ is an irreducible smooth divisor and it intersects $D_2^{(1)}$, $\pi_1^{-1}(Z(s_2^{(1)}))$ and $\pi_1^{-1}(F_1)$ transversally by Bertini's theorem and Corollary 1.7, where $D_3^{(1)}$ is the associated divisor to $s_3^{(1)} = \beta(f_1^*(s_3))$. Moreover, we can take $D_3^{(1)}$ and D_3 such that $D_3^{(1)}$ (resp. D_3) intersects $\pi_1^{-1}(F_1) \cap D_2^{(1)}$ (resp. $\pi^{-1}(Z(s_1)) \cap D_2$) transversally. In fact, $\dim \varphi_1(\pi_1^{-1}(F_1) \cap D_2^{(1)}) = \dim \varphi(\pi^{-1}(Z(s_1)) \cap D_2) = (n-r)+r-2$. If $n>r$, then they are obvious by Bertini's theorem. If $n=r$, then $\varphi_1(\pi_1^{-1}(F_1) \cap D_2^{(1)})= \varphi(\pi^{-1}(Zs_1)) \cap D_2)$ consists of finitely many linear subspaces P^{r-2} in P^{t-2} and hence we can take $D_3^{(1)}$ and D_3 satisfying the above condition. This implies that

$Z(s_2^{(1)} \wedge s_3^{(1)})$ has no irreducible components contained in F_1 and $\operatorname{codim}(Z(s_1) \cap Z(s_2 \wedge s_3)) \geqq 2r-1$. Now let $f_2: X_2 \to X_1$ be the blowing up of X_1 with center $Z(s_2^{(1)})$ and let F_2 be the exceptional divisor. Then we have the following exact sequence similarly:

$$(*)_2 \qquad 0 \longrightarrow O_{X_2}(F_2) \xrightarrow{\alpha_2} f_2^*(E_1) \xrightarrow{\beta_2} E_2 \longrightarrow 0,$$

where E_2 is a vector bundle on X_2 with rank $=r-2$. If we put $s_3^{(2)} = \beta_2(f_2^*(s_3^{(1)}))$, then $Z(s_3^{(2)})$ is a smooth subscheme of pure codim $=r-2$ and meets F_2 transversally. On the other hand, $Z(s_1 \wedge s_2 \wedge s_3)$ is a subscheme of pure codim $=r-2$ with $\operatorname{Sing}(Z(s_1 \wedge s_2 \wedge s_3)) = Z(s_2 \wedge s_3) \cap Z(s_1 \wedge s_3) \cap Z(s_1 \wedge s_2)$ and codim $(\operatorname{Sing}(Z(s_1 \wedge s_2 \wedge s_3))) \geqq 2(r-1)$ by Lemma 1.11. Moreover, we see that $Z(s_1 \wedge s_2 \wedge s_3)$ represents $c_{r-2}(E)$.

3) We can proceed with the above argument as follows. Let us suppose that we have $\{s_j\}$ $(1 \leqq j \leqq i,\ 1 \leqq i \leqq r-1)$, a set of global sections of E satisfying the followings: $Z(s_1)$ is a smooth subscheme of pure codim $=r$ and D_2 intersects D_1, and $\pi^{-1}(Z(s_1))$ transversally. We assume that we can define the blowing up of X_{j-1}, $f_j: X_j \to X_{j-1}$ $(1 \leqq j \leqq i)$ with smooth center $Z(s_j^{(j-1)})$ of pure codimension $r-j+1$ and $s_k^{(j)} = \beta_j(f_j^*(s_k^{(j-1)}))$ $(j+1 \leqq k \leqq i)$ inductively, where (a) $X_0 = X$ and $s_j^{(0)} = s_j$, (b) $(*)_j$: $0 \to O_{X_j}(F_j) \to f_j^*(E_{j-1}) \to E_j \to 0$ is an exact sequence of vector bundles on X_j (F_j being the exceptional divisor of f_j and E_j being a vector bundle with rank $=r-j$). Here let π_j: $P(E_j) \to X_j$ be the structure morphism and let $D_k^{(l)}$ be the divisor associated to the section $s_k^{(l)}$ $(0 \leqq l \leqq i-1,\ l+1 \leqq k \leqq i)$. With the above notation, we assume moreover that the following conditions hold: For every j $(1 \leqq j \leqq i-1)$, (i) $D_k^{(j-1)}$ $(j+1 \leqq {}^\forall k \leqq i)$ intersects $D_k^{(j-1)}$ and $\pi_{j-1}^{-1}(Z(s_j^{(j-1)}))$ transversally, (ii) $D_{j+1}^{(l)}$ $(0 \leqq {}^\forall l \leqq j-2)$ intersects $D_{l+1}^{(l)} \cap D_{l+2}^{(l)} \cap \cdots \cap D_j^{(l)}$ and $\pi^{-1}(Z(s_{l+1}^{(l)}) \cap D_{l+2}^{(l)} \cap \cdots \cap D_j^{(l)})$ transversally. Then we can take a sufficiently general global section s_{i+1} of E such that the conditions(i), (ii) hold also for the set $\{s_j\}$ $(1 \leqq j \leqq i+1)$. In fact, the proof is quite similar to the one given in 2). Therefore, E has sufficiently general global sections $\{s_1, \dots, s_r\}$ such that they satisfy the conditions (i), (ii). Hence for every i $(1 \leqq i \leqq r)$, $Z(s_1 \wedge \cdots \wedge s_i)$ is a subscheme of pure codim $=r-i+1$ with $\operatorname{Sing}(Z(s_1 \wedge \cdots \wedge s_i)) = \bigcap_{k=1}^{i} Z_k$, where $Z_k = Z(s_1 \wedge \cdots \hat{s}_k \cdots \wedge s_i)$ $(1 \leqq k \leqq i)$ and codim $(\operatorname{Sing}(Z(s_1 \wedge \cdots \wedge s_i))) \geqq 2(r-i+2)$. $Z(s_1 \wedge \cdots \wedge s_i)$ represents $c_{r-i+1}(E)$. Moreover, if we denote by g_{i-1} the restricted morphism of $f_1 \circ \cdots \circ f_{i-1}$: $X_{i-1} \to \cdots X_1 \to X_0$ to $Z(s_i^{(i-1)})$, then g_{i-1}: $Z(s_i^{(i-1)}) \to Z(s_1 \wedge \cdots \wedge s_i)$ is a desingularization.

Hence we get the following.

THEOREM 1.12. *We shall follow the above notations. Let X be a non-singular projective algebraic variety* ($\dim X \geqq 2$) *defined over an algebraically closed field of characteristic zero and let E be an ample vector bundle on X*

generated by global sections with rank $=r$ $(2\leqq r\leqq \dim X)$. *Then E has sufficiently general global sections $\{s_1,\dots, s_r\}$ satisfying the following properties: For every i $(1\leqq i\leqq r)$,*

(1) $Z(s_1\wedge\cdots\wedge s_i)$ *is a subscheme of pure* codim $=r-i+1$ *with* $\mathrm{Sing}(Z(s_1\wedge\cdots\wedge s_i))=\bigcap_{k=1}^{i} Z_k$ *and* $\mathrm{codim}(\bigcap_{k=1}^{i} Z_k)\geqq 2(r-i+2)$.

(2) $Z(s_1\wedge\cdots\wedge s_i)$ *represents* $c_{r-i+1}(E)$.

(3) *If we denote by g_{i-1} the restricted morphism of $f_1\circ\cdots\circ f_{i-1}: X_{i-1}\to\cdots\to X_0=X$ to $Z(s_i^{(i-1)})$, then $g_{i-1}: Z(s_i^{(i-1)})\to Z(s_1\wedge\cdots\wedge s_i)$ is a desingularization of $Z(s_1\wedge\cdots\wedge s_i)$ by successive blowing ups.*

(4) *There is a rational map $\xi_i: Z(s_1\wedge\cdots\wedge s_i)\to P^{i-1}$ whose regular domain coincides with $Z(s_1\wedge\cdots\wedge s_i)-\bigcap_{k=1}^{i} Z_k$ and every $Z_k=\xi^{-1}(H_k)$, where H_k is a hyperplane of P^{i-1}.*

In the proof of Theorem 1.12, Bertini's theorem has played a very important role. Though it fails in positive characteristic, Theorem 1.12 holds partially true in arbitrary characteristic if E is a very ample vector bundle. In fact, let E be a very ample vector bundle on X. Then there is a global s_1 of E such that the associated divisor D_1 to s_1 is smooth and $Z(s_1)\neq\phi$ because L_E is very ample. Moreover, there exists a sufficiently general global section s_2 of E such that D_2 intersects D_1 and $\pi^{-1}(Z(s_1))$ transversally, where D_2 is the associated divisor to s_2. If $r\geqq 3$, then we can take furthermore a sufficiently general global section s_3 of E satisfying the following conditions because L_E is very ample: (1) D_3 intersects D_1, $\pi^{-1}(Z(s_1))$, $D_1\cap D_2$, $\pi^{-1}(Z(s_1))\cap D_2$ and $\pi^{-1}(Z(s_1)\cap Z(s_2))$ transversally, (2) D_3 intersects $\pi^{-1}(Z(s_1\wedge s_2)-Z(s_1))\cap D_1$ transversally (by Lemma 1.9, $\pi^{-1}(Z(s_1\wedge s_2)-Z(s_1))\cap D_1$ is smooth). Now let $f_1: X_1\to X$ be the blowing up of X with center $Z(s_1)$, F_1 the exceptional divisor and let $s'_j=\beta_1(f_1^*(s_j))$, $D'_j=$the associated divisor to s'_j be as before $(j=2, 3)$. Then we the following.

LEMMA 1.13. *Under the above assumption,*

(1) *D'_3 intersects D'_2 and $\pi_1^{-1}(Z(s'_2))$ transversally.*

(2) *$D'_2\cap D'_3$ intersects $\pi_1^{-1}(F_1)$ transversally. Hence $\{D'_2, D'_3\}$ satisfies the equivalent condition in Lemma 1.10.*

PROOF. From our assumption, it is easily seen that we have only to prove that D'_3 intersects $\pi_1^{-1}(Z(s'_2))$ transversally. As for the transversality, it is enough to show that D'_3 meets $\pi_1^{-1}(Z(s'_2))$ transversally at the points lying over $F_1=f_1^{-1}(Z(s_1))$ because $f'_1: \pi_1^{-1}(Z(s'_2)-F_1)\cap D'_3\cong\pi^{-1}(Z(s_1\wedge s_2)-Z(s_1))\cap D_1\cap D_3$ is an isomorphism. Since the problem is local, we may assume that $X=U$ is an affine scheme with $E\mid U\simeq\oplus^r O_U$. Let us put $s_1\mid U=(x_1,\dots, x_r)$, $s_2\mid U=(y_1,\dots, y_r)$ and $s_3\mid U=(z_1,\dots, z_r)$. Without loss of generality, it is enough to check the transversality over the affine open subset U_1. On the open subset $\pi_1^{-1}(U_1)\simeq U_1\times P^{r-2}$, D'_3 is defined by the equation:

$$(z_2-(x_2/x_1)z_1)X_2+\cdots+(z_r-(x_r/x_1)z_1)X_r=0$$

and $\pi_1^{-1}(Z(s_2'))$ is defined by the equations:

$$(*)\qquad y_i-(x_i/x_1)y_1=0 \qquad (2\leqq i\leqq r),$$

where $\{X_2,\dots, X_r\}$ is a homogeneous coordinate of P^{r-2}. Let us fix a regular frame $\{x_1, x_2/x_1,\dots, x_r/x_1, u_1,\dots, u_s, X_2,\dots, X_r\}$ of $\pi_1^{-1}(U_1)$ at the point $(x', (\xi_2,\dots, \xi_r))$ of $D_3'\cap\pi_1^{-1}Z(s_2')$ where $\{x_1,\dots, x_r, u_1,\dots, u_r\}$ $(r+s=\dim X)$ is a regular system of parameters of X at $x=f_1(x')$.

Case i) $x\notin Z(s_2)$, i.e., $y_1\neq 0$. If we put $x_i'=x_i-(y_i/y_1)x_1$ $(2\leqq i\leqq r)$, then $Z(s_1)$ is defined in a neighbourhood of x by $x_1=x_2'=\cdots=x_i'=0$. Moreover $y_i/y_1-x_i/x_1=-x_r'/x_1$ and $z_i-(x_i/x_1)z_1=z_i-(y_i/y_1)z_1-(x_i'/x_1)z_1$ $(2\leqq i\leqq r)$. Hence we may assume that $\pi_1^{-1}(Z(s_2'))$ is defined by the equations: $x_i/x_1=0$ $(2\leqq i\leqq r)$ and so we have the following Jacobian matrix at $(x', (\xi_2,\dots, \xi_r))$;

$$\begin{bmatrix} * & *\cdots* & *\cdots* & z_2-(y_2/y_1)z_1\cdots z_r-(y_r/y_1)z_1 \\ 0 & 1\quad 0 & *\cdots* & 0\cdots\cdots\cdots 0 \\ \vdots & \ddots\ \vdots & & \vdots \qquad\qquad \vdots \\ 0 & 0\quad 1 & *\cdots* & 0\cdots\cdots\cdots 0 \end{bmatrix}.$$

This implies that if $x\notin Z(s_2\wedge s_3)$, then we can prove the transversality. Thus we assume $x\in Z(s_2\wedge s_3)-Z(s_2)$. Since $D_2\cap D_3$ meets $\pi^{-1}(Z(s_1))$ transversally from our assumption, $Z(s_2\wedge s_3)$ meets $Z(s_1)$ transversally at x by Lemma 1.9. Hence we can take $u_1=z_2-(y_2/y_1)z_1,\dots, u_{r-1}=z_r-(y_r/y_1)z_1$. Then the Jacobian matrix becomes the following one:

$$\begin{bmatrix} * & *\cdots* & X_2\cdots X_r & *\cdots* & 0\cdots 0 \\ 0 & 1\quad 0 & 0\cdots 0 & *\cdots* & 0\cdots 0 \\ \vdots & \ddots\ \vdots & \vdots\quad \vdots & \vdots\quad \vdots & \vdots\quad \vdots \\ 0 & 0\quad 1 & 0\cdots 0 & *\cdots* & 0\cdots 0 \end{bmatrix}$$

and hence we are done.

Case ii) $x\in Z(s_2)$, i.e., $y_1=0$. Since $Z(s_1)\cap Z(s_2)$ is a smooth subscheme of pure codim $=2r$, we can take $u_1=y_1,\cdots, u_r=y_r$. Thus we have the following Jacobian matrix in this case:

$$\begin{bmatrix} * & -z_1X_2\cdots -z_1X_r & * & *\cdots\quad\cdots* & z_2-(x_2/x_1)z_1\cdots z_r-(x_r/x_1)z_1 \\ 0 & 0\cdots\cdots 0 & -x_2/x_1 & 1\quad 0\ *\cdots* & 0\cdots\cdots\cdots 0 \\ \vdots & \vdots\qquad \vdots & \vdots & \ddots\ \vdots\quad \vdots & \vdots\qquad\qquad \vdots \\ 0 & 0\cdots\cdots 0 & -x_r/x_1 & 0\quad 1\ *\cdots* & 0\cdots\cdots\cdots 0 \end{bmatrix}.$$

If either $z_1\neq 0$, i.e., $x\notin Z(s_3)$, or $x'\notin Z(s_3')$, then we are done. Assume that $x\in Z(s_3)$ and $x'\in Z(s_3')$. Since $Z(s_1)\cap Z(s_2)\cap Z(s_3)$ is a smooth subscheme of pure codim $=3r$, we can take $u_{r+1}=z_1,\dots, u_{2r}=z_r$. Hence we can prove the transversality. q. e. d.

Therefore we get the following.

THEOREM 1.14. *Let X be a non-singular projective algebraic variety* ($\dim X \geqq 2$) *defined over an algebracically closed field of arbitrary characteristic and let E be a very ample vector bundle with* rank $=r$ ($2 \leqq r \leqq \dim X$). *Then there are sufficiently general global sections s_1, s_2, s_i($1 \leqq i \leqq \mathrm{Min}\,\{3, r\}$) which satisfy the properties* (1), (2), (3) *and* (4) *in Theorem* 1.12.

2. A theorem on splitting of vector bundles

The aim of this section is to prove the following theorem.

THEOREM 2.1. *Let X be a smooth quasi-projective k-algebraic scheme (k being an algebraically closed field of arbitary characteristic) and let E be an algebraic vector bundle on X. Then there is a quasi-projective smooth k-algebraic scheme X' over X satisfying the following conditions*:

(1) *$f: X' \to X$ is finite and faithfully flat.*

(2) *$f^*(E)$ has a splitting of line bundles, i.e., there is a sequence of subvector bundles of $f^*(E)=F_0 \supset F_1 \supset \cdots \supset F_r=\{0\}$ such that every quotient bundle F_i/F_{i+1} ($0 \leqq i \leqq r-1$) is a line bundle on X'* ($r=\mathrm{rank}\, E$).

We shall fix some notation and prepare elementary lemmas. Let X be a quasi-projective k-algebraic scheme, E (resp. L_E) a very ample vector bundle on X (resp. the tautological line bundle of E) and let $\pi: P(E) \to X$ be the structure morphism. Then for every positive integer n, $L_E^{\otimes n}$ gives an embedding of $P(E)$ into a projective space P^N because E is very ample. We shall denote an embedding by $\varphi_n: P(E) \to P^N$ (or, φ simply). Moreover, we shall denote by $[Y]$ the linear subspace of P^N spanned by Y for a closed integral subscheme Y of P^N. $(P^N)^*$ means the dual projective space of P^N.

LEMMA 2.2. *With the above notation, let x be a k-rational point of X, Y a closed irreducible subscheme in the fiber $\pi^{-1}(x) \cong P^{r-1}$ ($r=\mathrm{rank}\, E$) and let I be the defining ideal of Y_{red} in P^{r-1}. Then*

$$\dim\, [\varphi(Y_{red})] = {}_r\mathrm{H}_n - h^0(P^{r-1}, I(n)) - 1,$$

where ${}_r\mathrm{H}_n$ means multi-combination, $I(n)=I \otimes O_{P^{r-1}}(n)$ and $h^0(P^{r-1}, I(n))=$ $\dim H^0(P^{r-1}, I(n))$.

PROOF. Let J be the defining ideal of $\varphi(Y_{red})=Y_{red}$ in P^N. Then we have an exact sequence:

$$0 \longrightarrow J(1) \longrightarrow O_{P^N}(1) \longrightarrow O_{Y_{red}}(1) \longrightarrow 0.$$

Since we have the following exact sequence:

$$0 \longrightarrow H^0(J(1)) \longrightarrow H^0(O_{P^N}(1)) \longrightarrow H^0(O_{Y_{red}}(1)) \longrightarrow H^1(J(1)) \longrightarrow 0,$$

dim {hyperplanes of P^N containing $Y_{red}\} = h^0(J(1)) - 1$. On the other hand, there is an exact sequence:

$$0 \longrightarrow I_x(1) \longrightarrow J(1) \longrightarrow J/I_x(1) \longrightarrow 0.$$

where I_x = the defining ideal of $\varphi(\pi^{-1}(x))$ in P^N. Hence we have the exact sequence:

$$0 \longrightarrow H^0(I_x(1)) \longrightarrow H^0(J(1)) \longrightarrow H^0(J/I_x(1)) \longrightarrow H^1(I_x(1)) \longrightarrow \cdots.$$

Here the canonical map $H^0(O_{P^N}(1)) \to H^0(O_{P^{r-1}}(n))$ is surjective and $H^1(I_x)=0$. Thus $h^0(J(1)) = h^0(I_x(1)) + h^0(J/I_x(1)) = h^0(I_x(1)) + h^0(I(n)) = N+1-{}_r\mathrm{H}_n + h^0(I(n))$. Therefore, $\dim [(Y_{red})] = {}_r\mathrm{H}_n - h^0(I(n)) - 1$. q.e.d.

The following is a key lemma to prove our Theorem 2.1. Though Hironaka ([3]) has shown it in a more general form, we shall give here another simple proof.

LEMMA 2.3. *Let X ($\dim X \geqq 1$) be a quasi-projective smooth k-algebraic scheme, E a very ample vector bundle on X with* rank$=r(\geqq 2)$ *and let Y be a closed integral subscheme of $P(E)$ which is of pure relative dimension $d(\geqq 1)$ over X. Then there is a positive integer n_0 such that if we embed $P(E)$ into a projective space P^N by $L_E^{\otimes n}$ for $n \geqq n_0$, then there is a non-empty open subscheme U of $(P^N)^*$ satisfying the following: For a general member H of U, $H \cap Y$ is a closed integral subscheme which is of pure relative $(d-1)$-dimension over X. Moreover, if Y is smooth and flat over X, then $H \cap Y$ is smooth and flat over X.*

PROOF. For every positive integer n, we fix an embedding φ: $P(E) \to P^N$ by $L_E^{\otimes n}$. Let $\Gamma = \{(x, H) \in X \times (P^N)^* \mid H$ contains an irreducible component of $\pi^{-1}(x) \cap Y$, set-theoretically$\}$. Then Γ is a closed subscheme of $X \times (P^N)^*$. In fact let $\Delta = \{(z, H) \in P(E) \times (P^N)^* \mid z \in H\}$ and let θ: $\Delta \cap (Y \times (P^N)^*) \ni (z, H) \to (\pi(z) \times H) \in X \times (P^N)^*$. Then $\Gamma = \{(x, H) \in X \times (P^N)^* \mid \dim \theta^{-1}(x, H) = d\}$. Since θ is projective and is of relative dimension $\leqq d$, Γ is closed. Let p: $\Gamma \to X$ (resp. q: $\Gamma \to (P^N)^*$) be the first projection (resp. the second projection). By Lemma 2.2, for every k-rational point x of X, $\dim p^{-1}(x) = \mathrm{Max}\{N - {}_r\mathrm{H}_n + h^0(I(n))\}$, where the I's are the reduced defining ideals of irreducible components of $\pi^{-1}(x) \cap Y$ in P^{r-1}. On the other hand, the families of $O_{P^{r-1}}$-coherent sheaves $\{I\}$ and $\{O_{P^{r-1}}/I\}$ on the fibers of π: $P(E) \to X$ are limited families. In fact, let $\{Z_i\}$ be the set of irreducible components of $(Y \cap \pi^{-1}(x))_{red}$ for k-rational points x of X. Then, the degrees of Z_i's with respect to a hyperplane of P^{r-1} are bounded above. Thus the family $\{O_{P^{r-1}}/I\}$ is a limited family by Chow's theorem (cf. [5]). Therefore there is a positive integer m_0 such that all the ideals I are m_0-regular with respect to $O_{P^{r-1}}(1)$. Hence we have that for every $n \geqq m_0$, $H^i(I(n))=0$ for all $i>0$ and I. Thus $\dim \Gamma \leqq \dim X + N - {}_r\mathrm{H}_n + \mathrm{Max}\,\{\chi(I(n))\} = \dim X + N - {}_r\mathrm{H}_n + \chi(O_{P^{r-1}}(n)) - \mathrm{Min}\,\{\chi((O_{P^{r-1}}/I)(n))\}$ for all $n \geqq m_0$ and I. Since $\chi((O_{P^{r-1}}/I)(n)) = (a/d!)n^d + \cdots$ $(a>0, d\geqq 1)$, we can take a positive integer $n_0 \geqq m_0$ such that $\mathrm{Min}\,\{\chi((O_{P^{r-1}}/I)(n))\} > \dim X$ for all $n \geqq n_0$. Thus $\dim q(\Gamma) \leqq \dim \Gamma < N$ if we take $n \geqq n_0$. Therefore there is a non-empty open subset U of $(P^N)^*$ such that every member H of U does not contain any irreducible components of $Y \cap \pi^{-1}(x)$ for every k-rational point x of X, i.e., $H \cap Y$ is of pure relative $(d-1)$-dimension over X. If we take a sufficiently general member H of U, then $H \cap Y$ is integral. Moreover, if Y is smooth and flat over X, then $H \cap Y$ is smooth and flat over X. q.e.d.

We shall now prove Theorem 2.1. Since X is quasi-projective, there is an ample line bundle L on X such that $E \otimes L$ is very ample. Hence we may assume that E is very ample to prove our claim. Let π: $P(E) \to X$ be the structure morphism. Using Lemma 2.3 interatively, we see that there is a smooth closed subscheme X' of $P(E)$ such that $\pi \mid X'$: $X' \to X$ is finite and faithfully flat. On the other hand, it is well-known there is an exact sequence of vector bundles on $P(E)$.

$$0 \longrightarrow F \longrightarrow \pi^*(E) \longrightarrow L_E \longrightarrow 0,$$

where F is a vector bundle on $P(E)$ with rank $= r-1$. Hence if we put $f = \pi \circ i$ (i: $X' \to P(E)$ being the closed immersion), then we have an exact sequence of vector bundles on X'.

$$0 \longrightarrow F \mid X' \longrightarrow f^*(E) \longrightarrow L_E \mid X' \longrightarrow 0$$

Proceeding with the above argument to $F \mid X'$ if necessary, we can obtain a quasi-projective smooth k-algebraic scheme X' over X desired in Theorem 2.1. q.e.d.

REMARK 2.4. When X is projective, we can take an algebraic k-scheme X' satisfying $H^i(X, O_X) \simeq H^i(X', O_{X'})$ for $1 \leqq i \leqq \dim X - 1$ in addition to the conditions in Theorem 2.1.

3. Application

We shall show some applications of Theorem 2.1 in this section. When X is an affine variety, every vector bundle on X is associated to a finitely generated projective module and hence the following is easily seen from Theorem 2.1.

THEOREM 3.1. *Let A be a regular affine k-algebra and let P be a finitely* *generated projective A-module. Then there is a regular affine k-algebra B which is a finite and faithfully flat A-module such that* $P \otimes_A B$ *is a direct sum of projective B-modules of rank* 1.

When X is projective, the following implies that every algebraic cycle of X can be written as a sum of subvarieties which are complete intersections of divisors after a suitable multiplication of an integer and a pull-back of some finite faithfully flat morphism.

THEOREM 3.2. *Let X be a smooth integral projective algebraic k-scheme and let* $Z=\sum n_i Z_i$ *be an algebraic cycle of* codim $=p(\geqq 1)$ *on X. Then there is a finite and faithfully flat morphism* $f\colon X' \to X$, *where X' is smooth and integral, such that*

$$(p-1)!f^*(Z) = \sum \pm D_1 \cdots D_p \,(\textit{rat. equiv.}),$$

where D_i are divisors on X'. Hence in particular, $(p-1)!f^*(Z)$ *is smoothable.*

PROOF. We may assume that Z is a prime cycle to prove our claim. Let O_Z be the structure sheaf of Z. Then it is known that $c_p(O_Z)=(-1)^{p-1}(p-1)!Z$ (rat. equiv.) (cf. [1]). Let the following be the resolution of O_Z by vector bundles on X.

$$0 \longrightarrow E_n \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_1 \longrightarrow O_X \longrightarrow O_Z \longrightarrow 0 \quad (n = \dim X).$$

Then there is a finite faithfully flat morphism $f\colon X' \to X$ such that every $f^*(E_i)$ $(1 \leqq i \leqq n)$ has a splitting of line bundles on X' by Theorem 2.1. Then every chern class $c_j(f^*(E_i)) = \sum \pm D_1 \cdots D_j$ $(1 \leqq i, j \leqq n)$, where D_k are divisors on X'. Hence $(-1)^{p-1}(p-1)!f^*(Z) = c_p(f^*(O_Z)) = \sum \pm D_1 \cdots D_p$ for suitable divisors D_k on X'.

q.e.d.

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