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Derivatives of Stokes multipliers

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§1. In the paper [2] we dealt with the two point connection problem for the general system of linear differential equations

$$t\frac{dx}{dt} = (A_0 + A_1 t + \cdots + A_q t^q)x, \tag{1.1}$$

where t is a complex variable and the coefficients A_i $(i=0, 1,\dots, q)$ are n by n complex constant matrices. We shall briefly reconsider our theory of solving the connection problem in the case where there appear logarithmic solutions. And the purpose of this paper is to show that there holds the Frobenius theorem in a global sense, that is, concerning the Stokes multipliers of a set of logarithmic solutions, once the Stokes multipliers for the non-logarithmic solution are known, all the Stokes multipliers for its adjunct logarithmic solutions can be determined only by means of the differentiation with respect to the characteristic exponent.

Let

$$\begin{aligned} X(t) &= (x_0(t), x_1(t),\dots, x_\gamma(t)) \\ &= \textstyle\sum_{m=0}^{\infty} (G_0(m, \rho_0), G_1(m, \rho_0),\dots, G_\gamma(m, \rho_0))t^{m+\rho_0+J} \\ &= \textstyle\sum_{m=0}^{\infty} \mathfrak{G}(m, \rho_0)t^{m+\rho_0+J}, \end{aligned} \tag{1.2}$$

J being the $(\gamma+1)$ by $(\gamma+1)$ shifting matrix

$$J = \begin{pmatrix} 0 & 1 & & \text{\Large 0} \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & \ddots & 1 \\ \text{\Large 0} & & & & 0 \end{pmatrix},$$

be a matrix solution of (1.1) involving the logarithmic term t^J near the regular singularity $t=0$. Then the coefficient matrix $\mathfrak{G}(m, \rho_0)$ is given by a matrix solution of the following system of linear difference equations for $\rho=\rho_0$: Letting ρ be a parameter,

$$\begin{aligned} &(m+\rho-A_0)\mathfrak{G}(m, \rho) + \mathfrak{G}(m, \rho)J \\ &\quad = A_1\mathfrak{G}(m-1, \rho) + A_2\mathfrak{G}(m-2, \rho) + \cdots + A_q\mathfrak{G}(m-q, \rho). \end{aligned} \tag{1.3}$$

Putting $\mathfrak{G}(m, \rho)=(G_0(m, \rho), G_1(m, \rho),\dots, G_\gamma(m, \rho))$, we write down (1.3) in the column vectorial form

$$(1.4)\qquad (m+\rho-A_0)G_0(m, \rho) = A_1G_0(m-1, \rho) + A_2G_0(m-2, \rho) + \cdots + A_qG_0(m-q, \rho),$$

$$(1.5)\qquad (m+\rho-A_0)G_i(m, \rho) = A_1G_i(m-1, \rho) + A_2G_i(m-2, \rho) + \cdots + A_qG_i(m-q, \rho) - G_{i-1}(m, \rho) \quad (i=1, 2,\dots, \gamma).$$

Then from this it is easy to see that $G_i(m, \rho)$ $(i=1, 2,\dots, \gamma)$ can be given by the derivatives of $G_0(m, \rho)$ with respect to the parameter ρ. In fact, let us define the differential operator

$$(1.6)\qquad \partial^i = \frac{1}{i!}\frac{\partial^i}{\partial\rho^i}$$

and then the Leibniz rule of differentiation is written in the form

$$(1.7)\qquad \partial^i[uv] = \textstyle\sum_{l=0}^{i} \partial^{i-l}[u]\partial^l[v].$$

As an immediate consequence of (1.7), we have

$$(1.8)\qquad G_i(m, \rho) = \partial^i[G_0(m, \rho)] \qquad (i=1, 2,\dots, \rho)^{*)}.$$

We here assume, for simplicity, that A_q is similar to a nonsingular diagonal matrix diag $(\lambda_1, \lambda_2,\dots, \lambda_n)$. Then, near another singularity $t=\infty$, which is the irregular singulairty of rank q, there exists the formal matrix solution

$$Y(t) = (y^1(t), y^2(t),\dots, y^n(t)),$$

where the column vectors $y^k(t)$ are of the form

$$y^k(t) = \exp\left(\frac{\lambda_k}{q}t^q + \frac{\alpha_{q-1}^k}{q-1}t^{q-1} + \cdots + \alpha_1^k t\right)t^{\mu_k} \textstyle\sum_{s=0}^{\infty} H^k(s)t^{-s} \quad (k=1, 2,\dots, n).$$

The coefficient $H^k(s)$ satisfies the system of linear difference equations

$$(1.9)\qquad (A_q-\lambda_k)H^k(s) + \cdots + (A_1-\alpha_1^k)H^k(s-q+1) + (A_0+s-q-\mu_k)H^k(s-q) = 0.$$

Now we reduce the connection problem of deriving relations between $X(t)$

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*) The $G_i(m, \rho)$ are functions of $(m+\rho)$, and hence in the paper [2] we used the differentiation with respect to the variable m.

and $Y(t)$ to that for the difference equation (1.3).

Let $g_l^k(m, \rho)$ $(l=1, 2,\dots, q)$ be a fundamental set of solutions of the modified gamma equation

$$(1.10)\qquad (m+\rho-\mu_k)g^k(m, \rho) = \alpha_1^k g^k(m-1, \rho) + \cdots + \alpha_{q-1}^k g^k(m-q+1, \rho) + \lambda_k g^k(m-q, \rho)$$

and let us define the $(\gamma+1)$ by $(\gamma+1)$ matrix functions $\mathscr{G}_l^k(m, \rho)$ by

$$(1.11)\qquad \mathscr{G}_l^k(m, \rho) = \exp\left(J\frac{\partial}{\partial\rho}\right)[g_l^k(m, \rho)] = (1+J\partial^1+\cdots+J^\gamma\partial^\gamma)[g_l^k(m, \rho)] \qquad (k=1, 2,\dots, n;\ l=1, 2,\dots, q).$$

One can then verify

$$(1.12)\qquad \mathfrak{G}(m, \rho) = \textstyle\sum_{s=0}^{\infty}\sum_{k=1}^{n}\sum_{l=1}^{q} H^k(s)(T_{l0}^k(\rho),\dots, T_{l\gamma}^k(\rho))\mathscr{G}_l^k(m+s, \rho).$$

In fact, to prove this, we put

$$(1.13)\qquad F_{li}^k(m, \rho) = \textstyle\sum_{s=0}^{\infty} H^k(s)\partial^i[g_l^k(m, \rho)] \qquad (i=0, 1,\dots, \gamma)$$

and then, since the $H^k(s)$ are independent of the parameter ρ, we immediately see that

$$(1.14)\qquad F_{li}^k(m, \rho) = \partial^i[F_{l0}^k(m, \rho)] \qquad (i=1, 2,\dots, \gamma).$$

On the other hand, it can be proved from (1.9) and (1.10) that nq functions $F_{l0}^k(m, \rho)$ $(k=1, 2,\dots, n;\ l=1, 2,\dots, q)$ satisfy the same difference equation as (1.4):

$$(1.15)\qquad (m+\rho-A_0)F_{l0}^k(m, \rho) = A_1F_{l0}^k(m-1, \rho) + A_2F_{l0}^k(m-2, \rho) + \cdots + A_qF_{l0}^k(m-q, \rho)$$

and moreover they form a fundamental set of solutions. Hence, by the theory of linear difference equations, we have

$$(1.16)\qquad G_0(m, \rho) = \textstyle\sum_{k=1}^{n}\sum_{l=1}^{q} T_{l0}^k(m, \rho)F_{l0}^k(m, \rho),$$

where the $T_{l0}^k(m, \rho)$ are periodic functions of m with period 1, and however, when m takes integral values, they may be considered as constants. The formula (1.16) determines the constants $T_{l0}^k(m, \rho)\equiv T_{l0}^k(\rho)$ depending on ρ.

Next, for $i=1, 2,\dots, \gamma$ successively, we see from (1.14) and the differentiation of (1.15) that for the $T_{lj}^k(\rho)$ $(j=0, 1,\dots, i-1)$ already determined, the function

$$\textstyle\sum_{j=1}^{i}\sum_{k=1}^{n}\sum_{l=1}^{q} T_{li-j}^k(\rho)F_{lj}^k(m, \rho)$$

becomes a particular solution of (1.5) and hence we have

$$G_i(m, \rho) = \sum_{j=0}^{i} \sum_{k=1}^{n} \sum_{l=1}^{q} T_{li-j}^{k}(\rho) F_{lj}^{k}(m, \rho), \tag{1.17}$$

which determines the constants $T_{li}^k(\rho)$ depending on ρ. Consequently, we obtain

$$\begin{aligned}
&(G_0(m, \rho), G_1(m, \rho), \dots, G_\gamma(m, \rho)) \\
&\quad = \sum_{k=1}^{n} \sum_{l=1}^{q} (F_{l0}^k(m, \rho), F_{l1}^k(m, \rho), \dots, F_{l\gamma}^k(m, \rho)) \\
&\qquad\qquad \times (T_{l0}^k(\rho) + T_{l1}^k(\rho)J + \cdots + T_{l\gamma}^k(\rho)J^\gamma) \\
&\quad = \sum_{k=1}^{n} \sum_{l=1}^{q} \sum_{s=0}^{\infty} H^k(s)(g_l^k(m+s, \rho), \partial^1[g_l^k(m+s, \rho)], \dots, \partial^\gamma[g_l^k(m+s, \rho)]) \\
&\qquad\qquad \times (T_{l0}^k(\rho) + T_{l1}^k(\rho)J + \cdots + T_{l\gamma}^k(\rho)J^\gamma),
\end{aligned}$$

which just implies (1.12) since

$$(a_0, a_1, \dots, a_\gamma)(b_0 + b_1 J + \cdots + b_\gamma J^\gamma) = (b_0, b_1, \dots, b_\gamma)(a_0 + a_1 J + \cdots + a_\gamma J^\gamma).$$

Now, taking account of (1.8) and operating ∂^i on (1.16), we have by the Leibniz rule

$$\begin{aligned}
G_i(m, \rho) &= \sum_{k=1}^{n} \sum_{l=1}^{q} \sum_{j=0}^{i} \partial^{i-j}[T_{l0}^k(\rho)] \partial^j [F_{l0}^k(m, \rho)] \\
&= \sum_{k=1}^{n} \sum_{l=1}^{q} \sum_{j=0}^{i} \partial^{i-j}[T_{l0}^k(\rho)] F_{lj}^k(m, \rho).
\end{aligned}$$

We have thus obtained

THEOREM 1. *In* (1.12) *there hold*

$$T_{li}^k(\rho) = \partial^i [T_{l0}^k(\rho)] \qquad (i = 1, 2, \dots, \gamma). \tag{1.18}$$

We now come back to the connection problem between $X(t)$ and $Y(t)$. In the above consideration we put $\rho = \rho_0$. Defining

$$Z_l^k(t, s) = t^{\rho_0} \sum_{m=0}^{\infty} \mathscr{G}_l^k(m+s, \rho_0) t^m, \tag{1.19}$$

from (1.12) we have

$$X(t) = \sum_{s=0}^{\infty} \sum_{k=1}^{n} \sum_{l=1}^{q} H^k(s)(T_{l0}^k(\rho_0), T_{l1}^k(\rho_0), \dots, T_{l\gamma}^k(\rho_0)) Z_l^k(t, s) t^J.$$

From this expansion formula and the global behavior of $Z_l^k(t, s)$, we finally obtain the required result [2; Theorem 5.5]:

$$X(t) \sim Y(u) \mathscr{T}_{S_N} \quad \text{as} \quad t \longrightarrow \infty \text{ in } S_N,$$

where the sectors $S_N (N \in \boldsymbol{Z})$ cover the whole Riemann surface of logarithm and each matrix $\mathscr{T}_{S_N}$ consists of n vectors for $k = 1, 2, \dots, n$ chosen according to the sector S_N from the vectors $(T^k_{l0}(\rho_0), T^k_{l1}(\rho_0), \dots, T^k_{l\gamma}(\rho_0)) \exp(2\pi i p(J+\rho_0-\mu_k))$ $(p \in \boldsymbol{Z})$.

Lastly, it is remarked that the above consideration covers all the cases where A_0 has not only a finite number of multiple eigenvalues but also eigenvalues which are congruent modulo integers. For instance, assume that A_0 is similar to Jordan canonical matrix of the form

$$A_0 \sim \begin{pmatrix} \begin{array}{c|c} * & \\ \hline & A_{11} \end{array} & & & 0 \\ & A_{22} & & \\ & & \ddots & \\ 0 & & & \begin{array}{c|c} A_{pp} & \\ \hline & * \end{array} \end{pmatrix},$$

where the eigenvalue ρ_i of the Jordan block A_{ii} with size r_i differs by integers from the others, i.e., $\rho_1 - \rho_i \in \boldsymbol{Z}$,

$$0 \leqq \rho_1 - \rho_2 \leqq \rho_1 - \rho_3 \leqq \cdots \leqq \rho_1 - \rho_p$$

and for such a set of Jordan blocks there exists a set of logarithmic solutions of the form

$$X(t) = (\hat{X}_1(t), \hat{X}_2(t), \dots, \hat{X}_p(t))t^J, \tag{1.20}$$

where

$$\hat{X}_i(t) = t^{\rho_i} \textstyle\sum_{m=0}^{\infty} \mathfrak{G}_i(m) t^m \qquad (i=1, 2, \dots, p).$$

Then we can express $\mathfrak{G}_i(m)$ in terms of $G_i(m, \rho)$ defined in (1.8) as follows: Putting $\gamma_0 = 0$, $\gamma_j = r_1 + r_2 + \cdots + r_j$ $(j=1, 2, \dots, p)$, we have

$$\mathfrak{G}_i(m) = (G_{\gamma_{i-1}}(m-\rho_1+\rho_i, \rho_1), G_{\gamma_{i-1}+1}(m-\rho_1+\rho_i, \rho_1), \dots, G_{\gamma_i-1}(m-\rho_1+\rho_i, \rho_1))$$
$$(i=1, 2, \dots, p).$$

From this fact we can see again that all the Stokes multipliers of (1.20) are derived from those of the non-logarithmic solution

$$x_0(t) = t^{\rho_1} \textstyle\sum_{m=0}^{\infty} G_0(m, \rho_1) t^m$$

by means of (1.18).

In the next section, as an interesting example illustrating just this case, we treat of the connection problem for the extended Airy equations.

§2. Now we shall consider the connection problem for the extended Airy equation

$$z^n y^{(n)} - \delta z^q y = 0,$$

where $\delta=(-1)^n$. By the change of variables $z=t^n$ we can reduce this single linear differential equation to the system of the same type as (1.1), where $A_1=A_2=\cdots=A_{q-1}=0$:

$$t\frac{dx}{dt} = (A_0+A_q t^q)x. \tag{2.1}$$

Here the constant matrices A_0 and A_q are of the form

$$A_0 = \mathrm{diag}(\sigma_1, \sigma_2,\dots, \sigma_n) \qquad \sigma_j = (n-j)(n+q) \qquad (j=1, 2,\dots, n), \tag{2.2}$$

$$A_q = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & \ddots & 1 \\ \delta n^n & 0 & \cdots & \cdots & 0 \end{pmatrix}. \tag{2.3}$$

As we have seen in the paper [2], if $q\geqq n$, then we have a fundamental set of non-logarithmic solutions. In this case we completely solved the connection problem for (2.1). (See [3].) However, if $1\leqq q\leqq n-1$, then there appear logarithmic solutions. In fact, corresponding to the first q characteristic exponents $\sigma_j\,(j=1, 2,\dots, q)$ in (2.2), one can find non-logarithmic solutions

$$x_j(t) = t^{\sigma_j} \textstyle\sum_{m=0}^{\infty} \mathfrak{g}_j(m)t^m \qquad (j=1, 2,\dots, q) \tag{2.4}$$

and then, for each j $(1\leqq j\leqq q)$ there exist $\gamma_j=\max\{\gamma\geqq 0, j+q\gamma\leqq n\}$ linearly independent logarithmic solutions incidental to $x_j(t)$. That is, we have q sets of solutions

$$(x_j(t), x_{j+q}(t),\dots, x_{j+\gamma_j q}(t)) \qquad (j=1, 2,\dots, q), \tag{2.5}$$

which correspond to the characteristic exponents

$$(\sigma_j, \sigma_{j+q},\dots, \sigma_{j+\gamma_j q}), \tag{2.6}$$

respectively. So from now on we investigate the connection problem for one set of solutions of (2.5). Let us denote one of (2.6) by

$$(\rho_0, \rho_1, \rho_2,\dots, \rho_\gamma), \tag{2.7}$$

where $\rho_i=\rho_0-iq(n+q)$ $(i=1, 2,\dots, \gamma)$.
Then, corresponding to (2.7), we have a matrix solution

$$\begin{aligned} X(t) &= (x_0(t), x_1(t),\dots, x_\gamma(t)) \\ &= (\hat{x}_0(t), \hat{x}_1(t),\dots, \hat{x}_\gamma(t))t^J, \end{aligned} \tag{2.8}$$

where J denotes a $(\gamma+1)$ by $(\gamma+1)$ shifting matrix and

$$\hat{x}_i(t) = t^{\rho_i} \textstyle\sum_{m=0}^{\infty} \mathfrak{g}_i(m) t^m \qquad (i=0, 1,\dots, \gamma). \tag{2.9}$$

The coefficients $\mathfrak{g}_i(m)$ satisfy the following systems of linear difference equations

$$(m+\rho_0-A_0)\mathfrak{g}_0(m) = A_q \mathfrak{g}_0(m-q), \tag{2.10}$$

$$(m+\rho_i-A_0)\mathfrak{g}_i(m) = A_q \mathfrak{g}_i(m-q) - \mathfrak{g}_{i-1}(m+\rho_i-\rho_{i-1}) \qquad (i=1, 2,\dots, \gamma). \tag{2.11}$$

Using the functions $G_i(m, \rho)$ defined in §1, we can express $\mathfrak{g}_i(m)$ $(i=0, 1,\dots, \gamma)$ in the form

$$\mathfrak{g}_i(m) = G_i(m-\rho_0+\rho_i, \rho_0) \qquad (i=0, 1,\dots, \gamma). \tag{2.12}$$

On the other hand, since A_q is similar to $\operatorname{diag}(\lambda_1, \lambda_2,\dots, \lambda_n)$, where $\lambda_k = n\exp\left[\frac{2\pi i}{n}\left(-\frac{n}{2}+k-1\right)\right]$ $(k=1, 2,\dots, n)$, we have a formal matrix solution $Y(t)=(y^1(t), y^2(t),\dots, y^n(t))$, the column $y^k(t)$ being of the form

$$y^k(t) = \exp\left(\frac{\lambda_k}{q} t^q\right) t^{\mu_k} \textstyle\sum_{s=0}^{\infty} H^k(s) t^{-s} \qquad (k=1, 2,\dots, n). \tag{2.13}$$

The constants μ_k are equal to $(n+q)(n-1)/2$ and the coefficients $H^k(s)$ satisfy the system of linear difference equations (1.9) with $A_1=A_2=\cdots=A_{q-1}=0$ and $\alpha_1^k=\alpha_2^k=\cdots=\alpha_{q-1}^k=0$.

As for the modified gamma equation (1.10), in this case, it is exactly the gamma equation of q-th order

$$(m+\rho-\mu_k) g^k(m) = \lambda_k g^k(m-q). \tag{2.14}$$

We take as a fundamental set of solutions of (2.14)

$$g_l^k(m) = \frac{1}{q} \frac{\left\{\left(\frac{\lambda_k}{q}\right)^{1/q} \omega_q^{-(l-1)}\right\}^{m+\rho-\mu_k}}{\Gamma\left(\frac{m+\rho-\mu_k}{q}+1\right)} \tag{2.15}$$

$$(\omega_q = \exp(2\pi i/q);\ l=1, 2,\dots, q),$$

which then define the matrix functions $\mathscr{G}_l^k(m, \rho)$ by (1.11).

We are now in a position to apply the theory of the preceding section to the above problem. To this end, we first have to solve the linear difference equation (1.4) under the initial condition that $G_0(0, \rho) \neq 0$ and $G_0(r, \rho)=0$ $(r<0)$. It can be immediately seen that

$$G_0(qm, \rho) \neq 0, \quad G_0(m', \rho) = 0 \quad (m' \neq qm).$$

We put, denoting the column vector by the suffix $_*$,

$$G_0(qm, \rho) = (g^{(1)}(m, \rho), g^{(2)}(m, \rho), \dots, g^{(n)}(m, \rho))_*.$$

If for some ν $(1 \leqq \nu \leqq n)$, we impose the condition that $g^{(\nu)}(0, \rho) \neq 0$, $g^{(i)}(0, \rho) = 0$ $(i \neq \nu)$, then, as long as $(m+\rho-A_0)$ is non-singular, we can observe that

$$(2.16) \qquad G_0(q(nm+k), \rho) = (0, \dots, 0, g^{(k')}(nm+k, \rho), 0, \dots, 0)_* \quad k' = \nu - k \ (\mathrm{mod}\ n),$$

and moreover we can take, e.g.,

$$(2.17) \qquad g^{(1)}(nm+\nu-1, \rho) = \left(\frac{\delta}{q^n}\right)^m \prod_{j=1}^{n} \frac{1}{\Gamma\left(m+1+\frac{\nu-j}{n}+\frac{\rho-\rho_j}{nq}\right)}.$$

According to our theory, if we can know the explicit value or the asymptotic behavior of $G_0(m, \rho)$, then we can always determine the Stokes multipliers $T_{l0}^k(\rho)$ from (1.16) by means of the limiting method. (See [3]). In this case considered, we have

$$(2.18) \qquad T_{l0}^k(\rho) = \frac{(qn/\lambda_k)^{\nu-1}}{(2\pi)^{(n-1)/2} n^{1/2}} \left(\left(\frac{qn}{\lambda_k}\right)^{1/q} \omega_q^{l-1}\right)^{\rho-\mu_k} \quad (k=1, 2, \dots, n;\ l=1, 2, \dots, q).$$

Now we shall return to the original problem. Below ρ_0 is assumed to be σ_ν for some ν $(1 \leqq \nu \leqq q)$. From (2.12) and (2.16–7) we can know the explicit values of $\mathfrak{g}_i(m)$ and then from (1.17) we have

$$(2.19) \quad \mathfrak{g}_i(m) = \sum_{j=0}^{i} \sum_{k=1}^{n} \sum_{l=1}^{q} T_{l\,i-j}^k(\rho_0) F_{lj}^k(m-\rho_0+\rho_j, \rho_0) \qquad (i=0, 1, \dots, \gamma).$$

From this we obtain

$$\begin{aligned}(2.20) \qquad \hat{x}_i(t) &= t^{\rho_i} \sum_{m=0}^{\infty} \mathfrak{g}_i(m) t^m \\ &= t^{\rho_i} \sum_{m=0}^{\rho_0-\rho_i-1} \mathfrak{g}_i(m) t^m \\ &\quad + \sum_{j=0}^{i} \sum_{k=1}^{n} \sum_{l=1}^{q} T_{l\,i-j}^k(\rho_0) F_{lj}^k(m, \rho_0) t^{m+\rho_0} \\ &= t^{\rho_i} \sum_{m=0}^{\rho_0-\rho_i-1} \mathfrak{g}_i(m) t^m + \tilde{x}_i(t) \qquad (i=0, 1, \dots, \gamma),\end{aligned}$$

whence, putting

$$\tilde{X}(t) = (\tilde{x}_0(t), \tilde{x}_1(t), \dots, \tilde{x}_\gamma(t)),$$

we have the expansion formula in terms of the functions (1.19)

$$(2.21) \quad \tilde{X}(t) = \sum_{s=0}^{\infty} \sum_{k=1}^{n} \sum_{l=1}^{q} H^k(s)(T_{l0}^k(\rho_0), T_{l1}^k(\rho_0), \dots, T_{l\gamma}^k(\rho_0)) Z_l^k(t, s).$$

We here apply the following proposition to $\tilde{X}(t)$.

PROPOSITION [2; Theorem 3.1]

$$Z_l^k(t, s) \sim \delta_{ll'} \exp\left(\frac{\lambda_k}{q} t^q\right) t^{\mu_k - s - J} - t^{\rho_0}\{\mathscr{G}_l^k(s-1)t^{-1} + \mathscr{G}_l^k(s-2)t^{-2} + \cdots\}$$

as $t \to \infty$ in the sector

$$S_{l'}^k: \quad -\frac{3\pi}{q} + \frac{2\pi}{q} l' \leqq \arg \lambda_k^{1/q} t < -\frac{\pi}{q} + \frac{2\pi}{q} l',$$

where $\delta_{ll'}$ denotes the Kronecker delta.

Then we have

$$\text{(2.22)} \qquad \tilde{X}(t) \sim \textstyle\sum_{k=1}^n y^k(t)(T_{l_k 0}^k(\rho_0), T_{l_k 1}^k(\rho_0), \ldots, T_{l_k \gamma}^k(\rho_0)) t^{-J}$$
$$- \textstyle\sum_{m=1}^\infty (G_0(-m, \rho_0), G_1(-m, \rho_0), \ldots, G_\gamma(-m, \rho_0)) t^{\rho_0 - m}$$

as t tends to infinity in the sector

$$\text{(2.23)} \qquad S(l_1, l_2, \ldots, l_n) = \textstyle\bigcap_{k=1}^n S_{l_k}^k.$$

Taking account of

$$\textstyle\sum_{m=1}^\infty G_i(-m, \rho_0) t^{\rho_0 - m} = \sum_{m=1-\rho_0+\rho_i}^\infty \mathfrak{g}_i(-m) t^{\rho_i - m} \qquad (i=0, 1, \ldots, \gamma),$$

and applying (2.22) to the matrix solution $X(t)$ expressed, as is seen from (2.8) and (2.20), in the form

$$X(t) = \tilde{X}(t) t^J$$
$$+ (0, \textstyle\sum_{m=0}^{\rho_0-\rho_1-1} \mathfrak{g}_1(m) t^{m+\rho_1}, \ldots, \sum_{m=0}^{\rho_0-\rho_\gamma-1} \mathfrak{g}_\gamma(m) t^{m+\rho_\gamma}) t^J,$$

we consequently obtain the following

THEOREM 2. *As t tends to infinity in the sector* (2.23), *there holds*

$$X(t) \sim \textstyle\sum_{k=1}^n y^k(t)(T_{l_k 0}^k(\rho_0), T_{l_k 1}^k(\rho_0), \ldots, T_{l_k \gamma}^k(\rho_0))$$
$$- \textstyle\sum_{m=1}^\infty (\mathfrak{g}_0(-m) t^{\rho_0}, \mathfrak{g}_1(-m) t^{\rho_1}, \ldots, \mathfrak{g}_\gamma(-m) t^{\rho_\gamma}) t^{-m+J},$$

where the Stokes multipliers are explicitly given by (2.18) *and Theorem* 1, *i.e.,*

$$T_{li}^k(\rho_0) = \partial^i [T_{l0}^k(\rho_0)] = \frac{1}{i!}\left\{\log\left(\left(\frac{qn}{\lambda_k}\right)^{1/q} \omega_q^{l-1}\right)\right\}^i T_{l0}^k(\rho_0) \qquad (i=0, 1, \ldots, \gamma).$$

For the extended Airy equation, the same result as above has been derived by B. L. J. Braaksma [1]. Applying Theorem 1, we can also solve completely connection problems for the extended Bessel equation

$$z^n y^{(n)} + a_1 z^{n-1} y^{(n-1)} + \cdots + a_{n-1} z y' + (a_n - \delta z^\nu) y = 0$$

and moreover the generalized hypergeometric equation

$$z^n y^{(n)} + \sum_{l=1}^{n} (a_l + b_l z^v) z^{n-l} y^{(n-l)} = 0,$$

the a_l, the $b_l(b_l=0(l=1, 2,\dots, v-1),\ b_v \neq 0)$, δ and v being complex numbers, in case the characteristic equation is of the form

$$[\rho]_n + a_1[\rho]_{n-1} + \cdots + a_n = (\rho-\rho_0)^{h_0}(\rho-\rho_1)^{h_1}\cdots(\rho-\rho_\gamma)^{h_\gamma}\{\cdots\},$$

where the ρ_i differ by integers each other. See [3, 4].

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