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Two-step methods for ordinary differential equations

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1. Introduction

Consider the initial value problem

$$(1.1) \qquad y' = f(x, y), \quad y(x_0) = y_0,$$

where the function $f(x, y)$ is assumed to be sufficiently smooth. Let $y(x)$ be the solution of this problem and

$$(1.2) \qquad x_n = x_0 + nh \quad (n=1, 2,\dots; h>0),$$

where h is a stepsize. Let y_1 be an approximation of $y(x_1)$ obtained by some appropriate method. We are concerned with the case where the approximations y_j $(j=2, 3,\dots)$ of $y(x_j)$ are computed by two-step methods.

In our previous paper [1] we considered the methods of the form

$$(1.3) \qquad y_{n+1} = y_n + h \textstyle\sum_{j=0}^{r} p_j k_j,$$

where

$$(1.4) \qquad k_0 = f(x_{n-1}, y_{n-1}), \quad k_1 = f(x_n, y_n),$$

$$(1.5) \qquad k_i = f(x_n + a_i h, y_n + b_i(y_n - y_{n-1}) + h \textstyle\sum_{j=0}^{i-1} c_{ij} k_j),$$

$$(1.6) \qquad a_i = b_i + \textstyle\sum_{j=0}^{i-1} c_{ij}, \quad 0 < a_i \leqq 1 \quad (i=2, 3,\dots, r),$$

and a_i, b_i, c_{ij} $(i=2, 3,\dots, r; j=0, 1,\dots, i-1)$ and p_j $(j=0, 1,\dots, r)$ are constants. This method requires r function evaluations per step. It has been shown that for $r=2, 3, 4$ there exists a method (1.3) of order $r+2$.

In this paper we propose two-step methods of the form

$$(1.7) \qquad y_{n+1} = y_n + s(y_n - y_{n-1}) + h \textstyle\sum_{j=0}^{r} p_j k_j,$$

where

$$(1.8) \qquad k_i = f(x_n + a_i h, y_n + \textstyle\sum_{j=0}^{1} b_{ij}(y_{n+j} - y_{n+j-1}) + h \sum_{j=0}^{i-1} c_{ij} k_j),$$

$$(1.9) \qquad a_i = b_{i0} + b_{i1} + \textstyle\sum_{j=0}^{i-1} c_{ij}, \quad 0 < a_i \leqq 1 \quad (i=2, 3,\dots, r),$$

and b_{ij} $(i=2, 3,\dots, r; j=0, 1)$ and s are constants. The method (1.7) reduces to (1.3) for $b_{i1}=0$ $(i=2, 3,\dots, r)$ and $s=0$. It is called an explicit method if $b_{i1}=0$

($i=2, 3,\dots, r$) and an implicit one otherwise. The stepsize control is implemented by comparing (1.7) with the method

$$z_{n+1} = y_n + z(y_n - y_{n-1}) + h \sum_{j=0}^{r+1} w_j k_j, \tag{1.10}$$

where $k_{r+1} = f(x_{n+1}, y_{n+1})$ and w_j ($j=0, 1,\dots, r+1$) and z are constants.

It is shown that for $r=2, 3, 4$ there exist an explicit method (1.7) of order $r+2$ and a method (1.10) of order $r+1$ with $w_{r+1}=0$ ($r=2, 3$), and that for $r=2, 3$ there exist an implicit method (1.7) of order $r+3$ and a method (1.10) of order $r+2$. Predictors for implicit methods are constructed. The implicit method (1.7) can be used also as an explicit three- or four-step method of order $r+3$ with r function evaluations if y_{n+1} is predicted with sufficient accuracy and the corrector is applied only once per step.

2. Preliminaries

Let

$$y_{n+1} = y_n + s(y_n - y_{n-1}) + h \sum_{j=0}^{r} p_j k_{jn} \quad (r=2, 3, 4), \tag{2.1}$$

$$t_{n+1} = u(y_n - y_{n-1}) + h \sum_{j=0}^{r+1} v_j k_{jn}, \tag{2.2}$$

$$y_{n+1}^{*} = y_n + t(y_n - y_{n-1}) + h \sum_{j=0}^{3} q_j k_{1n-3+j} + h \sum_{i=2}^{r} \sum_{j=1}^{2} q_{2i+2-j} k_{in-j}, \tag{2.3}$$

$$z_{n+1} = y_{n+1} + t_{n+1}, \tag{2.4}$$

where

$$k_{1n} = f(x_n, y_n), \quad k_{0n} = k_{1n-1}, \quad k_{r+1n} = k_{1n+1}, \tag{2.5}$$

$$k_{in} = f(x_n + a_i h,\ y_n + \textstyle\sum_{j=0}^{1} b_{ij}(y_{n+j} - y_{n+j-1}) + h \sum_{j=0}^{i-1} c_{ij} k_{jn}), \tag{2.6}$$

$$a_i = b_{i0} + b_{i1} + \textstyle\sum_{j=0}^{i-1} c_{ij}, \quad 0 < a_i \leqq 1 \quad (i=2, 3,\dots, r). \tag{2.7}$$

The method (2.1) is stable if and only if $-1 \leqq s < 1$.

Denote by $y(x)$ the solution of (1.1) and let

$$a_0 = -1, \quad a_1 = 0, \quad a_{r+1} = 1, \tag{2.8}$$

$$\begin{aligned} &y(x) + s(y(x) - y(x-h)) + h \textstyle\sum_{j=0}^{r} p_j y'(x + a_j h) - y(x+h) \\ &\quad = \textstyle\sum_{j=1}^{7} S_j (h^j/j!) y^{(j)}(x) + O(h^8), \end{aligned} \tag{2.9}$$

$$u(y(x) - y(x-h)) + h \textstyle\sum_{j=0}^{r+1} v_j y'(x + a_j h) = \sum_{j=1}^{7} U_j (h^j/j!) y^{(j)}(x) + O(h^8), \tag{2.10}$$

$$\begin{aligned} &y(x) + t(y(x) - y(x-h)) + h \textstyle\sum_{j=0}^{3} q_j y'(x + (j-3)h) \\ &\quad + h \textstyle\sum_{i=2}^{r} \sum_{j=1}^{2} q_{2i+2-j} y'(x + (a_i - j)h) - y(x+h) \\ &\quad = \textstyle\sum_{j=1}^{7} T_j (h^j/j!) y^{(j)}(x) + O(h^8), \end{aligned} \tag{2.11}$$

(2.12) $$y(x) + \sum_{j=0}^{1} b_{ij}(y(x+jh)-y(x+(j-1)h)) + h \sum_{j=0}^{i-1} c_{ij}y'(x+a_jh) - y(x+a_ih) = \sum_{j=1}^{7} e_{ij}(h^j/j!)y^{(j)}(x) + O(h^8) \quad (i=2, 3, 4).$$

Then we have

(2.13) $$(-1)^{k-1}b_{i0} + b_{i1} + k \sum_{j=0}^{i-1} c_{ij}a_j^{k-1} - a_i^k = e_{ik} \quad (i=2, 3, 4;\ k=1, 2,\ldots, 7),$$

(2.14) $$(-1)^{k-1}s + k \sum_{j=0}^{r} a_j^{k-1}p_j - 1 = S_k,$$

(2.15) $$(-1)^{k-1}u + k \sum_{j=0}^{r+1} a_j^{k-1}v_j = U_k,$$

(2.16) $$(-1)^{k-1}t + k \sum_{j=0}^{3} (j-3)^{k-1}q_j + k \sum_{i=2}^{r} \sum_{j=1}^{2} (a_i-j)^{k-1}q_{2i+2-j} - 1 = T_k.$$

Let

(2.17) $$k_{0n}^* = y'(x_{n-1}), \quad k_{1n}^* = y'(x_n), \quad k_{r+1n}^* = y'(x_{n+1}),$$

(2.18) $$k_{in}^* = f(x_n+a_ih,\ y(x_n)+\sum_{j=0}^{1} b_{ij}(y(x_{n+j})-y(x_{n+j-1})) + h \sum_{j=0}^{i-1} c_{ij}k_{jn}^*) \quad (i=2, 3,\ldots, r),$$

(2.19) $$g(x) = f_y(x,\ y(x)),$$

(2.20) $$T(x_n) = y(x_n)+s(y(x_n)-y(x_{n-1})) + h \sum_{j=0}^{r} p_jk_{jn}^* - y(x_{n+1}),$$

(2.21) $$R(x_n) = u(y(x_n)-y(x_{n-1})) + h \sum_{j=0}^{r+1} v_jk_{jn}^*,$$

(2.22) $$T^*(x_n) = y(x_n) + t(y(x_n)-y(x_{n-1})) + h \sum_{j=0}^{3} q_jk_{1n-3+j}^* + h \sum_{i=2}^{r} \sum_{j=1}^{2} q_{2i+2-j}k_{in-j}^* - y(x_{n+1}),$$

(2.23) $$F_{k+1} = \sum_{i=2}^{r} e_{ik}p_i,\ G_{k+2} = \sum_{i=2}^{r} a_ie_{ik}p_i,\ H_{k+2} = \sum_{i=3}^{r} \sum_{j=2}^{i-1} c_{ij}e_{jk}p_i,$$
$$K_{k+1} = \sum_{i=2}^{r} e_{ik}v_i,\ M_{k+2} = \sum_{i=2}^{r} a_ie_{ik}v_i,\ N_{k+2} = \sum_{i=3}^{r} \sum_{j=2}^{i-1} c_{ij}e_{jk}v_i,$$
$$J_{k+1} = \sum_{i=2}^{r} e_{ik}(q_{2i}+q_{2i+1}) \quad (k=4, 5, 6).$$

(2.24) $$A_i = a_i(a_i+1),\quad B_i = (a_i-a_2)A_i,\quad C_i = (a_i-a_3)B_i,\quad D_i = a_i(a_i-1),$$
$$E_i = (a_i-1)A_i,\quad R_i = (2a_i+1)A_i \quad (i=1, 2, 3, 4).$$

Choosing $e_{ij}=0$ ($i=2, 3, 4$; $j=1, 2, 3$), we have

(2.25) $$b_{i0} = 5b_{i1} + 6 \sum_{j=2}^{i-1} A_jc_{ij} - a_i^2(2a_i+3),$$
$$c_{i0} = -2b_{i1} + \sum_{j=2}^{i-1} a_j(3a_j+2)c_{ij} + a_iA_i,$$
$$c_{i1} = -4b_{i1} - \sum_{j=2}^{i-1} (a_j+1)(3a_j+1)c_{ij} + (a_i+1)A_i,$$

(2.26) $$e_{i4} = 4b_{i1} + 2 \sum_{j=2}^{i-1} R_jc_{ij} - A_i^2,$$
$$e_{i5} = -4b_{i1} + \sum_{j=2}^{i-1} (a_j^2-5a_j-4)A_jc_{ij} - (a_i-2)A_i^2,$$
$$e_{i6} = 8b_{i1} + 6 \sum_{j=2}^{i-1} (a_j^3-a_j^2+a_j+1)A_jc_{ij} - (a_i^2-2a_i+3)A_i^2,$$

(2.27) $T(x) = \sum_{j=1}^{7} S_j(h^j/j!)y^{(j)}(x) + \sum_{k=4}^{6} (h^{k+1}/k!)[F_{k+1}g(x)y^{(k)}(x)$
$+G_{k+2}hg'(x)y^{(k)}(x)+H_{k+2}hg^2(x)y^{(k)}(x)+O(h^2)]$,

(2.28) $R(x) = \sum_{j=1}^{7} U_j(h^j/j!)y^{(j)}(x) + \sum_{k=4}^{6} (h^{k+1}/k!)[K_{k+1}g(x)y^{(k)}(x)$
$+M_{k+2}hg'(x)y^{(k)}(x)+N_{k+2}hg^2(x)y^{(k)}(x)+O(h^2)]$,

(2.29) $T^*(x) = \sum_{j=1}^{7} T_j(h^j/j!)y^{(k)}(x) + \sum_{k=4}^{6} (h^{k+1}/k!)\,[J_{k+1}g(x)y^{(k)}(x)+O(h)]$.

Put

(2.30) $X = a_2 + a_3,\ Y = a_2a_3,\ U = a_2 + a_3 + a_4,\ V = a_2a_3 + a_2a_4 + a_3a_4,$
$W = a_2a_3a_4,\ d = 2a_2+1,\quad m = 5a_2^2 - a_2 - 2,$

(2.31) $Q_1 = 2 + 3X + 5Y,\ Q_2 = 3 + 5X + 10Y,\ Q_3 = 5 + 8X + 15Y,$
$Q_4 = 22 - 27X + 35Y,\ Q_5 = 27 - 35X + 50Y,\ Q_6 = 10 + 14X + 21Y,$
$Q_7 = 130 - 154X + 189Y.$

The choice $S_i=0$ $(1\leqq i\leqq r+2)$ yields

(2.32) $\sum_{j=0}^{r} p_j = 1 - s,\quad 2\sum_{j=1}^{r} (a_j+1)p_j = 3 - s,\quad 6\sum_{j=2}^{r} A_jp_j = 5 + s,$
$12\sum_{j=3}^{r} B_jp_j = 7 - 10a_2 - ds,$
$60\sum_{j=3}^{r} a_jB_jp_j + 35a_2 - 27 - (5a_2+3)s = 12S_5,$
$60\sum_{j=4}^{r} a_jC_jp_j + Q_1s - Q_4 = 10S_6\quad (r\geqq 3),$
$(Q_6+7a_4Q_1)s - Q_7 + 7a_4Q_4 = 60S_7\quad (r=4).$

Setting $U_i=0$ $(1\leqq i\leqq r+1)$, we have

(2.33) $v_0 - \sum_{j=2}^{r+1} a_jv_j = -u/2,\quad v_1 + \sum_{j=2}^{r+1} (a_j+1)v_j = -u/2,$
$6\sum_{j=2}^{r+1} A_jv_j = u,\quad 12\sum_{j=3}^{r+1} B_jv_j + du = 3U_4,$
$60\sum_{j=4}^{r+1} C_jv_j - Q_2u = 12U_5 + 15(1-X)U_4,$
$60\sum_{j=5}^{r+1} (a_j-a_4)C_jv_j + (Q_1+a_4Q_2)u = 10U_6 + 12(1-U)U_5\quad (r\geqq 3),$
$420\sum_{j=5}^{r+1} a_j(a_j-a_4)C_jv_j - (Q_6+7a_4Q_1)u = 60U_7 + 70(1-U)U_6\quad (r=4).$

Choosing $T_i=0$ $(1\leqq i\leqq r+3)$, $q_4=-q_5$ and $q_6=-q_7$, we have

(2.34) $\sum_{i=0}^{3} q_i = 1 - t,\quad \sum_{j=0}^{2} (j-3)q_j + q_5 + q_7 = (t+1)/2,$
$3q_0 + q_1 + (a_2-1)q_5 + (a_3-1)q_7 = (t+5)/12,$
$-2q_0 + D_2q_5 + D_3q_7 = (t+9)/12,\quad 120\sum_{j=2}^{3} E_jq_{2j+1} = 19t + 251,$
$60\sum_{j=2}^{3} (5a_j-6)E_jq_{2j+1} + 17t - 367 = 10T_6,$
$70\sum_{j=2}^{3} (6a_j^2-15a_j+10)E_jq_{2j+1} - 11t - 339 = 10T_7 + 70T_6.$

3. Explicit methods

In this section we set $b_{i1}=e_{ij}=0\ (i=2, 3, 4; j=1, 2, 3)$ and show the following

THEOREM 1. *For $r=2, 3$ there exists an explicit method* (2.1) *of order $r+2$ which embeds a method of order $r+1$. For $r=4$ there exist an explicit method* (2.1) *of order $r+2$ and a method* (2.4) *of order $r+1$.*

3.1. Case r=2

Choosing $S_i=0\ (i=1, 2, 3, 4)$, $U_j=0\ (j=1, 2, 3, 4)$ and $v_3=0$, we have

$$b_{21} = -a_2^2(2a_2+3), \quad c_{20} = a_2A_2, \quad c_{21} = (a_2+1)A_2, \tag{3.1}$$

$$ds = 7 - 10a_2, \quad 6A_2p_2 = 5 + s, \quad p_1 + (a_2+1)p_2 = (3-s)/2, \tag{3.2}$$

$$p_0 + p_1 + p_2 = 1 - s,$$

$$6(a_2+1)v_0 = -(3a_2+2)u, \quad 6a_2v_1 = -(3a_2+1)u, \quad 6A_2v_2 = u, \tag{3.3}$$

$$dS_5 = 2m, \quad dF_5 = -2A_2, \quad 3U_4 = du, \tag{3.4}$$

$$6U_5 = (5a_2^2-4a_2-4)u, \quad 6K_5 = -A_2u.$$

For any given a_2 and $u\neq0$ other constants are determined uniquely. The method (2.1) is stable if and only if $1/2<a_2<1$. For instance the choice $a_2=7/10$ yields $s=0$.

3.2. Case r=3

Setting $S_i=0\ (i=1, 2,\dots, 5)$, $U_j=0\ (j=1, 2, 3, 4)$ and $F_5=v_4=0$, we have (3.1) and

$$b_{30} = -a_3^2(2a_3+3) + 6A_2c_{32}, \quad c_{30} = a_3A_3 - a_2(3a_2+2)c_{32}, \tag{3.5}$$

$$c_{31} = (a_3+1)A_3 - (a_2+1)(3a_2+1)c_{32},$$

$$mR_2c_{32} = -B_3(1+X+3Y), \tag{3.6}$$

$$Q_2s = -Q_5, \quad 12B_3p_3 = 7 - 10a_2 - ds, \quad A_2p_2 + A_3p_3 = (5+s)/6, \tag{3.7}$$

$$p_1 + (a_2+1)p_2 + (a_3+1)p_3 = (3-s)/2, \quad p_0 + p_1 + p_2 + p_3 = 1 - s,$$

$$12B_3v_3 = -du, \quad A_2v_2 + A_3v_3 = u/6, \tag{3.8}$$

$$v_1 + (a_2+1)v_2 + (a_3+1)v_3 = -u/2, \quad v_0 - a_2v_2 - a_3v_3 = -u/2,$$

$$10S_6 = Q_1s - Q_4, \quad 12U_5 = -Q_2u, \quad 10U_6 = -(Q_1+(X-1)Q_2)u, \tag{3.9}$$

$$12K_5 = 24R_2c_{32}v_3 + (1+X+2Y)u.$$

For any given $u\neq0$, a_2 and a_3 such that $a_2\neq a_3$ and $m\neq0$, other constants are determined uniquely. For example the choice $a_2=1/5$ and $a_3=4/5$ yields $s=0$.

3.3. Case r=4

Choosing $S_i=0$ $(i=1, 2,\dots, 6)$ and $F_5=F_6=G_6=H_6=0$, we have (3.1), (3.5) and

$$\text{(3.10)}\quad b_{40} = 6\textstyle\sum_{j=2}^{3} A_j c_{4j} - a_4^2(2a_4+3), \quad c_{40} = -\sum_{j=2}^{3} a_j(3a_j+2)c_{4j} + a_4A_4,$$
$$c_{41} = -\textstyle\sum_{j=2}^{3}(a_j+1)(3a_j+1)c_{4j} + (a_4+1)A_4,$$

$$\text{(3.11)}\quad (Q_1+a_4Q_2)s = Q_4 - a_4Q_5, \quad 60C_4p_4 = Q_2s + Q_5,$$
$$B_3p_3 + B_4p_4 = (7-10a_2-ds)/12, \quad 6\textstyle\sum_{j=2}^{4} A_jp_j = s+5,$$
$$p_1 + \textstyle\sum_{j=2}^{4}(a_j+1)p_j = (3-s)/2, \quad \sum_{i=0}^{4} p_i = 1-s,$$

$$\text{(3.12)}\quad 120(a_3-a_4)R_2c_{32}p_3 = 49 - 62a_4 + (2a_4+1)s,$$
$$120(a_4-a_3)\textstyle\sum_{j=2}^{3} R_jc_{4j}p_4 = 49 - 62a_3 + (2a_3+1)s,$$
$$60Q_2B_3c_{43}p_4 = Cs - D,$$

$$\text{(3.13)}\quad (3s-13)(a_4-1) = 0,$$

where

$$\text{(3.14)}\qquad C = 5a_2^2 + 5a_2 + 1, \quad D = 155a_2^2 - 5a_2 - 49.$$

From (3.13) we have $a_4=1$ by stability condition.

The choice $U_j=0$ $(j=1, 2,\dots, 5)$ and $K_5=0$ yields

$$\text{(3.15)}\quad 120(a_2-1)(a_3-1)(v_4+v_5) = Q_2u, \quad B_3v_3 + 2(1-a_2)(v_4+v_5) = -du/12,$$
$$A_2v_2 + A_3v_3 + 2(v_4+v_5) = u/6,$$
$$v_1 + \textstyle\sum_{j=2}^{3}(a_j+1)v_j + 2(v_4+v_5) = -u/2,$$
$$\textstyle\sum_{i=0}^{5} v_i = -u, \quad \sum_{i=3}^{4}\sum_{j=2}^{i-1} R_jc_{ij}v_i + 2v_5 = -u/60,$$

$$\text{(3.16)}\quad 60S_7 = 24 - 35X + 56Y - (24+35X+56Y)s, \quad 10U_6 = Q_3u,$$
$$60U_7 = -(Q_6+7Q_7+14Q_1-7UQ_3)u.$$

For any given $u\neq 0$, $a_2\neq 1$ and $a_3\neq 1$ such that $a_2\neq a_3$, $p_3\neq 0$, $p_4\neq 0$ and $\sum_{j=2}^{3} R_jc_{4j}\neq 2$ other constants are determined uniquely. For instance we have $s=0$ for $a_2=1/6$ and $a_3=2/3$.

4. Implicit methods

In this section we choose $e_{ij}=0$ $(j=1, 2, 3, 4;\ i=2, 3)$ and show the following

THEOREM 2. *For $r=2, 3$ there exist an implicit method* (2.1) *of order $r+3$ and a method* (2.4) *of order $r+2$.*

This method can be used also as an explicit method if the corrector is applied only once per step.

4.1. Case r = 2

Choosing $S_i = T_i = 0$ $(i = 1, 2, \ldots, 5)$, $U_j = 0$ $(j = 1, 2, 3, 4)$ and $q_4 = -q_5$, we have (3.2), (3.8) and

(4.1) $4b_{20} = a_2(5a_2^2 + 6a_2 - 7)$, $4b_{21} = A_2^2$, $2c_{20} = -a_2E_2$, $c_{21} = -(a_2+1)E_2$,

(4.2) $5a_2^2 - a_2 - 2 = 0$,

(4.3) $120E_2q_5 = 19t + 251$, $-2q_0 + (a_2 - 1)q_5 = (9+t)/12$, $5q_0 + q_1 = -1/3$,

$-q_1 - q_2 + a_2q_5 = (11+7t)/12$, $q_0 + q_1 + q_2 + q_3 = 1 - t$,

(4.4) $dS_6 = 4(a_2 - 1)(3a_2^2 - 1)$, $dF_6 = -2E_2$, $12U_5 = -(15a_2 + 8)u$,

$2U_6 = (1 - 3a_2^2)u$, $4K_6 = A_2u$, $4T_6 = (19a_2 - 16)t + 251a_2 - 448$, $J_6 = 0$.

For any given $a_2 \neq 1$, $u \neq 0$ and t, other constants are determined uniquely. For example the choice $a_2 = (1 + \sqrt{41})/10$ yields $s = 0$.

4.2. Case r = 3

Setting $S_i = T_i = 0$ $(i = 1, 2, \ldots, 6)$, $U_j = 0$ $(j = 1, 2, \ldots, 5)$, $q_{2i} = -q_{2i+1}$ $(i = 2, 3)$ and $F_6 = 0$, we have (4.1), (3.7) and

(4.5) $4b_{31} = -2R_2c_{32} + A_3^2$, $b_{30} = 5b_{31} + 6A_2c_{32} - a_3^2(2a_3 + 3)$,

$c_{30} = -2b_{31} - a_2(3a_2 + 2)c_{32} + a_3A_3$,

$c_{31} = -4b_{31} - (a_2 + 1)(3a_2 + 1)c_{32} + (a_3 + 1)A_3$,

$3A_2m^2c_{32} = (3 - X + 7Y)B_3$,

(4.6) $Q_1s = Q_4$,

(4.7) $120(a_2 - 1)(a_3 - 1)v_4 = Q_2u$, $B_3v_3 + 2(1 - a_2)v_4 = -du/12$,

$A_2v_2 + A_3v_3 + 2v_4 = u/6$, $v_1 + (a_2 + 1)v_2 + (a_3 + 1)v_3 + 2v_4 = -u/2$,

$v_0 + v_1 + v_2 + v_3 + v_4 = -u$,

(4.8) $120(a_3 - 1)B_3q_7 = (16 - 19a_2)t + 448 - 251a_2$,

$E_2q_5 + E_3q_7 = (19t + 251)/120$, $-2q_0 + D_2q_5 + D_3q_7 = (9+t)/12$,

$3q_0 + q_1 + (a_2 - 1)q_5 + (a_3 - 1)q_7 = (5+t)/12$,

$-q_1 - q_2 + a_2q_5 + a_3q_7 = (11+7t)/12$, $q_0 + q_1 + q_2 + q_3 = 1 - t$,

(4.9) $60S_7 = -Q_6s - Q_7$, $10U_6 = Q_3u$, $60K_6 = -(3 + Q_3)u + 60\,\mathrm{m}A_2c_{32}v_3$,

$60U_7 = -(24 + 56Y - 56X^2 - 105XY)u$,

$60T_7 = -(241 - 336X + 399Y)t - 16769 + 9408X - 5271Y.$

For any $u \neq 0$, t, $a_2 \neq 1$ and $a_3 \neq 1$ such that $a_2 \neq a_3$ and $m \neq 0$, other constants are determined uniquely. For instance we have $s=0$ for $a_2=(31-\sqrt{141})/50$ and $a_3=(31+\sqrt{141})/50$.

References

[1] H. Shintani, *On pseudo-Runge-Kutta methods of the third kind,* Hiroshima Math. J., **11** (1981), 247–254.

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