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Non-triviality of some compositions of β-elements in the stable homotopy of the Moore spaces

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§ 1. Introduction

Let S be the sphere spectrum and M the Moore spectrum modulo a prime $p \geqq 5$ given by the cofiber sequence $S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{\pi} \Sigma S$; and consider the stable homotopy rings $\pi_* S$ and $[M, M]_*$. Then, for $s \geqq 1$ and $t \geqq 2$, the β-elements

(1.1) $\quad \beta_{(s)}, \quad \beta_{(tp/p)}$ in $[M, M]_*$ and $\beta_s = \pi\beta_{(s)}i$,

$$\beta_{tp/p} = \pi\beta_{(tp/p)}i, \quad \beta_{tp^2/p,2} \text{ in } \pi_* S$$

are given by Smith [13] (see also [14], [16]) and Oka [7], [8].

Consider the Brown-Peterson spectrum BP at p, the Hopf algebroid $(A, \Gamma) = (BP_*, BP_*BP) = (Z_{(p)}[v_1, v_2, \cdots], BP_*[t_1, t_2, \cdots])$ and the Adams-Novikov spectral sequence:

$$E_2 = H^*A' = \mathrm{Ext}_\Gamma^*(A, A') \Longrightarrow \pi_* M \ (\text{resp. } \pi_* S) \qquad \text{for} \quad A' = A/(p) \ (\text{resp. } A).$$

Then, Miller-Ravenel-Wilson [4] proved the following:

(1.2) There are the β-elements

$$\beta'_s \quad \text{in} \quad H^1 A/(p) \ (\text{resp. } \beta_s, \beta_{tp/p}, \beta_{tp^2/p,2} \text{ in } H^2 A) \text{ (see (2.4.6))}$$

which converge to $\beta_{(s)}i$ in $\pi_* M$ (resp. the ones in $\pi_* S$ with the same notation).

The main purpose of this paper is to prove the following

THEOREM A. *In the E_2-term $H^3 A/(p)$, $\beta'_s \beta_{tp^2/p,2} = \beta'_{s+tp(p-1)} \beta_{tp/p}$ holds, and $\beta'_s \beta_{tp/p} = 0$ if and only if $p | st$.*

COROLLARY B. *In $[M, M]_*$, $\beta_{(s)}(\beta_{tp^2/p,2} \wedge 1_M)$, $\beta_{(s)}(\beta_{tp/p} \wedge 1_M)$ and $\beta_{(s)} \delta \beta_{(tp/p)}$ are all non-trivial if $p \nmid st$. Here $\delta = i\pi$ is the generator of $[M, M]_{-1}$.*

Corollary B is a consequence of Theorem A and is proved in Corollary 4.2. The equality and the triviality in Theorem A are in Theorem 2.7 which is valid for $p \geqq 3$ and can be proved easily by [4] and [9], and the non-triviality is in Theorem 4.1. We note that Theorems 2.7, 4.1 and Corollary 4.2 contain the (non-) triviality of some other compositions.

To show the non-triviality in Theorem 4.1, §3 is devoted to the study of $H^1M_1^1$ in the E_2-term of the chromatic spectral sequence [4] converging to $H^*A/(p)$, and forms the main part of this paper. By the change of rings theorem [3], we note that

$$H^1M_1^1 = \mathrm{Ext}_\Sigma^1(B, M_1^1 \otimes_A B)$$
$$\text{for} \quad (B, \Sigma) = (Z_{(p)}[v_1, v_2, v_2^{-1}], B[t_1, t_2, \cdots] \otimes_A B).$$

Then, by using some results in [4] and [13], some calculations give us suitable elements in Σ which satisfy good relations in the cobar complex $\Omega_\Sigma^* B$ (Lemma 3.4), and we can find generators of $H^1M_1^1$ given in Proposition 3.8 and Theorem 3.10. Theorem 4.1 is proved by these results.

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§2. Triviality in the E_2-term

Let p be an odd prime and BP the Brown-Peterson ring spectrum at p. Then, the following are due to Quillen [10] and Hazewinkel [2] (cf. also [1], [4]):

(2.1) $BP_* = \pi_* BP = Z_{(p)}[v_1, v_2, \cdots] \subset H_* BP = Z_{(p)}[m_1, m_2, \cdots]$,

$BP_*BP = BP_*[t_1, t_2, \cdots]$, $\deg v_n = \deg m_n = \deg t_n = 2(p^n - 1)$, and

(2.1.1) $\quad v_n = pm_n - \sum_{i=1}^{n-1} m_i v_{n-i}^{(i)}$ ($u^{(i)}$ denotes u^{p^i} in this paper),

where $BP_* \subset H_* BP$ by the Hurewicz map. Furthermore,

(2.1.2) $(BP_*, BP_*BP) = (A, \Gamma)$ (this abbreviation is used hereafter)

is a Hopf algebroid (cf. [3]), whose left unit η_L is the inclusion, and right unit η_R (denoted simply by η): $A \to \Gamma$ and diagonal $\Delta: \Gamma \to \Gamma \otimes_A \Gamma$ are given respectively by

(2.1.3) $\quad \eta m_n = \sum_{i=0}^n m_i t_{n-i}^{(i)}, \quad \sum_{i=0}^n m_i \Delta t_{n-i}^{(i)} = \sum_{i+j+k=n} m_i t_j^{(i)} \otimes t_k^{(i+j)}$,

where $m_0 = t_0 = 1$ and $v_0 = p$.

For a Γ-comodule M with coaction $\eta_M: M \to M \otimes_A \Gamma$, we study the homology

(2.2) (cf. [3]) $H^*M = \mathrm{Ext}_\Gamma^*(A, M)$ of the cobar complex $\Omega_\Gamma^* M = (\Omega_\Gamma^s M, d_s: \Omega_\Gamma^s M \to \Omega_\Gamma^{s+1} M)$ given by $\Omega_\Gamma^s M = M \otimes_A \Gamma \otimes_A \cdots \otimes_A \Gamma$ (s factors of Γ) and

$$d_s(m \otimes x) = \eta_M m \otimes x$$
$$+ \sum_{i=1}^s (-1)^i m \otimes x_1 \otimes \cdots \otimes \Delta x_i \otimes \cdots \otimes x_s - (-1)^s m \otimes x \otimes 1$$

for $m \in M$, $x_i \in \Gamma$ and $x = x_1 \otimes \cdots \otimes x_s$.

In particular, consider the case $M = A$ with $\eta_A = \eta$: $A \to A \otimes_A \Gamma = \Gamma$. Then:

(2.3) In the cobar complex $\Omega_\Gamma^* A$, $\Omega_\Gamma^0 A = A$, $\Omega_\Gamma^1 A = \Gamma$, $\Omega_\Gamma^2 A = \Gamma \otimes_A \Gamma$ and d_s: $\Omega_\Gamma^s A \to \Omega_\Gamma^{s+1} A$ for $s = 0, 1$ are given by

(2.3.1) $$d_0 u = \eta u - u \quad (u \in A) \quad \text{and}$$
$$d_1 x = \psi x - \Delta x, \quad \psi x = x \otimes 1 + 1 \otimes x \quad (x \in \Gamma).$$

Therefore, for any $u, v \in A$ and $x, y \in \Gamma$, we have the equalities

(2.3.2) $$d_0(uv) = d_0 u \eta v + u d_0 v; \quad d_1(xy) = d_1 x \Delta y + \psi x d_1 y - x \otimes y - y \otimes x,$$
$$d_1(uy) = d_0 u \otimes y + u d_1 y, \quad d_1(x \eta v) = d_1 x \Delta \eta v - x \otimes d_0 v.$$

Thus, by (2.1.1–3) and [11; Th. 7–8] for ηv_3 and Δt_3, and by considering

(2.3.3) the invariant ideal $J(n) = (p, v_1^n)$ of A,

direct calculations give us the following

(2.3.4) $$d_0 v_1 = \eta v_1 - v_1 = p t_1; \quad d_0 v_2 = \eta v_2 - v_2 \equiv v_1 t_1^p - v_1^p t_1 \bmod (p),$$
$$d_0(v_2^n) \equiv (v_2^{(i)} + v_1^{(i)} t_1^{(i+1)})^s - v_2^n \bmod (p^{j+1}, v_1^{(i+1)})$$
$$\text{if} \quad n = sp^i \quad \text{and} \quad p^j \mid s \quad (i, j \geqq 0),$$
$$d_0(v_3) \equiv v_2 t_1^{(2)} + v_1 t_2^p - t_1 \eta v_2^p + v_1^2 V \bmod J(p^2),$$

where $V = \{v_1^p t_1^{(2)} - v_1^{(2)} t_1^p + v_2^p - (v_1 t_1^p - v_1^p t_1 + v_2)^p\}/p v_1$

(2.3.5) $$d_1 t_1 = \psi t_1 - \Delta t_1 = 0, \quad d_1(t_1^{(i)}) \equiv p T^{(i-1)} \bmod (p^2) \qquad \text{for} \quad i \geqq 1;$$
$$d_1 t_2 = -t_1 \otimes t_1^p - v_1 T, \quad d_1 \tau = -t_1^p \otimes t_1 + v_1 T$$
$$\text{for} \quad \tau = t_1^{p+1} - t_2;$$
$$d_1 t_3 \equiv -g - v_2 T^p \bmod J(1) \qquad \text{for} \quad g = t_1 \otimes t_2^p + t_2 \otimes t_1^{(2)},$$

where $T = d_1(t_1^p)/p = \{\psi(t_1^p) - (\psi t_1)^p\}/p$.

We now consider the elements $x_i \in v_2^{-1} A$ given by

(2.4.1) $$x_0 = v_2, \quad x_i = v_2^{(i)} - v_1^{(i)}(v_2^{-1} v_3)^{(i-1)} - v_1^{a_i - a_{i-1}} \bar{x}_i \quad (i \geqq 1),$$
$$\bar{x}_1 = 0, \quad \bar{x}_2 = v_2^{1+c_2} + v_1^p v_2^{c_2 - p} v_3, \; \bar{x}_i = \bar{x}_{i-1}^p + 2 v_1^{a_{i-1} - p} v_2^{1+c_i} \quad (i \geqq 3),$$

where $a_0 = 1$, $a_i = p^i + p^{i-1} - 1$ and $c_i = p^i - p^{i-1}$ $(i \geqq 1)$. Then:

(2.4.2) x_i is equal (resp. congruent $\bmod (p)$) to x_i in [4; (2.4)] for $i = 0, 1$

(resp. $i \geqq 2$), and [4; Prop. 5.4, b)] says that in the cobar complex $\Omega_\Gamma^* v_2^{-1}A$,

$$d_0 x_0 \equiv v_1 t_1^p \bmod J(2), \quad d_0 x_i \equiv \varepsilon_i v_1^{a_i} v_2^{c_i} t_1 \bmod J(1+a_i) \quad \text{for} \quad i \geqq 1 \quad (\varepsilon_i = \min\{i, 2\}).$$

Therefore, by considering the inclusion $A/J \subset v_2^{-1}A/J$ for $J=(p^2, v_1^p)$ or $J(j)$.

(2.4.3) $x_2^s = v_2^n \in H^0(A/(p^2, v_1^p))$ for $s \geqq 1$ and $n = sp^2$; and

(2.4.4) x_i^s lies in $A/J(j)$ and $x_i^s \in H^0(A/J(j))$ for $(i, s, j) \in I$, where

(2.4.5) $I = \{(i, s, j) \in \boldsymbol{Z}^3 \mid i \geqq 0, s \geqq 1 \text{ and } 1 \leqq j \leqq a_i, \text{ with } j \leqq p^i \text{ if } s=1\}$.

In case of (2.4.4), we note that $x_i^s = x_{i+1}^{s'}$ if $s=s'p$. Thus, by using the boundary homomorphism δ_k (resp. $\delta'_{j,k}$) associated to the exact sequence

$$0 \longrightarrow A \xrightarrow{p^k} A \longrightarrow A/(p^k) \longrightarrow 0$$
$$(\text{resp. } 0 \longrightarrow A/(p^k) \xrightarrow{v_1^j} A/(p^k) \longrightarrow A/(p^k, v_1^j) \longrightarrow 0),$$

the β-elements in (1.2) can be defined (see [4; pp. 477–9]) by

(2.4.6) $\beta_{n/p,2} = \delta_2 \delta'_{p,2}(x_2^s) = \delta_2 \delta'_{p,2}(v_2^n) \in H^2 A$ for $n = sp^2 > 0$;

$\beta'_{n/j} = \delta'_{j,1}(x_i^s) \in H^1(A/(p))$, $\beta_{n/j} = \delta_1 \beta'_{n/j} \in H^2 A$

for $n = sp^i$ with $(i, s, j) \in I$. We abbreviate $\beta'_{n/1}$ to β'_n and $\beta_{n/1}$ to β_n, which can be defined for any $n \geqq 1$.

LEMMA 2.5. *In $\Omega_\Gamma^2 A$, the following hold* mod $J(1)$ *for* $s \geqq 1$:

(2.5.1) $\beta_{n/p,k} \equiv s v_2^{n-p} T^p$ *if* $n = sp^k$ *and* $k = 1, 2$ $(\beta_{n/p,1} = \beta_{n/p})$.

(2.5.2) $\beta_n \equiv \bar{\beta}_n = \binom{n}{2} v_2^{n-2}(2t_2 \otimes t_1^p + t_1 \otimes t_1^{2p}) + n v_2^{n-1} T$.

(2.5.3) $\beta_{n/j} \equiv - s v_2^{c(i,s)} t_1 \otimes \zeta$ *if* $n = sp^i, j = a_i$ $(s, i \geqq 2)$,

where $c(i, s) = sp^i - p^{i-1}$,

$\zeta = v_2^{-p-1}(v_2^p t_2 - v_2 \tau^p - v_3 t_1^p)$ $(\equiv \zeta_2$ *in* [4; p. 485] $\bmod (p)) \in v_2^{-1}\Gamma$.

PROOF. By (2.4.1), we see that

(2.5.4) $x_i^s = v_2^n$ in $A/(p^i, v_1^p)$ for $i = 1, 2$, $s \geqq 1$ and $n = sp^i$.

Therefore, the definition (2.4.6) and (2.3.2–5) imply directly (2.5.1). (2.5.2–3) are given in [9; Lemma 4.4 and the notice in §6]*). q. e. d.

*) We must replace the expression of $\beta_{p/p}$ in [9; Lemma 4.4(ii)] by the one in (2.5.1). We note that the results in [9] are valid by this replacement.

Lemma 2.6. *In the cobar complex $\Omega_\Gamma^* v_2^{-1}A$, the following hold for $s \in \boldsymbol{Z}$:*

(2.6.1) $d_1(x_0^s\zeta^p) \equiv sv_1v_2^{s-1}t_1^p \otimes \zeta^p \bmod J(2)$; *and for* $i \geqq 1$,

$$d_1(x_i^s\zeta^{(i+1)}) \equiv \varepsilon_i s v_1^{a_i} v_2^{c(i,s)} t_1 \otimes \zeta^{(i+1)} \bmod J(1+a_i) \ (\varepsilon_i = \min\{i, 2\}).$$

(2.6.2) $d_1(t_1\eta v_2^s - sv_1t_2\eta v_2^{s-1}) \equiv v_1^2\bar{\beta}_s \bmod J(3)$.

(2.6.3) $d_1(v_1v_2^{sp}V) \equiv v_1^pv_2^{sp}T^p + sv_1^{1+p}v_2^{sp-p}t_1^{(2)} \otimes V \bmod J(2p)$.

Proof. (2.6.1) is certified directly from (2.3.2), (2.4.2) and

(2.6.4) ([4; Prop. 3.18, c)]) $d_1\zeta \equiv 0 \bmod J(1)$ in $\Omega_\Gamma^* v_2^{-1}A$;

and so is (2.6.2) by (2.3.2–5). (2.6.3) is shown by calculating $d_1(pv_1v_2^{sp}V)$ using (2.3.2–5) in the range of the monomorphism $p\colon \Omega_\Gamma^* v_2^{-1}A/J(2p) \to \Omega_\Gamma^* v_2^{-1}A/(p^2, v_1^{2p})$. q.e.d.

Theorem 2.7. *The Yoneda product* $\beta'_m\beta_{n/j,k} \in H^3(A/(p)) = \mathrm{Ext}_\Gamma^3(A, A/(p))$ *of the β-elements given in* (2.4.6) *satisfies the following:*

(2.7.1) $\beta'_m\beta_{sp^2/p,2} = \beta'_{m+sp(p-1)}\beta_{sp/p}$ *for* $s \geqq 1$ *and* $m \geqq 1$.

(2.7.2) $\beta'_m\beta_{sp/p} = 0 = \beta'_m\beta_n$ *if* $p \mid ms$ *for* $s \geqq 1$ *and* $m \geqq 1$.

(2.7.3) *In case* $n = sp^i$, $j = a_i$ $(i, s \geqq 2)$ *and* $m \geqq 1$,

$\beta'_m\beta_{n/j} = 0$ *if* $m = c(e, u) - c(i, s)$ *for some* $e \geqq 1$ *and* $u \geqq 2$ *with* $p \nmid u$.

Proof. (2.5.1) shows $v_2^m\beta_{n/p,2} = v_2^{m+n-n'}\beta_{n'/p}$ in $H^2(A/J(1))$, whose image under $\delta'_{1,1}$ is (2.7.1).

$\beta'_m\beta_{n/j} = \delta'_{1,1}(v_2^m\beta_{n/j}) = \delta'_{k+1,1}(v_1^kv_2^m\beta_{n/j})$ by the definition of δ'. When $n = sp$, $v_1^pv_2^m\beta_{n/p} = sv_1^pv_2^{m+n-p}T^p$ in $H^2(A/J(p+1))$ by (2.5.1), which is 0 if $p|s$ or $p|m$ by (2.6.3). By (2.5.2), $v_2^m\beta_n = 0$ in $H^2(A/J(1))$ if $p|n$, and $v_1^2v_2^m\beta_n \equiv v_1^2\bar{\beta}_{n+m} \bmod J(3)$ if $p|m$, which is 0 in $H^2(A/J(3))$ by (2.6.2). In the last case, (2.5.3) and (2.6.1) show that

$$v_1^jv_2^m\beta_{n/j} = -sv_1^jv_2^{m+c(i,s)}t_1 \otimes \zeta = -sv_1^jv_2^{c(e,u)}t_1 \otimes \zeta^{(e+1)} = 0$$

in $H^2(A/J(j+1))$, because $\zeta^{(e+1)}$ is homologous to ζ in $\Omega_\Gamma^1 v_2^{-1}A/J(1)$ by [4; Lemma 3.19]. q.e.d.

By considering the δ_1-image of the elements in (2.7.1–3), we see the following

Corollary 2.8 (cf. [9; Prop. 6.1]). *For the product* $\beta_m\beta_{n/j,k} \in H^4A = \mathrm{Ext}_\Gamma^4(A, A)$, *Theorem 2.7 holds by replacing β'_m with β_m.*

§3. $H^1M_1^1=\mathrm{Ext}_\Gamma^1(A, M_1^1)$

Hereafter, assume that p is a prime $\geqq 5$. For the Hopf algebroid $(A, \Gamma)=(BP_*, BP_*BP)$ in (2.1.2), we recall the Γ-comodules N_1^s and M_1^s given in [4; §3], defined inductively by

(3.1.1) $N_1^0 = A/(p)$, $M_1^s = v_{s+1}^{-1}N_1^s$ and the exact sequence

$$0 \longrightarrow N_1^s \xrightarrow[\subset]{j} M_1^s \longrightarrow N_1^{s+1} \longrightarrow 0.$$

In this section, we compute $H^1M_1^1=\mathrm{Ext}_\Gamma^1(A, M_1^1)$ by using the following (3.1.2–6):

(3.1.2) [4; (3.10)] For $M_2^0=v_2^{-1}A/(p, v_1)$, $0 \longrightarrow M_2^0 \xrightarrow{1/v_1} M_1^1 \xrightarrow{v_1} M_1^1 \longrightarrow 0$ is exact.

(3.1.3) [3; §3] We can identify $H^*M=\mathrm{Ext}_\Gamma^*(A, M)$ as

$$H^*M = \mathrm{Ext}_\Gamma^*(A, M) = \mathrm{Ext}_\Sigma^*(B, M\otimes_A B) \qquad \text{for} \quad M = M_2^0 \quad \text{or} \quad M_1^1$$

by the isomorphism induced from the natural map, where

(3.1.4) (B, Σ) is the Hopf algebroid with $B=\boldsymbol{Z}_{(p)}[v_1, v_2, v_2^{-1}]$ acting v_n $(n\geqq 3)$ trivially and $\Sigma=B\otimes_A\Gamma\otimes_A B=B[t_1, t_2,\cdots]\otimes_A B$ such that the natural map $(A, \Gamma)\to(B, \Sigma)$ sending v_n $(n\geqq 3)$ to 0 is a map of Hopf algebroids. Thus, the relations in §2 for (A, Γ) are reduced to those for (B, Σ) by putting $v_n=0$ for $n\geqq 3$ and $\eta(v_2^{-1})\eta v_2=1$ in Σ.

(3.1.5) [13; Th. 3.2] $H^nM_2^0$ is spanned as the $\boldsymbol{F}_p[v_2, v_2^{-1}]$-vector space by

$$h_0 = t_1,\ h_1 = v_2^{-1}t_1^p \text{ and } \zeta \text{ in (2.5.3) for } n = 1, \text{ and}$$

$$h_0\zeta = t_1 \otimes \zeta, \quad h_1\zeta = v_2^{-1}t_1^p \otimes \zeta, \quad g_0 = v_2^{-p}g \quad (g \text{ in (2.3.5)}) \quad \text{and}$$

$$g_1 = v_2^{-1}g_0^p \qquad \text{for} \quad n = 2.$$

(3.1.6) [4; p. 500] The image of $1/v_1$: $H^1M_2^0\to H^1M_1^1$ induced by $1/v_1$ in (3.1.2) is spanned by h_0/v_1, $v_2^{sp}h_1/v_1$ for $s\in\boldsymbol{Z}$, $v_2^s\zeta/v_1$ for $s\in\boldsymbol{Z}$ and

$$v_2^m h_0/v_1 \text{ for } m = sp^i,\ i \geqq 0,\ s\in\boldsymbol{Z} \text{ with } p \nmid s(s+1) \text{ or } p^2 \mid s + 1.$$

LEMMA 3.2. *The following relations hold in Σ for $n\geqq 1$ and $i\geqq 0$:*

(3.2.1) $(v_2-v_1^pt_1)t_1^{(2)} + v_1t_2^p + v_1^2V - v_2^pt_1 \equiv 0 \bmod J(p^2)$ *for V in* (2.3.4).

(3.2.2) $v_2t_n^{(2)} + v_1t_{n+1}^p - v_2^{(n)}t_n \equiv 0 \bmod J(2)$.

(3.2.3) $v_2^{(i+n)}t_n^{(i)} \equiv v_2^{(i)}t_n^{(i+2)}$ *and* $v_2^{(i+2)}\tau^{(i)}\equiv v_2^{(i)}\tau^{(i+2)} \bmod J(p^i)$ *for τ in* (2.3.5).

(3.2.4) $\zeta^{(i)} \equiv (-v_2^{-1}\tau + v_2^{-p}t_2^p)^{(i)} \equiv \zeta^{(i+1)} \bmod J(p^i)$ *for* ζ *in* (2.5.3).

(3.2.5) $v_2^{(i+2)}T^{(i)} \equiv v_2^{(i+1)}T^{(i+2)} \bmod J(p^i)$ *for* T *in* (2.3.5).

PROOF. Since $v_3=0$ in B, (3.2.1) follows from (2.3.4). (3.2.2) holds for $n=1$ by (3.2.1) and is proved by induction on n as follows. Note that $m'_n = p^n m_n \in A$ and

(3.2.6) $m'_1 = v_1$ and $m'_n \equiv pm'_{n-2}v_2^{(n-2)} \bmod (p^n, v_1^{(n-1)})$ in B $(n \geqq 2)$,

by (2.1.1). Then, by (2.1.3), we see the following in Σ mod (p^{n+2}, v_1^p):

$$p^{n+1}(v_1 t_{n+1}^p + v_2 t_n^{(2)}) + \textstyle\sum_{i=1}^n p^{n+1-i}m'_i v_2^{(i)} t_{n-i}^{(i+2)} \equiv \sum_{j=0}^{n+2} p^{n+2-j}m'_j t_{n+2-j}^{(j)}$$
$$= \eta m'_{n+2} \equiv \eta(pm'_n v_2^{(n)}) = (p^{n+1}t_n + \textstyle\sum_{i=1}^n p^{n+1-i}m'_i t_{n-i}^{(i)})\eta v_2^{(n)}.$$

Here, by (2.3.4) for $d_0(v_2^{(n)}) = \eta v_2^{(n)} - v_2^{(n)}$ and the inductive hypothesis, we have

$$t_n \eta v_2^{(n)} \equiv v_2^{(n)} t_n \bmod J(p),\ t_{n-1}^p \eta v_2^{(n)} \equiv v_2^{(n)} t_{n-1}^p \equiv v_2^p t_{n-1}^{(3)} \bmod (p^2, v_1)$$

and $t_{n-i}^{(i)}\eta v_2^{(n)} \equiv v_2^{(n)} t_{n-i}^{(i)} \equiv v_2^{(i)} t_{n-i}^{(i+2)} \bmod (p^i, v_1^p)(\subset (p^i, v_1^2))$ for $1 \leqq i \leqq n$.

Therefore, we see $m'_i v_2^{(i)} t_{n-i}^{(i+2)} \equiv m'_i t_{n-i}^{(i)} \eta v_2^{(n)} \bmod (p^{i+1}, v_1^2)$ $(1 \leqq i \leqq n)$ by (3.2.6), which shows (3.2.2) since $p^{n+1}\colon \Sigma/J(2) \to \Sigma/(p^{n+2}, v_1^2)$ is monomorphic.

(3.2.2) implies (3.2.3–5) directly by definition. q.e.d.

We now define the elements Y_s, W_s, Z_s $(s \in \boldsymbol{Z})$ and X in Σ as follows:

(3.3.1) $Y_s = sv_2^{s-1}\tau + (s-1)v_2^s\zeta^p/2 + \binom{s}{2} v_1 v_2^{s-2} t_1^p(\tau + v_2\zeta^p) + sv_1 v_2^{s-1}\bar{t}_3^p$,

$W_s = v_2^{sp-1}t_1^p - v_1 v_2^{sp-p}\{\xi'_1 - (s-1)v_1^{p-1}\xi_2/2\}$, $Z_s = v_1 W_s + v_1^{p-1}v_2^{sp-p}(v_1^2\xi_2 - \xi_3)$,

where $\bar{t}_3 = v_2^{-p}t_3$, $\xi'_1 = V' + v_1^{p-2}\bar{t}_3^{(2)}$, $V' = (V + v_2^{p-1}t_1^p)/v_1$,

$$\xi_2 = v_2^{-1}\tau^p(2 - v_1 v_2^{-1}t_1^p) + v_2^{p-1}\zeta^p,\ \xi_3 = v_2^{-p}t_1^{(2)}(v_2 t_1^{(2)} + v_1 t_2^p) - v_1 t_1^{(2)}\zeta^p;$$

(3.3.2) $X = (t_1 - v_1^2\xi_1)\eta_1 - v_1 v_2^{1-p^2}t_2^{(2)}\eta_0 + v_1^p v_2^{-p}t_1^2 + v_1^{p+2}(\xi_4 + v_2^{-p}\xi_5)$,

where $\eta_0 = v_2^{-p} - v_1^p v_2^{-2p}t_1^{(2)}$, $\xi_1 = v_2^{-p}(V + v_1^{p-1}\bar{t}_3^{(2)}) = -v_2^{-1}t_1^p + v_1 v_2^{-p}\xi'_1$,

$$\eta_1 = v_2^{1-p} + v_1 v_2^{-(2)}t_1^{(3)} - v_1^p v_2^{-p}\sigma + v_1^{p+2}v_2^{-2p}V,\ \sigma = 2t_1 - v_1\zeta^p,$$
$$\xi_4 = v_2^{-2p}t_2^p(2 + v_1 v_2^{-1}t_1^p),\ \xi_5 = -\zeta^{2p}/2 + (v_2^{-p}t_2^p)^{p+1} + v_1 v_2^{-2p}\tau^p V.$$

Here, η_0 and η_1 satisfy the following by (2.3.4) for ηv_2, (3.2.1–2) and (2.3.2):

(3.3.3) $\eta_\varepsilon \equiv \eta v_2^{\varepsilon-p}$, $d_1(x\eta_\varepsilon) \equiv d_1 x \Delta\eta_\varepsilon - x \otimes (\eta_\varepsilon - v_2^{\varepsilon-p}) \bmod J(2p)$ $(\varepsilon = 0, 1)$.

LEMMA 3.4. *In the cobar complex $\Omega_\Sigma^* B$, we have the following*:

(3.4.1) $d_1Y_s \equiv -sv_2^{s-1}t_1^p \otimes t_1 - \binom{s}{2}v_1v_2^{s-2}t_1^{2p} \otimes t_1 - \binom{s+1}{2}v_1v_2^s g_1 \bmod J(2).$

(3.4.2) $d_1W_s \equiv v_1^{p-1}v_2^{sp}g_1^p - (s-1)v_1^{p+1}v_2^{sp-1}g_1/2 \bmod J(p+2).$

(3.4.3) $d_1Z_s \equiv v_1^{p-1}v_2^{sp-p}t_1^{(2)} \otimes \sigma - (s+1)v_1^{p+2}v_2^{sp-1}g_1/2 \bmod J(p+3).$

(3.4.4) $d_1X \equiv -v_1^2g_1^{(2)} - v_1^{p+3}v_2^{-p}g_1 \bmod J(p+4).$

PROOF. The calculations are based on (2.3.1–5) and Lemma 3.2. We have

$$d_1Y_s \equiv sd_0(v_2^{s-1}) \otimes \tau + sv_2^{s-1}d_1\tau + (s-1)d_0(v_2^s) \otimes \zeta^p/2 + \binom{s}{2}v_1v_2^{s-2}\{d_1(t_1^p\tau) + v_2d_1(t_1^p\zeta^p)\} + sv_1v_2^{s-1}d_1(\bar{t}_3^p) \bmod J(2)$$

by (2.6.4), which implies (3.4.1) since we see by (3.2.5) that

(3.4.5) $$d_1(\bar{t}_3^p) \equiv -v_2g_1 - T \bmod J(1).$$

$W_s = -v_2^{sp}\{\xi_1 - (s-1)v_1^pv_2^{-p}\xi_2/2\}$ by definition. By (2.6.3) for $s=-1$ and (3.4.5),

(3.4.6) $$-d_1\xi_1 \equiv v_1^{p-1}(-v_2^{-p}T^p + v_1v_2^{-2p}t_1^{(2)} \otimes V + g_1^p + v_2^{-p}T^p) \equiv A_1 = v_1^{p-1}g_1^p + v_1^pv_2^{-2p}t_1^{(2)} \otimes V \bmod J(2p-1).$$

Furthermore, we see that

(3.4.7) $$d_1\xi_2 \equiv 2v_2^{-p}t_1^{(2)} \otimes V - v_1v_2^{p-1}g_1 \bmod J(2) \text{ and}$$

$$d_1W_s \equiv -sv_1^pv_2^{sp-p}t_1^{(2)} \otimes \xi_1 + v_2^{sp}A_1 + (s-1)v_1^pv_2^{sp-p}d_1\xi_2/2 \bmod J(2p-1).$$

These imply (3.4.2). We see also (3.4.3) because

$$d_1\xi_3 \equiv -2v_2^{-p}(v_2+v_1t_1^p)t_1^{(2)} \otimes t_1^{(2)} + v_1t_1^{(2)} \otimes \zeta^p - 2v_1v_2^{-p}t_1^{(2)} \otimes t_2^p + v_1v_2^pg_1^p \equiv -t_1^{(2)} \otimes \sigma + 2v_1^2v_2^{-p}t_1^{(2)} \otimes V + v_1v_2^pg_1^p \bmod J(4).$$

Finally, we show (3.4.4). In the first place, we see that

$d_1(t_1 - v_1^2\xi_1) \equiv -v_1^2d_1\xi_1 \bmod (p)$ and $t_1 - v_1^2\xi_1 \equiv B_1 = v_2^{1-p}t_1^{(2)} + v_1v_2^{-(3)}t_2^{(3)} - v_1^pv_2^{-2p}t_1^{(2)}(v_2t_1^{(2)} + v_1t_2^p + v_1^2V) \bmod J(2p)$, and so

$$d_1((t_1 - v_1^2\xi_1)\eta_1) \equiv d_1(t_1 - v_1^2\xi_1)\Delta\eta_1 - (t_1 - v_1^2\xi_1) \otimes (\eta_1 - v_2^{1-p})$$
$$\equiv v_1^2A_1(v_2^{1-p} \otimes 1 + v_1v_2^{-(2)}\Delta t_1^{(3)}) + (t_1 - v_1^2\xi_1) \otimes v_1^pv_2^{-p}\sigma - B_1 \otimes (v_1v_2^{-(2)}t_1^{(3)} + v_1^{2+p}v_2^{-2p}V) \equiv v_1v_2^{-(2)}A_0 - v_1^2g_1^{(2)} + 2v_1^pv_2^{-p}t_1 \otimes t_1 - v_1^{2+p}v_2^{-2p}V \otimes \sigma$$

$+v_1^{2+p}v_2^{-2p}C_1$ mod $J(2p)$,

where $A_0 = -v_2^{1-p}t_1^{(2)} \otimes t_1^{(3)} + v_1 v_2^{-(2)} t_1^{(3)} \otimes t_2^{(2)} + v_1^p v_2^{1-2p} A'$, $A' = t_1^{(2)} \otimes \tau^{(2)} + t_1^{2p^2} \otimes t_1^{(3)} + t_2^{(2)} \otimes t_1^{(2)} \equiv t_1^{2p^2} \otimes t_1^{(3)} - v_2^{(2)} t_1^{(2)} \otimes \zeta^{(2)} + v_2^{p+p^2} g_1^p \bmod J(p)$ (by (3.2.4)), $C_1 = -(t_2^p + v_1 V) \otimes \zeta^p + v_2^{2p-p^2} g_1^p \Delta t_1^{(3)} + v_2^{-(2)} t_1^{(2)} t_2^p \otimes t_1^{(3)} + v_1 v_2^{-(2)} \{(t_1^{(2)} \otimes V)\Delta t_1^{(3)} + t_1^{(2)} V \otimes t_1^{(3)} - v_2^{p^2-p^3} t_2^{(3)} \otimes V\}$ and $2v_1^p v_2^{-p} t_1 \otimes t_1 \equiv -d_1(v_1^p v_2^{-p} t_1^2)$ mod $J(2p)$.

In the second place, we have

$$d_1(v_2 t_2^{(2)} \eta_0) \equiv \{d_0 v_2 \otimes t_2^{(2)} + v_2 d_1(t_2^{(2)})\}\Delta \eta_0 - v_2 t_2^{(2)} \otimes (\eta_0 - v_2^{-p})$$
$$\equiv A_0 + v_1^{1+p} v_2^{-2p} B_0 \bmod J(2p),$$

where $B_0 = (v_2^{p-p^2} t_2^{(2)} - t_1^{p+p^2} - t_2^p - v_1 V) \otimes t_2^{(2)} - t_1^p \otimes t_1^{(2)} t_2^{(2)}$.

Furthermore, $V \equiv -v_2^{p-1} t_1^p + v_1 v_2^{p-2} t_1^{2p}/2 \bmod J(2)$ by definition. Thus

$$(3.4.8) \quad d_1 \xi_4 \equiv v_2^{-2p}\{-2t_1^p \otimes t_1^{(2)} + v_1 v_2^{-1} d_1(t_1^p t_2^p)\}$$
$$\equiv v_2^{-2p} V \otimes \sigma - v_1 v_2^{-p} g_1 \bmod J(2),$$

since $d_1(t_1^p t_2^p) \equiv v_2^p t_1^p \otimes \zeta^p - v_2^{1+p} g_1 - t_1^{2p} \otimes t_1^{(2)} - 2t_1^p \otimes t_2^p \bmod J(p)$. Noting that $d_1 \zeta^p \equiv 0 \equiv v_1 d_1 V \bmod J(p)$ by (2.6.3–4), we have also

$$d_1 \xi_5 \equiv \zeta^p \otimes \zeta^p + v_2^{-p-p^2} d_1(t_2^{p+p^2}) + v_1 v_2^{-2p} d_1(\tau^p V)$$
$$\equiv v_2^{-p-p^2} B_0 - v_2^{-p} C_1 \bmod J(2)$$

by (3.2.1–4). These relations imply (3.4.4). q.e.d.

To give generators of $H^1 M_1^1 = \mathrm{Ext}_\Sigma^1(B, M_1^1 \otimes_A B)$, we write each integer $m \neq 0$ as

(3.5.1) $m = sp^\nu$ by integers $\nu = \nu(m) \geqq 0$ and $s = s(m) \not\equiv 0 \bmod p$ uniquely,

and define the integers $\bar{\nu} = \bar{\nu}(m)$, $\varepsilon = \varepsilon(m)$, s_m, $A(m)$ and $e(m)$ by

$$(3.5.2) \quad \bar{\nu} = \min\{\nu, 1\},$$

$$\varepsilon = \begin{cases} 0 & \text{if } s \not\equiv -1 \bmod p^2, \\ 1 & \text{otherwise,} \end{cases} \qquad s_m = (-1)^\nu (1+\bar{\nu})^{-1-\varepsilon} \binom{s+1}{2}^{1-\varepsilon},$$

$$A(m) = 2 + \varepsilon p^\nu (p^2-1) + (p+1)(p^\nu - 1)/(p-1),$$

$$e(m) = m - \varepsilon p^\nu (p-1) - (p^\nu - 1)/(p-1).$$

Furthermore, by using the elements in (3.3.1–2), we define the elements

(3.5.3) $\quad y_m$ and $\bar{y}_m$ in Σ with $y_m = v_2^m t_1 + v_1 \bar{y}_m$

for all integers $m = sp^\nu \neq 0$ in (3.5.1) inductively on $\nu \geqq 0$ as follows:

$$\bar{y}_s = Y_s \text{ and } \bar{y}_{sp} = -(v_2^{sp}\zeta^{(2)} + sZ_s)/2 \quad \text{if} \quad s \not\equiv -1 \bmod p^2, \text{ i.e., } \varepsilon = 0;$$

$$y_s = W_t^p + v_1^{p^2-p-2} v_2^{s+1} X \ (\equiv v_2^s t_1 \bmod J(1) \text{ by (3.2.3)}) \quad \text{if} \quad s = tp^2 - 1;$$

$$\bar{y}_{mp} = (\bar{y}_m^p - v_1^q \eta'_m + s_m v_1^{A(mp)-p-2} W_{e(m)})/(2-\bar{\nu}), \quad q = p^{\nu+1} - p - \bar{\nu},$$

for $m = sp^\nu \neq 0$ with $\nu \geqq 1-\varepsilon$, where $\eta'_m \in \Sigma$ is taken to satisfy

(3.5.4) $\quad v_1^{q+p+1}\eta'_m \equiv d_0(v_2^{1+mp}) - v_2^{mp}\{v_1 t_1^p - (2-\bar{\nu}) v_1^p t_1\} \bmod J(A(mp)+p+1)$

(the existence is certified by (2.3.1–4) and (3.2.1)).

LEMMA 3.6. *$d_1 y_m \equiv -s_m v_1^{A(m)} v_2^{e(m)} g_1 \bmod J(A(m)+1)$ in $\Omega_\Sigma^* B$.*

PROOF. The lemma for $m = sp^\nu$ with $\nu \leqq 1-\varepsilon$ is certified directly by (2.3.1–5), (2.6.4), (3.2.4) and (3.4.1–4), by noticing that $d_1(v_2^m t_1) = d_0(v_2^m) \otimes t_1$, $d_1(v_2^m \zeta^{(2)}) \equiv d_0(v_2^m) \otimes \zeta^p \bmod J(2p)$ if $\varepsilon = 0 = \nu - 1$, and that if $\varepsilon = 0 = \nu$, $\varepsilon = 0 = \nu - 1$ or $\varepsilon = 1 = \nu + 1$, then $s_m = \binom{s+1}{2}$, $-2^{-1}\binom{s+1}{2}$ or 1, $A(m) = 2$, $p+3$ or p^2+1, and $e(m) = m$, $m-1$ or $m-p+1$, respectively.

For $m = sp^\nu$ with $\nu \geqq 1-\varepsilon$, we note by definition that

$$v_1^{1+p}(\bar{y}_m^p - v_1^q \eta'_m) \equiv v_1 y_m^p - d_0(v_2^{1+mp}) - (2-\bar{\nu}) v_1^p v_2^{mp} t_1 \quad \text{and so}$$

$$d_1(v_1^p y_{mp}) \equiv d_1(v_1 y_m^p + s_m v_1^{A(mp)-1} W_{e(m)})/(2-\bar{\nu}) \bmod J(A(m)+p+1);$$

$$A(mp) = pA(m) - p + 3, \quad e(mp) = pe(m) - 1$$

$$\text{and} \quad s_{mp} \equiv (e(m)-1)s_m/2(2-\bar{\nu}) \bmod p.$$

Then, (3.4.2) implies the lemma by induction on ν, by noticing that $s^p \equiv s \bmod p$ and $v_1^p\colon \Omega_\Sigma^* B/J(n) \to \Omega_\Sigma^* B/J(n+p)$ is monomorphic. q.e.d.

By virtue of Lemmas 3.6 and 2.6, we have the cycles

(3.7.1) $\quad y_m/v_1^j \ (1 \leqq j \leqq A(m))$, $v_2^{sp} V/v_1^j \ (1 \leqq j < p)$, $x_n^s \zeta^{(n+1)}/v_1^j \ (1 \leqq j \leqq a_n)$

in $\Omega_\Sigma^1 M_1^1 \otimes_A B$ for any m, $s \in \boldsymbol{Z}$ and $n \geqq 0$; and we consider them the elements in $H^1 M_1^1 = \mathrm{Ext}_\Sigma^1(B, M_1^1 \otimes_A B)$ by (3.1.3). Now, consider the exact sequence

(3.7.2) $\quad \cdots \longrightarrow H^{n-1} M_1^1 \xrightarrow{\delta} H^n M_2^0 \xrightarrow{1/v_1} H^n M_1^1 \xrightarrow{v_1} H^n M_1^1 \xrightarrow{\delta} H^{n+1} M_2^0 \longrightarrow \cdots$

associated to the exact sequence in (3.1.2).

PROPOSITION 3.8. *$\delta\colon H^1 M_1^1 \to H^2 M_2^0$ (the range is given by (3.1.5)) satisfies the following for any m, $s \in \boldsymbol{Z}$ and $n \geqq 0$:*

(3.8.1) $\delta(y_m/v_1^{A(m)}) = -s_m v_2^{e(m)} g_1$ *for* s_m *with* $p\nmid s_m$ *and* $e(m)$ *in* (3.5.2).

(3.8.2) $\delta(v_2^{sp}V/v_1^{p-1}) = v_2^{sp}T^p = -v_2^{sp+p-1}g_0$.

$$(3.8.3)\quad \delta(x_n^s\zeta^{(n+1)}/v_1^{a_n}) = \begin{cases} sv_2^s h_1\zeta & \text{if } n=0, \\ \varepsilon_n s v_2^{c(n,s)} h_0\zeta & \text{if } n \geqq 1, \end{cases}$$

where $\varepsilon_n = \min\{n, 2\}$ *and* $c(n, s) = sp^n - p^{n-1}$.

PROOF. We note that $d_1(v_2^{sp-1}t_3) = -v_2^{sp+p-1}g_0 - v_2^{sp}T^p$ in $\Omega_\Sigma^* B/(p, v_1)$ by (2.3.1–5), which means the second equality in (3.8.2). By (3.1.3) and the definition of δ, the other equalities follow immediately from Lemma 3.6, (2.6.3) and (2.6.1). q.e.d.

LEMMA 3.9. *In* (3.7.2) *for* $n \geqq 1$, *assume that a submodule* $K \supset \mathrm{Im}(1/v_1)$ *of* $H^n M_1^1$ *is the direct sum of* $F_p[v_1]$*-submodules* $K_\lambda (\lambda \in \Lambda)$ *isomorphic to* $F_p[v_1, v_1^{-1}]/F_p[v_1]$ *and cyclic ones* $K_\mu (\mu \in M)$ *generated by* k_μ *such that* $\{\delta k_\mu | \mu \in M\}$ *is linearly independent. Then,* $K = H^n M_1^1$.

PROOF. By assumption, $H^n M_2^0 \xrightarrow{1/v_1} K \xrightarrow{v_1} K \xrightarrow{\delta} H^{n+1} M_2^0$ is exact, which together with (3.7.2) implies the lemma by [4; Remark 3.11]. In fact, for any $x = \sum_\lambda x_\lambda + \sum_\mu a_\mu k_\mu$ ($x_\lambda \in K_\lambda$, $a_\mu \in F_p[v_1]$), we have $x_\lambda \in v_1 K_\lambda$ and $\delta(a_\mu k_\mu) = 0$ if $v_1 | a_\mu$, and so $\delta x = 0$ implies $a_\mu = 0$ for $v_1 \nmid a_\mu$ and $x \in v_1 K$. The other parts of exactness are seen easily. q.e.d.

By these results, we have the following main result in this section:

THEOREM 3.10. $H^1 M_1^1 = \mathrm{Ext}_\Gamma^1(A, M_1^1) = \mathrm{Ext}_\Sigma^1(B, M_1^1 \otimes_A B)$ *is the direct sum of*

(3.10.1) *the* $F_p[v_1]$*-submodules* $F_p\{t_1/v_1^j | j \geqq 1\}$ *and* $F_p\{\zeta^{(j)}/v_1^j | j \geqq 1\}$, *which are both isomorphic to* $F_p[v_1, v_1^{-1}]/F_p[v_1]$, *and*

(3.10.2) *the cyclic ones* $F_p[v_1]\langle x \rangle$ *for* $x = x'/v_1^b \in \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$, *which are isomorphic to* $F_p[v_1]/(v_1^b)$, *where*

$$\Lambda_1 = \{y_m/v_1^{A(m)} | m = sp^\nu,\ \nu \geqq 0,\ s \in Z \text{ with } p \nmid s(s+1) \text{ or } p^2 | s+1\},$$

$$\Lambda_2 = \{v_2^{sp}V/v_1^{p-1} | s \in Z\},\ \Lambda_3 = \{x_n^s \zeta^{(n+1)}/v_1^{a_n} | n \geqq 0,\ s \in Z \text{ with } p \nmid s\}.$$

PROOF. We see that the direct sum K of the submodules in (3.10.1–2) satisfies the assumption in Lemma 3.9 for $n=1$ by (3.1.6), (3.5.3) and Proposition 3.8. Therefore, the theorem holds by Lemma 3.9. q.e.d.

§4. Non-triviality

Theorem A in the introduction is in (2.7.1) and the following (4.1.1):

THEOREM 4.1. *Let p be a prime* $\geqq 5$. *Then, the products* $\beta'_m\beta_{n/j} \in H^3(A/(p)) = \mathrm{Ext}^3_\Gamma(A, A/(p))$ *in* (2.7.2–3) *are non-trivial in the following cases*:

(4.1.1) $\beta'_m\beta_{sp/p} \neq 0$ *if and only if* $p \nmid ms$ *for* $s \geqq 1$ *and* $m \geqq 1$.

(4.1.2) $\beta'_m\beta_n \neq 0$ *if* $p|m+n$ *and* $p \nmid n$ *for* $n \geqq 1$ *and* $m \geqq 1$.

(4.1.3) *In case* $n = sp^i$, $j = a_i$ $(i, s \geqq 2)$ *and* $m \geqq 1$, $\beta'_m\beta_{n/j} \neq 0$ *if and only if* $m \neq c(e, u) - c(i, s)$ *for any* $e \geqq 1$ *and* $u \geqq 2$ *with* $p \nmid u$.

PROOF. The 'only if' parts are in (2.7.2–3). Consider the homomorphisms

$$H^1M^1_1 \xrightarrow{\delta} H^2M^0_2 \xrightarrow{1/v_1} H^2M^1_1 \xleftarrow{j_*} H^2N^1_1 \underset{\cong}{\xrightarrow{\delta'}} H^3N^0_1 = H^3(A/(p)),$$

where the first two are in (3.7.2) for $n=2$, j is the inclusion map in (3.1.1) for $s=1$ and δ' is the boundary associated to the exact sequence in (3.1.1) for $s=0$. Then, by the definition (2.4.6) and (2.5.4), $(1/v_1)^{-1}j_*\delta'^{-1}(\beta'_m\beta) = v_2^m\beta$ and so

(4.1.4) $v_2^m\beta \in \mathrm{Im}\,\delta = \mathrm{Ker}\,(1/v_1)$ if $\beta'_m\beta = 0$ for $\beta = \beta_{n/j} \in H^2A$.

Now, by (2.5.1), [9; Lemma 5.4] and (2.5.3), we have

$$v_2^m\beta_{sp/p} = sv_2^{m+sp-p}T^p, \quad -v_2^m\beta_n = \binom{n}{2}v_2^{n+m}h_1\zeta + \binom{n+1}{2}v_2^{n+m}g_1,$$

and $v_2^m\beta_{n/j} = -sv_2^{m+c(i,s)}h_0\zeta$ in case of (4.1.3), respectively. Thus, the assumptions in (4.1.1–3) imply $v_2^m\beta_{n/j} \notin \mathrm{Im}\delta$ by Proposition 3.8 and Theorem 3.10, and so $\beta'_m\beta_{n/j} \neq 0$ by (4.1.4). q.e.d.

COROLLARY 4.2. *On the compositions of the β-elements in* (1.1) *for* $s \geqq 1$ *and* $t \geqq 2$, $\beta_{(s)}(\beta_{tp^2/p,2} \wedge 1_M)$, $\beta_{(s)}(\beta_{tp/p} \wedge 1_M)$ *and* $\beta_{(s)}\delta\beta_{(tp/p)}$ *in* $[M, M]_*$ *are all non-trivial in* $[M, M]_*$ *if* $p \nmid st$, *and so are* $\beta_{(s)}(\beta_{s'} \wedge 1_M)$ *and* $\beta_{(s)}\delta\beta_{(s')}$ $(s' \geqq 1)$ *if* $p|s+s'$ *and* $p \nmid s'$. *Here* $\delta = i\pi$.

PROOF. Consider the Adams-Novikov spectral sequence with $E_2 = H^*N^0_1$ $(N^0_1 = A/(p))$ converging to π_*M, and the induced map $i^*\colon [M, M]_* \to \pi_*M$. Then, (1.2) shows that $\beta'_s\beta \in H^3N^0_1$ for $\beta = \beta_{tp^2/p,2}$, $\beta_{tp/p}$ or $\beta_{s'}$ converges to

$$\beta_{(s)}i\beta = \beta_{(s)}(\beta \wedge 1_M)i = i^*(\beta_{(s)}(\beta \wedge 1_M)) \in \pi_*M \quad \text{for the corresponding } \beta \text{ in } \pi_*S,$$

and $\beta_{(s)}i\beta_* = i^*(\beta_{(s)}\delta\beta_{(*)})$ if $\beta_* = \beta_{tp/p}$ or $\beta_{s'}$ by (1.1). Thus, we have the corollary by the non-triviality of $\beta'_s\beta$ in (4.1.1–2) and the sparseness of this spectral sequence. q.e.d.

REMARK. On the compositions $\beta_{(s)}\delta\beta_{(s')}$, we know some relations in [16; Th. 5.1] including

$$\beta_{(s)}\delta\beta_{(s')} = 0 \quad \text{if} \quad p \nmid s + s' \quad \text{and} \quad p | ss'.$$

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