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Existence and nonexistence of positive radial entire solutions of second order quasilinear elliptic systems

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Abstract. This paper treats the second order quasilinear elliptic system of the form $\Delta_p u = H(|x|)v^\alpha, \Delta_q v = K(|x|)u^\beta$ in $\mathbf{R}^N$ with nonnegative functions H, K. Sufficient conditions will be given to have positive radial entire solutions and to have no nonnegative nontrivial radial entire solutions under some restriction on p, q, α and β. When H and K behave like positive constant multiples of $|x|^\nu$, $\nu \in \mathbf{R}$, we can completely characterize the existence property of positive radial entire solutions.

1. Introduction and statement of results

This paper is concerned with second order quasilinear elliptic system of the form

$$(1) \qquad \begin{cases} \Delta_p u \equiv \operatorname{div}(|Du|^{p-2}Du) = H(|x|)v^\alpha \\ \Delta_q v \equiv \operatorname{div}(|Dv|^{q-2}Dv) = K(|x|)u^\beta \end{cases} \quad \text{in } \mathbf{R}^N,$$

where $N \geq 1$, $p > 1$, $q > 1$, α and β are positive constants satisfying $\alpha\beta > (p-1)(q-1)$, and $H, K : [0, \infty) \to [0, \infty)$ are continuous. An *entire solution* of (1) is defined to be a function $(u, v) \in C^1(\mathbf{R}^N) \times C^1(\mathbf{R}^N)$ such that $|Du|^{p-2}Du, |Dv|^{q-2}Dv \in C^1(\mathbf{R}^N)$ and satisfies (1) at every $x \in \mathbf{R}^N$. Such a solution of (1) is said to be radial if it depends only on $|x|$.

The problem of existence and nonexistence of positive radial entire solutions of *scalar* equations has been investigated by many authors under various situations. To illustrate some of typical known results let us consider the equation

$$(2) \qquad \Delta_p u = H(x)u^\sigma \quad \text{in } \mathbf{R}^N,$$

where $p > 1$, $\sigma > p - 1$, and H is a nonnegative continuous function in $\mathbf{R}^N$. The existence and nonexistence results of positive (radial) entire solutions of (2) may be described roughly as follows:

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THEOREM A ([8, Theorems 2.1, 3.1 and 3.2]). *If H has radial symmetry and*

$$\begin{cases} H(x) \le \dfrac{C}{|x|^{p+\varepsilon}}, & |x| \ge r_0 > 0, \qquad p < N; \\ H(x) \le \dfrac{C}{|x|^p(\log|x|)^{\sigma+1+\varepsilon}}, & |x| \ge r_0 > 1, \qquad p = N; \\ H(x) \le \dfrac{C}{|x|^{N+(\sigma(p-N)/(p-1))+\varepsilon}}, & |x| \ge r_0 > 0, \qquad p > N, \end{cases}$$

for some constants $C > 0$ and $\varepsilon > 0$, then (2) *has positive radial entire solutions.*

THEOREM B ([15, Theorems 1, 2 and 3]). *If*

$$\begin{cases} H(x) \ge \dfrac{C}{|x|^{p}}, & |x| \ge r_0 > 0, \qquad p < N; \\ H(x) \ge \dfrac{C}{|x|^p(\log|x|)^{\sigma+1}}, & |x| \ge r_0 > 1, \qquad p = N; \\ H(x) \ge \dfrac{C}{|x|^{N+(\sigma(p-N)/(p-1))}}, & |x| \ge r_0 > 0, \qquad p > N, \end{cases}$$

for some constants $C > 0$, then (2) *does not possess any positive entire solution.*

In [8], actually existence results are proved under weaker assumptions than above.

When H is a radial function and behaves like $c|x|^l$, $l \in \mathbf{R}$ and $c > 0$, as $|x| \to \infty$, Theorems A and B characterize the decaying order of H for (2) to admit positive entire solutions. Related results are found in [11, 12, 16].

The aim of this paper is to extend such results to elliptic system (1). As far as the author is aware, there are no results dealing with this subject except for the case $p = q = 2$ ([5, 20]).

Our results are as follows:

THEOREM 1. *Suppose that H and K satisfy*

$$H(|x|) \le \frac{L_1}{|x|^{\lambda}}, \qquad K(|x|) \le \frac{L_2}{|x|^{\mu}}, \qquad |x| \ge r_0 > 0, \tag{3}$$

where $L_1 > 0$, $L_2 > 0$, λ and μ are constants. Then, under one of the next four conditions, system (1) *has infinitely many positive radial entire solutions:*

(i) $p \le N,\ q \le N$ *and*

$$\begin{cases} \lambda > \dfrac{\alpha(q-\mu)}{q-1} + p \\ \mu > \dfrac{\beta(p-\lambda)}{p-1} + q \end{cases}$$

(ii) $p > N,\ q > N$ *and*

$$\begin{cases} \lambda > \dfrac{\alpha(q-\mu)}{q-1} + \dfrac{\alpha\beta(p-N)}{(p-1)(q-1)} + N \\ \mu > \dfrac{\beta(p-\lambda)}{p-1} + \dfrac{\alpha\beta(q-N)}{(p-1)(q-1)} + N \end{cases}$$

(iii) $p \le N,\ q > N$ *and*

$$\begin{cases} \lambda > \dfrac{\alpha(q-\mu)}{q-1} + p \\ \mu > \dfrac{\beta(p-\lambda)}{p-1} + \dfrac{\alpha\beta(q-N)}{(p-1)(q-1)} + N \end{cases}$$

(iv) $p > N,\ q \le N$ *and*

$$\begin{cases} \lambda > \dfrac{\alpha(q-\mu)}{q-1} + \dfrac{\alpha\beta(p-N)}{(p-1)(q-1)} + N \\ \mu > \dfrac{\beta(p-\lambda)}{p-1} + q \end{cases}$$

THEOREM 2. *Suppose that H and K satisfy*

$$H(|x|) \ge \frac{L_1}{|x|^{\lambda}}, \qquad K(|x|) \ge \frac{L_2}{|x|^{\mu}}, \qquad |x| \ge r_0 > 0, \tag{4}$$

where $L_1 > 0$, $L_2 > 0$, λ and μ are constants. Then, under one of the next four conditions, system (1) *does not possess any nonnegative nontrivial radial entire solutions:*

(i) $p \le N,\ q \le N$ *and*

$$\begin{cases} \lambda \le \dfrac{\alpha(q-\mu)}{q-1} + p \qquad \text{or} \\ \mu \le \dfrac{\beta(p-\lambda)}{p-1} + q \end{cases}$$

(ii) $p > N,\ q > N$ *and*

$$\begin{cases} \lambda \le \dfrac{\alpha(q-\mu)}{q-1} + \dfrac{\alpha\beta(p-N)}{(p-1)(q-1)} + N & \text{or} \\ \mu \le \dfrac{\beta(p-\lambda)}{p-1} + \dfrac{\alpha\beta(q-N)}{(p-1)(q-1)} + N \end{cases}$$

(iii) $p \le N,\ q > N$ *and*

$$\begin{cases} \lambda \le \dfrac{\alpha(q-\mu)}{q-1} + p & \text{or} \\ \mu \le \dfrac{\beta(p-\lambda)}{p-1} + \dfrac{\alpha\beta(q-N)}{(p-1)(q-1)} + N \end{cases}$$

(iv) $p > N,\ q \le N$ *and*

$$\begin{cases} \lambda \le \dfrac{\alpha(q-\mu)}{q-1} + \dfrac{\alpha\beta(p-N)}{(p-1)(q-1)} + N & \text{or} \\ \mu \le \dfrac{\beta(p-\lambda)}{p-1} + q \end{cases}$$

We note that, for the case where $p = q = 2$ (and $N \neq 2$), Theorem 1 reduces to Theorems 3.1 and 3.3 in [20], and Theorem 2 to Theorems 2.1 and 2.3 in [20].

We give an illustrative example to show the sharpness of our results.

Let us consider the elliptic system

$$(5) \qquad \begin{cases} \Delta_p u = \dfrac{C}{(1+|x|)^{\lambda}} v^{\alpha} \\ \Delta_q v = \dfrac{C}{(1+|x|)^{\mu}} u^{\beta} \end{cases} \quad \text{in } \mathbf{R}^N,$$

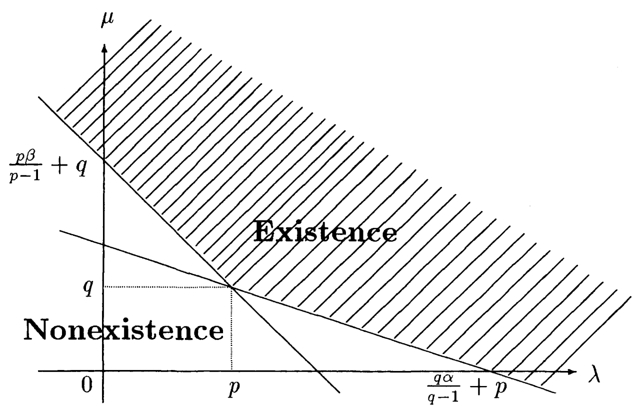
where $N \ge 2$, $N > p > 1$, $N > q > 1$, $\alpha\beta > (p-1)(q-1)$, $\lambda, \mu \in \mathbf{R}$, and C is a positive constant. We can completely characterize the existence of positive radial entire solutions of this system in terms of $p, q, \alpha, \beta, \lambda$ and μ. In fact, the inequalities

$$\frac{L_1}{|x|^{\lambda}} \le \frac{C}{(1+|x|)^{\lambda}} \le \frac{L_2}{|x|^{\lambda}}, \qquad |x| \ge 1$$

and

$$\frac{L_3}{|x|^{\mu}} \le \frac{C}{(1+|x|)^{\mu}} \le \frac{L_4}{|x|^{\mu}}, \qquad |x| \ge 1$$

hold, where L_i, $i = 1, \dots, 4$, are some positive constants. From Theorem 2, if $\lambda \le \dfrac{\alpha(q-\mu)}{q-1} + p$ or $\mu \le \dfrac{\beta(p-\lambda)}{p-1} + q$, then (5) does not admit any positive radial entire solutions. Conversely, from Theorem 1 if $\lambda > \dfrac{\alpha(q-\mu)}{q-1} + p$ and $\mu > \dfrac{\beta(p-\lambda)}{p-1} + q$, then (5) has infinitely many positive radial entire solutions. See the figure below.



For another case that H and K are nonpositive functions, there have been a great number of works on qualitative theory for solutions in the last three decades. We can find necessary and/or sufficient conditions to have positive entire solutions in this case with (or without) prescribed asymptotic forms near ∞; see [4, 9, 18]. For the scalar equation, we moreover know how oscillatory radial entire solutions behave near ∞.

As far as the author knows, the study for equation (2) was initiated essentially by J. B. Keller [10], who considered, for example, equation $\Delta u = u^\alpha$, $\alpha > 1$, in $\mathbf{R}^N$, and showed that this equation admits no positive entire solutions. In [17], equation (2) with $p = 2$ have been considered. It is known that there are some applications of qualitative theory for (2) to Riemannian geometry; see [17] and the references therein.

Equations of the type (2) have been investigated deeply not only in the entire space $\mathbf{R}^N$ but also in bounded domains. For example, the singular boundary value problem

$$(6) \qquad \begin{cases} \Delta u = u^\alpha & \text{in } D, \\ u \to \infty & \text{as } x \to \partial D, \end{cases}$$

where D is a bounded domain, has been treated by several authors. Problems of this type (in fact, (6) with u^α replaced by e^u) were firstly considered by Bieberbach [3]. In this case the problem plays an important role in the theory of Riemannian surfaces and in the theory of automorphic functions. Furthermore, according to [19] this problem arises in the study of the electric potential in a glowing hollow metal body. Related results on this topic are found in [2, 6, 13, 14, 21]. From these observations we do believe that considering system (1) is of practical interest as well as of theoretical interest.

Since for positive solutions (u, v) of (1), the functions $\max_{|x|=r} u(x)$ and $\max_{|x|=r} v(x)$, $r \geq 0$, are nondecreasing, it seems that the usual variational method does not always work effectively. Some of difficulties appearing in the analysis of (1) come from this fact. For non-symmetric solutions we refer to [1, 7].

The organization of the paper is as follows. The proofs of Theorems 1 and 2 are given in §2 and §3, respectively. In §4 we give existence and nonexistence theorems for the particular case $p = q = N$ which give stronger results than Theorems 1 and 2.

2. Proof of Theorem 1

In this section Theorem 1 is proved. We first observe that (u, v) is a positive radial entire solution of (1) if and only if the function $(y(r), z(r)) = (u(x), v(x))$, $r = |x|$, satisfies the system of second order ordinary differential equations

$$(7) \qquad \begin{cases} r^{1-N}(r^{N-1}|y'|^{p-2}y')' = H(r)z^\alpha, & r > 0, \ y'(0) = 0, \\ r^{1-N}(r^{N-1}|z'|^{q-2}z')' = K(r)y^\beta, & r > 0, \ z'(0) = 0, \end{cases}$$

where $' = d/dr$. Furthermore, integrating (7) twice, we obtain the following system of integral equations equivalent to (7):

$$(8) \qquad \begin{cases} y(r) = a + \displaystyle\int_0^r \left(s^{1-N} \int_0^s t^{N-1} H(t) z(t)^\alpha dt \right)^{1/(p-1)} ds, & r \geq 0, \\ z(r) = b + \displaystyle\int_0^r \left(s^{1-N} \int_0^s t^{N-1} K(t) y(t)^\beta dt \right)^{1/(q-1)} ds, & r \geq 0, \end{cases}$$

where $a = y(0)$, $b = z(0)$.

Proof of Theorem 1. Without loss of generality, we may assume that $r_0 = 1$ in (3). It suffices to solve (8). Choose constants $a > 0$ and $b > 0$ so that

$$(9)\qquad \begin{cases} \left((2b)^{\alpha}\int_0^1 H(t)dt\right)^{1/(p-1)} \le \dfrac{a}{2}, \\ M_1(N,p)\left((2b)^{\alpha}\max\left\{\int_0^1 H(t)dt, \dfrac{L_1}{N-\lambda+\alpha l}\right\}\right)^{1/(p-1)} \le \dfrac{a}{2}, \end{cases}$$

and

$$(10)\qquad \begin{cases} \left((2a)^{\beta}\int_0^1 K(t)dt\right)^{1/(q-1)} \le \dfrac{b}{2}, \\ M_2(N,q)\left((2a)^{\beta}\max\left\{\int_0^1 K(t)dt, \dfrac{L_2}{N-\mu+\beta k}\right\}\right)^{1/(q-1)} \le \dfrac{b}{2}, \end{cases}$$

where

$$k = \frac{(q-1)(\lambda-p)-\alpha(q-\mu)}{\alpha\beta-(p-1)(q-1)} > 0,$$

$$l = \frac{(p-1)(\mu-q)-\beta(p-\lambda)}{\alpha\beta-(p-1)(q-1)} > 0,$$

$$M_1(N,p) = \begin{cases} \dfrac{p-1}{p-\lambda+\alpha l} & \text{for } p \le N, \\ \dfrac{p-1}{p-N} & \text{for } p > N, \end{cases}$$

and

$$M_2(N,q) = \begin{cases} \dfrac{q-1}{q-\mu+\beta k} & \text{for } q \le N, \\ \dfrac{q-1}{q-N} & \text{for } q > N. \end{cases}$$

The inequalities $M_1(N,p) \ge 1$ and $M_2(N,q) \ge 1$ hold from the condition of λ and μ when $p \le N$ and $q \le N$, respectively. They are trivial when $p > N$ and $q > N$ respectively. It is possible to choose such a and b by our assumption $\alpha\beta > (p-1)(q-1)$. Define functions A and B by

$$A(r) = \begin{cases} 2a & \text{for } 0 \le r \le 1, \\ 2ar^k & \text{for } r \ge 1, \end{cases}$$

and

$$B(r) = \begin{cases} 2b & \text{for } 0 \le r \le 1, \\ 2br^l & \text{for } r \ge 1, \end{cases}$$

respectively. Put $\overline{\mathbf{R}}_+ = [0, \infty)$. We regard the space $C(\overline{\mathbf{R}}_+) \times C(\overline{\mathbf{R}}_+)$ as a Fréchet space equipped with the topology of uniform convergence of functions on each compact subinterval in $\overline{\mathbf{R}}_+$. Let $Y \subset C(\overline{\mathbf{R}}_+) \times C(\overline{\mathbf{R}}_+)$ denotes the subset defined by

$$Y = \{(y, z) \in C(\overline{\mathbf{R}}_+) \times C(\overline{\mathbf{R}}_+) : a \le y(r) \le A(r), b \le z(r) \le B(r), r \ge 0\}.$$

Obviously, Y is a non-empty closed convex subset of $C(\overline{\mathbf{R}}_+) \times C(\overline{\mathbf{R}}_+)$. Consider the mapping $\mathscr{F} : Y \to C(\overline{\mathbf{R}}_+) \times C(\overline{\mathbf{R}}_+)$ defined by $\mathscr{F}(y, z) = (\tilde{y}, \tilde{z})$, where

$$\tilde{y}(r) = a + \int_0^r \left(s^{1-N} \int_0^s t^{N-1} H(t) z(t)^\alpha dt \right)^{1/(p-1)} ds, \qquad r \ge 0,$$

and

$$\tilde{z}(r) = b + \int_0^r \left(s^{1-N} \int_0^s t^{N-1} K(t) y(t)^\beta dt \right)^{1/(q-1)} ds, \qquad r \ge 0.$$

In order to apply the Schauder-Tychonoff fixed point theorem, we will show that $\mathscr{F}$ is a continuous mapping from Y into itself such that $\mathscr{F}(Y)$ is relatively compact.

(I) *$\mathscr{F}$ maps Y into itself.* Let $(y, z) \in Y$. Clearly, $\tilde{y}(r) \ge a$ and $\tilde{z}(r) \ge b$. For $0 \le r \le 1$, we have

$$\begin{aligned}
\tilde{y}(r) &\le a + \int_0^r \left(\int_0^s H(t) z(t)^\alpha dt \right)^{1/(p-1)} ds \\
&\le a + \int_0^1 \left((2b)^\alpha \int_0^1 H(t) dt \right)^{1/(p-1)} ds \\
&\le a + \frac{a}{2} \le 2a.
\end{aligned}$$

When $p \le N$, for $r \ge 1$, we have

$$\begin{aligned}
\tilde{y}(r) &= a + \left(\int_0^1 + \int_1^r \right) \left(s^{1-N} \int_0^s t^{N-1} H(t) z(t)^\alpha dt \right)^{1/(p-1)} ds \\
&\le a + \frac{a}{2} + \int_1^r s^{(1-N)/(p-1)} \left((2b)^\alpha \int_0^1 H(t) dt + (2b)^\alpha L_1 \int_1^s t^{N-1-\lambda+\alpha l} dt \right)^{1/(p-1)} ds \\
&\le \frac{3}{2} a + \left((2b)^\alpha \max\left\{ \int_0^1 H(t) dt, \frac{L_1}{N - \lambda + \alpha l} \right\} \right)^{1/(p-1)} \int_1^r s^{((p-\lambda+\alpha l)/(p-1))-1} ds
\end{aligned}$$

$$\leq \frac{3}{2}a + M_1(N,p)\left((2b)^\alpha \max\left\{\int_0^1 H(t)dt, \frac{L_1}{N-\lambda+\alpha l}\right\}\right)^{1/(p-1)} r^{(p-\lambda+\alpha l)/(p-1)}$$

$$\leq \frac{3}{2}a + \frac{a}{2}r^k \leq 2ar^k.$$

When $p > N$, for $r \geq 1$, we have

$$\tilde{y}(r) = a + \left(\int_0^1 + \int_1^r\right)\left(s^{1-N}\int_0^s t^{N-1}H(t)z(t)^\alpha dt\right)^{1/(p-1)} ds$$

$$\leq a + \frac{a}{2} + \left(\int_1^r s^{(1-N)/(p-1)} ds\right)\left(\int_0^r t^{N-1}H(t)z(t)^\alpha dt\right)^{1/(p-1)}$$

$$\leq \frac{3}{2}a + M_1(N,p)r^{(p-N)/(p-1)}$$

$$\times \left((2b)^\alpha \int_0^1 H(t)dt + L_1(2b)^\alpha \int_1^r t^{N-1-\lambda+\alpha l}dt\right)^{1/(p-1)}$$

$$\leq \frac{3}{2}a + M_1(N,p)\left((2b)^\alpha \max\left\{\int_0^1 H(t)dt, \frac{L_1}{N-\lambda+\alpha l}\right\}\right)^{1/(p-1)} r^{(p-\lambda+\alpha l)/(p-1)}$$

$$\leq \frac{3}{2}a + \frac{a}{2}r^k \leq 2ar^k.$$

Thus we obtain

$$a \leq \tilde{y}(r) \leq A(r), \qquad r \geq 0.$$

A similar computation shows that

$$b \leq \tilde{z}(r) \leq B(r), \qquad r \geq 0.$$

Therefore $\mathscr{F}(Y) \subset Y$.

(II) *$\mathscr{F}$ is continuous.* Let $\{(y_m, z_m)\}$ be a sequence in Y which converges to $(y,z) \in Y$ uniformly on each compact subinterval of $\overline{\mathbf{R}}_+$. Put

$$\phi_m(r) = r^{1-N}\int_0^r s^{N-1}H(s)z_m(s)^\alpha ds.$$

Then we have

$$(11) \qquad |\phi_m(r) - \phi(r)| \leq \int_0^r H(s)|z_m(s)^\alpha - z(s)^\alpha| ds,$$

and

$$(12) \qquad |\tilde{y}_m(r) - \tilde{y}(r)| \leq \int_0^r |\phi_m(s)^{1/(p-1)} - \phi(s)^{1/(p-1)}| ds.$$

Let $R > 0$ be an arbitrary constant. Since $\{z_m\}$ converges to z uniformly on $[0, R]$, it follows from (11) that $\{\phi_m\}$ converges to ϕ uniformly on $[0, R]$; and hence $\{\phi_m^{1/(p-1)}\}$ converges to $\phi^{1/(p-1)}$ uniformly on $[0, R]$. From this fact and (12), we see that $\{\tilde{y}_m\}$ converges to $\tilde{y}$ uniformly on $[0, R]$. Similarly, $\{\tilde{z}_m\}$ converges to $\tilde{z}$ uniformly on each compact subinterval of $[0, \infty)$. These imply the continuity of $\mathscr{F}$.

(III) *$\mathscr{F}(Y)$ is relatively compact.* To see this, it suffices to verify the local equicontinuity of $\mathscr{F}(Y)$, since $\mathscr{F}(Y)$ is locally uniformly bounded by the fact that $\mathscr{F}(Y) \subset Y$. Let $(y, z) \in Y$ and $R > 0$. Then we have

$$\tilde{y}'(r) = \left(\int_0^r \left(\frac{s}{r}\right)^{N-1} H(s) z(s)^\alpha ds\right)^{1/(p-1)} \le \left(\int_0^R H(s) B(s)^\alpha ds\right)^{1/(p-1)}$$

and

$$\tilde{z}'(r) = \left(\int_0^r \left(\frac{s}{r}\right)^{N-1} K(s) y(s)^\beta ds\right)^{1/(q-1)} \le \left(\int_0^R K(s) A(s)^\beta ds\right)^{1/(q-1)}.$$

Obviously, these imply the local boundedness of the set $\{(\tilde{y}', \tilde{z}') | (y, z) \in Y\}$. Hence the relative compactness of $\mathscr{F}(Y)$ is shown by the Ascoli-Arzelà theorem.

Therefore, there exists $(y, z) \in Y$ such that $(y, z) = \mathscr{F}(y, z)$ by the Schauder-Tychonoff fixed point theorem, that is, (y, z) satisfies the integral equation (8). The function $(u(x), v(x)) = (y(|x|), z(|x|))$ then gives a solution of (1). Since infinitely many (a, b) satisfy (9) and (10), we can construct an infinitude of positive radial entire solutions of (1). This completes the proof.

3. Proof of Theorem 2

In this section, we prove Theorem 2. We give a preparatory observation as a first step.

Let (u, v) be a nonnegative nontrivial radial entire solution of (1). Then (u, v) satisfies the system of ordinary differential equation

$$(13) \qquad \begin{cases} (r^{N-1}|u'(r)|^{p-2}u'(r))' = r^{N-1}H(r)v(r)^\alpha, & r > 0, \ u'(0) = 0, \\ (r^{N-1}|v'(r)|^{q-2}v'(r))' = r^{N-1}K(r)u(r)^\beta, & r > 0, \ v'(0) = 0, \end{cases}$$

where $r = |x|$ and $' = d/dr$. Integrating (13) over $[0, r]$, we have

$$|u'(r)|^{p-2}u'(r) = r^{1-N} \int_0^r s^{N-1} H(s) v(s)^\alpha ds, \qquad r > 0.$$

Hence, we see that $u'(r) \ge 0$ for $r \ge 0$. Similarly we have $v'(r) \ge 0$ for

$r \geq 0$. Integrating (13) twice over $[R, r], R \geq 0$, we have

$$
(14) \qquad \begin{cases} u(r) \geq u(R) + \displaystyle\int_R^r \left(\int_R^s \left(\frac{t}{s}\right)^{N-1} H(t) v(t)^\alpha dt \right)^{1/(p-1)} ds, & r \geq R, \\ v(r) \geq v(R) + \displaystyle\int_R^r \left(\int_R^s \left(\frac{t}{s}\right)^{N-1} K(t) u(t)^\beta dt \right)^{1/(q-1)} ds, & r \geq R. \end{cases}
$$

Since the functions u and v are nondecreasing on $[0, \infty)$, there is an $r_* \geq 0$ such that $u(r_*) > 0$ or $v(r_*) > 0$. We see from (14) with $R = r_*$ that $u(r) > 0$ and $v(r) > 0$ for $r > r_*$. Let us fix $r_1 > \max\{r_0, r_*\}$.

Let $R > r_1$. Using (4) and inequality

$$
(15) \qquad \left(\frac{t}{s}\right)^{N-1} \geq \left(\frac{1}{3}\right)^{N-1} \qquad \text{for } R \leq t \leq s \leq 3R
$$

in (14), we have

$$
\begin{aligned}
u(r) &\geq u(R) + \int_R^r \left(\int_R^s \left(\frac{1}{3}\right)^{N-1} L_1 t^{-\lambda} v(t)^\alpha dt \right)^{1/(p-1)} ds \\
&\geq C_1 R^{-\lambda/(p-1)} \int_R^r \left(\int_R^s v(t)^\alpha dt \right)^{1/(p-1)} ds, \qquad R \leq r \leq 3R,
\end{aligned}
$$

and

$$
v(r) \geq C_2 R^{-\mu/(q-1)} \int_R^r \left(\int_R^s u(t)^\beta dt \right)^{1/(q-1)} ds, \qquad R \leq r \leq 3R,
$$

where C_1 and C_2 are some positive constants independent of r and R. Now, we fix $R > r_1$ arbitrary for a moment, and put

$$
(16) \qquad f(r; R) = C_1 R^{-\lambda/(p-1)} \int_R^r \left(\int_R^s v(t)^\alpha dt \right)^{1/(p-1)} ds, \qquad R \leq r \leq 3R,
$$

and

$$
(17) \qquad g(r; R) = C_2 R^{-\mu/(q-1)} \int_R^r \left(\int_R^s u(t)^\beta dt \right)^{1/(q-1)} ds, \qquad R \leq r \leq 3R.
$$

For simplicity of notation, we sometimes write $f(r; R) = f(r)$ and $g(r; R) = g(r)$ if there is no ambiguity. Then

$$
f(R) = f'(R) = g(R) = g'(R) = 0,
$$

$$f'(r) = C_1 R^{-\lambda/(p-1)} \left(\int_R^r v(s)^\alpha ds \right)^{1/(p-1)} \geq 0, \qquad R \leq r \leq 3R,$$

$$g'(r) = C_2 R^{-\mu/(q-1)} \left(\int_R^r u(s)^\beta ds \right)^{1/(q-1)} \geq 0, \qquad R \leq r \leq 3R,$$

$$f''(r) > 0, \qquad g''(r) > 0, \qquad R < r \leq 3R,$$

(18) $$(f'(r)^{p-1})' = C_3 R^{-\lambda} v(r)^\alpha \geq C_3 R^{-\lambda} g(r)^\alpha, \qquad R \leq r \leq 3R,$$

and

(19) $$(g'(r)^{q-1})' = C_4 R^{-\mu} u(r)^\beta \geq C_4 R^{-\mu} f(r)^\beta, \qquad R \leq r \leq 3R,$$

where $C_3 = C_1^{p-1}$ and $C_4 = C_2^{q-1}$. From now on, we use C to denote various positive constants independent of r and R, as we will have no confusion.

Multiplying (18) by $g' \geq 0$ and integrating by parts the resulting inequality on $[R+\varepsilon, r]$, $\varepsilon > 0$, we see that

$$f'(r)^{p-1} g'(r) \geq \frac{C_3}{\alpha+1} R^{-\lambda} \{ g(r)^{\alpha+1} - g(R+\varepsilon)^{\alpha+1} \}, \qquad R+\varepsilon \leq r \leq 3R.$$

Letting $\varepsilon \to 0$, we obtain

$$f'(r) g'(r)^{1/(p-1)} \geq C R^{-\lambda/(p-1)} g(r)^{(\alpha+1)/(p-1)}, \qquad R \leq r \leq 3R.$$

Multiplying this inequality by g' and integrating by parts, we see that

$$f(r) g'(r)^{p/(p-1)} \geq C R^{-\lambda/(p-1)} g(r)^{(\alpha+p)/(p-1)}, \qquad R \leq r \leq 3R.$$

From (19), we have

$$(g'(r)^{q-1})' g'(r)^{p\beta/(p-1)} \geq C R^{-(\lambda\beta+\mu(p-1))/(p-1)} g(r)^{\beta(\alpha+p)/(p-1)}, \qquad R \leq r \leq 3R.$$

Multiplying this relation by g' and integrating by parts, we obtain

$$g'(r) \geq C R^{-(\lambda\beta+\mu(p-1))/(\beta p+q(p-1))} g(r)^{(\beta(\alpha+p)+p-1)/(\beta p+q(p-1))}, \qquad R \leq r \leq 3R.$$

Since $\dfrac{\beta(\alpha+p)+p-1}{p\beta+q(p-1)} > 1$, we can set $\dfrac{\beta(\alpha+p)+p-1}{p\beta+q(p-1)} = \delta_1 + 1, \delta_1 > 0$. Thus

$$g'(r) g(r)^{-\delta_1 - 1} \geq C R^{-(\lambda\beta+\mu(p-1))/(\beta p+q(p-1))}, \qquad R \leq r \leq 3R.$$

Integrating over $[2R, 3R]$, we see that

(20) $$g(2R; R)^{-\delta_1} \geq C R^{\eta_1}, \qquad R > r_1.$$

Repeating similar argument as above by replacing $g(r)$ by $f(r)$, we obtain

(21) $$f(2R; R)^{-\delta_2} \geq C R^{\eta_2}, \qquad R > r_1.$$

Here the constants $\delta_1, \eta_1, \delta_2$, and η_2 are given, respectively, by

$$\delta_1 = \frac{\alpha\beta - (p-1)(q-1)}{p\beta + q(p-1)} > 0, \qquad \eta_1 = \frac{\beta(p-\lambda) + (p-1)(q-\mu)}{p\beta + q(p-1)},$$

$$\delta_2 = \frac{\alpha\beta - (p-1)(q-1)}{q\alpha + p(q-1)} > 0 \qquad \text{and} \qquad \eta_2 = \frac{\alpha(q-\mu) + (q-1)(p-\lambda)}{q\alpha + p(q-1)}.$$

Inequalities (20) and (21) play important role to prove Theorem 2.

On the other hand, from (17) and the monotonicity of u, we have

$$(22) \qquad \begin{aligned} g(2R;R) &= C_2 R^{-\mu/(q-1)} \int_R^{2R} \left(\int_R^s u(t)^\beta dt \right)^{1/(q-1)} ds \\ &\geq C_2 R^{-\mu/(q-1)} u(R)^{\beta/(q-1)} \int_R^{2R} (s-R)^{1/(q-1)} ds \\ &= C R^{(q-\mu)/(q-1)} u(R)^{\beta/(q-1)}. \end{aligned}$$

Similarly, from (16) and the monotonicity of v, we have

$$(23) \qquad f(2R;R) \geq C R^{(p-\lambda)/(p-1)} v(R)^{\alpha/(p-1)}.$$

Proof of Theorem 2. Suppose to the contrary that system (1) has a nonnegative nontrivial radial entire solution (u,v). From preceding observation, we see that $u(r), v(r) > 0$, $r \geq r_1$ for some $r_1 > r_0$, and inequalities (20)–(23) hold for $r \geq r_1$. If $\eta_1 \geq 0$ or $\eta_2 \geq 0$, then we show that $g(2R;R) \to \infty$ or $f(2R;R) \to \infty$ as $R \to \infty$ to get a contradiction. Otherwise, we show that $R^{\eta_1} g(2R;R)^{\delta_1} \to \infty$ or $R^{\eta_2} f(2R;R)^{\delta_2} \to \infty$ as $R \to \infty$. Through this proof, the letter $\tilde{C}$ will be used to denote various positive constants independent of r and R.

(i) Let $p \leq N$ and $q \leq N$. We may consider only the case that $\lambda \leq p$ and $\mu \leq \dfrac{\beta(p-\lambda)}{p-1} + q$. The other case that $\mu \leq q$ and $\lambda \leq \dfrac{\alpha(q-\mu)}{q-1} + p$ can be treated similarly. We easily see that $\eta_1 \geq 0$ in (20). So it suffices to show that $g(2R;R) \to \infty$ as $R \to \infty$.

We first show that, for $r \geq r_2 > r_1$,

$$(24) \qquad u(r) \geq \begin{cases} \tilde{C} r^{(p-\lambda)/(p-1)}, & \lambda < p \leq N, \\ \tilde{C} \log r, & \lambda = p < N, \\ \tilde{C} (\log r)^{p/(p-1)}, & \lambda = p = N. \end{cases}$$

Let $r \geq r_1$. Integrating (13) twice over $[r_1, r]$, we have

$$(25)\quad \begin{cases} u(r) \geq u(r_1) + \displaystyle\int_{r_1}^{r} s^{(1-N)/(p-1)} \left(\int_{r_1}^{s} t^{N-1} H(t) v(t)^{\alpha} dt \right)^{1/(p-1)} ds, & r \geq r_1, \\ v(r) \geq v(r_1) + \displaystyle\int_{r_1}^{r} s^{(1-N)/(q-1)} \left(\int_{r_1}^{s} t^{N-1} K(t) u(t)^{\beta} dt \right)^{1/(q-1)} ds, & r \geq r_1. \end{cases}$$

Then from (4) and (25) we observe that, for $\lambda < p \leq N$

$$\begin{aligned} u(r) &\geq (L_1 v(r_1)^{\alpha})^{1/(p-1)} \int_{r_1+1}^{r} s^{(1-N)/(p-1)} \left(\int_{r_1}^{s} t^{N-1-\lambda} dt \right)^{1/(p-1)} ds \\ &\geq \tilde{C} \int_{r_1+1}^{r} s^{(1-\lambda)/(p-1)} ds, \qquad r \geq r_1 + 1, \end{aligned}$$

for $\lambda = p < N$

$$\begin{aligned} u(r) &\geq (L_1 v(r_1)^{\alpha})^{1/(p-1)} \int_{r_1+1}^{r} s^{(1-N)/(p-1)} \left(\int_{r_1}^{s} t^{N-1-\lambda} dt \right)^{1/(p-1)} ds \\ &\geq \tilde{C} \int_{r_1+1}^{r} s^{-1} ds, \qquad r \geq r_1 + 1, \end{aligned}$$

and for $\lambda = p = N$

$$\begin{aligned} u(r) &\geq (L_1 v(r_1)^{\alpha})^{1/(p-1)} \int_{r_1+1}^{r} s^{-1} \left(\int_{r_1}^{s} t^{-1} dt \right)^{1/(p-1)} ds \\ &\geq \tilde{C} \int_{r_1+1}^{r} s^{-1} (\log s)^{1/(p-1)} ds, \qquad r \geq r_1 + 1. \end{aligned}$$

Thus we obtain (24).

Let us fix $R > r_2$ arbitrarily. From (22) and (24), we obtain

$$g(2R; R) \geq \begin{cases} \tilde{C} R^{\tau_1}, & \lambda < p, \\ \tilde{C} R^{(q-\mu)/(q-1)} (\log R)^{\beta/(q-1)}, & \lambda = p < N, \\ \tilde{C} R^{(q-\mu)/(q-1)} (\log R)^{\beta p/((p-1)(q-1))}, & \lambda = p = N, \end{cases}$$

where $\tau_1 = \dfrac{1}{q-1} \left(q - \mu + \dfrac{\beta(p-\lambda)}{p-1} \right)$. For $\lambda = p$, we see that $\mu \leq q$, which shows that $g(2R; R) \to \infty$ as $R \to \infty$. For $\lambda < p$, we observe that $g(2R; R) \to \infty$ if $\mu < \dfrac{\beta(p-\lambda)}{p-1} + q$.

It remains only to discuss the case where $\lambda < p$ and $\mu = \dfrac{\beta(p-\lambda)}{p-1} + q$. In this case we will show that $R^{\eta_2} f(2R; R)^{\delta_2} \to \infty$ as $R \to \infty$. To this end, we prove that, for $r \geq r_3 > r_2 + 1$,

$$v(r) \geq \begin{cases} \tilde{C} \log r, & q < N, \\ \tilde{C}(\log r)^{q/(q-1)}, & q = N. \end{cases} \tag{26}$$

Let $r > r_2 + 1$. Then from (24) and (25) we observe that, for $q < N$

$$\begin{aligned} v(r) &\geq (L_2 \tilde{C}^{\beta})^{1/(q-1)} \int_{r_2+1}^{r} s^{(1-N)/(q-1)} \left(\int_{r_2}^{s} t^{N-1-\mu+(\beta(p-\lambda)/(p-1))} \, dt \right)^{1/(q-1)} ds \\ &= \tilde{C} \int_{r_2+1}^{r} s^{(1-N)/(q-1)} \left(\int_{r_2}^{s} t^{N-1-q} \, dt \right)^{1/(q-1)} ds \\ &\geq \tilde{C} \int_{r_2+1}^{r} s^{-1} \, ds, \end{aligned}$$

and for $q = N$

$$\begin{aligned} v(r) &\geq (L_2 \tilde{C}^{\beta})^{1/(q-1)} \int_{r_2+1}^{r} s^{-1} \left(\int_{r_2}^{s} t^{-1} \, dt \right)^{1/(q-1)} ds \\ &\geq \tilde{C} \int_{r_2+1}^{r} s^{-1} (\log s)^{1/(q-1)} ds. \end{aligned}$$

Thus we obtain (26).

We fix $R > r_3$ arbitrarily. Then from (23) and (26), we have

$$f(2R; R) \geq \begin{cases} \tilde{C} R^{(p-\lambda)/(p-1)} (\log R)^{\alpha/(p-1)}, & q < N, \\ \tilde{C} R^{(p-\lambda)/(p-1)} (\log R)^{\alpha q/((p-1)(q-1))}, & q = N. \end{cases}$$

Hence, we see that

$$R^{\eta_2} f(2R; R)^{\delta_2} \geq \begin{cases} \tilde{C} R^{\tau_2} (\log R)^{\alpha\delta_2/(p-1)}, & q < N, \\ \tilde{C} R^{\tau_2} (\log R)^{\alpha q \delta_2/((p-1)(q-1))}, & q = N, \end{cases}$$

where $\tau_2 = \eta_2 + \dfrac{p-\lambda}{p-1}\delta_2$. An easy computation shows that $\tau_2 = 0$, and hence $R^{\eta_2} f(2R; R)^{\delta_2} \to \infty$ as $R \to \infty$. Thus the proof of (i) is complete.

(ii) Let $p > N$ and $q > N$. It suffices to treat the case that $\lambda \leq \dfrac{\alpha(q-N)}{q-1} + N$ and $\mu \leq \dfrac{\beta(p-\lambda)}{p-1} + \dfrac{\alpha\beta(q-N)}{(p-1)(q-1)} + N$. We show that $R^{\eta_1} g(2R; R)^{\delta_1} \to \infty$ or $R^{\eta_2} f(2R; R)^{\delta_2} \to \infty$ as $R \to \infty$. The proof is divided further into three cases.

(I) *The case* $\lambda < \dfrac{\alpha(q-N)}{q-1} + N$ *and* $\mu < \dfrac{\beta(p-\lambda)}{p-1} + \dfrac{\alpha\beta(q-N)}{(p-1)(q-1)} + N.$

First we show that

$$u(r) \geq \tilde{C} r^{((q-1)(p-\lambda)+\alpha(q-N))/((p-1)(q-1))}, \qquad r \geq r_2 > r_1. \tag{27}$$

From (25) we observe that, for $r > r_1 + 1$

$$\begin{aligned} v(r) &\geq v(r_1) + \int_{r_1+1}^{r} s^{(1-N)/(q-1)} \left(\int_{r_1}^{r_1+1} t^{N-1} K(t) u(t)^{\beta} dt \right)^{1/(q-1)} ds \\ &\geq \left(\int_{r_1}^{r_1+1} t^{N-1} K(t) u(t)^{\beta} dt \right)^{1/(q-1)} \int_{r_1+1}^{r} s^{(1-N)/(q-1)} ds. \end{aligned}$$

Thus, we obtain

$$v(r) \geq \tilde{C} r^{(q-N)/(q-1)}, \qquad r \geq \tilde{r}_1 > r_1 + 1. \tag{28}$$

From this estimate and (25), we have for $r \geq \tilde{r}_1 + 1$

$$\begin{aligned} u(r) &\geq (\tilde{C} L_1)^{1/(p-1)} \int_{\tilde{r}_1+1}^{r} s^{(1-N)/(p-1)} \left(\int_{\tilde{r}_1}^{s} t^{N-1-\lambda+(\alpha(q-N)/(q-1))} dt \right)^{1/(p-1)} ds \\ &\geq \tilde{C} \int_{\tilde{r}_1+1}^{r} s^{1/(p-1)\{1-\lambda+(\alpha(q-N)/(q-1))\}} ds. \end{aligned}$$

Thus we obtain (27).

Let us fix $R > r_2$ arbitrarily. From (22) and (27), we obtain

$$g(2R; R) \geq \tilde{C} R^{\gamma},$$

where $\gamma = \dfrac{1}{q-1}\left\{ q - \mu + \dfrac{\beta(p-\lambda)}{p-1} + \dfrac{\alpha\beta(q-N)}{(p-1)(q-1)} \right\}$. From this estimate, we have

$$R^{\eta_1} g(2R; R)^{\delta_1} \geq \tilde{C} R^{\tau_1},$$

where

$$\begin{aligned} \tau_1 &= \eta_1 + \gamma\delta_1 \\ &= \frac{\beta(p-\lambda) + (p-1)(q-\mu)}{p\beta + q(p-1)} \\ &\quad + \frac{\alpha\beta - (p-1)(q-1)}{(q-1)\{p\beta + q(p-1)\}} \left\{ q - \mu + \frac{\beta(p-\lambda)}{p-1} + \frac{\alpha\beta(q-N)}{(p-1)(q-1)} \right\} \\ &= \frac{\alpha\beta}{(q-1)\{p\beta + q(p-1)\}} \left\{ N - \mu + \frac{\beta(p-\lambda)}{p-1} + \frac{\alpha\beta(q-N)}{(p-1)(q-1)} \right\}. \end{aligned}$$

From the condition of μ we see that $\tau_1 > 0$, which shows that $R^{\eta_1} g(2R; R)^{\delta_1} \to \infty$ as $R \to \infty$.

(II) *The case* $\lambda < \dfrac{\alpha(q-N)}{q-1} + N$ *and* $\mu = \dfrac{\beta(p-\lambda)}{p-1} + \dfrac{\alpha\beta(q-N)}{(p-1)(q-1)} + N$. Let $r > r_2 + 1$. Then from (25) and (27), we have

$$(29) \quad v(r) \ge (\tilde{C}L_2)^{1/(q-1)} \int_{r_2+1}^{r} s^{(1-N)/(q-1)} \times \left(\int_{r_2}^{s} t^{N-1-\mu+(\beta(p-\lambda)/(p-1))+(\alpha\beta(q-N)/((p-1)(q-1)))} dt \right)^{1/(q-1)} ds$$

$$\ge \tilde{C} \int_{r_2+1}^{r} s^{(1-N)/(q-1)} (\log s)^{1/(q-1)} ds$$

$$\ge \tilde{C} r^{(q-N)/(q-1)} (\log r)^{1/(q-1)}, \qquad r \ge r_3 > r_2 + 1.$$

Here, the final inequality is given by integration by parts.

Let $R > r_3$ be large enough. From (23) and (29), we obtain

$$f(2R; R) \ge \tilde{C} R^{((p-\lambda)(q-1)+\alpha(q-N))/((p-1)(q-1))} (\log R)^{\alpha/((p-1)(q-1))}.$$

Hence, by this estimate we see that

$$R^{\eta_2} f(2R; R)^{\delta_2} \ge \tilde{C} R^{\eta_2 + (((p-\lambda)(q-1)+\alpha(q-N))/((p-1)(q-1)))\delta_2} (\log R)^{\alpha\delta_2/((p-1)(q-1))}.$$

By an easy computation we have $\eta_2 + \dfrac{(p-\lambda)(q-1)+\alpha(q-N)}{(p-1)(q-1)}\delta_2 = 0$. This shows that $R^{\eta_2} f(2R; R)^{\delta_2} \to \infty$ as $R \to \infty$.

(III) *The case* $\lambda = \dfrac{\alpha(q-N)}{q-1} + N$ *and* $\mu \le \dfrac{\beta(p-\lambda)}{p-1} + \dfrac{\alpha\beta(q-N)}{(p-1)(q-1)} + N = \dfrac{\beta(p-N)}{p-1} + N$. Let $r > \tilde{r}_1 + 1$. Then from (25) and (28), we have

$$(30) \quad u(r) \ge (\tilde{C}L_1)^{1/(p-1)} \int_{\tilde{r}_1+1}^{r} s^{(1-N)/(p-1)} \left(\int_{\tilde{r}_1}^{s} t^{N-1-\lambda+(\alpha(q-N)/(q-1))} dt \right)^{1/(p-1)} ds$$

$$\ge \tilde{C} \int_{\tilde{r}_1+1}^{r} s^{(1-N)/(p-1)} (\log s)^{1/(p-1)} ds$$

$$\ge \tilde{C} r^{(p-N)/(p-1)} (\log r)^{1/(p-1)}, \qquad r \ge r_4 > \tilde{r}_1 + 1.$$

Let $R > r_4$ be sufficiently large. From (22) and (30), we see that

$$g(2R; R) \ge \tilde{C} R^{((q-\mu)(p-1)+\beta(p-N))/((p-1)(q-1))} (\log R)^{\beta/((p-1)(q-1))}.$$

By this estimate, we obtain

$$R^{\eta_1} g(2R; R)^{\delta_1} \geq \tilde{C} R^{\tau_2} (\log R)^{\beta\delta_1/((p-1)(q-1))}, \qquad R \geq r_4,$$

where $\tau_2 = \dfrac{\alpha\beta}{(q-1)\{p\beta + q(p-1)\}} \left\{ N - \mu + \dfrac{\beta(p-N)}{p-1} \right\} \geq 0$. Hence, we see that $R^{\eta_1} g(2R; R)^{\delta_1} \to \infty$ as $R \to \infty$. Thus the proof of (ii) is completed.

(iii) Let $p \leq N$ and $q > N$. We consider the case (I) $\lambda \leq \dfrac{\alpha(q-N)}{q-1} + p$, $\mu \leq \dfrac{\beta(p-\lambda)}{p-1} + \dfrac{\alpha\beta(q-N)}{(p-1)(q-1)} + N$ first, and then the case (II) $\mu \leq N$, $\lambda \leq \dfrac{\alpha(q-\mu)}{q-1} + p$. The proofs of (I) and (II) are almost the same as those of (ii) and (i), respectively. So we leave the proof to the readers.

The proof of (iv) is essentially the same as that of (iii). Thus we may conclude the proof of Theorem 2.

4. A further study for the case of $p = q = N$

To begin with, we give an example for which we cannot apply Theorems 1 and 2 for the case $p = q = N$. Let us consider the elliptic system

$$\text{(31)} \qquad \begin{cases} \Delta_N u = \dfrac{1}{(|x|+2)^N \log(|x|+2)} v^\alpha, \\ \Delta_N v = \dfrac{1}{(|x|+2)^N \log(|x|+2)} u^\beta, \end{cases} \qquad x \in \mathbf{R}^N,$$

where $N \geq 2, \alpha > 0, \beta > 0$ and $\alpha\beta > (N-1)^2$. We can easily find that

$$\frac{C_1}{|x|^{N+\varepsilon}} \leq \frac{1}{(|x|+2)^N \log(|x|+2)} \leq \frac{C_2}{|x|^N}, \qquad |x| \geq 2,$$

where $\varepsilon > 0$, $C_1 = C_1(\varepsilon) > 0$ and $C_2 > 0$ are constants. We see therefore that neither the condition of (i) of Theorem 1 nor that of (i) of Theorem 2 holds. But, according to [20, Theorem 2.2], it is found that (31) has no positive radial entire solution when $N = 2$.

So, we will improve Theorems 1 and 2 for the case $p = q = N$ so that we can determine whether such systems have positive radial entire solutions or not.

Our results are as follows:

THEOREM 3. *Let $p = q = N$. Suppose that H and K satisfy*

$$H(|x|) \leq \frac{L_1}{|x|^N (\log|x|)^\lambda}, \qquad K(|x|) \leq \frac{L_2}{|x|^N (\log|x|)^\mu}, \qquad |x| \geq r_0 > 1,$$

where L_1 *and* L_2 *are positive constants and*

$$\begin{cases} \lambda > \dfrac{\alpha(\beta + N - \mu)}{N-1} + 1 \\ \mu > \dfrac{\beta(\alpha + N - \lambda)}{N-1} + 1. \end{cases}$$

Then (1) *has infinitely many positive radial entire solutions.*

THEOREM 4. *Let* $p = q = N$. *Suppose that* H *and* K *satisfy*

$$(32)\qquad H(|x|) \geq \frac{L_1}{|x|^N(\log|x|)^\lambda}, \qquad K(|x|) \geq \frac{L_2}{|x|^N(\log|x|)^\mu}, \qquad |x| \geq r_0 > 1,$$

where L_1 *and* L_2 *are positive constants and*

$$\begin{cases} \lambda < \dfrac{\alpha(\beta + N - \mu)}{N-1} + 1 \qquad \textit{or} \\ \mu < \dfrac{\beta(\alpha + N - \lambda)}{N-1} + 1. \end{cases}$$

Then (1) *does not possess any nonnegative nontrivial radial entire solutions.*

REMARK. From Theorem 4, we find that (31) has no nonnegative nontrivial radial entire solutions even when $N \geq 3$.

PROOF OF THEOREM 3. Without loss of generality, we may assume that $r_0 = e$. As in the proof of Theorem 1, it suffices to solve (8). Choose constants $a > 0$ and $b > 0$ so that

$$\begin{cases} e\left((2b)^\alpha \displaystyle\int_0^e H(t)dt\right)^{1/(N-1)} \leq \dfrac{a}{2}, \\ \left((2b)^\alpha \max\left\{e^{N-1}\displaystyle\int_0^e H(t)dt, \dfrac{L_1}{\alpha l - \lambda + 1}\right\}\right)^{1/(N-1)} \leq \dfrac{a}{2}, \end{cases}$$

and

$$\begin{cases} e\left((2a)^\beta \displaystyle\int_0^e K(t)dt\right)^{1/(N-1)} \leq \dfrac{b}{2}, \\ \left((2a)^\beta \max\left\{e^{N-1}\displaystyle\int_0^e K(t)dt, \dfrac{L_2}{\beta k - \mu + 1}\right\}\right)^{1/(N-1)} \leq \dfrac{b}{2}, \end{cases}$$

where $k = \dfrac{(N-1)(\lambda - N) - \alpha(N - \mu)}{\alpha\beta - (N-1)^2} > 0$, $l = \dfrac{(N-1)(\mu - N) - \beta(N - \lambda)}{\alpha\beta - (N-1)^2} > 0$. It is possible to choose such a and b by the assumption $\alpha\beta > (N-1)^2$. Put

$$A(r) = \begin{cases} 2a, & 0 \le r \le e \\ 2a(\log r)^k, & r \ge e, \end{cases}$$

and

$$B(r) = \begin{cases} 2b, & 0 \le r \le e \\ 2b(\log r)^l, & r \ge e. \end{cases}$$

Consider the set

$$Y = \{(y, z) \in C(\overline{\mathbf{R}}_+) \times C(\overline{\mathbf{R}}_+) : a \le y(r) \le A(r), b \le z(r) \le B(r), r \ge 0\},$$

which is a closed convex subset of $C(\overline{\mathbf{R}}_+) \times C(\overline{\mathbf{R}}_+)$. Define the mapping $\mathscr{F} : Y \to C(\overline{\mathbf{R}}_+) \times C(\overline{\mathbf{R}}_+)$ by $\mathscr{F}(y, z) = (\tilde{y}, \tilde{z})$, where

$$\tilde{y}(r) = a + \int_0^r \left(s^{1-N} \int_0^s t^{N-1} H(t) z(t)^\alpha dt \right)^{1/(N-1)} ds, \qquad r \ge 0,$$

and

$$\tilde{z}(r) = b + \int_0^r \left(s^{1-N} \int_0^s t^{N-1} K(t) y(t)^\beta dt \right)^{1/(N-1)} ds, \qquad r \ge 0.$$

First we show that $\mathscr{F}(Y) \subset Y$. Let $(y, z) \in Y$. Clearly, $\tilde{y}(r) \ge a$ and $\tilde{z}(r) \ge b$. For $0 \le r \le e$, we have

$$\begin{aligned} \tilde{y}(r) &= a + \int_0^r s^{-1} \left(\int_0^s t^{N-1} H(t) z(t)^\alpha dt \right)^{1/(N-1)} ds \\ &\le a + \int_0^r \left(\int_0^s H(t) z(t)^\alpha dt \right)^{1/(N-1)} ds \\ &\le a + e \left((2b)^\alpha \int_0^e H(t) dt \right)^{1/(N-1)} \\ &\le a + \frac{a}{2} \le 2a. \end{aligned}$$

For $r \ge e$, we have

$$\begin{aligned} \tilde{y}(r) &= a + \left(\int_0^e + \int_e^r \right) \left(s^{1-N} \int_0^s t^{N-1} H(t) z(t)^\alpha dt \right)^{1/(N-1)} ds \\ &\le a + \frac{a}{2} + \left(\int_e^r s^{-1} ds \right) \left(\int_0^r t^{N-1} H(t) z(t)^\alpha dt \right)^{1/(N-1)} \\ &\le \frac{3}{2} a + \left((2b)^\alpha e^{N-1} \int_0^e H(t) dt + L_1 (2b)^\alpha \int_e^r t^{-1} (\log t)^{\alpha l - \lambda} dt \right)^{1/(N-1)} \log r \end{aligned}$$

$$\leq \frac{3}{2}a + \left((2b)^{\alpha} \max\left\{e^{N-1}\int_0^e H(t)dt, \frac{L_1}{\alpha l - \lambda + 1}\right\}\right)^{1/(N-1)} (\log r)^{(N-\lambda+\alpha l)/(N-1)}$$

$$\leq \frac{3}{2}a(\log r)^k + \frac{a}{2}(\log r)^k = 2a(\log r)^k.$$

Thus we obtain

$$a \leq \tilde{y}(r) \leq A(r), \qquad r \geq 0.$$

A similar computation shows that

$$b \leq \tilde{z}(r) \leq B(r), \qquad r \geq 0.$$

Therefore $\mathscr{F}(y,z) \in Y$. The continuity of $\mathscr{F}$ and the relative compactness of $\mathscr{F}(Y)$ can be verified in a routine manner, and so by the Schauder-Tychonoff fixed point theorem there exists an element $(y,z) \in Y$ such that $(y,z) = \mathscr{F}(y,z)$. It is clear that this (y,z) gives rise to a positive radial entire solution $(u,v) = (y(|x|), z(|x|))$ of (1). The proof is completed.

Proof of Theorem 4. It suffices to treat the case that $\lambda < \alpha + 1$ and $\mu < \dfrac{\beta(\alpha + N - \lambda)}{N-1} + 1$. The proof is carried out by contradiction as before. Suppose to the contrary that system (1) admits a nonnegative nontrivial radial entire solution (u,v).

Step 1. As in the proof of Theorem 2, we may suppose that $u(r), v(r) > 0$, $r \geq r_1$ for some $r_1 > r_0$. Let $R > r_1$ be sufficiently large. As in the proof of Theorem 2, integrating (13) on $[R,r]$ and using (32) and inequality (15), we have

$$(33) \qquad u(r) \geq C_1 R^{-N/(N-1)}(\log R)^{-\lambda/(N-1)} \int_R^r \left(\int_R^s v(t)^{\alpha}dt\right)^{1/(N-1)} ds, \quad R \leq r \leq 3R,$$

and

$$(34) \qquad v(r) \geq C_2 R^{-N/(N-1)}(\log R)^{-\mu/(N-1)} \int_R^r \left(\int_R^s u(t)^{\beta}dt\right)^{1/(N-1)} ds, \quad R \leq r \leq 3R,$$

where C_1 and C_2 are some positive constants independent of r and R. Let us define the functions $f(r;R)$ and $g(r;R)$ for $R \leq r \leq 3R$ by the right hand sides of (33) and (34), respectively. Then using similar arguments as in the proof of Theorem 2, we see that

$$(35) \qquad C_3 \geq (\log R)^{\eta_1} g(2R;R)^{\delta_1}, \qquad R > r_1,$$

and

$$C_4 \geq (\log R)^{\eta_2} f(2R; R)^{\delta_2}, \qquad R > r_1, \tag{36}$$

where C_3, C_4, δ_1 and δ_2 are positive constants and η_1 and η_2 are constants given by $\eta_1 = -\frac{\beta\lambda + \mu(N-1)}{N\beta + N(N-1)}$ and $\eta_2 = -\frac{\alpha\mu + \lambda(N-1)}{N\alpha + N(N-1)}$. To get a contradiction to (35) or (36), we will show that $(\log R)^{\eta_1} g(2R; R)^{\delta_1} \to \infty$ or $(\log R)^{\eta_2} f(2R; R)^{\delta_2} \to \infty$ as $R \to \infty$. Note that, as before, we can get

$$f(2R; R) \geq C_5 (\log R)^{-\lambda/(N-1)} v(R)^{\alpha/(N-1)}, \qquad R > r_1, \tag{37}$$

and

$$g(2R; R) \geq C_6 (\log R)^{-\mu/(N-1)} u(R)^{\beta/(N-1)}, \qquad R > r_1, \tag{38}$$

where C_5 and C_6 are some positive constants independent of r and R.

Step 2. First we will obtain the estimates of u and v from below. Using the same letter C to denote various positive constants depending on $L_1, L_2, N, \alpha, \beta, \lambda$ and μ. From (25) we observe that, for $r > r_1 + 1$,

$$\begin{aligned} v(r) &\geq v(r_1) + \int_{r_1+1}^{r} s^{-1} \left(\int_{r_1}^{r_1+1} t^{N-1} K(t) u(t)^{\beta} dt \right)^{1/(N-1)} ds \\ &\geq \left(\int_{r_1}^{r_1+1} t^{N-1} K(t) u(t)^{\beta} dt \right)^{1/(N-1)} \int_{r_1+1}^{r} s^{-1}\, ds. \end{aligned}$$

Then, we obtain

$$v(r) \geq C \log r, \qquad r \geq r_2 > r_1 + 1.$$

From this estimate and (25), we have

$$\begin{aligned} u(r) &\geq (C^{\alpha} L_1)^{1/(N-1)} \int_{r_2+1}^{r} s^{-1} \left(\int_{r_2}^{s} t^{-1} (\log t)^{-\lambda+\alpha} dt \right)^{1/(N-1)} ds \\ &\geq C \int_{r_2+1}^{r} s^{-1} (\log s)^{(\alpha-\lambda+1)/(N-1)} ds \\ &\geq C (\log r)^{(\alpha+N-\lambda)/(N-1)}, \qquad r \geq r_3 > r_2 + 1. \end{aligned}$$

Again from this estimate and (25), we have

$$\begin{aligned} v(r) &\geq C \int_{r_3+1}^{r} s^{-1} \left(\int_{r_3}^{s} t^{-1} (\log t)^{-\mu+(\beta(\alpha+N-\lambda)/(N-1))} dt \right)^{1/(N-1)} ds \\ &\geq C \int_{r_3+1}^{r} s^{-1} (\log s)^{1/(N-1)\{-\mu+(\beta(\alpha+N-\lambda)/(N-1))+1\}} ds \\ &\geq C (\log r)^{1/(N-1)\{-\mu+(\beta(\alpha+N-\lambda)/(N-1))+1\}+1}, \qquad r \geq r_4 > r_3 + 1. \end{aligned}$$

By repeating this procedure, we can get inductively two positive sequences $\{C_m\}$ and $\{r_m\}$ satisfying

$$r_1 < r_2 < \cdots < r_m < \cdots,$$

(39) $$u(r) \geq C_m(\log r)^{\gamma+(\alpha\zeta/(N-1)^2)\tau_m}, \qquad r \geq r_m,$$

and

(40) $$v(r) \geq C_m(\log r)^{((\zeta/(N-1))\tau_m+1}, \qquad r \geq r_m,$$

where

$$\gamma = \frac{\alpha+N-\lambda}{N-1} > 0, \qquad \zeta = -\mu+\beta\gamma+1 > 0,$$

$$\tau_m = 1 + \frac{\alpha\beta}{(N-1)^2} + \cdots + \left\{\frac{\alpha\beta}{(N-1)^2}\right\}^{m-1}, \qquad m = 1, 2, \ldots.$$

Since $\alpha\beta > (N-1)^2$ by our fundamental assumption, we see that $\lim_{m\to\infty} \tau_m = \infty$.

From (37), (38), (39) and (40) we obtain

$$f(2R; R) \geq (C_m)^{\alpha/(N-1)}(\log R)^{((\alpha-\lambda)/(N-1))+(\alpha\zeta/(N-1)^2)\tau_m}, \qquad R \geq r_m,$$

and

$$g(2R; R) \geq (C_m)^{\beta/(N-1)}(\log R)^{((\beta\gamma-\mu)/(N-1))+(\alpha\beta\zeta/(N-1)^3)\tau_m}, \qquad R \geq r_m.$$

Therefore, we have

$$(\log R)^{\eta_1} g(2R; R)^{\delta_1} \geq (C_m)^{\alpha\delta_1/(N-1)}(\log R)^{M_1+M_2\tau_m}, \qquad R \geq r_m$$

and

$$(\log R)^{\eta_2} f(2R; R)^{\delta_2} \geq (C_m)^{\beta\delta_2/(N-1)}(\log R)^{M_3+M_4\tau_m}, \qquad R \geq r_m,$$

where M_i, $i = 1, \ldots, 4$ are constants and $M_2 > 0, M_4 > 0$. Since $\lim_{m\to\infty} \tau_m = \infty$, there exist m' and m'' such that $M_1 + M_2\tau_{m'} > 0$ and $M_3 + M_4\tau_{m''} > 0$. These imply that $(\log R)^{\eta_1} g(2R; R)^{\delta_1} \to \infty$ or $(\log R)^{\eta_2} f(2R; R)^{\delta_2}$ as $R \to \infty$, which contradicts (35) or (36). Thus the proof is completed.

Remark. Considering some results in [20], we conjecture that Theorem 4 is still true even though the condition for (λ, μ) is weakened to

$$\begin{cases} \lambda \leq \dfrac{\alpha(\beta+N-\mu)}{N-1} + 1 & \text{or} \\ \mu \leq \dfrac{\beta(\alpha+N-\lambda)}{N-1} + 1. \end{cases}$$

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