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THE SET OF RATIONAL HOMOTOPY TYPES WITH GIVEN COHOMOLOGY ALGEBRA

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Abstract

For a given commutative graded algebra A^*, we study the set $\mathcal{M}_{A^*} = \{\text{rational homotopy type of } X \mid H^*(X;Q) \cong A^*\}$. For example, we see that if A^* is isomorphic to $H^*(S^3 \vee S^5 \vee S^{16};Q)$, then $\mathcal{M}_{A^*}$ corresponds bijectively to the orbit space $P^3(Q)/Q^* \coprod\{*\}$, where $P^3(Q)$ is the rational projective space of dimension 3 and the point $\{*\}$ indicates the formal space.

1. Introduction

For a given graded algebra over the rationals (abbreviated to G.A.) A^*, there exists at least one rational homotopy type having A^* as a cohomology algebra, namely the formal space. In general there are many rational homotopy types having isomorphic cohomology algebras. In [**5**] it was shown that there are two rational homotopy types with isomorphic cohomology algebras and isomorphic homotopy Lie algebras, and in [**6**] it was shown that there are infinitely many rationally elliptic homotopy types having isomorphic cohomology algebras. Set

$$\mathcal{M}_{A^*} = \{\text{rational homotopy type of } X \mid H^*(X;Q) \cong A^*\}.$$

The set $\mathcal{M}_{A^*}$ was studied by several authors([**1**],[**2**],[**3**],[**7**],[**10**]). For example, Lupton ([**3**]) showed that for any positive integer n there is a G.A. A^* such that the cardinality of $\mathcal{M}_{A^*}$ is n. Halperin and Stasheff studied $\mathcal{M}_{A^*}$ by the set of perturbations of the differential of the formal differential graded algebra (abbreviated to D.G.A.). In particular they showed for $A^* = H^*((S^2 \vee S^2) \times S^3;Q)$, the set $\mathcal{M}_{A^*}$ consists of two points. This example is also caluculated from our view point (see Section 3(4)). Schlessinger and Stasheff ([**7**]) extended the arguments in [**2**].

We study $\mathcal{M}_{A^*}$ from a different point of view. Our strategy to study $\mathcal{M}_{A^*}$ is as follows. We construct inductively 1-connected minimal algebras m_{n-1} such that there is a G.A.map

$$\sigma_n : (H^*(m_{n-1})(n))^* \to A^*$$

so that σ^i is isomorphic for $i \leqslant n-1$ and monomorphic for $i = n$, where $(H^*(m_{n-1})(n))^*$ is the sub G.A. of $H^*(m_{n-1})$ generated by elements of degree

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$\leqslant n$. Suppose we have constructed the pair (m_{n-1}, σ_{n-1}). Then there is a unique minimal algebras m_D containing m_{n-1} and a G.A.map

$$\sigma_D : (H^*(m_D)(n))^* \to A^*$$

such that ${\sigma_D}^i$ is isomorphic for $i \leqslant n-1$, monomorphic for $i = n$ and moreover σ_D^{n+1} induces an isomorphism on the decomposable part

$${\sigma_D}^{n+1} : (H^*(m_D)(n))^{n+1} \to (A(n))^{n+1},$$

where $(A(n))^{n+1}$ is the degree $n+1$ part of the subalgebra $A(n)$ of A^* generated by elements of degree $\leqslant n$. To construct m_n we choose a subspace W of $H^{n+1}(m_D)$ satisfying certain conditions (see (2.3) and (2.4) in Section 2) so that $H^{n+1}(m_n) \oplus W = H^{n+1}(m_D)$.

Such a space W may be regarded as a rational point of a Grassmann manifold. The set of isomorphism classes of m_n containing m_{n-1} corresponds to the disjoint union of subsets of rational points of Grassmann manifolds modulo the action of D.G.A.automorphisms of m_D (see Theorem 2.1). We can show that any minimal algebra m with $H^*(m) \cong A^*$ is obtained in this way. For example if $A^* = H^*(S^3 \vee S^5 \vee S^{16}; Q)$, then $\mathcal{M}_{A^*}$ corresponds bijectively to $P^3(Q)/Q^* \coprod \{*\}$, where $P^3(Q)$ is the rational projective space of dimension 3 and the point $\{*\}$ corresponds to the formal space (see Section 3 (2)).

Throughout this paper we assume that G.A. A^* satisfies that $A^0 = Q$, $A^1 = 0$ and $\dim_Q A^i < \infty$ for any positive integer i.

2. Inductive construction of minimal models

In this section we construct inductively minimal algebras m_n and G.A. maps $\sigma_n : H^*(m_n)(n+1) \to A^*$ such that ${\sigma_n}^i$ is isomorphic for $i \leqslant n$ and monomorphic for $i = n+1$.

Suppose that we constructed a minimal algebra m_{n-1} satisfying the following conditions.

$\mathbf{(1)_{n-1}}$ m_{n-1} is generated by elements of degree $\leqslant n-1$.
$\mathbf{(2)_{n-1}}$ There is a G.A.-map

$$\sigma_{n-1} : (H^*(m_{n-1})(n))^* \to A^*$$

where ${\sigma_{n-1}}^i$ is isomorphic for $i \leqslant n-1$ and monomorphic for $i = n$.

Let m_D be the minimal algebra obtained by adding generators to m_{n-1} whose differentials form a basis for the kernel of ${\sigma_{n-1}}^{n+1}|(H(m_{n-1})(n))^{n+1}$ and $\sigma_D : (H(m_D)(n))^* \to A^*$ be the induced map. We set

$$\dim_Q A^{n+1} = u, \quad \dim_Q A^{n+1}/(A(n))^{n+1} = s$$

$$\dim_Q H^{n+1}(m_D) = v$$

and

$$\dim_Q \frac{H^{n+1}(m_D)}{(H^*(m_D)(n))^{n+1}} = t.$$

Then we have

$$u - s = v - t. \tag{2.1}$$

Let l be an integer satisfying

$$max(0, t - s) \leqslant l \leqslant t \tag{2.2}$$

and W be a l-dimensional subspace of $H^{n+1}(m_D)$ such that

$$W \cap (H^*(m_D)(n))^{n+1} = \{0\}. \tag{2.3}$$

Let m^W be the minimal algebra obtained by adding l generators whose differentials span W. Note that $H(m^W)(n) = H(m_D)(n)$, hence we have a G.A.map $\sigma_D : (H(m^W)(n))^* \to A^*$ and

$$H^{n+1}(m^W) \oplus W = H^{n+1}(m_D)$$

so that

$$\dim_Q \frac{H^{n+1}(m^W)}{(H(m^W)(n))^{n+1}} = t - l \leqslant s = \dim_Q \frac{A^{n+1}}{(A(n))^{n+1}}.$$

Let ${m^W}_n$ be a minimal algebra obtained by adding to m^W the cokernel of ${\sigma_D}^n : (H(m^W)(n))^n \to A^n$. Then we have a G.A. map

$$\sigma_n : (H({m^W}_n)(n))^* \to A^*$$

such that ${\sigma_n}^i$ is isomorphic for $i \leqslant n$. For a linear monomorphism

$$\psi : H^{n+1}(m^W)/(H(m^W)(n))^{n+1} \to A^{n+1}/(A(n))^{n+1},$$

if the map $\sigma_n \oplus \psi$ can be extend to a G.A. map

$${\sigma^W}_n : (H({m^W}_n)(n+1))^* \to A^*, \tag{2.4}$$

then the pair $({m^W}_n, {\sigma^W}_n)$ satisfies the condition $(1)_n$ and $(2)_n$. Remark that if we take W so that $\dim_Q W = t$ we can always construct a G.A. map (2.4).

Let m_n be a minimal algebra containg m_{n-1} (hence m_D) satisfying $(1)_n$ and $(2)_n$. Then m_n is constructed from m_D by taking W as the kernel of $i^* : H^*(m_D) \to H^*(m_n)$, where i is the inclusion.

By Plücker embedding Grassmann manifold is a projective variety defined over Q. Then the Q-subspace W corresponds to a rational point of the variety. Let $Gr(v, l)(Q)$ be the set of rational points of the Grassmann manifold of l-dimensional Q-subspaces in a v-dimensional space $H^{n+1}(m_D)$. Set

$$M_l = \{W \in Gr(v, l)(Q) |\ W \text{ satisfies (2.3)}\}$$

satisfying (2.3). We take bases for $H^{n+1}(m^W)/(H^*(m^W)(n))^{n+1}$ and $H^*(m^W)(n)^{n+1}$. If we write a basis for W as a linear combinations of those bases, we see that M_l is a Zariski open set of $Gr(v, l)(Q)$ (Compare with Example (3) in Section 3). Set

$$O_l = \{W \in M_l |\ \text{there is a G.A.map } {\sigma^W}_n \text{ satisfying (2.4) for some linear map } \psi\}.$$

Let G be the group of D.G.A.automorphisms of m_D. Then G acts on $H^{n+1}(m_D)$ and hence on $Gr(v, l)(Q)$. Let W be an element of O_l and Φ be an element of G.

Then it is easy to see that Φ can be extended to a D.G.A.isomorphism

$$\Phi : {m^{W}}_{n} \to {m^{\Phi(W)}}_{n}.$$

Hence G also acts on O_l.

Conversely let W_1, W_2 be l-dimensional subspaces of $H^{n+1}(m_D)$ such that there is a D.G.A.isomorphism

$$f : {m^{W_1}}_{n} \to {m^{W_2}}_{n}.$$

Then $f|m_D = \Phi$ is an element of G and

$$\Phi(W_1) = W_2.$$

Hence we have

Theorem 2.1. *The set of isomorphism classes of minimal algebras m_n containing a minimal algebra m_{n-1} and satisfying $(1)_n, (2)_n$ corresponds bijectively to the disjoint union of orbit spaces*

$$X_n = \coprod_{l=max(t-s,0)}^{t} O_l/G.$$

Note that X_n is not empty since O_t is not empty.

Definition 2.2. *A G.A. A^* is called k-intrinsically formal (abbreviated to k-I.F.) if for any minimal algebras m with $H^*(m) = A^*$, the sub D.G.A. $m(k)$ is unique up to isomorphism.*

Note that any G.A. A^* is at least 2-I.F..

Let A^* be $(n-1)$-I.F. and m be arbitrary minimal algebra with $H^*(m) \cong A^*$. Set $m_{n-1} = m(n-1)$ and $i_{n-1} : m_{n-1} \to m$ be the inclusion. Then we can construct minimal algebras m_D and ${m^{W_0}}_n$ as previous way where W_0 is the kernel of the induced map

$${i_D}^* : H^{n+1}(m_D) \to H^{n+1}(m).$$

The inclusion i_D can be extended to

$$i_n : {m^{W_0}}_n \to m$$

so that ${m^{W_0}}_n$ and ${i_n}^*$ satisfy $(1)_n, (2)_n$. Hence m can be constructed inductively as this way. Especially we have

Corollary 2.3. *If A^* is $(n-1)$-I.F. and $A^j = 0$ for $j > n+1$. Then $O_l = M_l$ and $\mathcal{M}_{A^*} = X_n = \coprod_{max(t-s,0) \leqslant l \leqslant t} M_l/G$.*

Suppose $A^i = 0$ for $i \leqslant n$. Then X_k is one point for $k < 3n+1$. Therefore m_{3n} is uniquely determined, i.e., A^* is $3n$-I.F.. This implies

Corollary 2.4. *Any n-connected k-dimensional finite CW complex is formal if $k \leqslant 3n+1$.*

This result was noticed by Stasheff [**8**]. We see that Corollary 2.4 is best possible by the example $A^* = H^*(S^3 \vee S^3 \vee S^8; Q)$.

The following examples are studied in the next section, where degree is denoted by suffix.

(1) $A^* = H^*(S^3 \vee S^7 \vee S^{22}; Q)$,which is 20-I.F. and $u = s = 1$, $v = t = 3$ at $n = 21$.

(2) $A^* = H^*(S^3 \vee S^5 \vee S^{16}; Q)$,which is 14-I.F. and $u = s = 1$, $v = t = 4$ at $n = 15$.

(3) $A^* = \wedge(x_3, y_5) \otimes Q[z_8]/(xy, xz^2, yz^2, z^3)$, which is 14-I.F. and $u = 1$, $s = 0$, $v = 5$, $t = 4$ at $n = 15$.

(4) $A^* = H^*((S^2 \vee S^2) \times S^3; Q)$, which is 3-I.F. and $u = 2$, $s = 0$, $v = 4$, $t = 2$ at $n = 4$.

(5) $A^* = H^*((S^3 \vee S^3) \times S^5; Q)$, which is 6-I.F. and $u = 2$, $s = 0$, $v = 4$, $t = 2$ at $n = 7$.

(6) $A^* = H^*(S^3 \vee S^5 \vee S^{10} \vee S^{16}; Q)$, which is 8-I.F. and $u = s = v = t = 1$ at $n = 9$.

(7) $A^* = H^*(S^5 \vee (S^3 \times S^{10}); Q)$, which is 8-I.F. and $u = s = v = t = 1$ at $n = 9$.

(8) $A^* = H^*((S^3 \times S^8)\sharp(S^3 \times S^8); Q)$, which is 6-I.F. and $u = s = v = t = 2$ at $n = 7$. Here $\sharp$ is connected sum.

3. Some examples

(1) $A^* = H^*(S^3 \vee S^7 \vee S^{22}; Q) = \wedge(x_3, y_7) \otimes Q[z_{22}]/(xy, xz, yz, z^2)$
Then A^* is 20-I.F. and by straightfoward calculation

$$m_{20} = (\wedge(x, y, \theta_9, \theta_{11}, \theta_{13}, \theta^1_{15}, \theta^2_{15}, \theta^1_{17}, \theta^2_{17}, \theta^1_{19}, \theta^2_{19}), d)$$

with the differential is as follows :
$d(x) = d(y) = 0$, $d\theta_9 = xy$, $d\theta_{11} = x\theta_9$, $d\theta_{13} = x\theta_{11}$, $d\theta^1_{15} = y\theta_9$, $d\theta^2_{15} = x\theta_{13}$, $d\theta^1_{17} = x\theta^1_{15} + y\theta_{11}$, $d\theta^2_{17} = x\theta^2_{15}$, $d\theta^1_{19} = x\theta^1_{17} + y\theta_{13}$, $d\theta^2_{19} = x\theta^2_{17}$.

Then at $n = 21$, $u = s = 1$ and $v = t = 3$. In fact $m_D = m_{20}$ and $H^{22}(m_D) = Q\{e_1, e_2, e_3\}$, where $e_1 = [x\theta^2_{19}]$, $e_2 = [x\theta^1_{19} + y\theta^2_{15}]$ and $e_3 = [y\theta^1_{15}]$. Let W be a 2 dimensional subspace of $H^{22}(m_D)$ spanned by

$$a_{1,i}e_1 + a_{2,i}e_2 + a_{3,i}e_3 \quad (i = 1, 2),$$

with

$$rank \begin{bmatrix} a_{1,1} & a_{2,1} & a_{3,1} \\ a_{1,2} & a_{2,2} & a_{3,2} \end{bmatrix} = 2.$$

Let $f \in Aut\ m_D = G$ be an element such that

$$f(x) = \lambda x, \quad f(y) = \mu y, \quad \lambda, \mu \in Q^*.$$

Then we have

$$f(e_1) = \lambda^7 \mu e_1,\ f(e_2) = \lambda^4 \mu^2 e_2,\ f(e_3) = \lambda \mu^3 e_3.$$

The set of W forms $Gr(3,2)(Q)$, the rational points of Grassmann manifold of 2-dimensional spaces in the 3-dimensional space $H^{22}(m(20))$. By the Plücker embedding $i : Gr(3,2)(Q) \to P^2(Q)$,

$$i(W) = [\begin{vmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \end{vmatrix}, \begin{vmatrix} a_{1,1} & a_{3,1} \\ a_{1,2} & a_{3,2} \end{vmatrix}, \begin{vmatrix} a_{2,1} & a_{3,1} \\ a_{2,2} & a_{3,2} \end{vmatrix}],$$

G acts on $P^2(Q)$ by $f[x_1, x_2, x_3] = [\lambda^{11}\mu^3 x_1, \lambda^8 \mu^4 x_2, \lambda^5 \mu^5 x_3] = [\rho x_1, x_2, \rho^{-1} x_3]$ with $\rho = \lambda^3 \mu^{-1}$. Hence by Corollary 2.3, we have

$$\mathcal{M}_{A^*} = M_2/G \coprod M_3 \simeq P^2(Q)/Q^* \ \coprod \{*\}.$$

(2) $A^* = H^*(S^3 \vee S^5 \vee S^{16}; Q) = \wedge(x_3, y_5) \otimes Q[z_{16}]/(xy, xz, yz, z^2)$

Then A^* is 14-I.F. and by straightfoward calculation

$$m_D = m_{14} = (\wedge(x, y, \theta_7, \theta_9, \theta_{11}^1, \theta_{11}^2, \theta_{13}^1, \theta_{13}^2), d) \qquad (*)$$

with the differential is as follows:
$d(x) = d(y) = 0$, $d\theta_7 = xy$, $d\theta_9 = x\theta_7$, $d\theta_{11}^1 = y\theta_7$, $d\theta_{11}^2 = x\theta_9$, $d\theta_{13}^1 = x\theta_{11}^2$, $d\theta_{13}^2 = x\theta_{11}^1 + y\theta_9$.

Then at $n = 15$, $u = s = 1$ and $H^{16}(m_D) = Q\{e_1, e_2, e_3, e_4\}$, where $e_1 = [x\theta_{13}^1]$, $e_2 = [y\theta_{11}^1]$, $e_3 = [x\theta_{13}^2 + \theta_7\theta_9]$ and $e_4 = [y\theta_{11}^2 + \theta_7\theta_9]$. Hence at $n = 15$, $v = t = 4$. Let W be a 3-dimensional subspace of $H^{16}(m_D)$ spanned by

$$a_{1,i}e_1 + a_{2,i}e_2 + a_{3,i}e_3 + a_{4,i}e_4 \quad (i = 1, 2, 3),$$

where $\mathrm{rank}(a_{j,i})_{1 \leqslant j \leqslant 4, 1 \leqslant i \leqslant 3} = 3$.

Let $f \in Aut\ m_D = G$ be an element such that

$$f(x) = \lambda x, \quad f(y) = \mu y, \quad \lambda, \mu \in Q^*.$$

Then we have

$$f(e_1) = \lambda^5 \mu e_1,\ f(e_2) = \lambda \mu^3 e_2,\ f(e_3) = \lambda^3 \mu^2 e_3,\ f(e_4) = \lambda^3 \mu^2 e_4.$$

The set of W forms $Gr(4,3)(Q)$, which is isomorphic to $P^3(Q)$ by the Plücker embedding $i : Gr(4,3)(Q) \to P^3(Q)$,

$$i(W) = \left[\begin{vmatrix} a_{1,1} & a_{2,1} & a_{3,1} \\ a_{1,2} & a_{2,2} & a_{3,2} \\ a_{1,3} & a_{2,3} & a_{3,3} \end{vmatrix}, \begin{vmatrix} a_{1,1} & a_{2,1} & a_{4,1} \\ a_{1,2} & a_{2,2} & a_{4,2} \\ a_{1,3} & a_{2,3} & a_{4,3} \end{vmatrix}, \begin{vmatrix} a_{1,1} & a_{3,1} & a_{4,1} \\ a_{1,2} & a_{3,2} & a_{4,2} \\ a_{1,3} & a_{3,3} & a_{4,3} \end{vmatrix}, \begin{vmatrix} a_{2,1} & a_{3,1} & a_{4,1} \\ a_{2,2} & a_{3,2} & a_{4,2} \\ a_{2,3} & a_{3,3} & a_{4,3} \end{vmatrix}\right].$$

Then G acts on $P^2(Q)$ by $f[x_1, x_2, x_3, x_4] = [\lambda^9\mu^6 x_1, \lambda^9\mu^6 x_2, \lambda^{11}\mu^5 x_3, \lambda^7\mu^7 x_4] = [\rho x_1, \rho x_2, \rho^2 x_3, x_4]$ by putting $\rho = \lambda^2\mu^{-1}$. Hence by Corollary 2.3, we have

$$\mathcal{M}_{A^*} = M_3/G \coprod M_4 \simeq P^3(Q)/Q^* \ \coprod \{*\}.$$

(3) $A^* = \wedge(x_3, y_5) \otimes Q[z_8]/(xy, xz^2, yz^2, z^3)$

Then A^* is 14-I.F. and at $n = 15$, $u = 1, s = 0$, and

$$m_D = m_{14} = m'_{14} \otimes Q[z],$$

where m'_{14} is isomorphic to m_{14} in the example (2) and $d(z) = 0$. Then $H^{16}(m_D) = Q\{e_1, e_2, e_3, e_4, f_1\}$, where $e_1 = [x\theta^1_{13}]$, $e_2 = [y\theta^1_{11}]$, $e_3 = [x\theta^2_{13} + \theta_7\theta_9]$, $e_4 = [y\theta^2_{11} + \theta_7\theta_9]$ and $f_1 = [z^2]$. Hence at $n = 15$, $v = 5$, $t = 4$. By Corollary 2.3,

$$\mathcal{M}_{A^*} = X_{15} = M_4/G.$$

Let W be an element of M_4 spanned by

$$a_{1,i}e_1 + a_{2,i}e_2 + a_{3,i}e_3 + a_{4,i}e_4 + a_{5,i}f_1 \quad (i = 1, 2, 3, 4),$$

with

$$rank\ (a_{j,i})_{1\leqslant j\leqslant 4, 1\leqslant i\leqslant 4} = 4 \quad (*).$$

By Plücker embedding, we see that the set of W satisfying $(*)$ corresponds bijectively to $A^4(Q) = \{[x_1, x_2, x_3, x_4, x_5] \in P^4(Q) | x_1 \neq 0\}$.

Let $f \in Aut\ m_D = G$ be an element such that

$$f(x) = \lambda x, \quad f(y) = \mu y, \quad f(z) = \kappa z, \quad \lambda, \mu, \kappa \in Q^*.$$

Then we have

$$f^*(e_1) = \lambda^5\mu e_1,\ f^*(e_2) = \lambda\mu^3 e_2,\ f^*(e_3) = \lambda^3\mu^2 e_3,$$

$$f^*(e_4) = \lambda^3\mu^2 e_4,\ f^*(f_1) = \kappa^2 f_1.$$

Hence G acts on $P^4(Q)$ by

$$f \cdot [x_1, x_2, x_3, x_4, x_5] = [\lambda^{12}\mu^8 x_1, \lambda^{11}\mu^5\kappa^2 x_2, \lambda^9\mu^6\kappa^2 x_3, \lambda^9\mu^6\kappa^2 x_4, \lambda^7\mu^7\kappa^2 x_5].$$

Hence G acts on $A^4(Q)$ by

$$f \cdot (y_1, y_2, y_3, y_4) = (\lambda^{-1}\mu^{-3}\kappa^2 y_1, \lambda^{-3}\mu^{-2}\kappa^2 y_2, \lambda^{-3}\mu^{-2}\kappa^2 y_3, \lambda^{-5}\mu^{-1}\kappa^2 y_4),$$

where $y_i = x_{i+1}/x_1$ for $i = 1, .., 4$. Then setting $\alpha = \lambda^{-7}\kappa^2$ and $\beta = \lambda^2\mu^{-1}$, G acts on $A^4(Q)$ by

$$f \cdot (y_1, y_2, y_3, y_4) = (\alpha\beta^3 y_1, \alpha\beta^2 y_2, \alpha\beta^2 y_3, \alpha\beta y_4).$$

Since α and β take any non-zero rational numbers independently, we have

$$\mathcal{M}_{A^*} \simeq A^4(Q)/(Q^* \times Q^*) \simeq P^3(Q)/Q^* \coprod \{*\},$$

where Q^* acts on $P^3(Q)$ by

$$\beta \cdot [z_1, z_2, z_3, z_4] = [\beta^2 z_1, \beta z_2, \beta z_3, z_4]$$

and the point $\{*\}$ corresponds $(0, 0, 0, 0)$ in $A^4(Q)$, which corresponds a formal model. Thus $\mathcal{M}_{A^*}$ is the same set as that of Example (2).

(4) $A^* = H^*((S^2 \vee S^2) \times S^3; Q) = Q[x_2, y_2] \otimes \Lambda(z_3)/(xy)$.

This example was studied by Halperin and Stasheff, see example 6.5 in [**2**]. It is 3-I.F. and at $n = 4$, $s = 0$ and $t = 2$. In fact

$$m_D = m_3 = (\wedge(x, y, \theta^1_3, \theta^2_3, \theta^3_3, z_3), d)$$

with $d(x) = d(y) = d(z) = 0$, $d\theta_3^1 = x^2$, $d\theta_3^2 = xy$, $d\theta_3^3 = y^2$ and $H^5(m_3) = Q\{e_1, e_2, f_1, f_2\}$, where $e_1 = [y\theta_3^1 - x\theta_3^2]$, $e_2 = [y\theta_3^2 - x\theta_3^3]$, $f_1 = [xz]$ and $f_2 = [yz]$. Then by Collorary 2.3,

$$\mathcal{M}_{A^*} = X_4 = M_2/G.$$

Let W in M_2 be spanned by

$$a_{1,i}e_1 + a_{2,i}e_2 + a_{3,i}f_1 + a_{4,i}f_2 \quad (i = 1, 2),$$

where

$$rank\ (a_{j,i})_{1 \leqslant j \leqslant 2, 1 \leqslant i \leqslant 2} = 2\ .$$

By Plücker embedding, the set of W forms

$$\{[x_1, x_2, x_3, x_4, x_5, x_6] \in P^5(Q) | x_1x_6 - x_2x_5 + x_3x_4 = 0, x_1 \neq 0\}$$

$$\simeq \{(X_1, X_2, X_3, X_4, X_5) \in A^5(Q) | X_5 - X_2X_5 + X_3X_4 = 0\}$$

$$\simeq \{(X_1, X_2, X_3, X_4) \in A^4(Q)\},$$

where $X_i = x_{i+1}/x_1$ $(i = 1, .., 5)$.

Let $f \in Aut\ m_D = G$ be an element such that

$$f(x) = x, \quad f(y) = y, \quad f(z) = \mu z \quad \mu \in Q^*$$

$$f(\theta_3^i) = \theta_3^i + \lambda_i z, \ \lambda_i \in Q, \quad i = 1, 2, 3.$$

Then we have

$$f^*(e_1) = e_1 - \lambda_2 f_1 + \lambda_1 f_2, \ f^*(e_2) = e_2 - \lambda_3 f_1 + \lambda_2 f_2,$$

$$f^*(f_1) = \mu f_1, \ f^*(f_2) = \mu f_2,$$

and f^* induces a map A_f defined by

$$A_f([x_1, .., x_6]) = [x_1, .., x_6] \begin{bmatrix} 1 & -\lambda_3 & \lambda_2 & \lambda_2 & -\lambda_1 & \lambda_1\lambda_3 - \lambda_2^2 \\ 0 & \mu & 0 & 0 & 0 & -\lambda_1\mu \\ 0 & 0 & \mu & 0 & 0 & -\lambda_2\mu \\ 0 & 0 & 0 & \mu & 0 & -\lambda_2\mu \\ 0 & 0 & 0 & 0 & \mu & -\lambda_3\mu \\ 0 & 0 & 0 & 0 & 0 & \mu^2 \end{bmatrix},$$

hence f^* induces a map $\tilde{A}_f$ from $A^4(Q)$ to itself defined by

$$\tilde{A}_f \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} \mu & & & \\ & \mu & & \\ & & \mu & \\ & & & \mu \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} -\lambda_3 \\ \lambda_2 \\ \lambda_2 \\ -\lambda_1 \end{bmatrix}.$$

From this we see by varing $\lambda_i \in Q$ $(i = 1, 2, 3)$ and $\mu \in Q^*$,

$$\tilde{A}_f \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \cup \tilde{A}_f \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = A^4(Q).$$

Hence $\mathcal{M}_{A^*}$ is at most two points.

Conversely, any element $g \in Aut\ m_D$ has the following form: $g(x) = a_1x + a_2y$, $g(y) = b_1x + b_2y$ and $g(z) = \mu z$ with

$$a_1, a_2, b_1, b_2 \in Q,\ D = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0, \mu \in Q^*$$

and then
$g(\theta_1) = a_1^2\theta_1 + 2a_1a_2\theta_2 + a_2^2\theta_3 + \lambda_1 z$,
$g(\theta_2) = a_1b_1\theta_1 + (a_1b_2 + a_2b_1)\theta_2 + a_2b_2\theta_3 + \lambda_2 z$,
$g(\theta_3) = b_1^2\theta_1 + 2b_1b_2\theta_2 + b_2^2\theta_3 + \lambda_3 z$
for some $\lambda_i \in Q$. By straightfoward calculations we see that $W_1 = \{e_1, e_2\}$, which corresponds to $(0,0,0,0)$ in $A^4(Q)$, can not be mapped to $W_2 = \{e_1, e_2 + f_2\}$ corresponding to $(0,1,0,0)$ in $A^4(Q)$ by $Aut\ m_D$. In fact,

$$\tilde{A}_g \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{D^2} \cdot \begin{bmatrix} -b_1^2\lambda_1 + 2a_1b_1\lambda_2 - a_1^2\lambda_3 \\ -b_1b_2\lambda_1 + (a_1b_2 + a_2b_1)\lambda_2 - a_1a_2\lambda_3 \\ -b_1b_2\lambda_1 + (a_1b_2 + a_2b_1)\lambda_2 - a_1a_2\lambda_3 \\ -b_2^2\lambda_1 + 2a_2b_2\lambda_2 - a_2^2\lambda_3 \end{bmatrix} = \begin{bmatrix} * \\ \alpha \\ \alpha \\ * \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Thus we see that $\mathcal{M}_{A^*}$ is just two points.

(5) $A^* = H^*((S^3 \vee S^3) \times S^5; Q) = \Lambda(x_3, y_3, z_5)/(xy)$.

This example was considered by Schlessinger and Stasheff, see section 8 in [**7**]. It is 6-I.F. and

$$m_D = m_6 = (\wedge(x_3, y_3, \theta_5, z_5), d)$$

with $d(x) = d(y) = d(z) = 0$ and $d\theta_5 = xy$. Then $H^8(m_D) = Q\{e_1, e_2, f_1, f_2\}$, where $e_1 = [x\theta_5]$, $e_2 = [y\theta_5]$, $f_1 = [xz]$ and $f_2 = [yz]$. Hence at $n = 7$, $s = 0$ and $t = 2$. By Corollary 2.3,

$$\mathcal{M}_{A^*} = X_7 = M_2/G.$$

Let W in M_2 be spanned by

$$a_{1,i}e_1 + a_{2,i}e_2 + a_{3,i}f_1 + a_{4,i}f_2 \qquad (i = 1, 2),$$

where $\mathrm{rank}(a_{j,i})_{1 \leqslant j \leqslant 2, 1 \leqslant i \leqslant 2} = 2$.

Let $f \in Aut\ m_D = G$ be an element such that $f(x) = a_1x + a_2y$, $f(y) = b_1x + b_2y$, $f(\theta_5) = D\theta_5 + \lambda z$ and $f(z) = \mu z$, where

$$D = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0, \lambda \in Q, \mu \in Q^*.$$

Then

$$f^*(e_1) = a_1De_1 + a_2De_2 + a_1\lambda f_1 + a_2\lambda f_2,$$

$$f^*(e_2) = b_1De_1 + b_2De_2 + b_1\lambda f_1 + b_2\lambda f_2,$$

$$f^*(f_1) = a_1\mu f_1 + a_2\mu f_2,\ f^*(f_2) = b_1\mu f_1 + b_2\mu f_2.$$

By Plücker embedding the set of W forms

$$\{[x_1,x_2,x_3,x_4,x_5,x_6]\in P^5(Q)|x_1x_6-x_2x_5+x_3x_4=0, x_1\neq 0\}$$

$$\simeq \{(X_1,X_2,X_3,X_4)\in A^4(Q)\},$$

where $X_i=x_{i+1}/x_1$ $(i=1,..,4)$. Then G acts on $A^4(Q)$ as follows:

$$\tilde{A}_f \begin{bmatrix} X_1\\ X_2\\ X_3\\ X_4\end{bmatrix} = \frac{\mu}{D^2}\begin{bmatrix} a_1^2 & a_1b_1 & a_1b_1 & b_1^2\\ a_1a_2 & a_1b_2 & a_2b_1 & b_1b_2\\ a_1a_2 & a_2b_1 & a_1b_2 & b_1b_2\\ a_2^2 & a_2b_2 & a_2b_2 & b_2^2\end{bmatrix}\begin{bmatrix} X_1\\ X_2\\ X_3\\ X_4\end{bmatrix} + \frac{\lambda}{D}\begin{bmatrix} 0\\ 1\\ -1\\ 0\end{bmatrix}.$$

First we show that any point (x_1,x_2,x_3,x_4) of $A^4(Q)$ lies in the union of the orbit of $(1,0,0,r)$ for some $r\in Q$ and that of $(0,0,0,0)$ by decomposing $A^4(Q)$ into the following pieces (a)∼(f).

(a) If $4x_1x_4\neq (x_2+x_3)^2$ and $x_1\neq 0$, set $a_1=0, a_2=-1, b_1=1, b_2=-\frac{x_2+x_3}{2x_1}$, $\mu=\frac{(x_2+x_3)^2-4x_1x_4}{4x_1}$, $r=\frac{4x_1^2}{(x_2+x_3)^2-4x_1x_4}$ and $\lambda=\frac{1}{2}(x_2-x_3)$. Then we have

$$\tilde{A}_f \begin{bmatrix} 1\\ 0\\ 0\\ r\end{bmatrix} = \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}. \tag{3.1}$$

(b) If $4x_1x_4\neq (x_2+x_3)^2$ and $x_4\neq 0$, set $a_1=1,\ a_2=0,\ b_1=-\frac{x_2+x_3}{2x_4}$, $b_2=1$, $\mu=\frac{(x_2+x_3)^2-4x_1x_4}{4x_4}$, $r=\frac{4x_4^2}{(x_2+x_3)^2-4x_1x_4}$ and $\lambda=\frac{1}{2}(x_2-x_3)$. Then we have (3.1).

(c) If $4x_1x_4\neq (x_2+x_3)^2$ and $x_1=x_4=0$, set $a_1=b_1=1,\ a_2=-\frac{1}{2},\ b_2=\frac{1}{2}$ $\mu=-\frac{x_2+x_3}{2}$, $r=-2$ and $\lambda=\frac{1}{2}(x_2-x_3)$. Then we have (3.1).

(d) If $4x_1x_4=(x_2+x_3)^2$ and $x_1\neq 0$, set $a_1=x_1,\ a_2=-\frac{x_2+x_3}{2},\ b_1=0,\ b_2=\frac{1}{x_1}$ $\mu=-\frac{1}{x_1}$, $r=0$ and $\lambda=\frac{1}{2}(x_2-x_3)$. Then we have (3.1).

(e) If $4x_1x_4=(x_2+x_3)^2$ and $x_4\neq 0$, set $a_1=-\frac{x_2+x_3}{2},\ a_1=x_4,\ b_1=-\frac{1}{x_1},\ b_2=0$, $\mu=-\frac{1}{x_4}$, $r=0$ and $\lambda=\frac{1}{2}(x_2-x_3)$. Then we have (3.1).

(f) If $x_1=x_4=0,\ x_2+x_3=0$, set $a_1=1,\ a_2=0,\ b_1=0,\ b_2=1,\ \mu=1$ and $\lambda=x_2$. Then we have

$$\tilde{A}_f \begin{bmatrix} 0\\ 0\\ 0\\ 0\end{bmatrix} = \begin{bmatrix} 0\\ x_2\\ x_3\\ 0\end{bmatrix}.$$

Thus we have a surjection

$$p: Q\coprod\{*\}\to \mathcal{M}_{A^*}\simeq A^4(Q)/G$$

defined by $p(*)=$ the class of $(0,0,0,0)$ and $p(r)=$ the class of $(1,0,0,r)$.

If $p(r_1) = p(r_2)$ then there is an element $f \in G$ such that

$$\tilde{A}_f \begin{bmatrix} 1 \\ 0 \\ 0 \\ r_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ r_2 \end{bmatrix}.$$

By straightfoward calculations we have $r_1 r_2 \in Q^{*2}$ if $r_1 r_2 \neq 0$. Thus we have

$$\mathcal{M}_{A^*} \simeq Q^*/Q^{*2} \coprod \{0\} \coprod \{*\},$$

where $\{0\}$ corresponds to $(1,0,0,0)$ and $\{*\}$ corresponds to the formal model. After tensoring with $\overline{Q}$ the set of isomorphism classes consists of three points.

(6) $A^* = H^*(S^3 \vee S^5 \vee S^{10} \vee S^{16}; Q) = \wedge(x_3, y_5) \otimes Q[v_{10}, z_{16}]/(xy, xv, xz, yv, yz, v^2, vz, z^2)$.

Then $m_D = m_8 = (\Lambda(x, y, \theta_7), d)$ with $d(\theta_7) = xy$. Since $H^{10}(m_8) = Q\{x\theta_7\}$, $s = t = 1$ at $n = 9$. Then since the condition $(2)_9$ is satisfied

$$X_9 = O_0 \coprod O_1 = M_0 \coprod M_1 \simeq \{p_0, p_1\},$$

where the corresponding model for p_0 is

$$m^{(0)}{}_9 = (\Lambda(x, y, \theta_7), d)$$

with $d(\theta_7) = xy$ and the corresponding model for p_1 is

$$m^{(1)}{}_9 = (\Lambda(x, y, \theta_7, \theta_9), d)$$

with $d(\theta_9) = x\theta_7$.

Next consider X_{15} over each point. The model containing $m^{(0)}{}_9$ is

$$m_D = m_{14} = (\wedge(x, y, \theta_7, \theta_{11}), d)$$

with $d(\theta_{11}) = y\theta_7$. Since $H^{16}(m_D) = Q\{y\theta_{11}\}$, $s = t = 1$ at $n = 15$. Hence X_{15} consists of two points.

The model containing $m^{(1)}{}_9$ is

$$m_D = m_{14} = (\Lambda(x, y, \theta_7, \theta_9, u_{10}, \theta^1_{11}, \theta^2_{11}, \theta^1_{13}, \theta^2_{13}), d) = (Q[u] \otimes m, d)$$

where $d(u_{10}) = 0$ for a basis u_{10} of $\mathrm{Coker}(\sigma_9^{\{x\theta_7\}})^{10}$ and m is the model $(*)$ in Example (2). Then $H^{16}(m_D) = Q\{e_1, e_2, e_3, e_4\}$ is same as that of the above Example (2). Hence we have in this case

$$X_{15} \simeq \mathcal{M}_{H^*(S^3 \vee S^5 \vee S^{16})}.$$

Since $A^{>16} = 0$, $\mathcal{M}_{A^*}$ is the disjoint union of two points and $P^3(Q)/Q^* \coprod \{*\}$. See Fig 1.

(7) $A^* = H^*(S^5 \vee (S^3 \times S^{10}); Q) = \Lambda(x_3, y_5) \otimes Q[z_{10}]/(xy, xz, z^2)$.

Then $m_D = m_8 = (\Lambda(x, y, \theta_7), d)$ with $d(\theta_7) = xy$. Since $H^{10}(m_8) = Q\{x\theta_7\}$, $W = 0$ or $W = Q\{x\theta_7\}$ at $n = 9$. If $W = \{0\}$, $(\sigma^W{}_9)^{13} : H^3(m^W{}_9) \cdot H^{10}(m^W{}_9) =$

$0 \to A^3 \cdot A^{10} \neq 0$ can not be a G.A.map. Hence the condition $(2)_9$ is not satisfied. Hence W must be $Q\{x\theta_7\}$.

Next consider X_{12}. Then

$$m_D = m_{12} = (\Lambda(x, y, \theta_7, \theta_9, u_{10}, {\theta^1}_{11}, {\theta^2}_{11}), d)$$

with $d(\theta_7) = xy, d(\theta_9) = x\theta_7, d({\theta^1}_{11}) = y\theta_7, d({\theta^2}_{11}) = x\theta_9$. Since $H^{13}(m_D) = (H^+(m_D)(12))^{13}$ and $A^{>13} = 0$, $\mathcal{M}_{A^*}$ is an one point.

(8) $A^* = H^*((S^3 \times S^8)\sharp(S^3 \times S^8); Q) = \Lambda(x_3, y_3) \otimes Q[u_8, w_8]/(xy, xu, xw + yu, yw, u^2, uw, w^2)$.

It is 6-intrinsically formal Poincaré algebra of formal dimension 11 such that $m_6 = (\Lambda(x, y, \theta_5), d)$ with $d(x) = d(y) = 0$ and $d(\theta_5) = xy$. There is a map $\sigma_6 : (H^*(m_6)(7))^* \to A^*$ given by $\sigma_6(x) = x$, $\sigma_6(y) = y$ and sending other elements to zero. Since $u = s = v = t = 2$ at $n = 7$, we have $0 \leqslant l \leqslant 2$. Consider the each cases of $l = 0, 1, 2$ at $n = 7$ in the followings.

Case of $l = 0$.

Since $W = 0$, $H^8(m^W) = H^8(m_6) = Q\{[x\theta_5], [y\theta_5]\}$. Put $\sigma^W(x) = x$, $\sigma^W(y) = y$, $\sigma^W([x\theta_5]) = u$ and $\sigma^W([y\theta_5]) = w$. Then the condition $(1)_7$ and $(2)_7$ are satisfied. Since $\sigma^W : H^*(m^W) \to A^*$ is isomorphic, this one point set $M_0 = O_0$, corresponding the elliptic model $(\Lambda(x, y, \theta_5), d)$, is a component of $\mathcal{M}_{A^*}$.

Case of $l = 1$.

For $H^8(m_6) = Q\{e_1 = [x\theta_5], e_2 = [y\theta_5]\}$, W is spanned by $ae_1 + be_2$ for $[a, b] \in P^1(Q) = M_1$. Then ${m^W}_8 = (\Lambda(x, y, \theta_5, \theta_7, u_8), d)$ where $d(\theta_7) = ae_1 + be_2$ and $d(u_8) = 0$. But $({\sigma^W}_8)^{11} : H^3({m^W}_8) \cdot H^8({m^W}_8) \to A^3 \cdot A^8$ can not be a G.A.map since $x \cdot (bx\theta_5 + ay\theta_5) = d(y\theta_7)$ and $y \cdot (bx\theta_5 + ay\theta_5) = d(x\theta_7)$. Hence the condition $(2)_7$ is not satisfied.

Case of $l = 2$.

Since $W = Q\{x\theta_5,\ y\theta_5\}$,

$$m^W = (\Lambda(x, y, \theta_5, {\theta^1}_7, {\theta^2}_7), d)$$

where $d({\theta^1}_7) = x\theta_5$ and $d({\theta^2}_7) = y\theta_5$ and

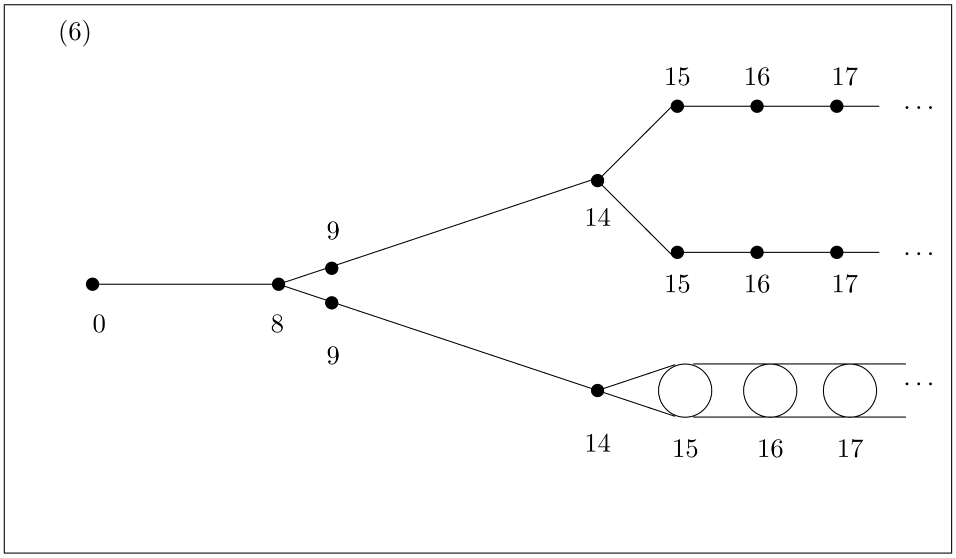
$${m^W}_8 = (\Lambda(x, y, \theta_5, {\theta^1}_7, {\theta^2}_7, {u^1}_8, {u^2}_8), d)$$

where $d{u^i}_8 = 0$ $(i = 1, 2)$. Since $t = 0$ at $8 \leqslant n \leqslant 11$ and $A^{>11} = 0$, it is one point corresponding to the formal model.

Thus $\mathcal{M}_{A^*}$ is two points. See Fig 2.

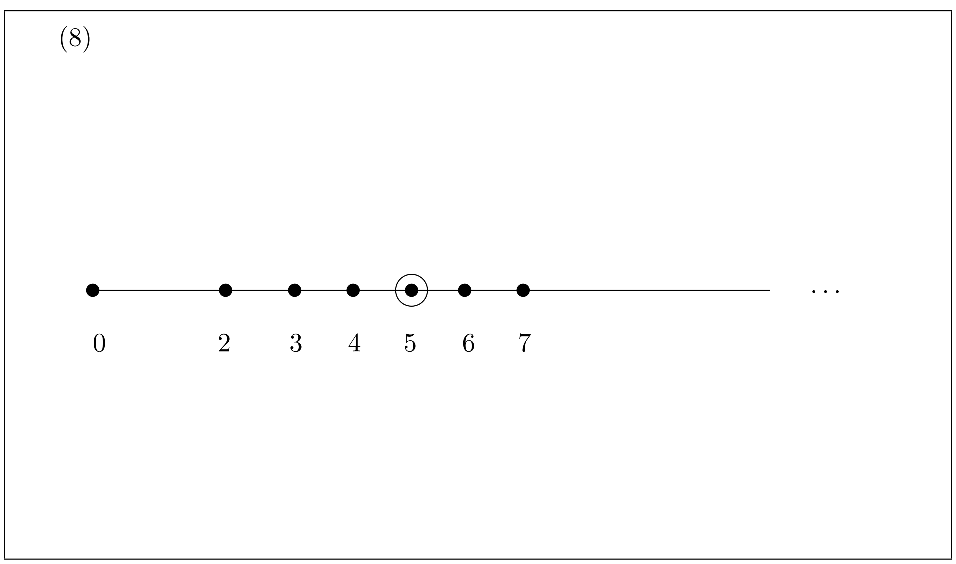
In the following figures, numbers mean degrees.

Fig 1



The set $P^3(Q)/Q^* \coprod \{*\}$ is indicated by one circle.

Fig 2



Here $\odot$ implies that there exists an elliptic minimal model generated by elements of degree $\leqslant 5$ satisfying $H^*(m) \cong A^*$.

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