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Contact integral geometry and the Heisenberg algebra

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Generalizing Weyl's tube formula and building on Chern's work, Alesker reinterpreted the Lipschitz–Killing curvature integrals as a family of valuations (finitely additive measures with good analytic properties), attached canonically to any Riemannian manifold, which is universal with respect to isometric embeddings. We uncover a similar structure for contact manifolds. Namely, we show that a contact manifold admits a canonical family of generalized valuations, which are universal under contact embeddings. Those valuations assign numerical invariants to even-dimensional submanifolds, which in a certain sense measure the curvature at points of tangency to the contact structure. Moreover, these valuations generalize to the class of manifolds equipped with the structure of a Heisenberg algebra on their cotangent bundle. Pursuing the analogy with Euclidean integral geometry, we construct symplectic-invariant distributions on Grassmannians to produce Crofton formulas on the contact sphere. Using closely related distributions, we obtain Crofton formulas also in the linear symplectic space.

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1 Introduction

1.1 Background and motivation

Intrinsic volumes first appeared in convex geometry through Steiner's formula: given a compact convex body $K \subset \mathbb{R}^n$, $\mathrm{vol}(K+\epsilon B^n) = \sum_{k=0}^n \omega_{n-k}\mu_k(K)\epsilon^{n-k}$, where B^j is the unit Euclidean ball in $\mathbb{R}^j$ and ω_j is its volume. The coefficient $\mu_k(K)$, the k^{th} intrinsic volume, can be written explicitly for smooth K as

$$\mu_k(K) = c_{n,k}\int_{\partial K}\sigma_{n-1-k}(\kappa_1,\dots,\kappa_{n-1})\,d\mathrm{Area}_K,$$

where κ_j are the principal curvatures of ∂K. Alternatively, $\mu_k(K)$ can be given by integral–geometric Crofton formulas: $\mu_k(K) = c'_{n,k}\int_{\mathrm{AGr}_{n-k}(\mathbb{R}^n)}\chi(K\cap E)\,dE$, where dE is the rigid motion-invariant measure on the affine Grassmannian, and χ is the Euler characteristic. A third, axiomatic, definition was given by Hadwiger, who described the intrinsic volumes as the unique rigid motion-invariant, continuous, finitely additive measures on compact convex sets.

A closely related famous result is Weyl's tube formula [54]. It asserts that the volume of an ϵ–tube around a Riemannian manifold M embedded isometrically in Euclidean space $\mathbb{R}^N$ is a polynomial in $\epsilon \ll 1$, whose coefficients are, remarkably, intrinsic invariants of the Riemannian manifold M, independent of the isometric embedding. These coefficients, now known as the intrinsic volumes of M, are intimately linked with the asymptotic expansion of the heat kernel; see Donnelly [27].

These results fall naturally in the domain of valuations on manifolds, a fairly young branch of valuation theory introduced by Alesker [4; 5; 6] and Alesker and Fu [10; 11], the last being a survey. Valuation theory itself is a mixture of convex and integral geometry, originating in the early 20th century in works of Steiner, Blaschke, Chern and Santalo, as well as in Dehn's solution to Hilbert's third problem. Generally speaking, valuations are finitely additive measures on some family of nice subsets. In this note, there is typically some analytic restriction on the nature of the valuation, such as smoothness or smoothness with singularities, and the subsets are manifolds with corners or differentiable polyhedra.

Building on results of Chern, Alesker noticed a natural extension of Weyl's theorem for valuations: restricting to M the intrinsic volumes of $\mathbb{R}^N$, (considered as valuations), yields an intrinsically defined family of valuations on M, now known as the Lipschitz–Killing valuations. Weyl's intrinsic volumes of M are then the integrals of the corresponding valuations. In recent work, Fu and Wannerer [30] characterized the Lipschitz–Killing valuations as the unique family of valuations attached canonically (in a sense made precise therein) to arbitrary Riemannian manifolds that are universal to isometric embeddings.

Other spaces whose (smooth) valuation theories were considered in recent years include complex space forms — see Bernig and Fu [20] and Bernig, Fu and Solanes [21] — the quaternionic plane — see Bernig and Solanes [22; 23] — the octonionic plane — see Bernig and Voide [24] — and exceptional spheres; see Solanes and Wannerer [51].

Numerous intriguing connections between convex and symplectic geometries are known to exist. To name a few: Viterbo's conjectured isoperimetric inequality for capacities of convex bodies [53], was later shown by Artstein-Avidan, Karasev and Ostrover to imply Mähler's famous conjecture [15]. Capacities have been successfully studied up to a bounded factor using convex techniques; see Artstein-Avidan, Milman and Ostrover [16] and Gluskin and Ostrover [32]. Álvarez Paiva and Balacheff [13] and Álvarez Paiva, Balacheff and Tzanev [14] establish links between systolic geometry, contact geometry, Mähler's conjecture and the geometry of numbers. Schäffer's dual girth conjecture for normed spaces has been proved by Álvarez Paiva using symplectic techniques [12], and generalized further by Faifman [28] using hamiltonian group actions. In a recent work of Abbondandolo, Bramham and Hryniewicz [1], some links are established between the geometry of the group of symplectomorphisms and systolic geometry of the 2–sphere. For an exposition of some of those connections, see Ostrover [44].

The main objective for this work is to further explore the convex-symplectic link by studying the valuation theory of contact manifolds, using the Riemannian case and Weyl's principle as guides.

1.2 Informal summary

We find that like in the Riemannian setting, contact manifolds possess a canonical family of valuations associated to them. We describe those valuations in two ways: geometrically through a curvature-type formula, and also dynamically through the invariants of a certain vector field at its singular points.

The contact valuations satisfy Weyl's principle of universality under embeddings, similarly to the valuation extension of the Weyl principle. Let us emphasize the role played by valuations in this phenomenon: It so happens that contact valuations only assume nonzero values on even-dimensional submanifolds, while they live on odd-dimensional contact manifolds. Thus unlike the Riemannian case, Weyl's principle in the contact setting is only manifested in its extended to valuations form, as the statement in the original form becomes vacuous: all integrals of the contact valuations vanish.

Contact valuations are in fact an instance of a natural collection of valuations associated to the larger class of manifolds whose cotangent spaces admit a smoothly varying structure of Heisenberg algebras. The Heisenberg algebra provides a unifying link between contact, symplectic and metric geometries.

This path leads us to consider the valuation theory of the dual Heisenberg algebra, invariant under the group of the automorphisms of the Heisenberg algebra, which is closely related to the symplectic group. From this perspective, this is another step in the study of the valuation theory of noncompact Lie groups, which up to now has only been considered for the indefinite orthogonal group by Alesker, Bernig and Faifman [9; 19; 29], and in a somewhat different framework for the special linear group; see Ludwig and Reitzner [41; 40].

Further similarity to the metric setting is exhibited by the contact sphere, where we prove Crofton formulas and a Hadwiger-type theorem, thus establishing an integral–geometric and an axiomatic description of the contact valuations.

Finally, in the last part we explore the valuation theory of linear symplectic spaces. We show that there are no nontrivial invariant valuations, but nevertheless one can write oriented Crofton formulas for the symplectic volume of submanifolds.

1.3 Main results

Let us very briefly recall or indicate the relevant notions. For precise definitions see Sections 2 and 3.

A contact manifold M^{2n+1} is given by a maximally nonintegrable hyperplane distribution, namely a smooth field of tangent hyperplanes $H \subset TM$ such that locally one can find $\alpha \in \Omega^1(M)$ with $H = \mathrm{Ker}(\alpha)$ and $d\alpha|_H$ a nondegenerate 2–form.

A smooth valuation ϕ on an orientable manifold M^n, written $\phi \in \mathcal{V}^\infty(M)$, is a finitely additive measure on the compact differentiable polyhedra of M, denoted by $\mathcal{P}(M)$, which has the form $\phi(X) = \int_X \mu + \int_{N^*X} \omega$ for some forms $\mu \in \Omega^n(M)$ and $\omega \in \Omega^{n-1}(S^*M)$. Here S^*M is the cosphere bundle, and N^*X is the conormal cycle of X, which is just the conormal bundle when X is a manifold. Orientability is not essential, and is only assumed to simplify the exposition.

There is a natural filtration $\mathcal{W}_n^\infty(M) \subset \cdots \subset \mathcal{W}_0^\infty(M) = \mathcal{V}^\infty(M)$. Very roughly speaking, $\mathcal{W}_j^\infty(M)$ consists of valuations which are locally homogeneous of degree at least j, for example $\mathcal{W}_n^\infty(M)$ are the smooth measures on M.

The generalized valuations $\mathcal{V}^{-\infty}(M)$ are, roughly speaking, distributional valuations: we allow ω and μ to be currents rather than smooth forms. Generalized valuations can be naturally evaluated on sufficiently nice subsets $X \in \mathcal{P}(M)$. The filtration $\mathcal{W}_j^\infty(M)$ on $\mathcal{V}^\infty(M)$ extends to a filtration $\mathcal{W}_j^{-\infty}(M)$ on $\mathcal{V}^{-\infty}(M)$.

Theorem 1.1 *To any contact manifold M^{2n+1} with contact distribution $H \subset TM$ there are canonically associated, linearly independent generalized valuations $\phi_{2k}^M \in \mathcal{V}^{-\infty}(M)$ for $0 \le k \le n$. They have the following properties:*

(i) *ϕ_0^M is the Euler characteristic, and $\phi_{2k}^M \in \mathcal{W}_{2k}^{-\infty}(M) \setminus \mathcal{W}_{2k+1}^{-\infty}(M)$.*

(ii) *ϕ_{2k}^M can be naturally evaluated on submanifolds in generic position relative to the contact structure* (see Definition 4.10). *For a generic closed hypersurface F,*

$$\phi_{2k}^M(F) = \sum_{T_pF = H_p} \phi_{2k}^M(F, p),$$

*where the **local contact area** $\phi_{2k}^M(F, p)$ only depends on the germ of F at p. It is described explicitly below in equations (1) and (2).*

(iii) **Universality to restriction under embedding** *If $i\colon N^{2m+1} \to M^{2n+1}$ is a contact embedding, then for $k \le m$ one has $i^*\phi_{2k}^M = \phi_{2k}^N$.*

(iv) *For a $2k$–dimensional submanifold in general position F, $\phi_{2k}^M(F) \ge 0$, with equality if and only if there are no contact tangent points.*

(v) *The space of generalized valuations on M invariant under all contactomorphisms of M is spanned by $(\phi_{2k}^M)_{k=0}^n$.*

Remark 1.2 The universality with respect to embeddings is sometimes referred to as the Weyl principle. Thus we recover a Weyl principle in the contact setting.

The local contact areas $\phi_{2k}^M(F, p)$ can be given explicitly in two different ways, through a geometric or a dynamical approach.

- From a dynamical point of view, $\phi_{2k}^M(F, p)$ encodes the invariants of the linearized vector field $B \in \mathcal{X}(F)$ representing the characteristic foliation. While there are many such vector fields, there is a distinguished choice, to linear order, at the critical points: choose an arbitrary contact form α near p and let B be given by $d\alpha|_F(B,\bullet)=\alpha|_F$. One then has

$$\phi_{2k}^M(F,p)=\frac{\operatorname{tr}\bigwedge^{2n-2k}d_pB}{|\det d_pB|}. \tag{1}$$

- From a geometric point of view,

$$\phi_k^M(F,p)=\binom{2n}{k}|\det(S-h)|^{-1}D(S-h[2n-k],J[k]). \tag{2}$$

  Here D denotes the mixed discriminant,

$$J=\begin{pmatrix}0 & -I_n\\ I_n & 0\end{pmatrix},$$

  S is the second fundamental form of F at p, and h the second fundamental form of the contact distribution at p (see Definition 4.20), both written with respect to a frame compatible with the contact structure in a natural way at p; see Definition 4.7. Thus they are reminiscent of (certain symmetric functions of) the principal radii of an embedded hypersurface in a Riemannian manifolds. This point of view is applicable in the wider setting of DH manifolds, as described below.

We may extend ϕ_{2k} to a nonnegative lower-semicontinuous functional on all $2k$–dimensional submanifolds with boundary, denoted by $\mathrm{CA}_{2k}(F)$, the contact area of F. We observe that $\mathrm{CA}_{2k}(F)=0$ if and only if F can be made nowhere tangent to the contact distribution by an arbitrarily small perturbation.

In Riemannian or Hermitian manifolds, a fair amount of the valuation theory appears already in the corresponding flat space, which can be thought of as the tangent space to the given manifold. In the contact setting, it is no longer true: the tangent space of a contact manifold does not itself inherit a contact structure.

The main observation guiding this paper is that every cotangent space of a contact manifold is canonically the Heisenberg Lie algebra. We are thus led to study the valuation theory of general manifolds with such structure.

Definition 1.3 A manifold X equipped with a hyperplane distribution H (called horizontal) and a smooth field of nowhere-degenerate forms

$$\omega\in\Gamma^\infty\big(X,\textstyle\bigwedge^2H_x^*\otimes(T_xX/H_x)\big)$$

will be called a dual Heisenberg (DH) manifold.

The space of valuations naturally associated to such manifolds turns out to resemble somewhat the Lipschitz–Killing space of valuations in Riemannian geometry; in particular, they exhibit universality with respect to embeddings.

Theorem 1.1 is then the contact instance of the following general result:

Theorem 1.4 *To any DH manifold M^{2n+1} with horizontal distribution $H \subset TM$ there are canonically associated generalized valuations $\phi_k^M \in \mathcal{W}_k^{-\infty}(M)$ for $0 \le k \le 2n$. They have the following properties:*

(i) *ϕ_0^M is the Euler characteristic, and $\phi_{2k}^M \in \mathcal{W}_{2k}^{-\infty}(M) \setminus \mathcal{W}_{2k+1}^{-\infty}(M)$.*

(ii) *ϕ_k^M can be naturally evaluated on submanifolds in generic position with respect to the horizontal distribution. For a generic closed hypersurface F,*

$$\phi_k^M(F) = \sum_{T_pF = H_p} \phi_k^M(F, p),$$

where $\phi_k^M(F, p)$ only depends on the germ of F at p.

(iii) *If $i\colon N^{2m+1} \to M^{2n+1}$ is a DH embedding, then $i^*\phi_k^M = \phi_k^N$ for $k \le 2m$.*

The local contact areas $\phi_k^M(F, p)$ are given by the same curvature-type formula (2) as in the contact case.

As an intermediate step of independent interest, we obtain a Hadwiger-type theorem for the dual Heisenberg algebra itself. We denote by $U = \mathbb{R}^{2n+1}$ the dual of the Heisenberg Lie algebra $\mathfrak{h}_{2n+1}$, and by $\mathrm{Sp}_H(U)$ its automorphism group.

Theorem 1.5 *It holds that $\mathrm{Val}^{-\infty}(U)^{\mathrm{Sp}_H(U)}$ consists of even valuations. For $0 \le k \le n$, $\mathrm{Val}_{2k+1}^{-\infty}(U)^{\mathrm{Sp}_H(U)} = \{0\}$ while $\dim \mathrm{Val}_{2k}^{-\infty}(U)^{\mathrm{Sp}_H(U)} = 1$. The corresponding Klain sections are zero-order distributions (that is, regular Borel measures).*

We also consider $\mathrm{Sp}_H^+(U)$, the connected component of the identity. We prove:

Theorem 1.6 *For $0 \le k \le n$, $\mathrm{Val}_{2k+1}^{-\infty}(U)^{\mathrm{Sp}_H^+(U)} = \{0\}$ and $\dim \mathrm{Val}_{2k}^{-\infty}(U)^{\mathrm{Sp}_H^+(U)} = 2$. In the latter space, the $\mathrm{Sp}_H(U)$–invariant valuations are complemented by a one-dimensional space of odd valuations.*

We then consider the standard contact sphere S^{2n+1}, which we identify with the oriented projectivization $\mathbb{P}_+(V)$ of a symplectic space $V = (\mathbb{R}^{2n+2}, \omega)$. We construct a canonical distribution $\mu_\omega \in \mathcal{M}^{-\infty}(\mathrm{Gr}_{2k}(V))^{\mathrm{Sp}(V)}$. We obtain the following Crofton formulas, establishing further common ground with the Riemannian setting.

Theorem 1.7 *Define for $0 \le i \le n$ the generalized valuations ψ_{2i} on S^{2n+1} given by the Crofton formula*

$$\psi_{2i} := \int_{\mathrm{Gr}_{2n+2-2i}(V)} \chi(\bullet \cap E)\, d\mu_\omega(E).$$

Then for certain explicit constants c_{ij}^n one has

$$\psi_{2i} = \sum_{j=i}^{n} c_{ij}^n \phi_{2j}.$$

We also establish a Hadwiger-type theorem for the contact sphere.

Theorem 1.8 *Both $(\phi_{2k})_{k=0}^n$ and $(\psi_{2k})_{k=0}^n$ are bases of $\mathcal{V}^{-\infty}(S^{2n+1})^{\mathrm{Sp}(2n+2)}$.*

Finally, we find that while symplectic space and manifolds do not possess interesting invariant valuations, one can nevertheless write certain integral geometric formulas for symplectic volumes of manifolds. We construct a canonical distribution on the affine oriented Grassmannian $\overline{\mu}_\omega \in \mathcal{M}^{-\infty}(\mathrm{AGr}_{2k}^+(\mathbb{R}^{2n}))$ which is translation-invariant and $\mathrm{Sp}(2n)$–invariant, and odd to orientation reversal. We prove the following Crofton formula on symplectic linear space:

Theorem 1.9 *Let $F^{2k} \subset \mathbb{R}^{2n}$ be a C^1 compact, oriented submanifold with boundary. Then*

$$\int_F \omega^k = (-1)^\kappa \binom{n}{k} \binom{2n}{2k}^{-1} \frac{(2n-1)^\kappa}{2^{\kappa+1}} \int_{\mathrm{AGr}_{2n-2k}^+(\mathbb{R}^{2n})} I(E,F)\, d\overline{\mu}_\omega(E),$$

where $\kappa = \min(k, n-k)$ and I is the oriented intersection index.

Plan of the paper

In Section 2 we introduce notation and present the basic geometric facts we will use. In Section 3 we recall the basics of valuation theory and prove some lemmas we will need. In Section 4 we construct the canonical valuations on general DH manifolds and establish their universality to embeddings, proving Theorem 1.4. We also explore some geometric properties of those valuations. In Section 5 we classify the translation-invariant valuations of the dual Heisenberg algebra, proving Theorems 1.5 and 1.6, and note their relation to gaussian curvature. Apart from its intrinsic interest, the linear classification is needed for the uniqueness statement in Theorem 1.1, as well as for

Theorem 1.8. In Section 6 we specialize the DH valuations to contact manifolds and prove Theorem 1.1. In particular, we give the dynamical description of the contact valuations. In Section 7 we construct symplectic-invariant distributions on linear and affine Grassmannians in symplectic space, which are used in the subsequent two sections. In Section 8 we consider the standard contact sphere. We produce Crofton formulas for ϕ_{2k} that are invariant under $\mathrm{Sp}(V)$, proving Theorems 1.7 and 1.8, and compute some examples explicitly. We also bound from below the contact valuations of a convex set. Finally, in Section 9, we study the integral geometry of linear symplectic space, proving Theorem 1.9.

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2 Preliminaries

2.1 Notation

We use σ_1 to denote the unit $\mathrm{SO}(n)$–invariant measure on various homogeneous spaces of the special orthogonal group. We write $\mathrm{Gr}_k(V)$ for the k–Grassmannian in V, Gr_k^+ for the oriented Grassmannian and $\mathrm{AGr}^{(+)}$ for the affine (oriented) Grassmannian. $\mathcal{K}(V)$ is the set of compact convex subsets of V.

The one-dimensional space of real-valued Lebesgue measures over V is denoted by $\mathrm{Dens}(V)$. For a manifold M, $|\omega_M|$ denotes the line bundle of densities, whose fiber over $x \in M$ is $\mathrm{Dens}(T_x M)$. We will write M^{tr} for the translation-invariant elements of a module M over V. For a group $G \subset \mathrm{GL}(V)$, $\overline{G}$ is the group generated by G and all translations in V.

We will write $\Omega_{-\infty}(M)$ for the space of currents on M, since we typically consider them as generalized differential forms.

Throughout the note,

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

is the standard real form of $\sqrt{-1}$.

2.2 The symplectic group action on the Grassmannian

This subsection is the symplectic version of the corresponding section in [19], where $\mathrm{O}(p,q)$ is considered.

Let $V=(\mathbb{R}^{2n},\omega)$ be a symplectic space. Let $X_r^k(V)$ for $0\le r\le\kappa:=\lfloor\frac{1}{2}\min(k,2n-k)\rfloor$ be the orbits of $\mathrm{Gr}_k(V)$ under the real symplectic group $\mathrm{Sp}(V)$, where

$$X_r^k(V)=\{E\in\mathrm{Gr}_k(V):\dim\operatorname{Ker}\omega|_E=\min(k,2n-k)-2r\}. \tag{3}$$

When no confusion can arise we write simply X_r^k. In the oriented Grassmannian $\mathrm{Gr}_k^+(\mathbb{R}^{2n})$, the double covers $X_r^{k,+}$ of X_r^k are orbits of $\mathrm{Sp}(V)$ when $0\le r\le\kappa-1$. For $r=\kappa$, there are two possibilities: for even k, the double cover of $X_\kappa^{k,+}$ splits into two open orbits, denoted by $X_\pm^k$, corresponding to the different orientations induced by ω on the subspace; for odd k, $X_\kappa^{k,+}$ is a single orbit.

We use the same notation for the corresponding $\overline{\mathrm{Sp}(V)}$–orbits in the (oriented) affine Grassmannian.

We will need a simple lemma from linear algebra, which is a linearized version of Witt's theorem.

Lemma 2.1 *Take $E\in X_r^k$. Write $E_0=E\cap E^\omega$. Let $\pi_0\colon E^*\otimes V/E\to E_0^*\otimes E_0^*$ denote the map $\pi_0(T)(x,y)=\omega(Tx,y)$. Then*

$$T_EX_r^k=\{T\colon E\to V/E\mid \pi_0(T)\in\operatorname{Sym}^2E_0^*\}.$$

Proof Clearly π_0 is onto. Denote by $\pi_E\colon V^*\otimes V\to E^*\otimes(V/E)$ the natural projection. Recall that $\mathfrak{sp}(V)=\{T\in\mathfrak{gl}(V):\omega(Tx,y)=\omega(Ty,x)\text{ for all }x,y\in V\}$. We have to show that $\pi_E\mathfrak{sp}(V)=\pi_0^{-1}\operatorname{Sym}^2E_0^*$.

The inclusion $\pi_E\mathfrak{sp}(V)\subset\pi_0^{-1}\operatorname{Sym}^2E_0^*$ is immediate. In the other direction, take $T\colon E\to V/E$ such that $\pi_0T\in\operatorname{Sym}^2E_0^*$. We should lift T to $\widetilde{T}\in\mathfrak{sp}(V)$.

Choose any subspaces $E'\subset E$ and $E''\subset E^\omega$ with $E=E_0\oplus E'$, $E^\omega=E_0\oplus E''$.

Write $\kappa=\min(k,2n-k)$, and note that $\dim E_0=\kappa-2r$. Then $\dim E'=k-\kappa+2r$ and $\dim E''=2n-k-\kappa+2r$, and they are both nondegenerate. Moreover, $E'\oplus E''\subset V$ is a nondegenerate subspace of dimension $2n-2\kappa+4r$, and E_0 is an isotropic subspace of the nondegenerate space $W:=(E'\oplus E'')^\omega$, and $\dim W=2\kappa-4r$. That is, $E_0\subset W$ is a Lagrangian subspace. Fix a Lagrangian complementing space $F\subset W$ such that $W=E_0\oplus F$. Then $V=E'\oplus E''\oplus E_0\oplus F$.

Let $T_1\in\mathrm{Hom}(E,V)$ be a lift of T such that $T_1(E')\subset E''\oplus F$, $T_1(E_0)\subset E''\oplus F\oplus E'$ and

$$\omega(\pi'(T_1x),e):=\omega(Te,x)\quad\text{for all } x\in E_0,\ e\in E',$$

where $\pi'(T_1x)$ is the E'–component of T_1x.

Note that ω gives identifications $E^*=E'\oplus F$, as well as $(E\oplus E'')^*=E'\oplus E''\oplus F$. We extend T_1 to a map $T_2\in\mathrm{Hom}(E\oplus E'',V)$ by requiring $T_2(E'')\subset E'\oplus F$ and

$$\omega(T_2e'',e):=\omega(T_1e,e'')\quad\text{for all } e''\in E'',\ e\in E.$$

Note that $\omega(T_2e_1,e_2)=0$ for all $e_1,e_2\in E''$.

Finally, we extend T_2 to a map $T'\in\mathrm{Hom}(V,V)$ by requiring $T'(F)\subset E'\oplus E''\oplus F$ and

$$\omega(T'f,x)=\omega(T_2x,f)\quad\text{for all } x\in E\oplus E'',\ f\in F.$$

Again we have $\omega(T'f,f')=0$ for $f,f'\in F$. Then $T'\in\mathfrak{sp}(V)$ lifts T, as required. □

Corollary 2.2 *There is a natural identification of $N_EX_r^k$ with $\bigwedge^2E_0^*$.*

Proof We have

$$\begin{aligned}N_EX_r^k&=T_E\,\mathrm{Gr}_k(V)/T_EX_r^k=E^*\otimes(V/E)/\pi_0^{-1}\,\mathrm{Sym}^2E_0^*\\&=(E_0^*\otimes E_0^*)/\mathrm{Sym}^2E_0^*=\textstyle\bigwedge^2E_0^*.\end{aligned}$$ □

Lemma 2.3 *Fix $L\in\mathbb{P}_+(V)$, and let $Z_L=\{E\in\mathrm{Gr}_k(V):L\subset E\}$. Then Z_L intersects X_r^k transversally for all r.*

Proof Take $E\in Z_L\cap X_r^k$; let $E_0=E\cap E^\omega$. Let $A\colon E\to V/E$ be an arbitrary linear map. We will find a representation $A=A_1+A_2$ with $A_1\in T_EZ_L$ (that is, $A_1|_L=0$) and $A_2\in T_EX_r^k$. Fix $l\in L$ and write $Al=v+E$.

Case 1 ($L\not\subset E_0$) We set $A_2|_L:=A|_L$ and $A_2|_{E_0}=0$.

Case 2 ($L\subset E_0$ and $\omega(v,E_0)=0$) Decompose $E_0=L\oplus F$ and set $A_2(l)=v+E$ and $A_2(f)=0$ for all $f\in F$.

Case 3 $(L \subset E_0$ and $\omega(L, AL) \neq 0)$ Since $\omega(l, v) \neq 0$, we may decompose $E_0 = E_0 \cap v^\omega \oplus L$. We then set $A_2(l) = v + E$ and $A_2(x) = 0$ for $x \in E_0 \cap v^\omega$.

Case 4 $(L \subset E_0,\ \omega(v, E_0) \neq 0$ and $\omega(L, AL) = 0)$ Then we decompose $E_0 = L \oplus F \oplus \mathrm{Span}(u)$, where $L \oplus F = v^\omega \cap E_0$ and $\omega(u, v) = 1$. Choose $w \in V$ such that $\omega(w, l) = -1$ and $w \in F^\omega$. This is possible since $l^\omega \neq F^\omega$. Then set $A_2 l = v + E$ and $A_2 f = 0$ for $f \in F$, and $A_2 u = w + E$.

In all cases, extend A_2 arbitrarily to E. Thus in all cases, by Lemma 2.1, $A_2 \in T_E X_r^k$ and $A_1 := A - A_2 \in T_E Z_L$. □

3 Valuation theory

3.1 Valuations on manifolds

For a manifold X, we let $\mathbb{P}_X := \mathbb{P}_+(T^*X)$ denote the oriented projectivization of its cotangent bundle and $\pi\colon \mathbb{P}_X \to X$ the projection. $\mathbb{P}_X$ has a canonical contact structure. A form $\omega \in \Omega(\mathbb{P}_X)$ that vanishes when restricted to the contact distribution is usually called vertical. However, we will have several different notions of verticality, so we will call such forms Legendrian.

Definition 3.1 We say that a form $\omega \in \Omega^d(\mathbb{P}_X)$ has horizontal degree at least k, written $\deg_H \omega \geq k$, if $\omega(v_1, \dots, v_d)$ vanishes whenever $d + 1 - k$ of the vectors v_j are vertical, that is, tangent to the fiber of $\pi\colon \mathbb{P}_X \to X$.

The following is a simple reformulation:

Lemma 3.2 *$\{\omega \in \Omega(\mathbb{P}_X) : \deg_H \omega \geq k\}$ is the ideal in $\Omega(\mathbb{P}_X)$ generated by $\pi^*\Omega^k(X)$.*

Proof Assume $\omega = \sum \pi^*\omega_j \wedge \eta_j$, where $\omega_j \in \Omega^k(X)$ and $\eta_j \in \Omega^{d-k}(\mathbb{P}_X)$. Clearly if $v_1, \dots, v_{d+1-k}$ are vertical vectors then $\pi^*\omega_j \wedge \eta_j(v_1, \dots, v_{d+1-k}, \dots) = 0$.

For the opposite direction, let us choose a Riemannian structure on X. Then $\mathbb{P}_X$ is the sphere bundle on X. Fix coordinates dx_j on $T_x X$ and $d\xi_j$ on $T_\xi S_x X$. Then $\pi^*\Omega^k(X)$ is spanned over $C^\infty(\mathbb{P}_X)$ by $\{\bigwedge_{i \in I} \pi^* dx_i : |I| = k\}$. Assume $\deg_H \omega \geq k$ and decompose $\omega = \sum_{I,J} f_{IJ} \pi^* dx_I \wedge d\xi_J$.

Assume a multi-index I appears in the sum with $|I| < k$, say $I = (i_1, \dots, i_l)$ with $l < k$, with corresponding $J = (j_1, \dots, j_{d-l})$. Let $e_1^H, \dots, e_n^H, e_1^V, \dots, e_{n-1}^V$ be dual to $\pi^* dx_1, \dots, \pi^* dx_n, d\xi_1, \dots, d\xi_{n-1}$. Then $d - l \geq d + 1 - k$ so, by assumption, $0 = \omega(e_{i_1}^H, \dots, e_{i_l}^H, e_{j_1}^V, \dots, e_{j_{d-l}}^V) = f_{IJ}(x, \xi)$, so $f_{IJ} = 0$. □

Let M be a smooth manifold, which we assume oriented for simplicity of exposition, and refer the reader to [5; 10] for the general case. Denote by $\mathcal{P}(M)$ the compact differentiable polyhedra of M. We remark that manifolds with corners are an example of differentiable polyhedra, and refer to [5] for the definition of differentiable polyhedra. The smooth valuations $\mathcal{V}^\infty(M)$ consist of functionals $\phi\colon \mathcal{P}(M) \to \mathbb{R}$ which can be presented in the form $\phi(X) = \int_X \mu + \int_{N^*X} \omega$ for some forms $\mu \in \Omega^n(M)$ and $\omega \in \Omega^{n-1}(\mathbb{P}_M)$. Here N^*X is the conormal cycle of X. It consists of codirections $\xi \in \mathbb{P}_+(T_x^*M)$ which are nonpositive on velocity vectors $\dot\gamma(0) \in T_xM$ of all curves $\gamma(t) \in X$ with $x = \gamma(0)$. The Euler characteristic χ is an important example of a smooth valuation. The smooth valuations over open subsets of X constitute a soft sheaf over X; see [5]. We denote the compactly supported valuations by $\mathcal{V}_c^\infty(M)$, and $\mathcal{W}_{i,c}^\infty(M) := \mathcal{V}_c^\infty(M) \cap \mathcal{W}_i^\infty(M)$. We remark that whenever a valuation is evaluated on a subset, the subset is assumed compact. There is a natural integration functional $\int_M\colon \mathcal{V}_c^\infty(M) \to \mathbb{R}$ which is essentially evaluation on M, but, more precisely, on a sufficiently large compact set. Both $\mathcal{V}^\infty(M)$ and $\mathcal{V}_c^\infty(M)$ inherit natural topologies from the corresponding spaces of pairs of forms.

There is a natural filtration $\mathcal{W}_n^\infty(M) \subset \cdots \subset \mathcal{W}_0^\infty(M) = \mathcal{V}^\infty(M)$, introduced by Alesker [5]. We will use an equivalent description, which is the content of Corollary 3.1.10 of [5].

Definition 3.3 $\mathcal{W}_k^\infty(M)$ consists of those valuations that can be represented by a pair (ω, μ) with $\deg_H \omega \geq k$.

In particular, $\mathcal{W}_n^\infty(M)$ are just the smooth measures on M, denoted by $\mathcal{M}^\infty(M)$.

Alesker defined a product structure $\mathcal{W}_i^\infty(M) \otimes \mathcal{W}_j^\infty(M) \to \mathcal{W}_{i+j}^\infty(M)$ which turns $\mathcal{V}^\infty(M)$ into a filtered algebra, whose unit is the Euler characteristic. It induces the following Alesker–Poincaré duality:

Theorem 3.4 (Alesker [6]) *The pairing $\mathcal{W}_i^\infty(M) \otimes \mathcal{W}_{n-i,c}^\infty(M) \to \mathbb{R}$ given by $(\phi, \psi) \mapsto (\phi \cdot \psi)(M)$ is nondegenerate.*

The presentation of $\phi \in \mathcal{V}^\infty(M)$ by a pair of forms is not unique. There is an alternative faithful description due to Bernig and Bröcker [18]. In the following, $a\colon \mathbb{P}_M \to \mathbb{P}_M$ is the antipodal map in every fiber, and D is the Rumin differential introduced in [49]. We recall that $D\omega$ is the unique Legendrian form $d(\omega + \eta)$, where $\eta \in \Omega^{n-1}(\mathbb{P}_M)$ ranges over all Legendrian forms.

Theorem 3.5 *If ϕ is represented by the pair $(\omega,\mu)\in\Omega^{n-1}(\mathbb{P}_M)\times\Omega^n(M)$, then*

$$(T,C):=(a^*(D\omega+\pi^*\mu),\pi_*\omega)\in\Omega^n(\mathbb{P}_M)\times C^\infty(M) \tag{4}$$

is determined by ϕ. They satisfy the relations $dT=0$ and $\pi_*T=(-1)^n dC$, and T is Legendrian. Moreover, any (T,C) with those properties corresponds to a valuation.

We refer to (T,C) as the defining currents of ϕ (and often we refer just to T as the defining current). The reason for this terminology will become evident once we introduce generalized valuations.

The Alesker–Poincaré duality is easy to describe using the defining currents.

Theorem 3.6 (Bernig [17]) *Let (ω,μ) represent ϕ_1, and let ϕ_2 have defining current (T_2,C_2). Then*

$$\int_M \phi_1\cdot\phi_2=\int_{\mathbb{P}_M}\omega_1\wedge T_2+\int_M C_2\mu_1. \tag{5}$$

Let us describe the filtration on $\mathcal{V}^\infty(M)$ through the defining currents.

Lemma 3.7 *Let $(T,C)\in\Omega^n(\mathbb{P}_M)\times C^\infty(X)$ be the defining current of $\phi\in\mathcal{V}^\infty(M^n)$. For $1\le k\le n$, $\phi\in\mathcal{W}_k^\infty(M^n)$ if and only if $C=0$ and $\deg_H T\ge k$.*

Proof Consider first $k=n$. If $\phi\in\mathcal{W}_n^\infty(M)$, we have $C=0$ and $T=\pi^*\mu$ for some $\mu\in\Omega^n(M)$, which is clearly horizontal of degree at least n. In the other direction, if $\deg_H T\ge n$, it follows that $T=f(\xi)\pi^*\mu$ for some $\mu\in\Omega^n(M)$ and $f\in C^\infty(\mathbb{P}_M)$. Now $0=dT=df\wedge\pi^*\mu$, that is df vanishes when restricted to the vertical fiber, so $f=\pi^*f_1$ for some $f_1\in C^\infty(M)$. Hence $T=\pi^*(f_1\mu)$, so ϕ is just the measure $f_1\mu$.

Assume now $1\le k\le n-1$. Recall that by Alesker–Poincaré duality, $\phi\in\mathcal{W}_k^\infty(M)$ if and only if $\int_M\phi\cdot\psi=0$ for all $\psi\in\mathcal{W}_{n+1-k,c}^\infty(M)$. Combining Definition 3.3 with equation (5), we conclude that $\phi\in\mathcal{W}_k^\infty(M)$ if and only if for all $\omega\in\Omega^{n-1}(\mathbb{P}_M)$ with $\deg_H\omega\ge n+1-k$ and all $\mu\in\Omega^n(M)$, it holds that $\int_M(\pi_*(\omega\wedge T)+C\mu)=0$.

Assume that $C=0$ and $\deg_H T\ge k$. Then, by Lemma 3.2, $\omega\wedge T=0$ for all ω as above, and hence $\phi\in\mathcal{W}_k^\infty(M)$.

In the other direction, assume $\phi\in\mathcal{W}_k^\infty(M)$. Taking $\omega=0$ and μ arbitrary, we deduce $C=0$. It then follows that $\pi_*(\omega\wedge T)=0$, and since ω can have an arbitrarily small support, also that $\omega\wedge T=0$ whenever $\deg_H\omega\ge n-k+1$, so that $\deg_H T\ge k$. □

The valuations that appear naturally in contact manifolds are not smooth. To formally study them we will need the larger family of generalized valuations.

Definition 3.8 The generalized valuations $\mathcal{V}^{-\infty}(M)$ are the continuous dual of $\mathcal{V}_c^\infty(M)$, equipped with the weak topology.

A generalized valuation is uniquely determined by its defining current $(T,C) \in \mathcal{D}_{n-1}(\mathbb{P}_M) \times \mathcal{D}_n(M)$; see [8]. It can be an arbitrary pair of currents satisfying the three properties: T is Legendrian, $\pi_* T = \partial C$ and $\partial T = 0$. If $\phi_2 \in \mathcal{V}^{-\infty}(M)$ has defining current (T_2, C_2), it acts on smooth valuations $\phi_1 \in \mathcal{V}_c^\infty(M)$ represented by forms (ω_1, μ_1) through equation (5). We will write $\phi_2 = [T_2, C_2]$, $T_2 = T(\phi_2)$ and $C_2 = C(\phi_2)$.

Example 3.9 (1) Given $X \in \mathcal{P}(M)$, the evaluation at X functional $\chi_X \colon \mathcal{V}^\infty(M) \to \mathbb{R}$ is a generalized valuation with defining currents $C = [\![X]\!]$ and $T = [\![N^* X]\!]$.

(2) A generalized valuation ϕ_1 can be represented by a pair of generalized forms $\omega_1 \in \Omega_{-\infty}^{n-1}(\mathbb{P}_M)$ and $\mu_1 \in \Omega_{-\infty}^n(M)$. It acts on $\phi_2 = [T_2, C_2] \in \mathcal{V}_c^\infty(M)$ through equation (5). It has defining currents given by $T_1 = a^*(D\omega_1 + \pi^*\mu_1)$ and $C_1 = \pi_*\omega_1$.

The filtration $\mathcal{W}_j^\infty(M)$ on $\mathcal{V}^\infty(M)$ extends to a filtration $\mathcal{W}_j^{-\infty}(M)$ on $\mathcal{V}^{-\infty}(M)$ by taking $\mathcal{W}_j^{-\infty}(M)$ to be the annihilator of $\mathcal{W}_{n+1-j,c}^{-\infty}(M)$.

We refer to [36; 34] for the notion of the wavefront set of a distribution. Let us only record that for an oriented submanifold $X \subset M$, $\mathrm{WF}([\![X]\!]) = N^* X$.

Definition 3.10 The wavefront $\mathrm{WF}(\phi)$ of $\phi \in \mathcal{V}^{-\infty}(M)$ is the pair of wavefront sets $\bigl(\mathrm{WF}(T(\phi)), \mathrm{WF}(C(\phi))\bigr)$. When $\mathrm{WF}(C(\phi)) = \varnothing$, we also write $\mathrm{WF}(\phi) = \mathrm{WF}(T(\phi))$.

Given a closed cone $\Gamma \subset T^*(\mathbb{P}_M) \setminus 0$, we let

$$\mathcal{V}_\Gamma^{-\infty}(M) := \{\phi \in \mathcal{V}^{-\infty}(M) : \mathrm{WF}(T(\phi)) \subset \Gamma,\ \mathrm{WF}(C(\phi)) = \varnothing\}.$$

It inherits a topology from Hörmander's topology on the corresponding space of currents $\mathcal{D}_{n-1,\Gamma}(\mathbb{P}_M)$.

The Alesker–Poincaré duality extends to a pairing of generalized valuations, as long as the wavefront sets are in good relative position. We will only need the following weak version of Theorem 8.3 in [8].

For a subset $T \in T^*\mathbb{P}_M$, we write $-T$ for its image under the antipodal map in every cotangent space $T^*_{x,\xi} T^*(\mathbb{P}_M)$. Define also $T^s := T \cup (-T) \cup aT \cup (-aT)$.

Theorem 3.11 *Let $\Gamma_1, \Gamma_2 \subset T^*(\mathbb{P}_M) \setminus 0$ be closed cones and $\Gamma_j = \Gamma_j^s$. Assume that $\Gamma_1 \cap \Gamma_2 = \varnothing$. Then there is a jointly sequentially continuous pairing*

$$\mathcal{V}_{\Gamma_1}^{-\infty}(M) \otimes \mathcal{V}_{\Gamma_2}^{-\infty}(M) \to \mathbb{R}$$

extending the Alesker–Poincaré duality on the dense subspaces of smooth valuations.

It follows that if $\phi = [T, C]$ with C smooth, and $X \in \mathcal{P}(M)$ such that

$$\mathrm{WF}[\![N^*X]\!] \cap \mathrm{WF}(T^s) = \varnothing,$$

then one can naturally evaluate $\phi(X) := \langle \phi, \chi_X \rangle$. In particular, if $X, Y \in \mathcal{P}(M)$ have disjoint conormal cycles, one can evaluate $\int_M \chi_X \cdot \chi_Y$. In all reasonable settings, the result should equal $\chi(X \cap Y)$. This was shown to be the case when X and Y are transversal submanifolds with corners in [8, Theorem 6].

The Euler–Verdier involution $\sigma\colon \mathcal{V}^{-\infty}(M) \to \mathcal{V}^{-\infty}(M)$ was introduced by Alesker in [5; 6]. It can be described through its action on the defining currents: $\sigma[T, C] = [(-1)^n a^* T, C]$.

We will need the following simple lemma, which we use in the proof of Proposition 6.3.

Lemma 3.12 *Take a generalized valuation $\phi \in \mathcal{V}^{-\infty}(M^n)$ satisfying $\sigma\phi = (-1)^{n+1}\phi$ and $C(\phi)$ smooth. Assume that $\phi(F) = 0$ for all closed hypersurfaces F for which $N^*(N^*F)$ is disjoint from $\mathrm{WF}(\phi)$. Assume moreover that M can be covered by open charts $U_\alpha \cong \mathbb{R}^n$ such that if $K \subset U_\alpha$ is a smooth, strictly convex body, then $N^*(N^*\partial K) \cap \mathrm{WF}(\phi) = \varnothing$. Then $\phi = 0$.*

Proof As generalized valuations form a sheaf over M [6, Proposition 7.2.2], we may assume $M = U_\alpha = \mathbb{R}^n$.

By [4], valuations of the form $\mu(\bullet - K)$, where K is a convex body with smooth support function and $\mu \in \mathcal{M}_c^\infty(\mathbb{R}^n)$, span a dense subspace in $\mathcal{V}_c^\infty(\mathbb{R}^n)$. It remains dense if we only take strictly convex K. Fix such μ and K.

By [6, Lemma 4.1.1] and approximating ϕ by smooth valuations in the Hörmander topology with the same Euler–Verdier eigenvalue, we have $(-1)^{n+1}\phi(K) = \sigma\phi(K) = (-1)^n(\phi(K) - \phi(\partial K))$. By assumption, $\phi(\partial K) = 0$, and hence $\phi(K) = 0$. It follows that

$$\langle \phi, \mu(\bullet - K) \rangle = \left\langle \phi, \int_{\mathbb{R}^n} \chi(\bullet \cap (K + x))\, d\mu(x) \right\rangle = \int_{\mathbb{R}^n} \phi(K + x)\, d\mu(x) = 0,$$

concluding the proof. □

3.2 Translation-invariant valuations

We will need the following standard facts when we study valuations in the dual Heisenberg algebra in Section 5.

The space of smooth translation-invariant valuations $\mathrm{Val}^\infty(\mathbb{R}^n)$ can be defined as $\mathcal{V}^\infty(\mathbb{R}^n)^{\mathrm{tr}}$. It is a Fréchet space. We will also consider generalized translation-invariant valuations, $\mathrm{Val}^{-\infty}(\mathbb{R}^n) := \mathcal{V}^{-\infty}(\mathbb{R}^n)^{\mathrm{tr}}$.

The even/odd valuations are $\mathrm{Val}^{\pm,\pm\infty}$ which have eigenvalue ± 1 under the antipodal map. The k–homogeneous valuations are those $\phi \in \mathrm{Val}^{\pm\infty}$ that have eigenvalue λ^k under all rescalings of $\mathbb{R}^n$ by $\lambda > 0$.

It is well known that $\mathrm{Val}_0^{-\infty}(\mathbb{R}^n) = \mathrm{Span}\{\chi\}$, and by Hadwiger's theorem [35], $\mathrm{Val}_n^{-\infty}(\mathbb{R}^n) = \mathrm{Span}\{\mathrm{vol}_n\}$. It follows from McMullen's work [42] that $\mathrm{Val}^{\pm\infty}(\mathbb{R}^n) = \bigoplus_{k=0}^n \mathrm{Val}_k^{\pm\infty}(\mathbb{R}^n)$.

The line bundle over $\mathrm{Gr}_k(\mathbb{R}^n)$ whose fiber over E is $\mathrm{Dens}(E)$ is called the Klain bundle. The Klain map $\mathrm{Kl}\colon \mathrm{Val}_k^{+,\infty} \to \Gamma^\infty(\mathrm{Gr}_k(V), \mathrm{Dens}(E))$ is given by $\mathrm{Kl}(\phi)(E) = \phi|_E$. It is well defined by Hadwiger's theorem, and injective by a theorem of Klain [37]. By [9], it admits an injective extension $\mathrm{Kl}\colon \mathrm{Val}_k^{+,-\infty} \to \Gamma^{-\infty}(\mathrm{Gr}_k(V), \mathrm{Dens}(E))$.

Dual to the Klain section is the Crofton map $\mathrm{Cr}\colon \mathcal{M}^\infty(\mathrm{AGr}_{n-k}(V))^{\mathrm{tr}} \to \mathrm{Val}_k^{+,\infty}(V)$, given by $\mathrm{Cr}(\mu)(K) = \int_{\mathrm{AGr}_{n-k}(V)} \chi(E \cap K)\, d\mu(E)$. It follows from Alesker's irreducibility theorem [2] that Cr is surjective. It was shown in [9] that it admits a surjective extension $\mathrm{Cr}\colon \mathcal{M}^{-\infty}(\mathrm{AGr}_{n-k}(V))^{\mathrm{tr}} \to \mathrm{Val}_k^{+,-\infty}(V)$.

The relation to valuations on manifolds is as follows:

Theorem 3.13 [5] *The quotient $\mathcal{W}_k^\infty(M)/\mathcal{W}_{k+1}^\infty(M)$ is naturally isomorphic to the space $\Gamma^\infty(M, \mathrm{Val}_k^\infty(TM))$ of smooth sections of the bundle with fiber $\mathrm{Val}_k^\infty(T_xM)$ over $x \in M$.*

Consequently, by Alesker–Poincaré duality we have

$$\mathcal{W}_k^{-\infty}(M)/\mathcal{W}_{k+1}^{-\infty}(M) \simeq (\mathcal{W}_{n-k}^\infty(M)/\mathcal{W}_{n-k+1}^\infty(M))^* \simeq \Gamma^\infty(M, \mathrm{Val}_{n-k}^\infty(TM))^*.$$

3.3 Crofton formulas on manifolds

Consider a double fibration

$$\begin{tikzcd} & W \arrow[dl, "\tau"'] \arrow[dr, "\pi"] & \\ Z & & X \end{tikzcd}$$

where π and τ are proper and $\tau \times \pi\colon W \to Z \times X$ is a closed embedding. We view Z as a space parametrizing a family of submanifolds $\hat{z} := \pi\tau^{-1}z \subset X$, of dimension $\dim \hat{z} = \dim W - \dim Z$. Assume furthermore that the map $Z \to \mathcal{D}(\mathbb{P}_X)$, $z \mapsto [\![N^*\hat{z}]\!]$, is smooth.

Given a distribution $\mu \in \mathcal{M}^{-\infty}(Z)$, define the generalized valuation $\mathrm{Cr}(\mu) \in \mathcal{V}^{-\infty}(X)$ by $\mathrm{Cr}(\mu) = \pi_*\tau^*\mu$, which is the Radon transform with respect to the Euler characteristic of μ in the terminology of Alesker [7]. Explicitly, for $\psi \in \mathcal{V}_c^\infty(X)$,

$$\langle \mathrm{Cr}(\mu), \psi\rangle = \int_Z \psi(\hat{z})\, d\mu(z).$$

Lemma 3.14 *Let X be a manifold and $\phi = \mathrm{Cr}(\mu) \in \mathcal{V}^{-\infty}(X)$ for $\mu \in \mathcal{M}^{-\infty}(Z)$. Then $\phi \in \mathcal{W}^{-\infty}_{\dim X - \dim W + \dim Z}(X)$ and $\sigma\phi = (-1)^{\dim X - \dim W + \dim Z}\phi$, where σ is the Euler–Verdier involution.*

Proof We use the Alesker–Poincaré duality: for every smooth, compactly supported $\psi \in \mathcal{W}^\infty_{\dim W - \dim Z + 1}(X)$, it holds that $\langle \psi, \mathrm{Cr}(\mu)\rangle = \int_Z \psi(\hat{z})\, d\mu(z) = 0$. Now [6, Proposition 7.3.2] implies the first statement. For the second statement, simply note that the defining current of ϕ is given by $\int_Z [\![N^*\hat{z}]\!]\, d\mu(z)$. □

4 Dual Heisenberg manifolds

4.1 Geometric preliminaries

Recall that the Heisenberg algebra is $\mathbb{R}^{2n+1}$ whose Lie bracket is given on the standard basis by $[e_i, e_{i+n}] = e_{2n+1}$ for $1 \le i \le n$, and $[e_i, e_j] = 0$ for all $i < j \neq i+n$. The center of the algebra is thus $Z := \mathrm{Span}(e_{2n+1})$.

The dual Heisenberg algebra U^{2n+1} is $\mathbb{R}^{2n+1}$, equipped with a distinguished hyperplane $H \subset U$, and $\omega \in \bigwedge^2 H^* \otimes U/H$ is a nondegenerate (twisted) form. Let us justify this terminology.

Lemma 4.1 *The space U^* is naturally the Heisenberg Lie algebra.*

Proof Write $Z = (U/H)^* = H^\perp \subset U^*$. There is a Z–valued symplectic form on the quotient $U^*/Z = H^*$: we may consider ω as the isomorphism $H \simeq H^* \otimes (U/H)$, or equivalently $H^* \simeq H \otimes (U/H)^*$, which is a nondegenerate bilinear form $\omega^* \in (H \otimes H) \otimes (U/H)^*$, which is clearly antisymmetric, that is, $\omega^* \in \bigwedge^2 (U^*/Z)^* \otimes Z$. The Lie bracket $[x, y] := \omega^*(x + Z, y + Z)$ then defines on U^* a Heisenberg Lie algebra structure with center Z. □

Definition 4.2 A manifold X equipped with a hyperplane distribution H and a smooth field of nowhere-degenerate forms $\omega \in \Gamma^\infty\big(X, \bigwedge^2 H_x^* \otimes (T_x X/H_x)\big)$ will be called a dual Heisenberg (DH) manifold. We call H the horizontal distribution and ω the DH form.

Remark 4.3 Ovsienko [45] defines the closely related notion of a local Heisenberg structure. Similar ideas lie in the foundation of Stein's Heisenberg calculus; see eg Ponge [46; 47].

Example 4.4 A contact manifold is naturally equipped with a DH structure. Let (X, H) be contact, and let α be a contact form defined locally. Then $\alpha\colon T_x X/H_x \xrightarrow{\sim} \mathbb{R}$ is an isomorphism, and we may define the 2–form $\omega(u, v) = \alpha^{-1}(d\alpha(u, v))$ for $u, v \in H_x$. It is independent of the choice of α.

Definition 4.5 Let (X, H_X, ω_X) be a DH manifold. A DH submanifold is a submanifold $Y^{2k+1} \subset X^{2n+1}$ which is nowhere tangent to H_X. It inherits the structure of a DH manifold with hyperplane distribution $H_Y = H_X \cap TY$, and DH form $\omega_Y = \omega_X|_{H_Y}$, where we naturally identify $TX/H_X = TY/H_Y$.

Definition 4.6 Let U^{2n+1} be the dual Heisenberg Lie algebra, with horizontal hyperplane $H \subset U$ and DH form $\omega \in \bigwedge^2 H^* \otimes U/H$. A Euclidean structure P on U is said to be compatible with the DH structure if one can find an orthonormal basis $\{X_j, Y_j, Z\}$ of U such that $\{X_j, Y_j\}$ form a symplectic basis of H with respect to Z: $\omega(X_i, Y_j) = 0$ and $\omega(X_j, Y_j) = Z + H \in U/H$ for all $1 \le i, j \le n$ and $i \ne j$. A compatible form $\alpha \in U^*$ is any form such that $\operatorname{Ker} \alpha = H$. P is compatible with α if $\alpha = P(Z, \bullet)$.

Definition 4.7 Let M^{2n+1} be a DH manifold with horizontal distribution $H \subset TM$ and DH forms $\omega \in \bigwedge^2 H^* \otimes TM/H$. A Riemannian metric g on M is compatible if g_x is compatible for $(T_x M, H_x, \omega_x)$ for every $x \in M$.

Remark 4.8 (1) A similar notion of adapted metric was used by Chern and Hamilton in [26] for contact 3–manifolds and in later works by other authors.

(2) In fact, for the purposes of this work we only need the metric we choose to be compatible with the contact structure at isolated points of interest.

Lemma 4.9 *For any DH manifold M, a globally defined compatible Riemannian metric exists. Moreover, if a compatible form $\alpha \in \Omega^1$ is given, one may choose the metric to be compatible with α.*

Proof Cover M by contractible open subsets U_i, and choose 1–forms α_i on U_i such that $\alpha_i|_{U_i\cap U_j} = \pm\alpha_j|_{U_i\cap U_j}$; this can be accomplished by fixing an arbitrary Riemannian metric and letting $\alpha_i(x) \in T_x^*M$ have unit norm for all $x \in U_i$. Set $\omega_i = \alpha_i \circ \omega \in \Gamma\big(U_i, \bigwedge^2 T^*M\big)$. Let us also fix a complementing line bundle L to H, $L \oplus H = TM$.

For each U_i we may choose a compatible complex structure J_i for ω_i on $H|_{U_i}$ such that $J_i|_{U_i\cap U_j} = \pm J_j|_{U_i\cap U_j}$. We thus obtain the corresponding Euclidean forms h_i on $H|_{U_i}$, which will satisfy $h_i = h_j$, and so can be patched to give a globally defined, positive definite quadratic form h on H. Now for every U_i, choose $Z_i \in \Gamma(U_i, L)$ satisfying $\alpha_i(Z_i) = 1$, and define the Riemannian metric g_i over U_i by $g_i|_H = h$, $g_i(L, H) = 0$ and $g_i(Z_i, Z_i) = 1$. Then clearly $g_i = g_j$ on $U_i \cap U_j$ and thus (g_i) define a Riemannian metric g on M. Over U_i we may locally choose an orthonormal symplectic basis $\{X_j^i, Y_j^i\}$ for ω_i, and then $\omega(X_j^i, Y_j^i) = \alpha_i^{-1}(1) = Z_i$ for all j. This shows g is a compatible metric.

Finally, if α is given, we just take $\alpha_i = \alpha|_{U_i}$ in the construction above. □

Let M^{2n+1} be a DH manifold with horizontal distribution H and DH form ω. Set $M_H := \{(x,\xi) : x \in M,\ \xi \perp H_x\} \subset \mathbb{P}_M\ (= \mathbb{P}_+(T^*M))$.

Definition 4.10 A C^2 boundaryless submanifold $F \subset M$ is said to be normally transversal (to H) if N^*F intersects M_H transversally.

More generally, a C^2 submanifold with boundary $F \subset M$ is normally transversal if $T_xF \not\subset H_x$ for all $x \in \partial F$ and both $N^*(\mathrm{int} F)$ and $N^*\partial F$ intersect M_H transversally.

Note that if F is normally transversal then so is ∂F.

Lemma 4.11 *If a submanifold with boundary $F \subset M$ intersects H normally transversally, then* $\mathrm{WF}([\![N^*F]\!]) \cap N^*M_H = \varnothing$.

Proof For a boundaryless submanifold this is immediate: N^*F is a boundaryless submanifold, and $\mathrm{WF}([\![N^*F]\!]) = N^*(N^*F)$.

Now assume F has boundary. Consider the sets $N_{\partial F}^*F = \bigcup_{x\in\partial F} N_x^*F$ and $N_{\mathrm{int}}^*F := \bigcup_{x\in b}\{\xi \in \mathbb{P}_+(T_x^*M) : \xi \in (T_xF)^\perp\}$. Both are manifolds with the same boundary $B = \{(x,\xi) : x \in \partial F,\ \xi \in (T_xF)^\perp\}$, and $N^*F = N_{\partial F}^*F \cup N_{\mathrm{int}}^*F$. Also, $N_{\partial F}^*F \subset N^*\partial F$ is a submanifold with the same boundary B.

By assumption, M_H intersects the interior points of $N^*_{\partial F} F$ and $N^*_{\mathrm{int}} F$ transversally, and does not intersect B. It also holds for $(x,\xi) \notin B$ that $\mathrm{WF}_{x,\xi}([\![N^*F]\!])$ coincides with $N^*_{x,\xi}(N^*_{\partial F} F)$ if $x \in \partial F$, and with $N^*_{x,\xi}(N^*_{\mathrm{int}} F)$ if $x \in \mathrm{int}\, F$. □

Normal transversality is of course generic:

Lemma 4.12 *Any hypersurface with boundary $F \subset M$ can be perturbed by an arbitrarily C^∞–small amount to become normally transversal.*

Proof First, we may perturb F so that it is tangent to the contact distribution at isolated points. Now near interior contact points, we only need to perturb F locally near those points to get normal transversality. Next, we may perturb F near the boundary to have no contact points of F near the boundary, and isolated contact points of ∂F. We then perturb F near those contact points such that ∂F has normal transversality at all its contact points. □

Normal transversality is universal to embeddings, as the following lemma shows.

Lemma 4.13 *Let $(Y, H^Y) \subset (X, H^X)$ be DH manifolds and $F \subset Y$ a normally transversal submanifold with boundary. Then $F \subset X$ is also normally transversal.*

Proof We will distinguish the conormal bundles of F in the different ambient manifolds by writing eg $N^*_Y F \subset \mathbb{P}_Y$. Recall that $H^Y_x = T_x X \cap H^X_x$. Thus, for $x \in \partial F$, $T_x F \not\subset H^X_x$, since $T_x F \subset H^Y_x$.

Next take $(x,\xi) \in N^*_X F$. For $x \in \mathrm{int}\, F$ we should check $T_{x,\xi} N^*_X F \cap T_{x,\xi} X_{H^X} = \{0\}$. Consider the natural surjective map $\beta\colon \mathbb{P}_X|_Y \setminus N^*X \to \mathbb{P}_Y$. It holds that $\beta(N^*_X F) = N^*_Y F$. Denote $X_{H^X}|_Y := X_{H^X} \cap \pi_X^{-1}(Y)$, where $\pi_X\colon \mathbb{P}_X \to X$. Since $X \subset Y$ is a DH submanifold, it holds that $\beta\colon X_{H^X}|_Y \to Y_{H^Y}$ is a diffeomorphism. If $v \in T_{x,\xi} N^*_X F \cap T_{x,\xi} X_{H^X}|_Y$ then $d\beta(v) \in T_{x,\xi} Y_{H^Y} \cap T_{x,\xi} N^*_Y F$ is nonzero, a contradiction to the assumption of normal transversality of $F \subset M$. The case of $x \in \partial F$ is virtually identical. □

4.2 Constructing the valuations

Consider a point $q = (p,\xi) \in M_H$. The vertical subspace is $\mathrm{Ker}(d\pi) = \xi^* \otimes T^*_p M/\xi = (H_p^\perp)^* \otimes H_p^*$. Recall that $\omega_p \in \bigwedge^2 H_p^* \otimes T_p M/H_p$ is nondegenerate, so that $H_p^* \simeq H_p \otimes H_p^\perp$. Thus $\mathrm{Ker}(d\pi) \simeq H_p$. Note also that $d\pi\colon T_q M_H \to T_p M$ is an isomorphism, so that we have a natural decomposition

$$T_q \mathbb{P}_M \simeq T_p M \oplus H_p. \tag{6}$$

Denoting the contact hyperplane of $\mathbb{P}_M$ by $\widehat{H}_q$, we get

$$\widehat{H}_q \simeq H_p \oplus H_p. \tag{7}$$

We will define certain $\eta_k \in \Gamma^{-\infty}_{M_H}(\mathbb{P}_M, \Omega^{2n+1} \otimes \pi^* o_M)$, $1 \le k \le 2n$ (twisted by the pullback of the orientation bundle of M) supported on M_H.

Take $\widetilde{\psi} \in \Omega^{2n+1}_c(\mathbb{P}_M)$ a Legendrian form. We have

$$\widetilde{\psi}|_q \in \bigwedge^{2n+1} T^*_q(\mathbb{P}_M) \simeq \bigoplus_{k=0}^{2n} \bigwedge^{2n+1-k} T^*_p M \otimes \bigwedge^k H^*_p.$$

Moreover, since $\widetilde{\psi}$ is Legendrian, its first factor in the k^{th} component belongs to the kernel of the restriction map $\bigwedge^{2n+1-k} T^*_p M \to \bigwedge^{2n+1-k} H^*_p$, which is naturally isomorphic to $H^\perp_p \otimes \bigwedge^{2n-k} H^*_p$. Thus the k^{th} component of $\widetilde{\psi}$, denoted by $\widetilde{\psi}^{2n+1-k,k}$, lies in $H^\perp_p \otimes \bigwedge^{2n-k} H^*_p \otimes \bigwedge^k H^*_p$.

Wedging the last two factors, we then get an element $(\widetilde{\psi})_k \in \bigwedge^{2n} H^*_p \otimes (T_p M/H_p)^* \simeq \bigwedge^{2n+1} T^*_p M \simeq \bigwedge^{2n+1} T^*_q M_H$. Now set

$$\langle \widetilde{\eta}_k, \widetilde{\psi} \otimes \epsilon_M \rangle := \int_{M_H} (\widetilde{\psi})_k \otimes \epsilon_M,$$

where ϵ_M is a section of $\pi^* o_M$, so that $\widetilde{\psi}_k \otimes \epsilon_M \in \operatorname{Dens}(T_q M_H)$. Thus $\widetilde{\eta}_k$ is a linear functional on compactly supported, Legendrian, $\pi^* o_M$–twisted $(2n+1)$–forms, and it is supported on M_H. For $k = 0$, $\widetilde{\eta}_0 = [\![M_H]\!]$ is in fact a well-defined (twisted) generalized $2n$–form.

Finally, for $1 \le k \le 2n$, define $\eta_k \in \Omega^{2n+1}_{-\infty}(\mathbb{P}_M) \otimes \pi^* o_M$ by setting, for $\psi \in \Omega^{2n}_c(\mathbb{P}_M)$,

$$\langle \eta_k, \psi \otimes \epsilon_M \rangle := \langle \widetilde{\eta}_k, D\psi \otimes \epsilon_M \rangle,$$

where D denotes the Rumin differential. It is clear that η_k is a (twisted) Legendrian cycle, and $\pi_* \eta_k = 0$.

Definition 4.14 Define $\phi_k \in \mathcal{V}^{-\infty}(M)$ for $0 \le k \le 2n$ as follows: for $1 \le k \le 2n$, $\phi_k = \left[\frac{1}{2}\eta_k, 0\right]$; for $k = 0$, ϕ_0 is represented by $\left(\frac{1}{2}\widetilde{\eta}_0, 0\right)$.

Remark 4.15 It is in fact possible to define natural curvature measures globalizing to ϕ_k: extend $\widetilde{\eta}_k$ arbitrarily as a functional on all forms, that is, $\widetilde{\eta}_k \in \Omega^{2n}_{-\infty}(\mathbb{P}_M)$, and define the generalized curvature measure Φ_k represented by the pair of forms $\left(\frac{1}{2}\widetilde{\eta}_k, 0\right)$. The resulting curvature measure is independent of the extension, as $\widetilde{\eta}_k$ is only applied to subsets of conormal cycles.

Note that $\phi_0(X) = \int_{N^*X} \frac{1}{2}\tilde{\eta}_0 = \frac{1}{2}\int_{N^*X}[\![M_H]\!]$ is one-half the intersection index of M_H and N^*X, both oriented locally by a fixed local orientation on M. In the following proofs, we often assume for simplicity M is oriented. They are easily adjusted for the general case.

Lemma 4.16 *ϕ_0 is the Euler characteristic.*

Proof Note that $[\![M_H]\!]$ is a closed current, and, writing $\pi\colon \mathbb{P}_M \to M$, clearly $\pi_*[\![M_H]\!] = 2$. It follows that $\phi_0 = \chi$ by an obvious extension to $\mathcal{V}^{-\infty}$ of Corollary 1.5 of [18]. □

Recall that $a\colon \mathbb{P}_M \to \mathbb{P}_M$ is the fiberwise antipodal map, and $\sigma\colon \mathcal{V}^{-\infty}(M) \to \mathcal{V}^{-\infty}(M)$ denotes the Euler–Verdier involution.

Proposition 4.17 *For all $0 \le k \le 2n$ it holds that $\sigma\phi_k = \phi_k$.*

Proof For $\psi \in \Omega^{2n+1}(\mathbb{P}_M)$ it holds that $\int_{M_H} \psi = \int_{M_H} a^*\psi$, since $a\colon M_H \to M_H$ is clearly orientation-preserving. Note also that $a\colon \mathbb{P}_M \to \mathbb{P}_M$ is orientation-reversing as the fibers of $\mathbb{P}_M \to M$ are even-dimensional spheres. Hence, for a test form $\psi \in \Omega_c^{2n}(\mathbb{P}_M)$,

$$\begin{aligned}\langle a^*\eta_k, \psi\rangle &= -\langle \eta_k, a^*\psi\rangle = -\int_{M_H} [Da^*\psi]_k = -\int_{M_H} a^*[D\psi]_k = -\int_{M_H} [D\psi]_k \\ &= -\langle \eta_k, \psi\rangle.\end{aligned}$$

That is, $a^*\eta_k = -\eta_k$ for all k, and hence $\sigma\phi_k = (-1)^{2n+1}(-\phi_k) = \phi_k$, as asserted. □

Proposition 4.18 *It holds that $\phi_k \in \mathcal{W}_k^{-\infty}(M)$.*

Proof Take a closed Legendrian form $\psi \in \Omega^{2n+1}(\mathbb{P}_M)$ defining a smooth valuation $\Psi \in \mathcal{W}_{2n+2-k}^{\infty}(M)$. We ought to show that $\phi_k \cdot \Psi = 0$, equivalently $\langle \tilde{\eta}_k, \psi\rangle = 0$. But $\Psi \in \mathcal{W}_{2n+2-k}^{\infty}(M)$ implies that ψ has a horizontal degree at least $2n+2-k$, and the claim follows from the definition of $\tilde{\eta}_k$. □

Lemma 4.19 *The wavefront set of ϕ_k for $1 \le k \le 2n$ is N^*M_H.*

Proof This is immediate from definition: the wavefront set of $[\![M_H]\!]$ is N^*M_H, and restriction to M_H is the only source of singularities of η_k. □

It follows by Lemma 4.11 that we may evaluate ϕ_k on any normally transversal submanifold with boundary.

Let $F \subset M$ be a smooth hypersurface in a DH manifold M^{2n+1}; assume that F is normally transversal to the horizontal distribution and tangent to it at $p \in M$.

Working in a small open ball U near p with no other contact points, let us fix a 1–form $\alpha \in \Omega^1(U)$ defining the horizontal distribution, which also trivializes the DH form: $\omega \in \Gamma\big(U, \bigwedge^2 H^*\big)$. Let g be a Riemannian metric on U which is compatible with α and ω on T_pM (but not necessarily elsewhere), with Levi-Civita connection ∇. Let $R\colon U \to SM$ be the vector field given by $R(x) \perp H_x$, $g(R,R)=1$ and $\alpha(R)>0$. Let $(X_j)_{j=0}^{2n}$ be an orthonormal frame in U with $X_0|_F = \nu\colon F \cap U \to SM$ the unit normal oriented by $\alpha(\nu) > 0$, and $(X_j(p))_{j=1}^{2n}$ a symplectic basis of H_p. Let $\theta_j \in \Omega^1(U)$ be the dual coframe to X_j. Let $S = (s_{ij})_{i,j=1}^{2n}$ with $s_{ij} = \theta_i(\nabla_{X_j}\nu)$ be the second fundamental form given by $\nabla_{X_j}\nu = \sum_{i=0}^{2n} s_{ij} X_i$.

Definition 4.20 Let h_{ij} for $1 \le i \le 2n$ and $0 \le j \le 2n$ be given by $h_{ij} = \theta_i(\nabla_{X_j} R)$. The matrix $h = (h_{ij})_{i,j=1}^{2n}$ is the second fundamental form of the contact structure.

Remark 4.21 This is a slightly different definition than the one in [48], where the second fundamental form is symmetrized.

We define the matrix $A_p = (h_{ij} - s_{ij})_{i,j=1}^{2n}$.

Proposition 4.22 *Let F be a normally transversal closed hypersurface. It then holds that*

$$\phi_k(F) = \binom{2n}{k} \sum_{\substack{p \\ T_pF = H_p}} |\det A_p|^{-1} D(A_p[2n-k], J[k]), \tag{8}$$

where D is the mixed discriminant.

Remark 4.23 We thus see that, geometrically, ϕ_k is reminiscent of the k^{th} elementary symmetric polynomial in the principal radii of a hypersurface in a Riemannian manifold. In particular, ϕ_{2n}^{-1} plays the role of the absolute value of the gaussian curvature. In fact, for $M = U$ the dual Heisenberg algebra, $\phi_{2n}(F)$ is precisely the inverse absolute value of the gaussian curvature, summed over all contact points.

Proof Write ω_i for the Ehresmann connection on SU, namely $\omega_i = \pi_V^*\theta_i$, where $\pi_V\colon T_{p,\xi}SU \to T_pM$ is the projection to the vertical tangent space. It then holds that $R^*\omega_i = \sum_{j=0}^{2n} h_{ij}\theta_j$, and $\nu^*\omega_i = \sum_{j=1}^{2n} s_{ij}\theta_j$. Recall that (s_{ij}) is symmetric. For notational simplicity, we write θ_j also for $\pi^*\theta_j \in \Omega^1(SU)$.

First we describe the normal cycle $NF = \{(x, \pm\nu(x)) : x \in F\} \subset SM$ near $(p,\xi) = X_0(p) \in M_H$ explicitly as a generalized form $\omega_{NF} := [\![NF]\!] \in \Omega_{-\infty}(SM)$. Let $W \subset \pi^{-1}U$ be a small neighborhood of (p,ξ). Write $\sigma_{SM} := \bigwedge_{i=0}^{2n}\theta_i \wedge \bigwedge_{i=1}^{2n}\omega_i$, and $\theta_F = \bigwedge_{i=1}^{2n}\theta_i \in \Omega^{2n}(F \cap U)$.

Claim *$\omega_{NF}^W := \omega_{NF}|_W \in \Omega_{-\infty}(W)$ is given by*

$$\omega_{NF} = \theta_0 \wedge \bigwedge_{i=1}^{2n}\Big(\omega_i - \sum_{j=1}^{2n} s_{ij}\theta_j\Big)\delta_{NF}^W,$$

where

$$\langle \delta_{NF}^W, \mu(x,\zeta)\sigma_{SM}\rangle = \int_{F\cap\pi(W)} \mu(x,\nu(x))\theta_F.$$

Proof of Claim Take $\psi \in \Omega_c^{2n}(W)$. We should check that

$$\int_{NF}\psi = \int_F \nu^*\left(\frac{\psi\wedge\theta_0\wedge\bigwedge_{i=1}^{2n}\big(\omega_i - \sum_{j=1}^{2n} s_{ij}\theta_j\big)}{\sigma_{SM}}\right)\theta_F,$$

which reduces to the pointwise verification

$$\frac{\nu^*\psi}{\theta_F} = \nu^*\left(\frac{\theta_0\wedge\psi\wedge\bigwedge_{i=1}^{2n}\big(\omega_i - \sum_{j=1}^{2n} s_{ij}\theta_j\big)}{\sigma_{SM}}\right).$$

We will check this equality for a basis of the $2n$–forms, which is given by $2n$ wedges of $(\theta_i)_{i=0}^{2n}$ and $(\omega_j)_{j=1}^{2n}$. If ψ contains a θ_0 factor, clearly both sides vanish.

Assume $\psi = \theta_1\wedge\cdots\wedge\theta_k\wedge\omega_{k+1}\wedge\cdots\wedge\omega_{2n}$. The left-hand side is easily seen to be equal $\det(s_{ij})_{i,j=k+1}^{2n}$. The right-hand side is

$$\nu^*\left(\frac{(-1)^{2n-k}\det(s_{ij})_{i,j=k+1}^{2n}\theta_0\wedge\cdots\wedge\theta_k\wedge\omega_{k+1}\wedge\cdots\wedge\omega_{2n}\wedge\omega_1\wedge\cdots\wedge\omega_k\wedge\theta_{k+1}\wedge\cdots\wedge\theta_{2n}}{\sigma_{SM}}\right)$$

and after reordering we get $(-1)^{2n-k+k(2n-k)}\det(s_{ij})_{i,j=k+1}^{2n} = \det(s_{ij})_{i,j=k+1}^{2n}$, concluding the proof of the claim. $\square$

We wish to apply $\widetilde{\eta}_k$ to ω_{NF}^W. Since NF intersects M_H transversally at the isolated point p, we need only look at the value of ω_{NF}^W at $(p, \pm X_0(p))$.

Recall the decomposition $T_{p,\xi}\mathbb{P}_M = T_{p,\xi}M_H \oplus T_\xi(\mathbb{P}_+(T_p^*M))$. We denote by $W_{p,\xi} = T_{p,\xi}M_H = T_pM$ the contact-horizontal subspace, and by $V_{p,\xi} = T_\xi(\mathbb{P}_+(T_p^*M)) = (H_p\otimes H_p^\perp)^*$ the vertical subspace. Using the Riemannian structure, we get a decomposition $T_{p,\xi}(SM) = W_{p,\xi}\oplus V_{p,\xi}$. The DH form $\omega \in \bigwedge^2 H_p^*\otimes T_pM/H_p$ can be recast as the isomorphism $\Omega\colon H_p\otimes H_p^\perp \to H_p^*$, that is, $\Omega\colon V_{p,\xi}^* \to H_p^*$.

We first need to identify the $(2n+1-k,k)$–component (with respect to the decomposition (6)) of ω_{NF}^W over $(p,\xi)\in M_H$. The forms θ_i vanish on the vertical subspace. The contact-horizontal component of ω_i is given by $R^*\omega_i=\sum_{j=0}^{2n}h_{ij}\theta_j$. We will write the resulting decomposition as $\omega_i=R^*\omega_i\oplus\omega_i^V$, where $R^*\omega_i$ is the contact-horizontal component, while ω_i^V, the restriction of ω_i to $V_{p,\xi}$, is the vertical component.

Define $(\gamma_{ij})_{i,j=1}^{2n}$ by $\Omega(\omega_i^V)=\sum_{j=1}^{2n}\gamma_{ij}\theta_j\in H_p^*$. Note that under the Euclidean identification, $V_{p,\xi}=H_p$ and $\omega_i^V=\theta_i$ for $i=1,\dots,2n$. Compute $\gamma_{ij}=\Omega(\theta_i)(X_j)=\omega(X_i,X_j)$. By assumption, X_j is a symplectic basis of H_p, so that γ_{ij} is the standard $2n\times 2n$ matrix J representing $\sqrt{-1}$.

Write $a_{ij}=h_{ij}-s_{ij}$ for $1\le i,j\le 2n$. Then

$$\omega_{NF}^W=\delta_{NF}^W\cdot\theta_0\wedge\bigwedge_{i=1}^{2n}\left(\omega_i^V+\sum_{j=1}^{2n}a_{ij}\theta_j\right),$$

so that

$$(\omega_{NF}^W)^{2n+1-k,k}=\delta_{NF}^W\cdot\theta_0\wedge\sum_{|I|=k}\epsilon_I\bigwedge\nolimits_{i\notin I}\left(\sum_{j=1}^{2n}a_{ij}\theta_j\right)\wedge\bigwedge\nolimits_{i\in I}\omega_i^V,$$

where $\epsilon_I=(-1)^{i_2-i_1+\cdots+i_{2j}-i_{2j-1}-j}$ is the sign of the permutation

$$\sigma_I=\binom{1\dots 2n}{I^c,I},$$

where $I=\{i_1<\cdots<i_k\}$, $I^c=\{1,\dots,2n\}\setminus I$ is ordered increasingly and $j=\lfloor\frac{1}{2}k\rfloor$.

Applying Ω to the last k factors and subsequently wedging all the factors to get a top form on T_pM, we get

$$\begin{aligned}\langle\tilde\eta_k,\omega_{NF}^W\rangle&=\int_{M_H}\theta_0\wedge\sum_{|I|=k}\epsilon_I\bigwedge\nolimits_{i\notin I}\left(\sum_{j=1}^{2n}a_{ij}\theta_j\right)\wedge\bigwedge\nolimits_{i\in I}\left(\sum_{j=1}^{2n}\gamma_{ij}\theta_j\right)\cdot\delta_{NF}^W\\&=\sum_{|I|=k}\sum_{\sigma\in S_{2n}}\operatorname{sign}\sigma\prod_{i\notin I}a_{i\sigma(i)}\prod_{i\in I}\gamma_{i\sigma(i)}\int_{M_H}\theta_0\wedge\cdots\wedge\theta_{2n}\cdot\delta_{NF}^W.\end{aligned}$$

Recall $A_p=(a_{ij})=h-S$, where $S=(s_{ij})$ and $h=(h_{ij})$ for $1\le i,j\le 2n$, and let B^α be the α–row of a matrix B. For an ordered subset $I\subset\{1,\dots,2n\}$, let $(A_p^{I^c},J^I)$ denote the matrix with the corresponding columns. Then

$$\langle\tilde\eta_k,\omega_{NF}^W\rangle=\sum_{|I|=k}\det(A_p^{I^c},J^I)\int_{M_H}\theta_0\wedge\cdots\wedge\theta_{2n}\cdot\delta_{NF}^W.$$

Define

$$B(\alpha) = \begin{cases} A_p & \text{if } \alpha \le 2n-k, \\ J & \text{if } \alpha > 2n-k. \end{cases}$$

The mixed discriminant is given by

$$\begin{aligned} D(A_p[2n-k], J[k]) &= \frac{1}{(2n)!} \sum_{\tau \in S_{2n}} \det(B(\tau(i))_{i,j}) \\ &= \frac{1}{(2n!)} \sum_{|I|=k} k!(2n-k)! \det(A_p^{I^c}, J^I), \end{aligned}$$

that is,

$$\langle \widetilde{\eta}_k, \omega_{NF}^W \rangle = \binom{2n}{k} D(A_p[2n-k], J[k]) \int_{M_H} \theta_0 \wedge \cdots \wedge \theta_{2n} \cdot \delta_{NF}^W.$$

Recall that we should fix a section of the orientation bundle of M over SM to get numerical values for the integral. Let us verify that for a choice of $\epsilon_\theta \in \pi^* o_M$ corresponding to $\theta_0 \wedge \cdots \wedge \theta_{2n}$ we get the identity

$$\int_{M_H} \theta_0 \wedge \cdots \wedge \theta_{2n} \cdot \delta_{NF}^W \otimes \epsilon_\theta = |\det A_p|^{-1}.$$

One can compute it directly, but we can do something simpler: Observe that $\langle \widetilde{\eta}_0, \omega_{NF}^W \rangle$ is just the intersection index $I_{p,\xi}$ of NF and M_H at (p, ξ). The order of intersection is not important as $\dim NF$ is even, while the orientations of M_H and NF are determined by ϵ_θ.

By what we have proved, we see that

$$I_{p,\xi}(NF, M_H) = \langle \widetilde{\eta}_0, \omega_{NF}^W \rangle = \det A_p \int_{M_H} \theta_0 \wedge \cdots \wedge \theta_{2n} \delta_{NF}^W.$$

It thus remains to verify that $I_{p,\xi}(NF, M_H) = \operatorname{sign} \det A_p$.

The positive orientation on $T_{p,\xi} SM$ is given by the dual basis $\theta_0, \dots, \theta_{2n}, \omega_1, \dots, \omega_{2n}$. To see that, consider a homotopy of the Riemannian metric between our metric and a flat one, and some corresponding homotopy of the orthonormal frame X_j. The dual basis above remains a basis throughout the homotopy, and clearly defines the positive orientation in the flat case.

Considering $\nu\colon F \to SM$ and $R\colon F \to SM$ as maps, we get a positive basis of $T_{p,\xi} M_H$ given by $D_p R(X_0), \dots, D_p R(X_{2n})$, and a positive basis of $T_{p,\xi} NF$ given by $D_p \nu(X_1), \dots, D_p \nu(X_{2n})$.

Form the $(4n+1)\times(4n+1)$ matrix

$$B=\begin{pmatrix} \theta_i(D_pR(X_j))_{i=0\dots 2n}^{j=0\dots 2n} & \theta_i(D_p\nu(X_j))_{i=0\dots 2n}^{j=1\dots 2n} \\ \omega_i(D_pR(X_j))_{i=1\dots 2n}^{j=0\dots 2n} & \omega_i(D_p\nu(X_j))_{i=1\dots 2n}^{j=1\dots 2n}\end{pmatrix}=\begin{pmatrix}\delta_{ij} & \delta_{ij}\\ h_{ij} & s_{ij}\end{pmatrix}.$$

By definition, $I_{p,\xi}(NF,M_H)=\operatorname{sign}\det B$. Now, for $1\le i\le 2n$, subtract column $2n+1+i$ from column $i+1$. Finally, interchange each column of index $2,\dots,2n+1$, with the respective column from the last $2n$ columns, resulting in $4n^2$ swaps. The resulting matrix is block-triangular, and has determinant $\det B=\det(h_{ij}-s_{ij})_{i=1\dots 2n}^{j=1\dots 2n}=\det A_p$, verifying our assertion.

Accounting also for the point $\xi=H^\perp$ with the opposite orientation, we conclude that

$$\langle\tilde{\eta}_k,\omega_{NF}\rangle=2\binom{2n}{k}\sum_{\substack{p\\ T_pF=H_p}}|\det A_p|^{-1}D(A_p[2n-k],J[k]),$$

and the statement follows. □

Definition 4.24 For a normally transversal contact point p of F, the local contact areas are

$$\phi_k(F,p):=\binom{2n}{k}|\det A_p|^{-1}D(A_p[2n-k],J[k]).$$

Corollary 4.25 *When $k=2n$, equation (8) remains valid also for a normally transversal hypersurface with boundary F.*

Proof By Lemma 4.11, $\phi_{2n}(F)$ is well defined. Using the notation of the proof of Lemma 4.11, we may write

$$\phi_{2n}(F)=\tfrac{1}{2}\langle\tilde{\eta}_{2n},[\![N^*_{\mathrm{int}}F]\!]\rangle+\tfrac{1}{2}\langle\tilde{\eta}_{2n},[\![N^*_{\partial F}F]\!]\rangle.$$

The first summand is computed in Proposition 4.22. The second summand trivially vanishes. □

Corollary 4.26 *For a normally transversal hypersurface with boundary F, we have $\phi_{2n}(F)\ge 0$, with equality if and only if it is nowhere tangent to H.*

Proof This is immediate from Corollary 4.25 and equation (8). □

It will be convenient to extend ϕ_{2n} to general hypersurfaces by setting $\phi_{2n}(F)=\infty$ when F is not normally transversal.

Next we establish the universality of ϕ_k with respect to embeddings.

Let (X^{2n+1}, H^X, ω) be a DH manifold, and $Y^{2m+1} \subset X^{2n+1}$ a DH submanifold. Let $i\colon Y \to X$ denote the embedding. We will write η_{2k}^X and ϕ_{2k}^X for the generalized forms and valuations defined over X, and similarly for Y.

Theorem 4.27 *It holds that $i^*\phi_k^X = \phi_k^Y$.*

Proof The following constructions appeared in [7], and we refer therein for more details. Consider the natural submersion $\beta\colon \mathbb{P}_X|_Y \setminus N^*Y \twoheadrightarrow \mathbb{P}_Y$ and the inclusion $\alpha\colon \mathbb{P}_X|_Y \hookrightarrow \mathbb{P}_X$. Let $\widetilde{\pi}\colon W \to \mathbb{P}_X|_Y$ be the oriented blow-up of $\mathbb{P}_X|_Y$ along the conormal bundle $N^*Y \subset \mathbb{P}_X$, and $\widetilde{\alpha}\colon W \to \mathbb{P}_X$ the corresponding lift. Let $\widetilde{\beta}\colon W \to \mathbb{P}_Y$ be induced by the restriction map $X \times_Y T^*Y \to T^*X$. For a valuation $\Psi \in \mathcal{V}^\infty(Y)$ defined by the closed Legendrian form $\psi \in \Omega^{2m+1}(\mathbb{P}_Y)$, $i^*\Psi$ is given by the current $\widetilde{\alpha}_*\widetilde{\beta}^*\psi$. Now $\langle \phi_k^Y, \Psi\rangle = \langle \widetilde{\eta}_k^Y, \psi\rangle$ and similarly for X. We should thus verify that

$$\langle \widetilde{\eta}_k^Y, \psi\rangle = \langle \widetilde{\eta}_k^X, \widetilde{\alpha}_*\widetilde{\beta}^*\psi\rangle.$$

Note that $\widetilde{\eta}_k^X$ is supported on $X_{H^X} \subset \mathbb{P}_X$, which by assumption is disjoint from N^*Y. Hence the right-hand side can be replaced by $\langle \widetilde{\eta}_k^X, \alpha_*\beta^*\psi\rangle$.

Next take $p \in Y$, $\xi = (H_p^Y)^\perp \in \mathbb{P}_Y$ and $(p, \xi') \in \mathbb{P}_X|_Y$ such that $\beta(p,\xi') = (p,\xi)$. We may decompose $T_{p,\xi'}\mathbb{P}_X|_Y = T_pY \oplus (H_p^X)^* \otimes (H_p^X)^{\perp *}$. Consider $d\beta\colon T_{p,\xi'}\mathbb{P}_X|_Y \to T_{p,\xi}\mathbb{P}_Y$, so that

$$d\beta^*\colon T_p^*Y \oplus H_p^Y \otimes (T_pY/H_p^Y)^* \to T_p^*Y \oplus H_p^X \otimes (T_pX/H_p^X)^*$$

acts as the identity on the first summand. On the second summand, recalling we have the canonical identification $T_pY/H_p^Y \simeq T_pX/H_p^X$ induced by the inclusion $T_pY \subset T_pX$, $d\beta$ simply acts as the inclusion $H_p^Y \hookrightarrow H_p^X$. Let us denote by $\Omega_k^Y\colon \bigwedge^k H_p^Y \otimes (H_p^Y)^\perp \to \bigwedge^k (H_p^Y)^*$ the isomorphism induced by the DH form ω, and similarly for X. It follows that the following diagram commutes:

$$\begin{tikzcd}
H_p^Y \otimes (H_p^Y)^\perp \arrow[r, "d\beta^*"] \arrow[d, "\Omega_Y"] & H_p^X \otimes (H_p^X)^\perp \arrow[d, "\Omega_X"] \\
(H_p^Y)^* & (H_p^X)^* \arrow[l, "\mathrm{res}"']
\end{tikzcd}$$

We are left with verifying the identity

$$\int_{Y_H} \psi_0 \otimes \epsilon_Y = \int_{X_H} \alpha_*\beta^*\psi_0 \otimes \epsilon_X$$

for an arbitrary Legendrian form $\psi_0 \in \Omega^{2m+1}(\mathbb{P}_Y)$. But this is now equivalent to the statement $i^*\phi_0^X = \phi_0^Y$, which holds as both sides are just the Euler characteristic. $\square$

Weyl's principle for DH manifolds, which we just established, is readily applicable in conjunction with the following technical lemma.

Lemma 4.28 *Consider a compact submanifold with boundary $F^{2k} \subset M^{2n+1}$ of a DH manifold (M, H, ω).*

(i) *If $2k \le n$, F lies in fact inside a DH submanifold $N^{2k+1} \subset M$.*

(ii) *For arbitrary $k < n$, one may find a pair of DH manifolds $N^{2k+1} \subset X^{4n+1}$ such that we get a commuting diagram of DH manifolds*

$$\begin{tikzcd} & N^{2k+1} \arrow[dr, hook] & \\ F^{2k} \arrow[ur, hook] \arrow[r, hook] & M^{2n+1} \arrow[r, hook] & X^{4n+1} \end{tikzcd}$$

where all inclusions are DH embeddings.

Proof Assume first $2k \le n$. Choose a Riemannian metric g on M, and let $L\colon F \to \mathbb{P}(TM)|_F$ be given by $L(x) = H_x^\perp$, the orthogonal complement with respect to g. Note that $L(F)$ does not intersect $\mathbb{P}(H)|_F \subset \mathbb{P}(TM)|_F$. By the transversality theorem, and since

$$\dim L(F) + \dim \mathbb{P}(TF) < \dim \mathbb{P}(TM)|_F \iff 2k + 4k - 1 < 2n + 2k \iff 2k \le n,$$

we may perturb g so that $L(F)$ avoids $\mathbb{P}(TF) \subset \mathbb{P}(TM)$. We then take N to be the image under the exponential map of a small neighborhood of the zero section in $TF \oplus L(F) \subset TM|_F$. It is clearly a DH submanifold containing F.

Now in the general case, consider $\widehat{M}_H \subset \mathbb{P}(T^*M)$, which is the quotient of $M_H \subset \mathbb{P}_M$ under the two-covering map $\mathbb{P}_M \to \mathbb{P}(T^*M)$. Define a DH structure on a neighborhood of $\widehat{M}_H$ as follows: the horizontal structure $\widehat{H}$ will be the canonical contact structure of $\mathbb{P}(T^*M)$. For $(x,\xi) \in \widehat{M}_H$, by equations (6) and (7), $\widehat{H}_{x,\xi}$ is canonically identified with $H_x \oplus H_x \subset T_xM \oplus H_x$. Noting that $T_{x,\xi}\mathbb{P}(T^*M)/\widehat{H}_{x,\xi} \simeq T_xM/H_x$, define $\omega_X \in \bigwedge^2 \widehat{H}^*_{x,\xi} \otimes T_{x,\xi}\mathbb{P}(T^*M)/\widehat{H}_{x,\xi}$ by

$$\omega_X((u_1,v_1),(u_2,v_2)) = \omega(u_1,v_1) + \omega(u_2,v_2) \quad \text{for all } (u_1,v_1),(u_2,v_2) \in \widehat{H}_{x,\xi}.$$

Extend ω_X arbitrarily to a global section of $\bigwedge^2 \widehat{H}^* \otimes T\mathbb{P}(T^*M)/\widehat{H}$. Take X to be a small neighborhood of $\widehat{M}_H$ in which ω_X is nondegenerate. Then $(X, \widehat{H}, \omega_X)$ is a DH manifold, and the DH submanifold $\widehat{M}_H \subset X$ is clearly isomorphic to M. It remains to note that $\dim X = 2 \cdot 2n + 1$ and $2k < 2n$, and so we may choose the desired $N^{2k+1} \subset X$ by the first case. $\square$

Corollary 4.29 *Let $F^{2k} \subset M$ be a submanifold with boundary, normally transversal to the horizontal distribution H. Then $\phi_{2k}(F) \geq 0$, with equality if and only if it is nowhere tangent to H.*

Proof This follows at once from Lemmas 4.28 and 4.13, Corollary 4.26 and Theorem 4.27. $\square$

Finally, we consider the relations between the valuations we constructed.

Proposition 4.30 (i) *On any DH manifold, $\phi_{2b} \in \mathcal{W}_{2b}^{-\infty} \setminus \mathcal{W}_{2b+1}^{-\infty}$ as $0 \leq b \leq n$. In particular, ϕ_{2b} for $0 \leq b \leq n$ are linearly independent valuations.*

(ii) *On a generic DH manifold* (in a sense to be made precise in the proof), $(\phi_b)_{b=0}^{2n}$ *are all linearly independent.*

Proof By Proposition 4.18, $\phi_k \in \mathcal{W}_k^{-\infty}(M)$. For a normally transversal hypersurface F write

$$\begin{aligned}\phi_k(F, p) &= |\det A_p|^{-1} D(A_p[2n-k], J[k]) \\ &= (-1)^k |\det A_p|^{-1} \sum_{j=0}^{2n-k} \binom{2n-k}{j} D(S_p[2n-k-j], h_p[j], J[k]).\end{aligned}$$

For even $k = 2b$, the term of highest order in S_p (corresponding to $j = 0$) is nonzero for generic S_p. In particular, $\phi_{2n}(F) \neq 0$ for generic hypersurfaces F. Now choose any DH submanifold $N \subset M$ of dimension $2b+1$. Using Theorem 4.27, Lemma 4.13 and the last observation, we may find a $2b$–dimensional submanifold $F^{2b} \subset N$ such that $\phi_{2b}^M(F) = \phi_{2b}^N(F) \neq 0$. Thus $\phi_{2b} \notin \mathcal{W}_{2b+1}^{-\infty}(M)$, proving (i).

For (ii), it now suffices to check that ϕ_{2b-1} and ϕ_{2b} are linearly independent in $\mathcal{W}_{2b}^{-\infty}(M)/\mathcal{W}_{2b+1}^{-\infty}(M)$ for a generic M. In turn this is implied by the following statement: as a function on normally transversal closed submanifolds of dimension $2b$ lying inside a fixed DH submanifold $N^{2b+1} \subset M$, ϕ_{2b-1} and ϕ_{2b} are linearly independent. By Theorem 4.27, we may assume $b = n$. Now note that for ϕ_{2n-1}, the summand of degree 1 in S_p vanishes, as S_p is symmetric while J is antisymmetric. The statement now follows by examining the summand whose numerator contains no S_p for both valuations, which has different coefficients (depending on p for a general horizontal distribution). $\square$

4.3 Extending ϕ_{2k} to arbitrary $2k$–submanifolds

Define the contact area $\mathrm{CA}_{2k}(F) \in [0,\infty]$ for any C^2, $2k$–dimensional submanifold with boundary $F \subset M$ by

$$\mathrm{CA}_{2k}(F) := \liminf_{F_\epsilon \to F} \phi_{2k}(F_\epsilon),$$

where F_ϵ is a C^2, normally transversal submanifold with boundary that C^2–converges to F. Note that Proposition 4.22 and Theorem 4.27, and the proof of Lemma 4.28, imply that $\phi_{2k}(F)$ is C^2–continuous on normally transversal submanifolds F. Hence, on such submanifolds, $\mathrm{CA}_{2k}(F) = \phi_{2k}(F)$.

We now show that the vanishing of $\mathrm{CA}_{2k}(F^{2k})$ is a necessary condition for the existence of an arbitrarily small perturbation with no contact points. For closed F this supplements the topological necessary condition $\chi(F) = 0$.

Proposition 4.31 *For a $2k$–dimensional submanifold with boundary $F \subset M$, we have $\mathrm{CA}_{2k}(F^{2k}) = 0$ if and only if there is an arbitrarily small C^2–perturbation of F which is nowhere tangent to the horizontal structure.*

Proof The "if" direction follows from Corollary 4.29. For the other direction, let us assume $\mathrm{CA}_{2k}(F) = 0$. Again by Corollary 4.29, F cannot be normally transversal to the horizontal distribution unless it is nowhere tangent to it. Assuming the contrary to the assertion, we conclude F is not normally transversal. Let $F_\epsilon \to F$ for $\epsilon \to 0$ be a normally transversal family of smooth perturbations of F such that $\phi_{2k}(F_\epsilon) \to 0$. If a normally transversal contact point exists for F (which is then necessarily isolated), it persists to F_ϵ for small ϵ, and examining equation (8), one can find $c > 0$ such that $\phi_{2k}(F_\epsilon) \geq c$ for all small ϵ, a contradiction. Thus F is not normally transversal to the horizontal distribution at any of its contact points. It follows that a sequence x_j of contact points of F_{ϵ_j} must approach a necessarily degenerate tangency point, and hence, by equation (8), $\phi_{2k}(F_{\epsilon_j}) \to \infty$. This again is a contradiction, implying that for small ϵ, F_ϵ has no horizontal tangent spaces, as claimed. $\square$

5 The dual Heisenberg Lie algebra

5.1 Linear algebra

Recall that U^{2n+1} denotes the dual Heisenberg algebra, with $H \subset U$ a fixed linear hyperplane, and $\omega \in \bigwedge^2 H^* \otimes U/H$ is a nondegenerate (twisted) form. It is the simplest DH manifold.

Let the group of automorphisms of the Heisenberg algebra be denoted by $\mathrm{Sp}_H(U)$ or $\mathrm{Sp}_H(2n+1)$. There is an $\mathrm{Sp}_H(U)$–equivariant isomorphism $\mathrm{Dens}(H) \simeq \mathrm{Dens}(U/H)^n$.

Define the subgroups

$$\begin{aligned} \mathrm{Sp}_H^+(U) &= \{g \in \mathrm{Sp}_H(U) : g|_{U/H} > 0\} \subset \mathrm{Sp}_H(U), \\ \mathrm{Sp}_H^1(U) &= \{g \in \mathrm{Sp}_H(U) : g|_{U/H} = \mathrm{Id}\} \subset \mathrm{Sp}_H^+(U), \\ \mathrm{Scal}_H(U) &= \{g \in \mathrm{Sp}_H(U) : g|_H \in \mathbb{R}^*\} \subset \mathrm{Sp}_H^+(U). \end{aligned}$$

Note that for $g \in \mathrm{Sp}_H^1(U)$, clearly $\det g = 1$. For an $\mathrm{Sp}_H(U)$–module X and $x \in X$, we will write $\mathrm{Stab}(x)$, $\mathrm{Stab}^+(x)$ and $\mathrm{Stab}^1(x)$ for its stabilizer in the corresponding subgroup. Let us record the following trivial fact:

Lemma 5.1 $\mathrm{Sp}_H^+(U)$ *is generated by* $\mathrm{Sp}_H^1(U)$ *and* $\mathrm{Scal}_H(U)$*.*

Corollary 5.2 *Let* $\delta_\lambda \in \mathrm{Scal}_H(U)$ *act by* $\lambda \in \mathbb{R}$ *on* H *and by* λ^2 *on a fixed direction* L *complementing* H*. Then* $\mathrm{Sp}_H^1(U)$ *and* $(\delta_\lambda)_{\lambda \neq 0}$ *generate* $\mathrm{Sp}_H^+(U)$*.*

Proof Take any $g \in \mathrm{Scal}_H(U)$; set $\lambda := g|_H$. Then $\delta_\lambda^{-1} \circ g \in \mathrm{Sp}_H^1$. The claim now follows from Lemma 5.1. □

Lemma 5.3 *For any subspace* $E \subset H$*, set* $E_0 = E \cap E^\omega$*. Then one can find:*

(i) $S \in \mathrm{Stab}^1(E)$ *such that* $S(E_0) = E_0$ *and* $S|_{E_0} = 2$*.*

(ii) $T \in \mathrm{Stab}^+(E)$ *such that* $T(E_0) = E_0$*,* $T|_{E_0} = 1$ *and* $\det T < 1$*.*

Proof (i) Decompose $E \cap H = E_0 \oplus F$, where $F \subset H$ is a nondegenerate subspace. Fix a vector $z \in E \setminus H$. There is then an induced symplectic form ω_z on H given by $\omega_z(u,v)(z+H) = \omega(u,v)$. Define $S \in \mathrm{GL}(E)$ by setting $S|_{E_0} = 2$ and $S|_F = 1$, and note that S leaves $\omega_z|_E$ invariant. By Witt's extension theorem, we can find an extension $S \in \mathrm{Sp}(H, \omega_z)$, and finally setting $S(z) = z$ yields $S \in \mathrm{Stab}^1(E)$, as required.

(ii) Using S from (i), define $T \in \mathrm{Sp}_H^+(U)$ by $T|_H = \frac{1}{2} S|_H$ and $Tz = \frac{1}{4} z$. □

The orbits $Y^k_{\epsilon,r}$ of $\mathrm{Sp}^+_H(U)$ in $\mathrm{Gr}_k(U)$ are classified by the pairs (ϵ, r) as follows:

$$E \in Y^k_{\epsilon,r} \quad \Longleftrightarrow \quad \epsilon = k - \dim H \cap E \in \{0,1\} \text{ and } 2r = \kappa - \dim\ker(\omega|_{E\cap H}),$$

where $\epsilon \in \{0,1\}$, and $r \in \{0,\dots,\lfloor \frac{1}{2}\kappa \rfloor\}$, $\kappa = \min(k-\epsilon, 2n-(k-\epsilon))$. The unique open orbit has $\epsilon = 1$ and $r = \lfloor \frac{1}{2}\kappa \rfloor$; the unique closed orbit has $\epsilon = 0$ and $r = 0$.

5.2 Translation-invariant valuations on the dual Heisenberg algebra

We start by classifying the $\mathrm{Sp}^+_H(U)$–invariant Klain sections.

Proposition 5.4 *Fix $1 \le k \le 2n$.*

(i) *For even k, there is at most a one-dimensional space of $\mathrm{Sp}^+_H(U)$–invariant generalized Klain sections over $\mathrm{Gr}_k(U)$.*

(ii) *For odd k, there are no $\mathrm{Sp}^+_H(U)$–invariant generalized Klain sections over $\mathrm{Gr}_k(U)$.*

Proof We will make repeated use of Lemma B.1 without explicit mention, wherein also the bundle F^α_Y of principal symbols transversal to Y is defined.

Take $E \in Y^k_{\epsilon,r}$, and denote $E_0 = (E\cap H) \cap (E\cap H)^\omega$, $\dim E_0 = \kappa - 2r$.

Step 1 $(\epsilon = 1)$ We consider first the open orbit, $r = r_{\max}$. If k is odd, $E \cap H$ is nondegenerate, and ω gives an isomorphism $\mathrm{Dens}(E\cap H) = \mathrm{Dens}(U/H)^{(k-1)/2}$. Thus

$$\begin{aligned}\mathrm{Dens}(E) &= \mathrm{Dens}(E/E\cap H) \otimes \mathrm{Dens}(E\cap H) = \mathrm{Dens}(U/H) \otimes \mathrm{Dens}(U/H)^{(k-1)/2} \\ &= \mathrm{Dens}(U/H)^{(k+1)/2} = \mathrm{Dens}(U)^{(k+1)/2(n+1)}.\end{aligned}$$

It follows that for odd k there are no invariant sections on the open orbit.

For even k, let $E_0 \subset E \cap H$ be the kernel of $\omega|_{E\cap H}$, which is a line. Then

$$\begin{aligned}\mathrm{Dens}(E) &= \mathrm{Dens}(E/E\cap H) \otimes \mathrm{Dens}(E\cap H) \\ &= \mathrm{Dens}(U/H) \otimes \mathrm{Dens}(E_0) \otimes \mathrm{Dens}((E\cap H)/E_0) \\ &= \mathrm{Dens}(U/H)^{k/2} \otimes \mathrm{Dens}(E_0) = \mathrm{Dens}(U)^{k/2(n+1)} \otimes \mathrm{Dens}(E_0).\end{aligned}$$

By Lemma 5.3, we may find $S \in \mathrm{Stab}^1(E)$ acting trivially on the first factor and rescaling the second factor. It follows that there are no invariant sections on the open orbit $Y^k_{1,r_{\max}}$.

Now let us consider the orbit $Y = Y^k_{1,r}$ with $r < r_{\max}$. We have

$$N_E Y = N_{E\cap H} X^{k-1}_r(H) = \textstyle\bigwedge^2 E_0^*.$$

Consider the bundle over Y with fiber

$$\begin{aligned} F^\alpha_Y|_E &= \mathrm{Dens}(E) \otimes \mathrm{Dens}^*(N_E Y) \otimes \mathrm{Sym}^\alpha(N_E Y) \\ &= \mathrm{Dens}(E) \otimes \mathrm{Dens}\big(\textstyle\bigwedge^2 E_0\big) \otimes \mathrm{Sym}^\alpha\big(\textstyle\bigwedge^* E_0\big). \end{aligned}$$

Since $E/E\cap H$ is $\mathrm{Stab}(E)$–isomorphic to U/H,

$$\begin{aligned} \mathrm{Dens}(E) &= \mathrm{Dens}(E\cap H) \otimes \mathrm{Dens}(U/H) \\ &= \mathrm{Dens}((E\cap H)/E_0) \otimes \mathrm{Dens}(E_0) \otimes \mathrm{Dens}(U/H). \end{aligned}$$

Now $\dim(E\cap H)/E_0 = 2r$, and ω readily yields a nondegenerate form

$$\widetilde{\omega} \in \textstyle\bigwedge^2((E\cap H)/E_0)^* \otimes U/H,$$

so that there is a $\mathrm{Stab}(E)$–isomorphism $\mathrm{Dens}((E\cap H)/E_0) = \mathrm{Dens}(U/H)^r$. Thus

$$F^\alpha_Y|_E = \mathrm{Dens}(U/H)^{r+1} \otimes \mathrm{Dens}(E_0) \otimes \mathrm{Dens}\big(\textstyle\bigwedge^2 E_0\big) \otimes \mathrm{Sym}^\alpha\big(\textstyle\bigwedge^2 E_0^*\big).$$

Again by Lemma 5.3, there are no $\mathrm{Stab}^+(E)$–invariants in $F^\alpha_Y|_E$ when $r < r_{\max}$ (and so $E_0 \neq \{0\}$).

We conclude that for no k are there invariant generalized sections whose support intersects $Y^k_{1,r}$ for any r. We assume from now on that $\epsilon = 0$, so $E \subset H$.

Step 2 Consider $Y = Y^k_{0,r}$ with $r = r_{\max}$. Then $N_E Y = E^* \otimes U/H$, and

$$F^0_Y|_E = \mathrm{Dens}(E) \otimes \mathrm{Dens}^*(N_E Y) = \mathrm{Dens}(E)^2 \otimes \mathrm{Dens}^*(U/H)^k.$$

Taking $g \in \mathrm{Scal}_H(U)$ with $g|_H = \lambda$, we see that it acts on $F^0_E \otimes \mathrm{Sym}^\alpha(N_E Y) = \mathrm{Dens}(E)^2 \otimes \mathrm{Dens}^*(U/H)^k \otimes (U/H)^\alpha \otimes \mathrm{Sym}^\alpha E^*$ by $\lambda^{-2k+2k+2\alpha-\alpha} = \lambda^\alpha$. Thus $\alpha = 0$ is the only possible transversal order of an invariant section. We now consider separately the different parities of k.

If k is odd, set $E_0 = E \cap E^\omega$, so $\dim E_0 = 1$. Then

$$\begin{aligned} F^0_E &= \mathrm{Dens}(E_0)^2 \otimes \mathrm{Dens}(E/E_0)^2 \otimes \mathrm{Dens}^*(U)^{k/(n+1)} \\ &= \mathrm{Dens}(E_0)^2 \otimes \mathrm{Dens}^*(U)^{(k+1)/2(n+1)} \end{aligned}$$

since $\mathrm{Dens}(E/E_0)^2 \simeq \mathrm{Dens}(U/H)^{k-1} = \mathrm{Dens}(U)^{(k-1)/(n+1)}$. Thus, by Lemma 5.3, the action of $\mathrm{Stab}^+(E)$ on F^0_E is clearly nontrivial, and so there are no invariant sections whose support intersects Y.

If k is even, the restriction of ω to E gives an isomorphism $\mathrm{Dens}(E)=\mathrm{Dens}(U/H)^{k/2}$, so that $\mathrm{Stab}(E)$ acts trivially on F_E^0. We know by now that all invariant sections are supported inside $\overline{Y}$. Thus we conclude that the space of restrictions of the space of invariant sections to $\mathrm{Gr}_k(U)\setminus(\overline{Y}\setminus Y)$ is at most one-dimensional.

Step 3 It remains to show there are no invariant sections supported on the closure of either of the orbits $Y_{0,r}^k(U)$ with $r<r_{\max}$ for any k. In particular, we have $1<k<2n$ and $\kappa-2r=\dim E_0\geq 2$, with $\kappa=\min(k,2n-k)$.

One has the chain of inclusions $T_E Y_{0,r}^k(U)=T_E X_r^k(H)\subset T_E\,\mathrm{Gr}_k(H)\subset T_E\,\mathrm{Gr}_k(U)$. Hence $N_E Y_{0,r}^k(U)$ fits into the exact sequence

$$0\to T_E\,\mathrm{Gr}_k(H)/T_E X_r^k(H)\to N_E Y_{0,r}^k(U)\to T_E\,\mathrm{Gr}_k(U)/T_E\,\mathrm{Gr}_k(H)\to 0,$$

which is $\mathrm{Stab}(E)$–isomorphic to

$$0\to \textstyle\bigwedge^2 E_0^*\to N_E Y_{0,r}^k(U)\to E^*\otimes U/H\to 0.$$

Write $Y=Y_{0,r}^k$. For $\alpha=0$,

$$\begin{aligned}
F_Y^0|_E &= \mathrm{Dens}(E)\otimes \mathrm{Dens}^*(N_E Y_{0,r}^k(U))\\
&= \mathrm{Dens}(E_0)^2\otimes \mathrm{Dens}(E/E_0)^2\otimes \mathrm{Dens}^*(U/H)^k\otimes \mathrm{Dens}^*\big(\textstyle\bigwedge^2 E_0^*\big)\\
&= \mathrm{Dens}(E_0)^2\otimes \mathrm{Dens}^*(U/H)^{k-r}\otimes \mathrm{Dens}^*\big(\textstyle\bigwedge^2 E_0^*\big).
\end{aligned}$$

Thus $(F_Y^0|_E)^{\mathrm{Stab}^+(E)}=\{0\}$ since one can find $g\in\mathrm{Stab}^1(E)$ with $g|_{E_0}=\lambda\neq 1$.

When $\alpha>0$, a $\mathrm{Stab}^+(E)$–invariant element of $F_Y^\alpha|_E$ would imply the existence of an invariant element in

$$F_Y^0|_E\otimes\big(\textstyle\bigwedge^2 E_0^*\big)^a\otimes(E^*)^b\otimes(U/H)^b,$$

which in turn implies the existence of an invariant element in

$$\mathrm{Dens}(E_0)^2\otimes \mathrm{Dens}\big(\textstyle\bigwedge^2 E_0\big)\otimes\big(\textstyle\bigwedge^2 E_0^*\big)^a\otimes(E_0^*)^{b'}\otimes((E/E_0)^*)^{b''}\otimes\mathrm{Dens}(U)^\lambda$$

for some nonnegative integers a, b, b' and b'' and $\lambda\in\mathbb{R}$.

By the proof of Lemma 5.3, we can find $S\in\mathrm{Stab}^1(E)$ such that $S|_{E_0}=2$ and $S\colon E/E_0\to E/E_0$ is the identity. Thus there are no invariants in this space. □

For a vector bundle E over a manifold B, we let $\Gamma_m(B,E):=\Gamma_c(B,E^*\otimes|\omega_B|)^*$ denote the space of generalized sections that are given locally by a regular Borel measure.

Proposition 5.5 *There are $\mathrm{Sp}_H^+(U)$–invariant generalized Klain sections κ_k realizing the upper bounds obtained in Proposition 5.4. Moreover, $\kappa_k\in\Gamma_m(\mathrm{Gr}_k(U),\mathrm{Dens}(E))$, they are supported on $Y_H^k:=\{E\subset H\}$ and are $\mathrm{Sp}_H(U)$–invariant.*

Proof Take $f(E)=|\omega^{\wedge k/2}|^{\otimes 2}\in\mathrm{Dens}(E)^2\otimes\mathrm{Dens}^*(U/H)^k$, which is a continuous section over $\mathrm{Gr}_k(H)$. We may rewrite f as an absolutely continuous measure on $\mathrm{Gr}_k(H)$ with values in the bundle $\mathrm{Dens}^2(E)\otimes\mathrm{Dens}^*(U/H)^k\otimes\mathrm{Dens}^*(T_E\,\mathrm{Gr}_k(H))=\mathrm{Dens}^*(U)^k\otimes\mathrm{Dens}(E)^{2n+2}$. Writing $i\colon\mathrm{Gr}_k(H)\to\mathrm{Gr}_k(U)$ for the natural embedding, we get

$$\begin{aligned}i_*f\in\mathcal{M}_{\mathrm{Gr}_k(H)}&(\mathrm{Gr}_k(U),\mathrm{Dens}^*(U)^k\otimes\mathrm{Dens}(E)^{2n+2})\\ &\simeq\Gamma_m\big(\mathrm{Gr}_k(U),\mathrm{Dens}^*(U)^k\otimes\mathrm{Dens}(E)^{2n+2}\otimes\mathrm{Dens}(T_E\mathrm{Gr}_k(U))\big).\end{aligned}$$

The latter bundle is just the Klain bundle:

$$\mathrm{Dens}^*(U)^k\otimes\mathrm{Dens}(E)^{2n+2}\otimes\mathrm{Dens}(E^*\otimes U/E)=\mathrm{Dens}(E),$$

and it remains to note that we only used $\mathrm{Sp}_H(U)$–equivariant identifications. □

Let us fix an involution $R\in\mathrm{Sp}_H(U)$ acting by -1 on U/H. Since $R^{-1}\,\mathrm{Sp}_H^+(U)R=\mathrm{Sp}_H^+(U)$, it follows that R acts on the space of $\mathrm{Sp}_H^+(U)$–invariants in any $\mathrm{Sp}_H(U)$–module M. We will call the ±1 eigenspaces of R in $M^{\mathrm{Sp}_H^+(U)}$ R–even and R–odd.

Proposition 5.6 *For $0\le k\le n$ it holds that*

$$\dim\mathrm{Val}_{2k}^{-\infty}(U)^{\mathrm{Sp}_H^+(U)}\le 2\quad\textit{and}\quad\dim\mathrm{Val}_{2k+1}^{-\infty}(U)^{\mathrm{Sp}_H^+(U)}=0.$$

For every k, the spaces of R–even and R–odd invariants are each at most one-dimensional. Moreover, any R–even invariant valuation is even, and any R–odd invariant valuation is odd.

In particular, any $\mathrm{Sp}_H(U)$–invariant valuation must be even.

Proof We proceed in four steps.

Step 0 In the following, all forms are translation-invariant. For $\xi\in\mathbb{P}_+(U^*)$, let $\xi_\perp\subset U$ be its annihilator. There is a natural identification of $\Omega_{-\infty}^{j,2n+1-j}(U\times\mathbb{P}_+(U^*))^{\mathrm{tr}}$ with the generalized sections of the bundle over $\mathbb{P}_+(U^*)$ with fiber

$$\textstyle\bigwedge^j U^*\otimes\bigwedge^{2n+1-j}\xi_\perp\otimes\xi^{2n+1-j}$$

over ξ. A Legendrian form corresponds to a section of the subbundle

$$\xi\otimes\bigwedge^{j-1}(U^*/\xi)\otimes\bigwedge^{2n+1-j}\xi_\perp\otimes\xi^{2n+1-j},$$

which we then call a Legendrian section. We will find the $\mathrm{Sp}_H^+(U)$–invariant generalized Legendrian sections. Let $\psi(\xi)$ be such a section.

There are three orbits under $\mathrm{Sp}_H^+(U)$: the open orbit $X_o=\{\xi\neq H^\perp\}$ and two closed orbits $X_c^\pm=\{\pm H^\perp\}$.

Step 1 Let us first show that ψ vanishes when restricted to the open orbit. Note that $\psi|_{X_o}$ is smooth by $\mathrm{Sp}_H^+(U)$–invariance, and fix $\xi\in X_o$.

Consider the stabilizer $G_\xi:=\mathrm{Stab}^+(\xi)\subset\mathrm{Sp}_H^+(U)$, and $G_\xi^1=\{g\in G_\xi : g|_\xi=1\}$. Then $\psi(\xi)$ is a G_ξ–invariant element in $\bigwedge^{j-1}(U^*/\xi)\otimes\bigwedge^{2n+1-j}\xi_\perp\otimes\xi^{2n+2-j}$. By considering the various invariant subquotients, we deduce the existence of an invariant element in one of the spaces

$$V_1=(H^\perp\oplus\xi)/\xi\otimes\bigwedge^{j-2}(U^*/(\xi\oplus H^\perp))\otimes\bigwedge^{2n+1-j}(\xi_\perp\cap H)\otimes\xi^{2n+2-j},$$

$$V_2=(H^\perp\oplus\xi)/\xi\otimes\bigwedge^{j-2}(U^*/(\xi\oplus H^\perp))\otimes\bigwedge^{2n-j}(\xi_\perp\cap H)\otimes(\xi_\perp/\xi_\perp\cap H)\otimes\xi^{2n+2-j},$$

$$V_3=\bigwedge^{j-1}(U^*/(\xi\oplus H^\perp))\otimes\bigwedge^{2n+1-j}(\xi_\perp\cap H)\otimes\xi^{2n+2-j},$$

$$V_4=\bigwedge^{j-1}(U^*/(\xi\oplus H^\perp))\otimes\bigwedge^{2n-j}(\xi_\perp\cap H)\otimes(\xi_\perp/\xi_\perp\cap H)\otimes\xi^{2n+2-j}.$$

Now take $g_\lambda\in G_\xi$ such that $g_\lambda|_H=\lambda$. Since g_λ has a $2n$–dimensional eigenspace of eigenvalue λ on U, the action of g_λ on U^*, which is by $(g_\lambda^{-1})^*$, has a $2n$–dimensional invariant subspace of eigenvalue λ^{-1}. In the following we will simply write g_λ for this action. We may choose g_λ such that $g_\lambda|_\xi=\lambda^{-1}$ and $g_\lambda|_{H^\perp}=\lambda^{-2}$. Since $\xi_\perp/(\xi_\perp\cap H)\simeq U/H$, g_λ acts on it by λ^2.

The action of g_λ on V_i is as follows: $g_\lambda|_{V_1}=\lambda^{-2-(j-2)+2n+1-j-(2n+2-j)}=\lambda^{-j-1}$, $g_\lambda|_{V_2}=\lambda^{-j}$, $g_\lambda|_{V_3}=\lambda^{-(j-1)+2n+1-j-(2n+2-j)}=\lambda^{-j}$ and $g_\lambda|_{V_4}=\lambda^{-j+1}$.

Since $j\geq 1$, we conclude that a G_ξ–invariant can only exist in V_4, and only if $j=1$. However, $\bigwedge^{2n}\xi_\perp\otimes\xi^{2n+1}$ has no G_ξ–invariants: one may choose an element $g\in G_\xi$ with $g|_\xi=1$, while $\det g\neq 1$. Then $\det(g\colon \xi_\perp\to\xi_\perp)\neq 1$, and thus no such invariant exists.

Step 2 We conclude that ψ is supported on $X_c:=\{\pm H^\perp\}$. Since $-\mathrm{Id}$ commutes with $\mathrm{Sp}_H^+(U)$, it acts on any space of $\mathrm{Sp}_H^+(U)$–invariants, which thus decomposes into a sum of even and odd invariants. Now $\det(-\mathrm{Id})=(-1)^{2n+1}=-1$, so that odd closed Legendrian forms correspond to even valuations and vice versa.

Let us show that for odd j there are no $\mathrm{Sp}_H^+(U)$–invariant closed Legendrian forms. Otherwise by what we proved, there is an $\mathrm{Sp}_H^+(U)$–invariant such form T supported on $\xi = H^\perp$ with a fixed orientation. Then $T_s := T - (-\mathrm{Id})^* T$ is odd, closed, Legendrian and $\mathrm{Sp}_H^+(U)$–invariant. Thus T_s defines an even $\mathrm{Sp}_H^+(U)$–invariant generalized valuation which is j–homogeneous. This contradicts Proposition 5.4.

Step 3 The principal symbol of ψ, denoted by $F^\alpha_{X_c}|_\xi$ (see Appendix B), over each point $\xi \in X_c$ is an element of

$$\begin{aligned} W &:= H^\perp \otimes \textstyle\bigwedge^{j-1} H^* \otimes \bigwedge^{2n+1-j} H \otimes (H^\perp)^{2n+1-j} \otimes \mathrm{Dens}^*(T_\xi \mathbb{P}_+(U^*)) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \otimes \mathrm{Sym}^\alpha(T_\xi \mathbb{P}_+(U^*)) \\ &= \textstyle\bigwedge^{j-1} H^* \otimes \bigwedge^{2n+1-j} H \otimes (H^\perp)^{2n+2-j-\alpha} \otimes \mathrm{Dens}(H) \otimes \mathrm{Dens}(H^\perp)^{2n} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \otimes \mathrm{Sym}^\alpha(H^*). \end{aligned}$$

Take $\delta_\lambda \in \mathrm{Scal}_H(U)$ acting by $\lambda \in \mathbb{R}$ on H and by λ^2 on some fixed vector $w \in U \setminus H$. Then δ_λ acts on $H^\perp$ by λ^{-2}, and

$$\delta_\lambda|_W = \lambda^{-(j-1)} \lambda^{2n+1-j} \lambda^{-2(2n+2-j-\alpha)} \lambda^{-2n} \lambda^{4n} \lambda^{-\alpha} = \lambda^{\alpha-2},$$

hence $\alpha = 2$ is necessary for an invariant to exist.

We may identify $\bigwedge^{2n+1-j} H \simeq \bigwedge^{j-1} H^* \otimes \bigwedge^{2n} H$. Thus

$$\begin{aligned} W = \textstyle\bigwedge^{j-1} H^* \otimes \bigwedge^{j-1} H^* \otimes \mathrm{Sym}^2(H^*) \otimes \bigwedge^{2n} H \otimes (H^\perp)^{2n-j} \otimes \mathrm{Dens}(H) \\ \otimes \mathrm{Dens}(H^\perp)^{2n}. \end{aligned}$$

Let us find all $w \in W$ that are invariant under $\mathrm{Sp}_H^1(U)$. We may fix an $\mathrm{Sp}_H^1(U)$–invariant symplectic form ω_H on H, so that, $\mathrm{Sp}_H^1(U)$–equivariantly,

$$W \simeq \textstyle\bigwedge^{j-1} H^* \otimes \bigwedge^{j-1} H^* \otimes \mathrm{Sym}^2(H^*).$$

By the fundamental theorem of invariant theory, an invariant element of W is given by fixing pairings of all the factors using ω, and then symmetrizing/antisymmetrizing accordingly (some pairings could give zero). There would be j pairings.

Note that $R^*\omega_H = -\omega_H$. Hence $\det R|_H = (-1)^n$ and $\det R = (-1)^{n+1}$. We see that $Rw = (-1)^j (-1)^n (-1)^j = (-1)^n w$ for any $w \in W^{\mathrm{Sp}_H^1(U)}$.

Step 4 Now take $\phi \in \mathrm{Val}_j^{-\infty}(U)^{\mathrm{Sp}_H^+(U)}$ which is either even or odd, and write $T = T(\phi)$. Assume first that ϕ is R–even, that is, $\mathrm{Sp}_H(U)$–invariant. Then $R^*T = \det \cdot T$. Thus its principal symbol is $\sigma(T)(+H^\perp, -H^\perp) = (w_0, -w_0)$ for some $w_0 \in W^{\mathrm{Sp}_H^1(U)}$. Noting that $-\mathrm{Id}$ acts by $(-1)^j$ on W, it follows that ϕ is an even valuation if and only if j is even, which we may assume by Step 2. By Proposition 5.4, $\dim \mathrm{Val}_j^{-\infty}(U)^{\mathrm{Sp}_H(U)} \le 1$.

Similarly if ϕ is R–odd, its principal symbol is $\sigma(T) = (w_0, w_0)$ for some $w_0 \in W^{\mathrm{Sp}_H^1(U)}$, and since j is even, ϕ must be odd. It remains to prove the dimension of the space of R–odd, j–homogeneous valuations is at most 1. It suffices to show by Theorem 3.5 that the space of $\mathrm{Sp}_H^+(U)$–invariant, R–odd, j–homogeneous, closed Legendrian generalized forms supported on X_c is at most 1–dimensional. Any such form has the form $T_+ - R^*T_+$ with T_+ supported on $+H^\perp$ and $\mathrm{Sp}_H^+(U)$–invariant. Clearly $T_+ - R^*T_+ \mapsto T_+ + R^*T_+$ is a bijection onto the corresponding R–even forms, which in turn bijectively correspond to $\mathrm{Sp}_H(U)$–invariant valuations. The R–even case now concludes the proof. □

The following proposition completes the proof of Theorems 1.5 and 1.6.

Proposition 5.7 *For* $0 \le k \le n$, *it holds that* $\dim \mathrm{Val}_{2k}^{-\infty}(U)^{\mathrm{Sp}_H(U)} = 1$ *while* $\dim \mathrm{Val}_{2k}^{-\infty}(U)^{\mathrm{Sp}_H^+(U)} = 2$.

Proof The first assertion follows from the general construction of valuations on DH manifolds, as follows. By Proposition 4.30, we have at least $n+1$ linearly independent valuations $\phi_{2j} \in \mathcal{W}_{2j}^{-\infty}(U)$ which are invariant under the symmetries of U as a DH manifold, in particular under translations and $\mathrm{Sp}_H(U)$. Now Proposition 5.6 concludes the proof.

The second assertion follows from the first one: by the last paragraph of the proof of Proposition 5.6, there is a bijection between the $\mathrm{Sp}_H(U)$–invariant valuations and the R–odd $\mathrm{Sp}_H^+(U)$–invariant valuations. □

Remark 5.8 (1) Alternatively, one may use the Corfton valuations ψ_{2j} on the contact sphere constructed in Section 8 to prove the statement, by considering the translation-invariant valuations they define on the tangent spaces by Theorem 3.13.

(2) Examining the construction of ϕ_{2k} for general DH manifolds, we immediately see that ϕ_{2k}^U is in fact $2k$–homogeneous.

Example 5.9 Let us describe explicitly the valuation ϕ_{2n}. Consider $U = \mathbb{R}^{2n+1}$ with the standard Euclidean structure, and take $H = \{x_{2n+1} = 0\}$. For $u, v \in H$ we let $\omega(u,v) = \sum_{j=1}^n (u_{2j-1}v_{2j} - u_{2j}v_{2j-1})$ be the standard symplectic form. Identifying $U/H = H^\perp = \mathbb{R}e_{2n+1} = \mathbb{R}$, we recover the DH structure on U.

Examining the proof of Proposition 5.5, we see that, for convex $K \in \mathcal{K}(V)$, $\phi_{2n}(K) = \frac{1}{2}(\sigma_K(H^\perp)+\sigma_K(-H^\perp))$, where $\sigma_K d\theta \in \mathcal{M}(S^{2n})$ stands for the surface area measure of K, which is essentially just the pushforward of the $2n$–dimensional Hausdorff measure on ∂K to S^{2n}; see [50] for the exact definition.

5.3 Homogeneity $2n$

The valuation of homogeneity $2n$ in U, which is just the surface area measure at the points of tangency to $H \subset U$, also has a different natural setting, valid in dimension of either parity.

Let $H^m \subset V^{m+1}$ be a hyperplane, and let vol_H and vol_V be fixed volume forms on H and V, respectively. Consider $\mathrm{SL}_H(V) := \{g \in \mathrm{SL}(V) : g(H) = H,\ g|_H \in \mathrm{SL}(H)\}$. Fix a Euclidean product P on V inducing the given Lebesgue measures on V and H. Assume for simplicity $m \geq 2$.

Proposition 5.10 $\phi_m(K) := \frac{1}{2}(\sigma_K(H^\perp)+\sigma_K(-H^\perp))$ *is the unique m–homogeneous, even $\mathrm{SL}_H(V)$–invariant generalized valuation.*

Proof The Klain section of ϕ_m is a delta measure on $\mathrm{Gr}_m(V)$ supported on H. It is given by an element of $\mathrm{Dens}(H)\otimes\mathrm{Dens}^*(T_H \mathrm{Gr}_m(V)) = \mathrm{Dens}(H)^{m+2}\otimes\mathrm{Dens}^*(V)^m$. Clearly $\mathrm{SL}_H(V)$ acts trivially on this space.

The proof of uniqueness is similar to Proposition 5.6. We first note that there are no invariant sections of the Klain bundle over the open orbit $\{E : E \neq H\} \subset \mathrm{Gr}_m(V)$, since for such E there is a natural isomorphism $E/E \cap H = V/H$, so a $\mathrm{Stab}(E)$–invariant density on E would produce a $\mathrm{Stab}(E)$–invariant density on $E \cap H$, which clearly does not exist since $\mathrm{Stab}(E)$ can rescale this space. Next we use Lemma B.1 to study Klain sections supported on the closed orbit $\{H\}$: for $\alpha \geq 0$,

$$F^\alpha_{\{H\}} = \mathrm{Dens}(H) \otimes \mathrm{Dens}^*(H^* \otimes V/H) \otimes \mathrm{Sym}^\alpha(H^* \otimes V/H),$$

which is $\mathrm{SL}_H(V)$–equivariantly isomorphic to $\mathrm{Sym}^\alpha(H^*)$. There are no invariant polynomials on H, so we must have $\alpha = 0$, which readily yields a one-dimensional space of invariants. □

Next, we write an invariant Crofton formula for this valuation.

Proposition 5.11 *There is an invariant Crofton measure μ_{SL} over $\mathrm{Gr}_1(V)$ which defines ϕ_m.*

Proof Let $\theta(E)\in[0,\pi]$ be the Euclidean angle between $E\in\mathrm{Gr}_1^+(V)$ and $H^\perp$ with fixed orientation. Define the meromorphic family of functions $f_s(E)=(\cos\theta)^*|t|^s$, where $|t|^s\in C^{-\infty}(\mathbb{R})$ is the standard meromorphic family of even homogeneous generalized functions. Note that $\cos\theta$ is submersive whenever $t=\cos\theta=0$, so the pullback is well defined. Since $f_s(E)$ is invariant to orientation reversal, we get a generalized family on $\mathrm{Gr}_1(V)$, still denoted by $f_s(E)$.

Let us identify translation-invariant measures (distributions) on $\mathrm{AGr}_1(V)$ with (generalized) sections of the Crofton bundle Cr_m over $\mathrm{Gr}_1(V)$, whose fiber over $E\in\mathrm{Gr}_1(V)$ is $\mathrm{Dens}(V/E)\otimes\mathrm{Dens}(T_E\,\mathrm{Gr}_1(V))$.

Define a generalized section of Cr_m over $\mathrm{Gr}_1(V)\setminus\mathrm{Gr}_1(H)$ by $\mu_{\mathrm{SL}}:=f_{-m-2}(E)$ when m is even, and $\mu_{\mathrm{SL}}:=\mathrm{Res}_{s=-m-2}f_s$ when m is odd.

Let us check μ_{SL} is $\mathrm{SL}_H(V)$–invariant. For $g\in\mathrm{GL}(V)$ consider the Jacobian $\psi_g(E)=\mathrm{Jac}(g\colon E\to gE)^{-2}$, where E and gE are endowed with the volume induced by the Euclidean product P. It follows that, for $g\in\mathrm{SL}_H(V)$ and $E\not\subset H$,

$$\psi_g(E)=\frac{\cos^2\angle(gE,H^\perp)}{\cos^2\angle(E,H^\perp)}.$$

Now $f(E)$ represents (with respect to the Euclidean trivialization) an $\mathrm{SL}_H(V)$–invariant Crofton measure precisely if $g^*f(E)=\psi_g(E)^{-(m+2)/2}f(E)$. Since $g^*f_s=\psi_g(E)^{s/2}f_s(E)$, we conclude μ_{SL} is indeed $\mathrm{SL}_H(V)$–invariant.

It remains to verify these Crofton measures define nonzero valuations. For this, we evaluate $\phi_m(B)$ for the unit ball B^{m+1}. By definition, $\phi_m(B)=1$. On the other hand, writing ω_m for the volume of the Euclidean ball B^m, we get

$$\begin{aligned}\mathrm{Cr}(f_s(E)dE)(B)&=\omega_m\int_{\mathrm{Gr}_1(V)}|\cos\theta(E)|^s\,dE\\&=\omega_m\frac{\int_0^\pi|\cos\theta|^s\sin^{m-1}\theta\,d\theta}{\int_0^\pi\sin^{m-1}\theta\,d\theta}=\omega_m\frac{\int_0^1 t^s(1-t^2)^{(m-2)/2}\,dt}{\int_0^1(1-t^2)^{(m-2)/2}\,dt}\\&=\omega_m\frac{\int_0^1 u^{(s-1)/2}(1-u)^{(m-2)/2}\,du}{\int_0^1 u^{-1/2}(1-u)^{(m-2)/2}\,du}=\omega_m\frac{B\left(\frac{s+1}{2},\frac{m}{2}\right)}{B\left(\frac12,\frac{m}{2}\right)}.\end{aligned}$$

Thus

$$C_m=\frac{\pi^{m/2}}{\Gamma\left(1+\frac{m}{2}\right)}\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+1}{2}+\frac{m}{2}\right)}\frac{\Gamma\left(\frac{m}{2}+\frac12\right)}{\Gamma\left(\frac12\right)}=\frac{\pi^{m/2-1/2}}{\Gamma\left(1+\frac{m}{2}\right)}\frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{m}{2}+\frac12\right)}{\Gamma\left(\frac{s+1}{2}+\frac{m}{2}\right)}.$$

When $m = 2n$ is even, $s = -2n-2$ and

$$C_{2n} = -\frac{1}{2}\frac{\pi^{n-1}}{n!}\Gamma\bigl(n+\tfrac{1}{2}\bigr)\Gamma\bigl(-n-\tfrac{1}{2}\bigr).$$

When $m = 2n-1$ is odd, $s = -2n-1$, $\Gamma\bigl(\frac{1+s}{2}\bigr)$ has a simple pole with residue $2\operatorname{Res}_{z=-n}\Gamma(z) = 2(-1)^n/n!$, and

$$C_{2n-1} = 2(-1)^n \frac{1}{n!}\frac{\pi^{n-1}}{\Gamma\bigl(n+\frac{1}{2}\bigr)}\frac{\Gamma(n)}{\Gamma\bigl(-\frac{1}{2}\bigr)} = (-1)^{n-1}\frac{\pi^{n-3/2}}{n\Gamma\bigl(n+\frac{1}{2}\bigr)}. \qquad \square$$

We can now recover effortlessly the inverse form of the Koldobsky–Ryabogin–Zvavitch formula [38].

Corollary 5.12 *Let $K \subset \mathbb{R}^{m+1}$ be a smooth, symmetric convex body, and let $\kappa(x)$ be its gaussian curvature and $\nu(x)$ the unit normal at $x \in \partial K$. Then there is a universal explicit constant C_m* (computed in the proof of Proposition 5.11) *such that for even $m = 2n$,*

$$\kappa(x)^{-1} = C_m^{-1}\int_{S^m} \operatorname{vol}_m(\operatorname{Pr}_{\theta^\perp}(K))\frac{d\sigma_1(\theta)}{\langle\theta,\nu(x)\rangle^{m+2}},$$

while for odd m,

$$\begin{aligned}\kappa(x)^{-1} &= C_m^{-1}\bigl\langle\delta^{(m+1)}(\langle\nu(x),\theta\rangle), \operatorname{vol}_m(\operatorname{Pr}_{\theta^\perp}(K))\bigr\rangle \\ &= \frac{2}{(m+1)!}C_m^{-1} \\ &\qquad \times \frac{d^{m+1}}{dt^{m+1}}\bigg|_{t=0}\biggl((1-t^2)^{(m-2)/2}\int_{\substack{\theta \\ \langle\theta,\nu(x)\rangle=\cos t}} \operatorname{vol}_m(Pr_{\theta^\perp}(K))\, d\sigma_1(\theta)\biggr).\end{aligned}$$

Remark 5.13 Since $\operatorname{vol}_m(\operatorname{Pr}_{\eta^\perp}(K))$ is the cosine transform of $\sigma_K(\theta)d\sigma_1(\theta)$, one can think of this formula as simply the inversion of the cosine transform. The integral can be viewed as the $(-2n-2)$–cosine transform of the support function of the projection body.

6 Contact manifolds

6.1 Specializing from general DH manifolds.

Recall (Example 4.4) that a contact manifold has a canonical structure of a DH manifold. Let us recall a few basic facts.

The standard contact structure on $\mathbb{R}^{2n+1}=\{(x_1,\dots,x_n,y_1,\dots,y_n,z)\}$ is given by $dz=\sum_{j=1}^n x_j\,dy_j-y_j\,dx_j$. Its symmetry group contains translations along the z–axis and rotations in the (x_j,y_j)–planes.

Theorem 6.1 (Darboux and Pfaff) *Take a contact manifold M^{2n+1}. Then any $x\in M$ has a neighborhood contactomorphic to an open subset of the standard contact $\mathbb{R}^{2n+1}$.*

We will make use of the following simple observation:

Lemma 6.2 *In the standard contact space $\mathbb{R}^{2n+1}$, it holds that a closed, smooth, strictly convex hypersurface $F\subset\mathbb{R}^{2n+1}$ is normally transversal.*

Proof We use the standard Euclidean structure to identify $\mathbb{P}_{\mathbb{R}^{2n+1}}$ with $\mathbb{R}^{2n+1}\times S^{2n}$. Let p be a contact point of F. We may assume $p=(a_1,0,\dots,a_n,0,0)$. As $T_pF=H_p=\{dz=\sum a_j\,dy_j\}$, $(x_j,y_j)_{j=1}^n$ form a system of coordinates for F near p, and so locally F is the graph $z=f(x_1,\dots,y_n)$ of a strictly convex or strictly concave function f. We will write $w=w(x)=(x_1,\dots,y_n)$ and $x=(w,f(w))$. Let $\nu_w=\nu_x$ be the normal to F at x, and $h_w=h_x$ the normal to H_x, both normalized to have z coordinate 1 (the other pair of normals has $z=-1$ and is treated identically). Thus $\nu_w=(-f'_{x_1},\dots,-f'_{y_n},1)$ and $h_w=(y_1,-x_1,\dots,y_n,-x_n,1)$.

Claim *There exists $c=c(f,p)>0$ such that $\|\nu_w-h_w\|\ge c\|w-w(p)\|$ in a neighborhood of p.*

Proof Write

$$E=\|\nu_w-h_w\|=\sum(x_j-f'_{y_j})^2+(y_j+f'_{x_j})^2.$$

Replacing f with $f-\sum a_jy_j$ and x_j with x_j-a_j, E remains unchanged, and the convexity of f is retained. We thus may assume $p=0$, and $\nabla f(p)=0$. Hence

$$\begin{aligned}E&=\|w\|^2+\|\nabla f(w)\|^2-2\sum_j(x_jf'_{y_j}(w)-y_jf'_{x_j}(w))\\&=\|w\|^2+\|\nabla f(w)\|^2-2\omega(w,\nabla f(w)),\end{aligned}$$

where $\omega(w,w')=\sum(x_jy'_j-x'_jy_j)$ is the standard symplectic form on $\mathbb{R}^{2n}$. We clearly may replace f with its quadratic approximation at p, namely $f(w)=\frac{1}{2}\langle Aw,w\rangle$ for

$A = \operatorname{Hess} f(p)$. Then $\nabla f(w) = Aw$, and, by Writinger's inequality, $|\omega(w, \nabla f(w))| \le \|w\| \|Aw\| \sin \angle(w, Aw)$. We conclude that

$$\begin{aligned} E &\ge \|w\|^2 + \|Aw\|^2 - 2\|w\| \|Aw\| \sin \angle(w, Aw) \\ &= (\|w\| - \|Aw\|)^2 + 2\|w\| \|Aw\| (1 - \sin \angle(w, Aw)). \end{aligned}$$

Since A is sign-definite, it holds that $\lambda_{\min}\|w\| \le \|Aw\| \le \lambda_{\max}\|w\|$ and $|\langle Aw, w\rangle| \ge \lambda_{\min}\|w\|^2$, where $\lambda_{\min/\max}$ is the minimal/maximal eigenvalue of $|A|$. It follows that $|\cos \angle(w, Aw)| = |\langle w, Aw\rangle|/(\|w\| \|Aw\|) \ge \lambda_{\min}/\lambda_{\max}$, so $1 - \sin \angle(w, Aw) > 1 - \sqrt{1 - \lambda_{\min}^2/\lambda_{\max}^2} =: c_0$, and hence $E \ge 2c_0 \lambda_{\min}\|w\|^2$, as claimed. □

Recall $M_H = \{(x, h_x) : x \in \mathbb{R}^{2n+1}\}$ and $NF = \{(x, \nu_x) : x \in F\}$ have dimension $2n+1$ and $2n$, respectively. Thus we ought to show $T_{(p,\nu_p)} M_H \cap T_{(p,\nu_p)} NF = \{0\}$. If not, there is a curve $x(t) \subset F$, $x(0) = p$ with nonzero velocity vector, such that $h_{x(t)}$ and $\nu_{x(t)}$ have equal velocity vectors at $t = 0$. But then we should have $\|h_{x(t)} - \nu_{x(t)}\| = O(t^2)$ as $t \to 0$, contradicting the claim. □

Proposition 6.3 *Assume M^{2n+1} is a contact manifold. Then ϕ_{2j} for $0 \le j \le n$ are all linearly independent, while ϕ_{2j-1} is a linear combination of ϕ_{2i} for $j \le i \le n$.*

Proof The first statement is just Proposition 4.30(i). For the second statement, we will use the same notation as in the proof of Proposition 4.22.

Write $h = (h_{ij})_{i,j=1}^{2n} = H_s + H_a$, the symmetric and antisymmetric part.

First, we note that in a contact manifold, the relation between the symplectic and horizontal structures is given by $H_a = -J$. Indeed, $\alpha(v) = g(R, v)$, so we have

$$\begin{aligned} d\alpha(u, v) &= L_u\alpha(v) - L_v\alpha(u) - \alpha([u, v]) = L_u g(R, v) - L_v g(R, u) - \alpha([u, v]) \\ &= g(\nabla_u R, v) + g(R, \nabla_u v) - g(\nabla_v R, u) - g(R, \nabla_v u) - g(R, [u, v]) \\ &= g(\nabla_u R, v) - g(\nabla_v R, u). \end{aligned}$$

Hence $h_{ji} - h_{ij} = \theta_j(\nabla_{X_i} R) - \theta_i(\nabla_{X_j} R) = d\alpha(X_i, X_j) = J_{ij}$.

Take F a closed hypersurface which is normally transversal. Introduce the notation $\Psi_k(F, p) := \binom{2n}{k}^{-1} |\det A_p| \phi_{2n-k}(F, p)$. Then

$$\begin{aligned} \Psi_k(F, p) &= D(H - S[k], J[2n-k]) = (-1)^k D(S - H_s + J[k], J[2n-k]) \\ &= (-1)^k \sum_{i=0}^{k} \binom{k}{i} D(S - H_s[i], J[2n-i]). \end{aligned}$$

Observe that for $2n\times 2n$ matrices X and Y for which $X^T=X$ and $Y^T=-Y$ it holds that $D(X[2n-i],Y[i])=0$ for odd i. We conclude

$$\Psi_k(F,p)=(-1)^k\sum_{i=0}^{\lfloor k/2\rfloor}\binom{k}{2i}D(S-H_s[2i],J[2n-2i]),$$

that is,

$$\phi_k(F,p)=(-1)^k\binom{2n}{k}|\det A_p|^{-1}\sum_{i=0}^{\lfloor (2n-k)/2\rfloor}\binom{2n-k}{2i}D(S-H_s[2i],J[2n-2i]). \tag{9}$$

Thus for a normally transversal closed hypersurface F, ϕ_{2j-1} is a fixed linear combination of the ϕ_{2i} with $j\le i\le n$. By Darboux–Pfaff, we may cover M by charts U_α that are contactomorphic to open subsets V_α of the standard contact space $\mathbb{R}^{2n+1}$. By Lemma 6.2, a closed, smooth, strictly convex hypersurface in V_α is normally transversal to the contact structure. The statement now follows from Lemma 4.19, Proposition 4.17 and Lemma 3.12. □

We now prove a Hadwiger theorem for contact manifolds.

Theorem 6.4 *Let M^{2n+1} be a contact manifold. Then $\mathcal{V}^{-\infty}(M)^{\mathrm{Cont}(M)}$ is spanned by ϕ_{2k} for $0\le k\le n$.*

Proof For an element $\phi\in\mathcal{W}_k^{-\infty}$, we will write $[\phi]_k$ for its image in $\mathcal{W}_k^{-\infty}/\mathcal{W}_{k+1}^{-\infty}$. We will show by induction on $2n+1-k$ that $\mathcal{W}_k^{-\infty}(M)^{\mathrm{Cont}(M)}$ is spanned by $(\phi_{2j})_{2j\ge k}$. Take $\phi\in\mathcal{W}_{2n+1}^{-\infty}(M)^{\mathrm{Cont}(M)}=\mathcal{M}^{-\infty}(M)^{\mathrm{Cont}(M)}=\mathcal{M}^{\infty}(M)^{\mathrm{Cont}(M)}$, where the latter equality holds since $\mathrm{Cont}(M)$ acts transitively on M, so that an invariant distribution is automatically smooth. There is no such invariant measure, hence $\phi=0$.

Assume now that $\phi\in\mathcal{W}_{2n+1-k}^{-\infty}(M)^{\mathrm{Cont}(M)}$. Recall

$$\mathcal{W}_{2n+1-k}^{-\infty}(M)/\mathcal{W}_{2n+2-k}^{-\infty}(M)=(\mathcal{W}_{k,c}^{\infty}(M)/\mathcal{W}_{k+1,c}^{\infty}(M))^*=\Gamma_c(M,\mathrm{Val}_k^{\infty}(TM))^*.$$

Hence, by Proposition B.2,

$$[\phi]_{2n+1-k}\in(\Gamma_c(M,\mathrm{Val}_k^{\infty}(TM))^*)^{\mathrm{Cont}(M)}=\Gamma^{\infty}(M,\mathrm{Val}_{2n+1-k}^{-\infty}(TM))^{\mathrm{Cont}(M)},$$

where we used the Alesker–Poincaré isomorphism

$$\mathrm{Val}_k^{\infty}(T_xM)^*=\mathrm{Val}_{2n+1-k}^{-\infty}(T_xM)\otimes\mathrm{Dens}(T_xM).$$

For a point $x \in M$, its stabilizer in $\mathrm{Cont}(M)$ is $\mathrm{Stab}(x) = \mathrm{Sp}_H(T_x M)$. Hence $\Gamma^\infty(M, \mathrm{Val}^{-\infty}_{2n+1-k}(TM))^{\mathrm{Cont}(M)} = \mathrm{Val}^{-\infty}_{2n+1-k}(T_x M)^{\mathrm{Sp}_H T_x M}$. By Proposition 5.7, the latter space of invariants is trivial if k is even, and one-dimensional if k is odd.

Thus, if k is even, $\phi \in \mathcal{W}^{-\infty}_{2n+2-k}(M)$. If k is odd, we use Proposition 4.30(i) to find a multiple of ϕ^M_{2n+1-k} such that

$$[\phi]_{2n+1-k} = c[\phi^M_{2n+1-k}]_{2n+1-k} \implies \phi - c\phi^M_{2n+1-k} \in \mathcal{W}^{-\infty}_{2n+2-k}(M)^{\mathrm{Cont}(M)}.$$

The induction assumption now completes the proof. □

Remark 6.5 Unlike the contact case, we do not have a uniqueness result in the DH category, where the symmetry group is in general trivial. In the Riemannian setting, uniqueness of isometry-invariant valuation assignment can be deduced from the classical Hadwiger theorem in conjunction with the Nash embedding theorem. This last piece is missing in the DH setting. A different type of uniqueness in terms of the Cartan frame apparatus, which is again tailored to the Riemannian setting, was established by Fu and Wannerer [30].

6.2 A dynamical point of view

Let $F \subset M^{2n+1}$ be a hypersurface, and let $p \in F$ be a normally transversal contact point: $T_p F = H_p$. Denote by F_H the singular hyperplane field on F given by $F_H|_p = H_p \cap T_p F$. When $\dim M = 3$, this field integrates to the characteristic foliation.

One can describe $\phi_{2k}(F, p)$ explicitly through the singular bundle H_x near x.

Let $\beta \in \Omega^1(F)$ be a form defined near p such that $\operatorname{Ker}\beta = F_H$. Since M is contact, we may assume $d\beta \neq 0$ near p (eg by taking $\beta = \alpha|_F$ for some contact form α on M), and there is a unique vector field $B \in \mathfrak{X}(F)$ near p such that $i_B\, d\beta = \beta$. In particular, $B(p) = 0$ and $B(x)$ is tangent to the characteristic foliation. If $\beta' = f\beta$ is a different form with $d_p\beta' \neq 0 \iff f(p) \neq 0$, the corresponding vector field is

$$B' = \left(1 + \frac{df(B)}{f}\right)^{-1} B.$$

Since $df(B)(p) = df(B(p)) = 0$, the differential $d_p B \in \mathfrak{gl}(T_p B)$ only depends on F_H.

Remark 6.6 The sign of $\det d_p B$ determines whether p is an elliptic or a hyperbolic singular point of the characteristic foliation.

Proposition 6.7 *F is normally transversal at p if and only if d_pB is nonsingular. In that event,*

$$\phi_k(F,p)=|\det d_pB|^{-1}\operatorname{tr}\bigl(\textstyle\bigwedge^{2n-k}d_pB\bigr).$$

Proof Again we use notation from the proof of Proposition 4.22.

Since all contact manifolds are locally isomorphic, we work in $\mathbb{R}^{2n+1}$ with coordinates $(x_1,\dots,x_n,y_1,\dots,y_n,z)$, and contact form $\alpha=-dz+\sum x_j\,dy_j$. We may assume further that $p=0$ and $T_pF=\{z=0\}$. Then (x_j,y_j) are local coordinates on F, and $\beta=\sum-(\partial f/\partial x_j)dx_j+(x_j-\partial f/\partial y_j)dy_j$, with $d\beta|_0=\sum dx_j\wedge dy_j$. It follows that $B(x,y)^T=(x,0)^T+J\nabla f$, where $\nabla f=(\partial f/\partial x_j,\partial f/\partial y_j)^T$. Thus

$$d_0B=\begin{pmatrix} I_n & 0\\ 0 & 0\end{pmatrix}+JH^2f.$$

On the other hand, one immediately computes that

$$h=\begin{pmatrix} 0 & 0\\ I_n & 0\end{pmatrix}$$

and $S=H^2f$, hence $h-S=J\,d_0B$. This readily shows that F is normally transversal at p — which is equivalent to the nonsingularity of $h-S$ — if and only if $\det d_pB\neq 0$; and

$$D(h-S[2n-k],J[k])=\det J\cdot D(d_0B[2n-k],I_n[k])=\binom{2n}{k}^{-1}\operatorname{tr}\bigl(\textstyle\bigwedge^{2n-k}(d_0B)\bigr).$$

By Definition 4.24, we are done. □

Computatiton of ϕ_{2k} is straightforward with this approach. Here is a simple proof of a well-known fact:

Corollary 6.8 *In the standard contact space $\mathbb{R}^3$ with contact structure given by $dz=x\,dy-y\,dx$, spheres of different radii are not equivalent through a contactomorphism of the ambient space.*

Proof One computes that $\phi_2(S_R)=8\bigl(1+\frac{1}{4}R^{-2}\bigr)^{-1}$. □

Example 6.9 (contact sphere) Let us compute $\phi_{2k}(S^{2m})$ in S^{2n+1}. By Theorem 4.27, we may assume $n=m$. Consider $S^{2m+1}\subset\mathbb{C}^{m+1}$, with coordinates $x_1,y_1,\dots,x_{m+1},y_{m+1}$. The contact form is given by

$$\alpha_p(v)=\langle v,\sqrt{-1}p\rangle=\sum_{j=1}^{m+1}(-y_j\,dx_j+x_j\,dy_j).$$

so that $d\alpha = 2\sum_{j=1}^{m+1} dx_j \wedge dy_j$. Fix $S^{2m} = \{y_{m+1} = 0\}$. Then the two unique contact points of S^{2m} are given by $x_{m+1} = \pm 1$, and we use the coordinates $(x_j, y_j)_{j=1}^m$ near those points. In those coordinates, $\beta = \alpha|_{S^{2m}} = \sum_{j=1}^m (-y_j\, dx_j + x_j\, dy_j)$, and $d\beta = 2\sum_{j=1}^m dx_j \wedge dy_j$. Then

$$i_B d\beta = \beta \implies 2\sum (dx_j(B) dy_j - dy_j(B) dx_j) = \sum (x_j\, dy_j - y_j\, dx_j),$$

so that $B(p) = \frac{1}{2}p$. Thus $d_0 B = \frac{1}{2} I_{2m}$, and

$$\phi_{2k}(S^{2m}) = 2\,\mathrm{tr}\big(\textstyle\bigwedge^{2m-2k} I_{2m}\big) = 2\binom{2m}{2k}. \tag{10}$$

7 Symplectic-invariant distributions

7.1 Linear algebra

The real antisymmetric matrices of size $2N \times 2N$ will be denoted by Alt_{2N}. Define $\mathrm{SDiag}(\lambda_1, \dots, \lambda_N) \in \mathrm{Alt}_{2N}(\mathbb{R})$ to be the block-diagonal matrix consisting of the 2×2 blocks

$$\mathrm{SDiag}(\lambda_j) = \begin{pmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{pmatrix}.$$

The following is a standard fact from linear algebra:

Lemma 7.1 *For $A \in \mathrm{Alt}_{2N}$, there are $B \in \mathrm{O}_{2N}(\mathbb{R})$ and $D = \mathrm{SDiag}(\lambda_1, \dots, \lambda_N)$ such that $A = B^T DB$. The vector $(\lambda_1, \dots, \lambda_N)$ is uniquely defined up to permutations and signs of the λ_j.*

Write $\Delta_N = \{\lambda_1 \geq \cdots \geq \lambda_N \geq 0\}$. For $A \in \mathrm{Alt}_{2N}$, let $\Lambda(A) \in \Delta_N$ be the unique vector such that $A = B^T \cdot \mathrm{SDiag}(\Lambda(A)) \cdot B$ for some $B \in \mathrm{O}_{2N}(\mathbb{R})$.

Recall the multi-Kähler angles $0 \leq \theta_1 \leq \cdots \leq \theta_\kappa \leq \frac{\pi}{2}$ of $E \in \mathrm{Gr}_{2k}^{\mathbb{R}}(\mathbb{C}^n)$ introduced by Tasaki [52], where $\kappa = \min(k, n-k)$. They are defined as follows: Choose an orthonormal basis $(e_i)_{i=1}^{2k}$ of E, and define the symplectic Gram matrix $A = \omega(e_i, e_j)$. Then $\Lambda(A) = (\cos\theta_1, \dots, \cos\theta_\kappa)$.

Proposition 7.2 *Let θ_i for $i = 1, \dots, \kappa$ be the multi-Kähler angles of a subspace $E \in \mathrm{Gr}_{2k}^{\mathbb{R}}(\mathbb{C}^n)$ chosen at random (with respect to the $\mathrm{SO}(2n)$–invariant probability measure). Then the probability distribution of $(\cos\theta_i)_{i=1}^{\kappa}$ is uniform in Δ_κ.*

Proof We may assume $2k \le n$ and so $\kappa = k$. We first observe that the distribution is independent of n. Indeed we may condition on the event $E \subset F$ where F is any fixed complex k–dimensional subspace in $\mathbb{C}^n$, but the distribution of the multi-Kähler angles is clearly independent of F. Thus we assume $n = 2k$.

Next notice that for a symplectic subspace E, there is a unique (up to order) decomposition $E = E_1 \oplus \cdots \oplus E_k$, where $\dim E_j = 2$ and all $F_j := \mathbb{C} E_j$ are pairwise orthogonal. This decomposition can be found as follows: for an orthonormal basis $e = (e_j)_{j=1}^{2k}$ of E, consider the matrix $M(E, e) = (\omega(e_i, e_j))_{i,j=1}^{2k} \in \mathrm{Alt}_{2k}$. If (e') is a different orthonormal basis, there is an equality of row vectors $(e'_i) = (e_j) B$ for some $B \in \mathrm{O}(2k)$, and one checks that $M(E, e') = B^T M(E, e) B$. Thus for generic E there is a unique (up to order) orthonormal basis e such that $M(E, e) = \mathrm{SDiag}\big(\Lambda(M(E, e))\big)$. We then set $E_j = \mathrm{Span}(e_{2j-1}, e_{2j})$ for $1 \le j \le k$.

For a given decomposition $\mathbb{C}^{2k} = F_1 \oplus \cdots \oplus F_k$ into orthogonal copies of $\mathbb{C}^2$, the multi-Kähler angles of E are the collection of Kähler angles of $E \cap F_j \in \mathrm{Gr}_2^{\mathbb{R}}(F_j)$. Thus, conditioning on the decomposition, we conclude the multi-Kähler angles of E are independent and identically distributed, and it remains to find the distribution of the Kähler angle of a real 2–plane $E \subset \mathbb{C}^2$.

We will work with the oriented Grassmannian. Consider $\mathbb{R}^4 = \mathbb{C}^2$ with

$$\omega((x_1 + i x_3, x_2 + i x_4), (x'_1 + i x'_3, x_2 + i x'_4)) = x_1 x'_3 - x'_1 x_3 + x_2 x'_4 - x'_2 x_4.$$

Let (u, v) be an orthonormal basis for $E \in \mathrm{Gr}_2^+(\mathbb{R}^4)$. We make the identification $\mathrm{Gr}_2^+(\mathbb{R}^4) = S^2 \times S^2$, $E \mapsto (z, w)$, using the standard Euclidean structure and the Plücker embedding $\mathrm{Gr}_2^+(\mathbb{R}^4) \subset S\big(\bigwedge^2 \mathbb{R}^4\big)$, $E \mapsto u \wedge v = (x_{12}, \dots, x_{34})$, where $x_{kl} = u_k v_l - u_l v_k$, followed by the change of coordinates

$$x_{12} := \tfrac{1}{2}(w_1 + z_1), \quad x_{34} := \tfrac{1}{2}(w_1 - z_1), \quad x_{13} := -\tfrac{1}{2}(w_2 + z_2),$$
$$x_{24} := \tfrac{1}{2}(w_2 - z_2), \quad x_{14} := \tfrac{1}{2}(w_3 + z_3), \quad x_{23} := \tfrac{1}{2}(w_3 - z_3).$$

The corresponding measure on $S^2 \times S^2$ is the standard one. Then, for $E = (z, w) \in S^2 \times S^2$ and $\{u, v\}$ an oriented orthonormal basis of E, $\omega(u, v) = x_{13} + x_{24} = -z_2$. Denoting by θ_E the Kähler angle, we conclude that $\cos\theta_E = |\Lambda(E)| = |\omega(u, v)| = |z_2|$ is distributed uniformly in $[0, 1]$ by the theorem of Archimedes. □

Let $\Delta^1_\kappa = \{1 \ge \lambda_1 \ge \cdots \ge \lambda_\kappa \ge 0\}$. Denote by $\Lambda \colon \mathrm{Gr}_{2k}^{\mathbb{R}}(\mathbb{C}^n) \to \Delta^1_\kappa$ the vector $(\cos\theta_i)_{i=1}^{\kappa}$. We conclude that $\Lambda_*(d\sigma_1) = \kappa! \prod_{i=1}^{\kappa} d\lambda_i$.

7.2 Powers of the Pfaffian

Consider the meromorphic families of generalized functions $|\mathrm{Pf}|_{\pm}^{s} \in C^{-\infty}(\mathrm{Alt}_{2N}(\mathbb{R}))$ for $s \in \mathbb{C}$. They can be constructed by first considering $\mathrm{Re}\, s > -1$, whence $|\mathrm{Pf}|_{\pm}^{s}$ is an integrable function, and then using the Cayley-type identity (see eg [25])

$$\mathrm{Pf}(\partial)\,\mathrm{Pf}(X)^{s+1} = (s+1)(s+3)\cdots(s+2N-1)\,\mathrm{Pf}(X)^{s}$$

for a meromorphic extension to $s \in \mathbb{C}$. Thus the poles of both families are at $s = -1, -2, \dots$.

Theorem 7.3 (Muro [43]) *The linear combination* $|\mathrm{Pf}(X)|^{s} := |\mathrm{Pf}(X)|_{+}^{s} + |\mathrm{Pf}(X)|_{-}^{s}$ *is analytic at even* $s \in \mathbb{Z}$ *and has a simple pole at odd* $s \le -1$. *The linear combination* $\mathrm{sign}\,\mathrm{Pf}(X)\,|\mathrm{Pf}(X)|^{s} := |\mathrm{Pf}(X)|_{+}^{s} - |\mathrm{Pf}(X)|_{-}^{s}$ *is analytic at odd* $s \in \mathbb{Z}$ *and has a simple pole at even* $s \le -2$.

Let (V,ω) be a $2n$–dimensional symplectic space and P a compatible Euclidean structure with corresponding complex structure J, so that $\omega(u,v) = P(Ju,v)$. Then P induces a Lebesgue measure $\mathrm{vol}_P(E)$ on all subspaces $E \subset V$. Define $\sigma_{\omega,P}\colon \mathrm{Gr}_{2k}^{+}(V) \to [-1,1]$ by $\sigma_{\omega,P}(E) = \omega^k|_E/\mathrm{vol}_P(E)$. We will often omit P from the index when no confusion can arise.

We will now define meromorphic families of distributions on $\mathrm{Gr}_{2k}^{+}(\mathbb{R}^{2n})$ given by $\mu_{\pm}(s) = |\sigma_\omega(E)|_{\pm}^{s}\, d\sigma_1(E)$ for large $\mathrm{Re}(s)$. The construction is virtually identical to the one carried out in [29], with $\mathrm{Sp}(2n)$ replacing $\mathrm{O}(p,q)$. We present it here for the reader's convenience.

We will assume for now that $2k \le n$.

Let $U \subset \mathrm{Gr}_{2k}(V)$ be an open set. Let $B_E = (u_1(E),\dots,u_{2k}(E))\colon U \to V^k$ be a smooth field of P–orthonormal bases of $E \in U$. Define the function $M_P\colon U \to \mathrm{Alt}_{2k}(\mathbb{R})$ by $M_P(E) = \omega(u_i(E),u_j(E))_{i,j=1}^{2k}$. Note that $\sigma_\omega(E) = \mathrm{Pf}\, M_P(E)$.

Denote by $U_P \subset \mathrm{Gr}_k(V)$ the open, dense subset of subspaces $E \in \mathrm{Gr}_{2k}(V)$ for which $E \cap JE = \{0\}$. Clearly $E \in U_P$ if and only if $1 \notin \Lambda(M_P(E))$.

Lemma 7.4 M_P *is a proper submersion at every* $E \in U \cap U_P$.

Proof Consider a curve γ_1 through E given by

$$\gamma_1(t) = \mathrm{Span}(u_1(t), u_2, \dots, u_{2k})$$

with $u_2,\dots,u_{2k}$ fixed and $\xi=\dot u_1(0)\in E^P$ arbitrary. It follows that

$$D_E M_P(\dot\gamma_1(0))=\begin{pmatrix} 0 & \omega(\xi,u_2) & \cdots & \omega(\xi,u_{2k})\\ -\omega(\xi,u_2) & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -\omega(\xi,u_{2k}) & 0 & \cdots & 0\end{pmatrix}.$$

Since $E\in U_P$, the $\omega(u_j,\bullet)_{j=2}^{2k}$ are linearly independent functionals in $\xi\in E^P$, and so the first row of $D_\Lambda M_P(\dot\gamma_1)$ is arbitrary, while the other entries in the upper triangle vanish. Replacing γ_1 with γ_j in the obvious way, we conclude $D_E M_P\bigl(\sum\alpha_j\dot\gamma_j(0)\bigr)$ can be arbitrary, thus concluding the proof. □

Lemma 7.5 *One can choose finitely many ω–compatible Euclidean structures P_i such that $\{U_{P_i}\}$ cover $\mathrm{Gr}_{2k}(V)$.*

Proof Given $E\in\mathrm{Gr}_{2k}(V)$, a generic choice of an ω–compatible (P,J) would have $E\cap JE=\{0\}$ by a trivial dimension count. Fixing one such $J=J(E)$ with corresponding $P(E)$ for every E, we get an open cover $(U_{P(E)})_{E\in\mathrm{Gr}_{2k}(V)}$ of $\mathrm{Gr}_{2k}(V)$. The claim now follows by the compactness of $\mathrm{Gr}_{2k}(V)$. □

We now explain how to pull back $|\mathrm{Pf}(X)|_\pm^s$ to $\mathrm{Gr}_{2k}^+(V)$, using the locally defined submersion M_P.

Definition 7.6 For $s\in\mathbb{C}$, let $\mathcal{D}^s$ be the line bundle of s–densities over $\mathrm{Gr}_{2k}^+(V)$, which has fiber $\mathrm{Dens}^s(E)$ over $E\in\mathrm{Gr}_{2k}^+(V)$. We say that a choice of generalized section $f(s)\in\Gamma^{-\infty}(U,\mathcal{D}^s)$ over $U\subset\mathrm{Gr}_{2k}^+(V)$ for $s\in\Omega\subset\mathbb{C}$ is meromorphic in s if, having fixed a Euclidean metric P and using it to identify all bundles $\mathcal{D}^s$, one obtains a map $f_P\colon\Omega\to C^{-\infty}(U)$ which is meromorphic in s.

We denote by $\mathfrak{M}^{-\infty}(\mathcal{D}^s)$ the sheaf for which $\Gamma(U,\mathfrak{M}^{-\infty}(\mathcal{D}^s))$ is the space of meromorphic in s maps $\mathbb{C}\to\Gamma^{-\infty}(U,\mathcal{D}^s)$.

Recall the orbits $X_\pm^{2k}$ and $(X_r^{2k,+})_{r=0}^{\kappa-1}$ of $\mathrm{Gr}_{2k}^+(V)$ under $\mathrm{Sp}(V)$ defined in the paragraph following equation (3) on page 3050, where $\kappa=\min(k,n-k)$. In terms of multi-Kähler angles, $E\in X_r^{2k,+}$ precisely when exactly r of the angles are distinct from $\frac{\pi}{2}$.

We are now ready to construct the meromorphic families.

Proposition 7.7 *There are global sections $f_\pm(s) = M_P^*|\mathrm{Pf}|_\pm^s$ of $\mathfrak{M}^{-\infty}(\mathcal{D}^s)$ supported on $\overline{X_\pm^{2k}}$ such that whenever s is not a pole of $f_\pm$, $f_\pm(s)$ is $\mathrm{Sp}(V)$–invariant.*

Proof Assume first $2k \le n$. Let P_i be a finite collection of ω–compatible Euclidean structures as in Lemma 7.5, and let $U_i = U_{P_i} \subset \mathrm{Gr}_{2k}^+(V)$ be the corresponding open sets of generic subspaces. For each i, cover U_i by open sets $U_{ij} \subset U_i$ so that $M_{ij} = M_{P_i}\colon U_{ij} \to \mathrm{Alt}_{2k}$ can be defined by some smooth field of orthonormal bases of E over U_{ij}. Now since M_{ij} is a proper submersion, one obtains a meromorphic in s family of functions $\widetilde{f}_{ij}^\pm(E;s) \in C^{-\infty}(U_{ij})$ given by $\widetilde{f}_{ij}^\pm(\bullet;s) = M_{ij}^*|\mathrm{Pf}|_\pm^s$.

It then obviously holds that on $U_{ij} \cap U_{ij'}$, $\widetilde{f}_{ij}^\pm(\Lambda;s)$ and $\widetilde{f}_{ij'}^\pm(\Lambda;s)$ coincide as continuous functions for $\mathrm{Re}(s) > 0$. Therefore, they coincide on $U_{ij} \cap U_{ij'}$ as meromorphic functions, and we may merge all $\widetilde{f}_{ij}^\pm$ into one meromorphic family $\widetilde{f}_i^\pm(\bullet;s) \in C^{-\infty}(U_i)$. The corresponding (through P_i) section $f_i^\pm \in \Gamma(U_i, \mathfrak{M}^{-\infty}(D^s))$ is obviously $\mathrm{Sp}(V) \cap \mathrm{O}(P_i)$–invariant. Moreover, it is $\mathfrak{sp}(V)$–invariant.

Next, we claim that $f_i^\pm$ and $f_{i'}^\pm$ coincide on $U_i \cap U_{i'}$. Since both are meromorphic, we may assume in the following that $\mathrm{Re}(s) > 0$.

It is easy to see, using Lemma B.1 as in the proof of Proposition 5.4, that for $\mathrm{Re}(s) > 0$, no $\mathrm{Sp}(V)$–invariant generalized sections of $\mathcal{D}^s$ can be supported on a set of positive codimension: using Lemma B.1, we consider for $\alpha \ge 0$ the bundle $F^\alpha_{X_r^{2k,+}}$ over $X_r^{2k,+}$ where $r > 0$, that has fiber $\mathrm{Dens}(E)^s \otimes \mathrm{Dens}^*(N_E X_r^{2k,+}) \otimes \mathrm{Sym}^\alpha(N_E X_r^{2k,+})$ over E. Denoting $E_0 = E \cap E^\omega$, by Corollary 2.2,

$$F^\alpha_{X_r^{2k,+}}|_E = \mathrm{Dens}(E)^s \otimes \mathrm{Dens}\big(\textstyle\bigwedge^2 E_0\big) \otimes \mathrm{Sym}^\alpha\big(\textstyle\bigwedge^2 E_0^*\big),$$

which clearly contains no $\mathrm{Sp}(V)$–invariants. It follows that the space of $\mathrm{Sp}(V)$–invariants in $\Gamma^{-\infty}(\mathrm{Gr}_{2k}^+(V), \mathcal{D}^s)$ supported on $\overline{X_\epsilon^{2k}}$ is at most 1–dimensional for each $\epsilon \in \{\pm\}$.

Since $U_i \subset \mathrm{Gr}_{2k}^+(V)$ is dense, it follows by construction that for $\mathrm{Re}(s) > 0$, $f_i^\pm(\bullet;s)$ extends by continuity to an $\mathrm{Sp}(V)$–invariant section of $\mathcal{D}^s$ over $\mathrm{Gr}_{2k}^+(V)$ supported on $\overline{X_\pm^{2k}}$, and by the previous paragraph we can find meromorphic functions $c_i(s)$ such that $c_1(s) = 1$ and the sections $c_i(s) f_i^\pm(\Lambda;s)$ coincide for all i. Denoting by $p_i(s) \in \Gamma^\infty(\mathrm{Gr}_{2k}^+(V), \mathcal{D}^s)$ the Euclidean section defined by P_i, it holds that

$$f_i^\pm(E;s) = |\mathrm{Pf}\, M_{P_i}(E)|^s p_i(s)$$

for $E \in X_\pm^{2k}$, so that

$$|\mathrm{Pf}\, M_{P_1}(E)|^s p_1(s) = c_i(s) |\mathrm{Pf}\, M_{P_i}(E)|^s p_i(s),$$

implying

$$c_i(s)=\left(\frac{|\mathrm{Pf}\, M_{P_1}(E)|}{|\mathrm{Pf}\, M_{P_i}(E)|}\frac{p_1(1)}{p_i(1)}\right)^s$$

for all $E\in X^{2k}_\pm$. Since $c_i(s)$ is independent of E, one has $c_i(s)=c_i^s$ for some $c_i>0$.

Finally, for $s=1$, $\mathcal{D}^1$ is the Klain bundle of Lebesgue measures, and it is easy to see that all $f_i^\pm(1)$ represent the Liouville measure induced by the symplectic form on every symplectic subspace E. It follows that $c_i=1$ and so $c_i(s)\equiv 1$. Thus we have shown that $f_i^\pm$ and $f_{i'}^\pm$ coincide in $\Gamma(U_i\cap U_{i'},\mathfrak{M}^{-\infty}(\mathcal{D}^s))$.

We conclude there is a globally defined section $f_\pm$ of $\mathfrak{M}^{-\infty}(\mathcal{D}^s)$ which is $\mathfrak{sp}(V)$–invariant and supported on $\overline{X^{2k}_\pm}$, respectively. For $\mathrm{Sp}(V)$–invariance, we observe it holds for $\mathrm{Re}(s)>0$ and then invoke uniqueness of meromorphic continuation.

This concludes the proof when $2k\le n$. For the case $2k>n$, we simply use the oriented skew-orthogonal complement map $\mathrm{Gr}^+_{2k}(V)\to\mathrm{Gr}^+_{2n-2k}(V)$ to pull back f_s. This is a valid operation since we have the equivariant identification $E^\omega\simeq (V/E)^*$, which implies $\mathrm{Dens}(E^\omega)\simeq\mathrm{Dens}(E)$. □

Definition 7.8 Set $|\sigma_\omega|^s_\pm\in C^{-\infty}(\mathrm{Gr}^+_{2k}(V))$ to be the value of $f_\pm(s)$ under the Euclidean trivialization.

Lemma 7.9 *Write* $\kappa=\min(k,n-k)$. *One has*

$$\int_{\mathrm{Gr}_{2k}(\mathbb{R}^{2n})}|\sigma_\omega|^s_\pm\, d\sigma_1(E)=\frac{1}{2\kappa!\,(s+1)^\kappa}.$$

Proof Using the cosines of the multi-Kähler angles in decreasing order, denoted by $\Lambda(E)=(\lambda_1,\dots,\lambda_\kappa)$ with $\lambda_j=\cos\theta_j$, we have $|\sigma_\omega(E)|=\prod_{j=1}^\kappa\lambda_j$. Then, for $\mathrm{Re}(s)>0$, using Proposition 7.2 we get

$$\int_{\mathrm{Gr}_{2k}(\mathbb{R}^{2n})}|\sigma_\omega|^s\, d\sigma_1(E)=\frac{1}{\kappa!}\int_{[0,1]^\kappa}\prod_{j=1}^{\kappa}\lambda_j^s\, d\lambda_j=\frac{1}{\kappa!\,(s+1)^\kappa}$$

and the result follows by uniqueness of meromorphic extension. □

The two values $s=-2n,-(2n+1)$ are of particular interest, as evidenced in the two theorems below. The first theorem concerns the linear Grassmannian:

Theorem 7.10 *The distribution $\mu_\omega^+ \in \mathcal{M}^{-\infty}(\mathrm{Gr}_{2k}^+(V))$ given by*

$$\mu_\omega^+ := |\sigma_\omega(E)|^{-2n}\, d\sigma_1(E)$$

is $\mathrm{Sp}(V)$–invariant, has full support and $\int_{\mathrm{Gr}_{2k}(V)} \mu_\omega^+ \neq 0$. It is even with respect to orientation reversal.

In particular, we get a canonically normalized $\mathrm{Sp}(V)$–invariant distribution $\mu_\omega := \pi_* \mu_\omega^+$ on $\mathrm{Gr}_{2k}(2n)$, where $\pi\colon \mathrm{Gr}_{2k}^+(V) \to \mathrm{Gr}_{2k}(V)$ is the double cover map.

Proof A distribution over $\mathrm{Gr}_{2k}^+(V)$ is a generalized section of the bundle with fiber $\mathrm{Dens}(T_E \,\mathrm{Gr}_{2k}^+(V))$ over E, which is $\mathrm{Sp}(V)$–isomorphic to $\mathcal{D}^{-2n}$. All statements follow immediately from Proposition 7.7, Theorem 7.3 and Lemma 7.9. □

We conjecture that μ_ω is the unique $\mathrm{Sp}(V)$–invariant distribution on $\mathrm{Gr}_{2k}(V)$. This was shown by Gourevitch, Sahi and Sayag in [33] for $k=n$ when n is even.

Similarly, we have a statement for the affine Grassmannian. We define $|\sigma_\omega(E)|^s \in C^{-\infty}(\mathrm{AGr}_{2k}^+(V))^{\mathrm{tr}}$ by pulling back by the projection map $\mathrm{AGr}_{2k}^+(V) \to \mathrm{Gr}_{2k}^+(V)$. Let dE be the measure on $\mathrm{AGr}_{2k}^+(V)$ which is built out of $d\sigma_1$ on $\mathrm{Gr}_{2k}^+(V)$ and the Euclidean measure on translations. Define the odd distribution

$$\bar{\mu}_\omega := \mathrm{sign}(\sigma_\omega)|\sigma_\omega(E)|^{-2n-1} dE \in \mathcal{M}^{-\infty}(\mathrm{AGr}_{2k}^+(V)) \tag{11}$$

and the even distribution

$$\bar{\mu}_0 := \mathrm{Res}_{s=-2n-1}\, |\sigma_\omega(E)|_+^s\, dE \in \mathcal{M}^{-\infty}(\mathrm{AGr}_{2k}^+(V)). \tag{12}$$

Theorem 7.11 *The distributions $\bar{\mu}_\omega$ and $\bar{\mu}_0$ are $\overline{\mathrm{Sp}(V)}$–invariant and $\bar{\mu}_\omega$ has full support.*

The proof is as in the linear case. In particular, there is a an even, canonically normalized $\overline{\mathrm{Sp}(V)}$–invariant distribution $\bar{\mu}_0 := \pi_* \mu_0^+$ on $\mathrm{AGr}_{2k}(V)$, supported on the ω–degenerate subspaces.

We will need the following a priori information about the wavefront set of $\bar{\mu}_\omega$:

Proposition 7.12 *The wavefront set of $\bar{\mu}_\omega$ belongs to $\bigcup_r N^* X_r^{2k}$.*

Proof This is immediate from $\overline{\mathrm{Sp}(V)}$–invariance. □

8 The contact sphere

8.1 A Crofton basis for S^{2n+1}

In this section, $V=\mathbb{R}^{2n+2}$ and $M=S^{2n+1}=\mathbb{P}_+(V)$. Take $\xi\in\mathbb{P}_+(V)$ such that $T_\xi S^{2n+1}=\xi^*\otimes V/\xi$. The contact hyperplane is $H_\xi=\xi^*\otimes\xi^\omega/\xi\subset T_\xi\mathbb{P}_+(V)$. Then $T_\xi S^{2n+1}/H_\xi=\xi^*\otimes V/\xi^\omega\simeq(\xi^*)^{\otimes 2}$ $\operatorname{Stab}(\xi)$–equivariantly. The symplectic form on ξ^ω/ξ defines a form $\omega_\xi\in\bigwedge^2 H_\xi^*\otimes(\xi^*)^{\otimes 2}=\bigwedge^2 H_\xi^*\otimes T_\xi S^{2n+1}/H_\xi$. We remark that the form ω_ξ is determined by the contact distribution alone, without reference to the form ω on V; see Example 4.4. The linear symplectic group $\operatorname{Sp}(V)$ acts on S^{2n+1} by contactomorphisms. The stabilizer of $\xi\in S^{2n+1}$ is easily seen to act on $T_\xi S^{2n+1}$, which is a dual Heisenberg algebra, by its full group of automorphisms $\operatorname{Sp}_H(2n+1)$.

Now $\dim\operatorname{Sp}(2n+2)-\dim\operatorname{Sp}_H(2n+1)=(n+1)(2n+3)-(n+1)(2n+1)=2n+2=\dim S^{2n+1}+1$, so that $\dim\operatorname{Stab}(\xi)=\dim\operatorname{Sp}_H(2n+1)+1$, and we write $\operatorname{Stab}(\xi)=\operatorname{Sp}_H^*(2n+1)$. The kernel of the restriction homomorphism $\operatorname{Sp}_H^*(2n+1)\to\operatorname{Sp}_H(2n+1)$ consists of the linear maps that fix all $v\in\xi^\omega$ and acts on some $w\in V\setminus\xi^\omega$ by $w\mapsto w+\lambda\xi$ for some $\lambda\in\mathbb{R}$.

Thus $S^{2n+1}=\operatorname{Sp}(2n+2)/\operatorname{Sp}_H^*(2n+1)$. Inspired by the analogy with the Riemannian symmetric space presentation $S^{2n+1}=\operatorname{SO}(2n+2)/\operatorname{SO}(2n+1)$, we look for Crofton-type formulas for the contact valuations on the sphere.

Consider the double fibration

$$\begin{tikzcd} & W \arrow[dl, "\tau"'] \arrow[dr, "\pi"] & \\ \operatorname{Gr}_{2k}(V) & & S^{2n+1} \end{tikzcd}$$

where W is the partial flag manifold $\{(E,\theta)\in\operatorname{Gr}_{2k}(V)\times S^{2n+1}:\theta\in E\}$.

Definition 8.1 For $0\le k\le n$, define the generalized valuation ψ_{2k} through the Crofton formula (in the sense of Section 3.3)

$$\psi_{2k}:=\pi_*\tau^*(\mu_\omega)=\int_{\operatorname{Gr}_{2n+2-2k}(V)}\chi(\bullet\cap E)\sigma_\omega(E)^{-2n-2}\,d\sigma_1(E)\in\mathcal{V}^{-\infty}(S^{2n+1})^{\operatorname{Sp}(V)}.$$

It follows from Lemma 3.14 that $\psi_{2k}\in\mathcal{W}_{2k}^{-\infty}(S^{2n+1})$. Moreover, since $\psi_{2k}(S^{2k})\neq 0$ by Theorem 7.10 (see the computation preceding equation (14) below for its precise value), we conclude $\psi_{2k}\in\mathcal{W}_{2k}^{-\infty}(S^{2n+1})\setminus\mathcal{W}_{2k+1}^{-\infty}(S^{2n+1})$, and in particular all ψ_{2k} are linearly independent.

Proposition 8.2 *$\mathcal{V}^{-\infty}(S^{2n+1})^{\mathrm{Sp}(V)}$ is spanned by $\chi, \psi_2, \dots, \psi_{2n}$.*

Proof The proof is identical to that of Theorem 6.4, with $\mathrm{Sp}(V)$ replacing the full group of contactomorphisms. The proof remains valid since $\mathrm{Sp}(V)$ acts transitively, with the same action of the stabilizer on the tangent space by $\mathrm{Sp}_H(2n+1)$. □

This completes the proof of Theorem 1.8. In light of Theorem 6.4 we get:

Corollary 8.3 *For $0 \le k \le n$, the ψ_{2k} are linear combinations of ϕ_{2j} for $0 \le j \le n$. In particular, ψ_{2k} is invariant under all contactomorphisms of $\mathbb{P}_+(V)$.*

Thus we establish Theorem 1.7, except for the explicit determination of the coefficients which is deferred to the next subsection.

It follows also that $\psi_{2k}(F)$ is well defined for subsets $F \subset S^{2n+1}$ normally transversal to the contact distribution. We will make use of the following lemma:

Lemma 8.4 *Assume that a closed submanifold $F \subset S^{2n+1}$ is normally transversal, and $\chi_F := \chi(F \cap \bullet) \in C^{-\infty}(\mathrm{Gr}_{2n+2-k}(V))$ has wavefront disjoint from $\mathrm{WF}(\mu_\omega)$. Then*

$$\psi_{2k}(F) = \int_{\mathrm{Gr}_{2n+2-2k}(V)} \chi(F \cap E)\, d\mu_\omega(E).$$

Proof Choose an approximate identity ρ_ϵ on $\mathrm{GL}(V)$, and define

$$\phi_\epsilon = \int_{\mathrm{GL}(V)} g^*\chi_F \cdot \rho_\epsilon(g)\, dg \in \mathcal{V}^\infty(S^{2n+1}).$$

Then

$$\psi_{2k}(F) = \lim_{\epsilon\to 0}\langle \psi_{2k}, \phi_\epsilon\rangle = \lim_{\epsilon\to 0}\int_{\mathrm{Gr}_{2n+2-2k}(V)} \phi_\epsilon(E)\, d\mu_\omega(E).$$

Now $\phi_\epsilon(E) = \int_{\mathrm{GL}(V)} \chi_F(gE)\rho_\epsilon(g)\, dg$ converges to $\chi_F \in C^{-\infty}(\mathrm{Gr}_{2n+2-2k}(V))$ in Hörmander's topology on the space of generalized functions with wavefront set contained in $\mathrm{WF}(\chi_F)$. It follows that $\psi_{2k}(F) = \int_{\mathrm{Gr}_{2n+2-2k}(V)} \chi_F(E)\, d\mu_\omega(E)$. □

8.2 Integral geometry of the contact sphere

Here we determine the coefficients in Theorem 1.7. We have two different bases of contact-invariant valuations on the contact sphere, indexed by $0 \le k \le n$. Namely, we have ϕ_{2k} defined in terms of curvature at the contact points, and ψ_{2k} given by

Crofton integrals. Since $\phi_{2k}, \psi_{2k} \in \mathcal{W}_{2k}^{-\infty}(S^{2n+1})$, they are related by a triangular matrix, that is,

$$\psi_{2k} = \sum_{j=k}^{n} c_{kj}^n \phi_{2j}. \tag{13}$$

We will compute c_{kj}^n by evaluating both bases on all great spheres S^{2m}.

Take $F = S^{2n} = S^{2n+1} \cap \Pi$, where $\Pi \subset S^{2n+1}$ is a fixed hyperplane. It is normally transversal. Note also that $\chi(F \cap E) = 2$ for a generic $E \in \mathrm{Gr}_{2n+2-2k}(\mathbb{R}^{2n+2})$, that is, χ_F is the constant 2 on $\mathrm{Gr}_{2n+2-2k}(\mathbb{R}^{2n+2})$. Thus by Lemma 8.4 we may compute $\psi_{2k}(F)$ using the explicit Crofton formula. Denote $\kappa_k = \min(k, n+1-k)$. By Lemma 7.9,

$$\begin{aligned}\psi_{2k}(S^{2n}) &= \int_{\mathrm{Gr}_{2n+2-2k}(\mathbb{R}^{2n+2})} \chi(S^{2n} \cap E)\sigma_\omega(E)^{-2n-2}\, d\sigma_1(E)\\ &= 2\int_{\mathrm{Gr}_{2n+2-2k}(\mathbb{R}^{2n+2})} \sigma_\omega(E)^{-2n-2}\, d\sigma_1(E) = (-1)^{\kappa_k}\frac{2}{\kappa_k!}\frac{1}{(2n+1)^{\kappa_k}}.\end{aligned}$$

Considering spheres of lower dimension, we see that $\psi_{2k}(S^{2m}) = \psi_{2k}(S^{2n})$ if $m \geq k$ and zero otherwise. Thus

$$\psi_{2k}(S^{2m}) = \begin{cases} (-1)^{\kappa_k}\dfrac{2}{\kappa_k!}\dfrac{1}{(2n+1)^{\kappa_k}} & \text{if } k \leq m,\\ 0 & \text{if } k > m.\end{cases} \tag{14}$$

On the other hand, by equation (10),

$$\phi_{2j}(S^{2m}) = 2\binom{2m}{2j}.$$

We now plug those values into (13). Define the lower-triangular matrix A by $A(m,j) = \binom{2m}{2j}$ for $0 \leq m, j \leq n$. Its inverse is given by $A^{-1}(j,m) = \binom{2j}{2m}E_{2j-2m}$, where E_i is the i^{th} Euler (secant) number. Set $b_k = (-1)^{\kappa_k}(1/\kappa_k!)/(2n+1)^{\kappa_k}$. Then $c_{kj}^n = b_k\sum_{m=k}^{j} A^{-1}(j,m)$. In particular, $c_{kk}^n = b_k$, $c_{k,k+1}^n = b_k\bigl(1-\binom{2k+2}{2}\bigr)$.

8.3 Contact curvature of convex sets

Here we prove an upper bound on the contact valuations of a convex set.

Theorem 8.5 *Let $K \subset S^{2n+1}$ be a convex subset with C^2 boundary. Then for all $0 \leq k \leq n$, $\phi_{2k}(\partial K) \leq \phi_{2k}(\partial K_0)$, where K_0 is the hemisphere.*

Proof We will use an auxiliary complex structure, $V = \mathbb{C}^{n+1}$, where S^{2n+1} is identified with the unit sphere therein. We may assume that ∂K is normally transversal.

First, let us verify that ∂K has exactly two contact tangent hyperplanes. Since the Euler characteristic is $\chi(\partial K)=2$, this amounts to verifying that the intersection index of N^*K and M_H in $\mathbb{P}_{S^{2n+1}}$ at every contact point is $+1$. We will refer to this number as the contact index, denoted by $I_H(\partial K,x)$.

Consider a sphere $\mathcal{E}=S^{2n}\subset S^{2n+1}$ given by a quadratic equation in V. It is easy to check by an explicit computation that $\mathcal{E}$ has exactly two contact tangent points. Since $\chi(\mathcal{E})=2$, we conclude that the contact index of $\mathcal{E}$ at each of those points is $+1$. Now let $x\in\mathcal{K}$ be a contact tangent point, and let $\mathcal{E}$ be the osculating sphere at x. Then $I_H(\partial K,x)=I_H(\mathcal{E},x)=1$.

Next, let $p\in\partial K$ be a point where $T_p\partial K=H_p$. We now project the hemisphere U centered at p to $\mathbb{R}^{2n+1}=T_pS^{2n+1}$ by a central projection π from the origin π, so that p is mapped to the origin. Clearly $\pi(K\cap U)$ is convex near the origin: if $K=S^{2n+1}\cap C$ where C is a convex cone, then $\pi(K\cap U)=C\cap T_pS^{2n+1}$. By a standard computation and assuming $p=\{y_{n+1}=1\}$, the resulting contact structure in $\mathbb{R}^{2n+1}$ is given by the contact form $\alpha=-dx_{n+1}+\sum_{j=1}^n(-y_j\,dx_j+x_j\,dy_j)$. We will write $z=-x_{n+1}$.

We thus consider a convex body K with C^2 boundary in the contact space $(\mathbb{R}^{2n+1},\alpha)$. We assume K is tangent to the contact distribution at the origin, which is $\mathbb{R}^{2n}=\{z=0\}$, and further assume without loss of generality that K lies below it. The normal to the contact distribution is $\nu_H=(-y_1,\dots,-y_n,x_1,\dots,x_n,1)$, the normal to ∂K is ν_K. Then, by Proposition 4.22,

$$\phi_{2n}(\partial K,0)=|\det(d_0\nu_K-d_0\nu_H)|^{-1}D(d_0\nu_K-d_0\nu_H[2n-2k],J[2k]),$$

where $d_0\nu_K,d_0\nu_H\colon\mathbb{R}^{2n}\to T_{e_{2n+1}}S^{2n}$, the latter space identified with $\mathbb{R}^{2n}$. Using the coordinates $x_1,\dots,x_n,y_1,\dots,y_n$ on $\mathbb{R}^{2n}$ we get

$$d_0\nu_H=\begin{pmatrix}0&-I_n\\ I_n&0\end{pmatrix}=J.$$

Write also $S=d_0\nu_K$. Thus $\phi_{2n}(\partial K,0)=|\det(S-J)|^{-1}D(S-J[2n-2k],J[2k])=|\det(I+SJ)|^{-1}D(I+SJ[2n-2k],I[2k])=|\det(I+SJ)|^{-1}\operatorname{tr}\bigwedge^{2n-2k}(I+SJ)$.

Note that $S\geq 0$ since K is convex. Then JS and $\sqrt{S}J\sqrt{S}$ have the same characteristic polynomial. The latter matrix is antisymmetric, hence the roots of its characteristic polynomial appear in purely imaginary pairs $\pm i\lambda_j$ for $j=1,\dots,n$. Let us write $\mu_1,\dots,\mu_{2n}$ for these eigenvalues in some order. Note that for K_0, all $\lambda_j=0$. Writing

$m=2n-2k$, we ought to show that

$$\operatorname{tr}\textstyle\bigwedge^m(I+SJ)\le\binom{2n}{m}\det(I+SJ)\iff\displaystyle\sum_{|T|=m}\prod_{t\in T}(1+\mu_t)\le\binom{2n}{m}\prod_{j=1}^n(1+\lambda_j^2).$$

We will use induction on n. For $n=0,1$ the verification is trivial. Partition the sum over subsets T as $S_0+S_1+S_2$, where S_j for $j=0,1,2$ is composed of those summands where $\pm i\lambda_1$ appears j times inside $\{\mu_t:t\in T\}$. By the induction assumption, $S_0\le\binom{2n-2}{m}\prod_{j=2}^n(1+\lambda_j^2)$, $S_1\le 2\binom{2n-2}{m-1}\prod_{j=2}^n(1+\lambda_j^2)$ and $S_2\le\binom{2n-2}{m-2}\prod_{j=1}^n(1+\lambda_j^2)$. It remains to check that

$$\binom{2n-2}{m}+2\binom{2n-2}{m-1}+\binom{2n-2}{m-2}(1+\lambda_1^2)\le\binom{2n}{m}(1+\lambda_1^2),$$

which clearly follows from the equality

$$\binom{2n-2}{m}+2\binom{2n-2}{m-1}+\binom{2n-2}{m-2}=\binom{2n}{m},$$

concluding the induction and the proof. □

Remark 8.6 The case of equality is far from unique: any convex subset which is flat to second order at its two contact points would have the same values of ϕ_{2k}.

9 Symplectic integral geometry

9.1 Symplectic space

Let us first show there is no interesting symplectic valuation theory.

Theorem 9.1 *There is no* $\mathrm{Sp}(2n)$*–invariant, translation-invariant generalized valuation except for linear combinations of* χ *and* vol_{2n}.

Proof We find all invariant k–homogeneous valuations: Let $N=\lfloor\frac12\min(k,2n-k)\rfloor$ be the number of multi-Kähler angles for $E\in\mathrm{Gr}_k(\mathbb{R}^{2n})$. Since $\mathrm{U}(n)\subset\mathrm{Sp}(2n)$, an $\mathrm{Sp}(2n)$–invariant valuation ϕ would also be $\mathrm{U}(n)$–invariant. In particular, it is smooth by Alesker's theorem [3], as $\mathrm{U}(n)$ acts transitively on the unit sphere. We consider two separate cases. When $k=2l+1$ and $E\in\mathrm{Gr}_k(V)$ is maximally nondegenerate, there is no $\mathrm{Stab}(E)$–invariant Lebesgue measure on E. Since such subspaces are dense in $\mathrm{Gr}_k(V)$, we conclude $\mathrm{Kl}(\phi)=0$, and hence $\phi=0$.

When $k=2l$, an $\mathrm{Sp}(2n)$–invariant section of the Klain bundle should be proportional to $\left|\omega^l|_E\right|$, that is, after Euclidean trivialization it is proportional to $|\sigma(E)|=\prod_{i=1}^N \cos\theta_i$. But by [20], the Klain section of a $\mathrm{U}(n)$–invariant valuations should be given by a symmetric polynomial of $\cos^2\theta_i$. Thus there can be no nontrivial $\mathrm{Sp}(2n)$–invariant valuations unless $k=0,2n$. □

Remark 9.2 Instead of using the description of Bernig and Fu, one can simply notice that $\prod_{i=1}^N \cos\theta_i \in C(\mathrm{Gr}_k(V))$ is not smooth, violating the smoothness of ϕ.

Recall the distributions $\bar\mu_\omega, \bar\mu_0 \in \mathcal{M}^{-\infty}(\mathrm{AGr}^+_{2n-2k}(V))^{\overline{\mathrm{Sp}(V)}}$ given by equations (11) and (12).

Corollary 9.3 *For $1\le k\le n-1$ and a smooth convex body $K\subset V$ it holds that*

$$\int_{\mathrm{AGr}^+_{2k}(V)} \chi(K\cap E)\, d\bar\mu_0(E)=0.$$

Put differently, $\bar\mu_0$ lies in the kernel of the cosine transform.

Nevertheless, we can write symplectic Crofton formulas with the oriented valuation theory approach detailed in Appendix A. For a compact oriented C^1–submanifold with boundary $F\subset V$ of codimension $2k$, set $\mathrm{ind}_F(E)=I(E,F)$ for $E\in \mathrm{AGr}^+_{2k}(V)$, which is well defined whenever E and F intersect transversally.

Lemma 9.4 *For F as above, $\mathrm{ind}_F\in C^{-\infty}(\mathrm{AGr}^+_{2k}(V))$ and $\mathrm{WF}(\mathrm{ind}_F)\cap N^*X_r^{2k,+}=\varnothing$ for all $r<\min(k,n-k)$.*

Proof Let $Z=\{(x,E): x\in E\}\subset V\times \mathrm{AGr}^+_{2k}(V)$ be the incidence manifold, which has a natural orientation, and denote by $\tau\colon Z\to V$ and $\pi\colon Z\to \mathrm{AGr}^+_{2k}(V)$ the obvious submersions. Consider $Z_F=\tau^{-1}(F)$, which is a C^1 oriented submanifold of Z of codimension $2k$. Define $\delta_{Z_F}:=\tau^*[\![F]\!]=[\![Z_F]\!]$. Note that $\pi_*\tau^*[\![F]\!]\in C^{-\infty}(\mathrm{AGr}^+_{2k}(V))$, and $\pi_*\tau^*[\![F]\!](E)=I(E,F)$ whenever $E\pitchfork F$. The first statement follows. Note that $\mathrm{WF}(\delta_{Z_F})\subset \mathrm{Im}(\tau^*)$, and therefore also $\pi^*\,\mathrm{WF}(\mathrm{ind}_F)\subset\mathrm{Im}(\tau^*)$. As π^* is injective, the statement of the lemma would follow from $\mathrm{Im}(\tau^*)\cap\pi^*N^*X_r^{2k,+}=\{0\}\iff \mathrm{Im}(\tau^*)\cap N^*\pi^{-1}X_r^{2k,+}=\{0\}$.

Take $(x,E)\in Z_F\cap\pi^{-1}X_r^{2k,+}$. Define $\widetilde Z_x:=\tau^{-1}(x)\subset Z$. Then $\mathrm{Im}(\tau^*)\cap T^*_{x,E}=\tau^*(T^*_xV)=N^*\widetilde Z_x$. Let us check that $\widetilde Z_x$ and $\pi^{-1}X_r^{2k,+}$ intersect transversally at (x,E). For a tangent vector $(v,\Xi)\in T_{x,E}Z\subset T_xV\times T_E\,\mathrm{AGr}^+_{2k}(V)$, use Lemma 2.3

to decompose $\Xi = \Xi_x + \Xi_L$ with $\Xi_x \in T_E Z_x$, where $Z_x := \pi \widetilde{Z}_x$ and $\Xi_L \in T_E X_r^{2k,+}$. In the product manifold $V \times \mathrm{AGr}_{2k}^+(V)$ we get the equality $(v, \Xi) = (0, \Xi_x) + (v, \Xi_L)$. Since $(0, \Xi_x) \in TZ$ while by assumption $(v, \Xi) \in TZ$, we conclude $(v, \Xi_L) \in TZ$. Thus $T_{x,E} Z = T_{x,E} \widetilde{Z}_x + T_{x,E} \pi^{-1} X_r^{2k,+}$ and therefore $N^* Z_F \cap N^* \pi^{-1} X_r^{2k,+} = \{0\}$, concluding the proof. □

Theorem 9.5 *Let $F \subset V$ be a C^1 compact, oriented submanifold with boundary. Then*

$$\int_F \omega^k = \frac{(-1)^\kappa}{2} \binom{n}{k} \binom{2n}{2k}^{-1} n^\kappa \int_{\mathrm{AGr}_{2n-2k}^+(V)} I(E, F)\, d\overline{\mu}_\omega(E),$$

where $\kappa = \min(k, n-k)$.

Proof Considered as a function of F, the integral on the right-hand side is a Crofton integral as in Proposition A.1. Hence it defines a closed, $2k$–form on V which is $\overline{\mathrm{Sp}(V)}$–invariant. By the fundamental theorem of invariant theory, it is a multiple of ω^k, that is,

$$\int_{\mathrm{AGr}_{2n-2k}^+(V)} I(E, F)\, d\overline{\mu}_\omega(E) = C \int_F \omega^k. \tag{15}$$

It remains to find the constant C. We will use a compatible Euclidean structure. Let B_W be the unit Euclidean ball in the ω–positively oriented subspace $W \in X_+^{2k}$. We will average the integral over X_+^{2k} with respect to the probability measure dW that is invariant under $\mathfrak{so}(2n)$. For ease of computation, we replace the exponent of σ_ω in $\overline{\mu}_\omega(E) = \operatorname{sign} \sigma_\omega(E) |\sigma_\omega(E)|^{-2n-1}\, dE$ with the meromorphic variable $s \in \mathbb{C}$ and compute, for real s which is sufficiently large so that all integrands are continuous,

$$\begin{aligned} A_s &:= \int_{X_+^{2k}} dW \int_{\mathrm{AGr}_{2n-2k}^+(V)} I(E, B_W) \operatorname{sign} \sigma_\omega(E) |\sigma_\omega(E)|^s\, dE \\ &= \int_{\mathrm{Gr}_{2n-2k}^+(V)} \operatorname{sign} \sigma_\omega(E) |\sigma_\omega(E)|^s\, d\sigma_1(E) \int_{X_+^{2k}} dW \int_{V/E} I(E + x, B_W)\, dx. \end{aligned}$$

Here the inner integral is with respect to dx, the Euclidean Lebesgue measure under the identification $V/E = E^\perp$. We may write

$$\int_{V/E} I(E + x, B_W)\, dx = \operatorname{sign} \sigma_\omega(E) \operatorname{vol}_{2k}(\mathrm{Pr}_{E^\perp}(B_W)).$$

Hence,

$$A_s = \int_{\mathrm{Gr}_{2n-2k}^+(V)} |\sigma_\omega(E)|^s\, d\sigma_1(E) \int_{X_+^{2k}} \operatorname{vol}_{2k}(\mathrm{Pr}_{E^\perp}(B_W))\, dW.$$

The inner integral is independent of E and can be computed using the Kubota formula:

$$\int_{\mathrm{Gr}_{2n-2k}(V)} \mathrm{vol}_{2k}(\mathrm{Pr}_{F^\perp}(B^{2k}))\,d\sigma_1(F) = c_0\,\mathrm{vol}_{2k}(B^{2k}) = c_0\frac{\pi^k}{k!},$$

where B^{2k} is any fixed $2k$–dimensional Euclidean ball and

$$c_0 = \begin{bmatrix} 2n \\ 2k \end{bmatrix} = \binom{2n}{2k}\frac{\mathrm{vol}_{2n}(B^{2n})}{\mathrm{vol}_{2k}(B^{2k})\,\mathrm{vol}_{2n-2k}(B^{2n-2k})} = \binom{2n}{2k}\binom{n}{k}^{-1}.$$

We get

$$A_s = c_0\frac{\pi^k}{k!}2\int_{[0,1]^\kappa}(\lambda_1\cdots\lambda_\kappa)^s\,d\lambda_1\cdots d\lambda_\kappa = 2c_0\frac{\pi^k}{k!}\frac{1}{(s+1)^\kappa}$$

and, taking $s=-(2n+1)$, we conclude $A = 2c_0(-1)^\kappa(\pi^k/k!)/(2n)^\kappa$. Averaging the right-hand side of equation (15), we get

$$C\int_{X_{2k}^+}\mathrm{vol}(B^{2k})|\sigma_\omega(W)|\,dW = C\frac{\pi^k}{k!}\int_{[0,1]^\kappa}\lambda_1\cdots\lambda_\kappa\,d\lambda_1\cdots d\lambda_\kappa = C\frac{\pi^k}{k!2^\kappa}.$$

Summing up, $2c_0(\pi^k/k!)(-1)^\kappa/(2n)^\kappa = (\pi^k/k!2^\kappa)C$, so that

$$C^{-1} = (-1)^\kappa\frac{(2n)^\kappa}{2^{\kappa+1}c_0} = \frac{(-1)^\kappa}{2}\binom{n}{k}\binom{2n}{2k}^{-1}n^\kappa. \qquad \square$$

Appendix A Oriented valuation theory

In this appendix we draw a common thread between valuation theory and the much simpler theory of closed differential forms and linking integrals.

Let X^n be an oriented manifold. We will think of the closed k–forms, denoted by $\mathcal{Z}_k(X)$, as smooth oriented valuations of degree k, and consider them as functions on k–dimensional oriented submanifolds of X with boundary, given by integration: $\omega(A) = \int_A \omega$. The form is clearly determined by its value on all submanifolds, which is analogous to the Klain embedding. Moreover, $\omega(A)$ only depends on ∂A.

The wedge product on $\mathcal{Z}(X)$ turns it into an algebra. When X is compact, we have Poincaré duality, namely that $\mathcal{Z}_k(X)\otimes\mathcal{Z}_{n-k}(X)\to\mathcal{Z}_n(X)=\Omega^n(X)\to\mathbb{R}$ is nondegenerate, where the last arrow is given by $\int_X$.

We also consider the closed currents, which are analogous to the generalized valuations. We will denote them by $\mathcal{Z}^{-\infty}(X)$. We will sometimes write $\mathcal{Z}^\infty$ instead of $\mathcal{Z}$.

Now assume $X = V = \mathbb{R}^n$. The translation-invariant (smooth or generalized) oriented valuations $\mathcal{Z}^{\pm\infty}(V)^{\mathrm{tr}}$ are just $\bigwedge^\bullet V^*$. The following construction appears in [31].

For oriented manifolds with boundary $A, B \subset V$ of complementary dimensions, at least one of which is compact, let $I(A, B)$ denote their intersection index. It is well defined when A and B are in general position. Note that for a closed (as a subset) $(n-k)$–dimensional submanifold $E \subset V$, $I(\bullet, E) \in \mathcal{Z}_k^{-\infty}(V)$.

Given a distribution $\mu \in \mathcal{M}^{-\infty}(\mathrm{AGr}^+_{n-k}(V))$, we define $\mathrm{Cr}(\mu) \in \mathcal{Z}_k^{-\infty}(V)$ by $\mathrm{Cr}(\mu) = \int_{\mathrm{AGr}^+_{n-k}(V)} I(\bullet, E)\, d\mu(E)$. Clearly the even measures (with respect to the orientation-reversing map) lie in the kernel of Cr, so we restrict our attention to odd measures.

Proposition A.1 *The map* $\mathrm{Cr}\colon \mathcal{M}^{\pm\infty}(\mathrm{AGr}^+_{n-k}(V))^{\mathrm{tr}} \to \mathcal{Z}_k(V)^{\mathrm{tr}}$ *is surjective.*

Proof The $\mathrm{GL}(V)$–module $Z_k(V)^{\mathrm{tr}} = \bigwedge^k V^*$ is irreducible. Since the Crofton map is $\mathrm{GL}(V)$–equivariant, it suffices to show Cr is nonzero. This is not hard to see, for instance $\mathrm{Cr}(\delta_E - \delta_{-E})(A) = 2I(A, E)$, where $\langle \delta_E, f\rangle = \int_{V/E} f(x+E)\, dx$ on a compactly supported test function f. For a smooth example, one could convolve with an approximate identity on $\mathrm{GL}(V)$. □

The analogues of the Alesker–Poincaré and Alesker–Fourier dualities coincide in this setting: The Alesker–Poincaré pairing is the wedge product $\bigwedge^k V^* \otimes \bigwedge^{n-k} V^* \to \bigwedge^n V^*$. The Alesker–Fourier duality operation is given by the Hodge star, $*\colon \bigwedge^k V^* \to \left(\bigwedge^{n-k} V^*\right)^* \otimes \bigwedge^n V^*$.

The following easy statement is the analogue for oriented valuations of the principal kinematic formula. Due to the finite-dimensionality of the space of translation-invariant forms, one can average over translations alone.

Let a top form $\mathrm{vol}_n \in \bigwedge^n V^*$ be fixed on V. Let $\Omega \in \bigwedge^\bullet V^* \otimes \bigwedge^\bullet V^* = \left(\bigwedge^\bullet V \otimes \bigwedge^\bullet V\right)^*$ be the wedge product, that is, $\langle \xi \wedge \eta, \mathrm{vol}_n\rangle = \Omega(\xi, \eta)$ for $\xi, \eta \in \bigwedge^\bullet V$.

For oriented compact manifolds with boundary A and B we write $\Omega(A, B) := \int_{A\times B} \Omega$, which can be written more explicitly by representing $\Omega = \sum \omega_i \otimes \omega_i'$, then $\Omega(A, B) = \sum \int_A \omega_i \int_B \omega_i'$ (the integrals with mismatched dimension vanish by definition). Consider the kinematic operator

$$K_V(A, B) := \int_V I(A, B + x)\, d\mathrm{vol}_n(x),$$

which is well defined, since the integrand is compactly supported.

Proposition A.2 *Let $A^{n-k}, B^k \subset V$ be compact, oriented submanifolds with boundary, of complementary dimension. Then $K_V(A,B)=\Omega(A,B)$.*

Proof Let $[\![B]\!]$ be the current defined by B. Then $[B]_V := \int_V [\![B+x]\!]\, d\,\mathrm{vol}_n(x)$ is translation-invariant, that is, $[B]_V \in \bigwedge^k V$. Now $I(A,B)=[\![A]\!]\cap[\![B]\!]$, so that $K_V(A,B)=[\![A]\!]\cap[B]_V=\mathrm{vol}_n([A]_V\wedge[B]_V)=\Omega(A,B)$, as claimed. □

We remark that this formula is also reminiscent of the Bezout formula in complex algebraic geometry.

Appendix B Invariant sections

We will need two technical lemmas concerning invariants of group actions.

The first goes back to Kolk and Varadarajan [39], and appeared in a form most suitable for our needs in [19]. Let us quote the result in its simplest sufficient form.

Take a Lie group G acting on a manifold X with finitely many orbits, all locally closed submanifolds. Let E be a G–vector bundle over X. Define for integer $\alpha\geq 0$ and a submanifold $Y\subset X$ the G–bundle F_Y^α over Y by

$$F_Y^\alpha|_y = E|_y\otimes \mathrm{Dens}^*(N_yY)\otimes \mathrm{Sym}^\alpha(N_yY).$$

A generalized section $s\in\Gamma_{\overline{Y}}^{-\infty}(X,E)$ has a certain transversal order α along Y, and a transversal principal symbol $\sigma(s)\in\Gamma^{-\infty}(Y,F_Y^\alpha)$. For details, see eg [9, Section 4.4].

Lemma B.1 *Let $Z\subset X$ be a closed G–invariant subset. Decompose into G–orbits: $Z=\bigcup_{j=1}^J Y_j$. Then*

$$\dim \Gamma_Z^{-\infty}(X,E)^G \leq \sum_{\alpha=0}^{\infty}\sum_{j=1}^{J} \dim \Gamma^\infty(F_{Y_j}^\alpha)^G.$$

The second statement is surely well known, but we were not able to locate a reference. For completeness, we include the proof.

Proposition B.2 *Let G be a (possibly infinite-dimensional) Lie group acting on a manifold M^n transitively, and let $\mathcal{E}$ be an infinite-dimensional G–bundle of Fréchet spaces over X. Then the space of G–invariants of $\Gamma_c^\infty(M,\mathcal{E})^*$ is naturally isomorphic to $\Gamma^\infty(M,\mathcal{E}^*\otimes|\omega_M|)^G$.*

Remark B.3 In the infinite-dimensional case, we assume that every T_xM admits a basis of infinitesimal generators of 1–parametric subgroups of G. This is certainly the case for $\mathrm{Diff}(M)$, $\mathrm{Symp}(M)$ and $\mathrm{Cont}(M)$.

First we consider the wavefront set of the average of a distribution along a flow.

Lemma B.4 *Let M be a manifold. Assume $\mathbb{R}$ acts on M, and let the curve $C \subset M$ be an orbit. Let $\phi \in C^{-\infty}(M)$ be a generalized function, and $\mu \in C_c^\infty(\mathbb{R})$. Define $\phi * \mu := \int_{\mathbb{R}} t^*\phi \cdot \mu(t)\,dt$. Then for all $p \in C$ we have $\mathrm{WF}(\phi * \mu) \cap T_p^*M \subset N_p^*C$.*

Proof Fix a small neighborhood V of $p \in C$ which is $\mathbb{R}$–equivariantly identified with a neighborhood $U \subset \mathbb{R}^n$ of $p = 0$, with $\mathbb{R}$ acting by translations along the x_1–axis and C coinciding with the x_1–axis. Choose a partition of unity ρ_i on M and w_j on $\mathbb{R}$. Writing $\phi_i := \rho_i \cdot \phi$, $\mu_j := w_j \cdot \mu$ we have $\phi * \mu = \sum_{i,j} \phi_i * \mu_j$. We may assume that each ϕ_i has finite order, and that each $\psi = \phi_i * \mu_j$ whose support contains p is in fact supported inside V. Taking such ψ, we may write $\psi = \Phi * \pi_1^*\nu$ for some $\nu \in C_c^\infty(\mathbb{R})$ and $\Phi \in C_c^{-\infty}(\mathbb{R}^n)$ of order k, where $\pi_1\colon \mathbb{R}^n \to \mathbb{R}$ is the projection to the first coordinate. Then the Fourier transform satisfies $\widehat{\psi}(\xi) = \widehat{\Phi}(\xi)\widehat{\nu}(\xi_1)$. Now $|\widehat{\Phi}(\xi)| \le C(1+|\xi|)^k$ while $|\widehat{\nu}(\xi_1)| \le C_N(1+|\xi_1|)^{-N}$ for all N. It follows that for all $\epsilon > 0$, the cone $C_\epsilon := \{\xi_1^2 > \epsilon(\xi_2^2 + \cdots + \xi_n^2)\}$ falls outside the wavefront set of ψ, so that $\mathrm{WF}(\psi) \subset \{\xi_1 = 0\}$. That concludes the proof. $\square$

Clearly the same statement holds also for S^1–actions, and with ϕ a generalized section of an arbitrary $\mathbb{R}$–equivariant vector bundle over M.

Proof of Proposition B.2 Take $s \in \Gamma_c^\infty(M,\mathcal{E})^*$. For $\phi \in \Gamma_c^\infty(M,\mathcal{E})$ one can define $s\cdot\phi \in \mathcal{M}^{-\infty}(M)$ by $\int f \cdot d(s\cdot\phi) := s(f\phi)$ for $f \in C_c^\infty(M)$. Moreover, the map $\Gamma_c^\infty(M,\mathcal{E}) \to \mathcal{M}^{-\infty}(M)$ given by $\phi \mapsto s\cdot\phi$ is G–equivariant. Fix ϕ, and let us verify that $s\cdot\phi$ is in fact a smooth measure. Consider a 1–parametric subgroup $H \subset G$, which can be either $\mathbb{R}$ or S^1. By the Dixmier–Mallavin theorem, we may find smooth probability measures $\mu_1,\dots,\mu_N \in \mathcal{M}_c^\infty(H)$ and sections $\psi_1,\dots,\psi_N \in \Gamma_c^\infty(M,\mathcal{E})$ such that

$$\phi = \sum_j \int_H g\psi_j\,d\mu_j(g) \implies s\cdot\phi = \sum_j \int_H g(s\cdot\psi_j)\,d\mu_j(g).$$

It follows by Lemma B.4 that the wavefront set $\mathrm{WF}(s\cdot\phi)$ lies in the conormal bundle to the orbits of H on M. Since G acts transitively, we conclude $\mathrm{WF}(s\cdot\phi) = \varnothing$, that is, $s\cdot\phi \in \mathcal{M}^\infty(M)$.

In particular, we can consider the density on every tangent plane, $(s\cdot\phi)(x)\in\mathrm{Dens}(T_xM)$. We next claim that $(s\cdot\phi)(x)$ only depends on $\phi(x)$. Indeed, if $\phi(x)=0$, we may represent $\phi=f\cdot\psi$ for some $\psi\in\Gamma_c^\infty(M,\mathcal{E})$ and $f_c\in C^\infty(M)$ with $f(0)=0$. Then $(s\cdot\phi)=f\cdot(s\cdot\psi)$, and thus $(s\cdot\phi)(x)=0$. That is, we get an element $s(x)\in\mathcal{E}^*|_x\otimes\mathrm{Dens}(T_xM)$. By G–invariance, $x\mapsto s(x)$ is a smooth section of $\mathcal{E}^*\otimes|\omega_M|$, which clearly defines the same functional as s on $\Gamma_c^\infty(M,\mathcal{E})$. $\square$

References

[1] **A Abbondandolo**, **B Bramham**, **U L Hryniewicz**, **P A S Salomão**, *A systolic inequality for geodesic flows on the two-sphere*, Math. Ann. 367 (2017) 701–753 MR

[2] **S Alesker**, *Description of translation invariant valuations on convex sets with solution of P McMullen's conjecture*, Geom. Funct. Anal. 11 (2001) 244–272 MR

[3] **S Alesker**, *The multiplicative structure on continuous polynomial valuations*, Geom. Funct. Anal. 14 (2004) 1–26 MR

[4] **S Alesker**, *Theory of valuations on manifolds, I: Linear spaces*, Israel J. Math. 156 (2006) 311–339 MR

[5] **S Alesker**, *Theory of valuations on manifolds, II*, Adv. Math. 207 (2006) 420–454 MR

[6] **S Alesker**, *Theory of valuations on manifolds, IV: New properties of the multiplicative structure*, from "Geometric aspects of functional analysis" (V D Milman, G Schechtman, editors), Lecture Notes in Math. 1910, Springer (2007) 1–44 MR

[7] **S Alesker**, *Valuations on manifolds and integral geometry*, Geom. Funct. Anal. 20 (2010) 1073–1143 MR

[8] **S Alesker**, **A Bernig**, *The product on smooth and generalized valuations*, Amer. J. Math. 134 (2012) 507–560 MR

[9] **S Alesker**, **D Faifman**, *Convex valuations invariant under the Lorentz group*, J. Differential Geom. 98 (2014) 183–236 MR

[10] **S Alesker**, **J H G Fu**, *Theory of valuations on manifolds, III: Multiplicative structure in the general case*, Trans. Amer. Math. Soc. 360 (2008) 1951–1981 MR

[11] **S Alesker**, **J H G Fu**, *Integral geometry and valuations*, Springer (2014) MR

[12] **J C Álvarez Paiva**, *Dual spheres have the same girth*, Amer. J. Math. 128 (2006) 361–371 MR

[13] **J C Álvarez Paiva**, **F Balacheff**, *Contact geometry and isosystolic inequalities*, Geom. Funct. Anal. 24 (2014) 648–669 MR

[14] **J C Álvarez Paiva**, **F Balacheff**, **K Tzanev**, *Isosystolic inequalities for optical hypersurfaces*, Adv. Math. 301 (2016) 934–972 MR

[15] **S Artstein-Avidan**, **R Karasev**, **Y Ostrover**, *From symplectic measurements to the Mahler conjecture*, Duke Math. J. 163 (2014) 2003–2022 MR

[16] **S Artstein-Avidan**, **V Milman**, **Y Ostrover**, *The M–ellipsoid, symplectic capacities and volume*, Comment. Math. Helv. 83 (2008) 359–369 MR

[17] **A Bernig**, *A product formula for valuations on manifolds with applications to the integral geometry of the quaternionic line*, Comment. Math. Helv. 84 (2009) 1–19 MR

[18] **A Bernig**, **L Bröcker**, *Valuations on manifolds and Rumin cohomology*, J. Differential Geom. 75 (2007) 433–457 MR

[19] **A Bernig**, **D Faifman**, *Valuation theory of indefinite orthogonal groups*, J. Funct. Anal. 273 (2017) 2167–2247 MR

[20] **A Bernig**, **J H G Fu**, *Hermitian integral geometry*, Ann. of Math. 173 (2011) 907–945 MR

[21] **A Bernig**, **J H G Fu**, **G Solanes**, *Integral geometry of complex space forms*, Geom. Funct. Anal. 24 (2014) 403–492 MR

[22] **A Bernig**, **G Solanes**, *Classification of invariant valuations on the quaternionic plane*, J. Funct. Anal. 267 (2014) 2933–2961 MR

[23] **A Bernig**, **G Solanes**, *Kinematic formulas on the quaternionic plane*, Proc. Lond. Math. Soc. 115 (2017) 725–762 MR

[24] **A Bernig**, **F Voide**, *Spin-invariant valuations on the octonionic plane*, Israel J. Math. 214 (2016) 831–855 MR

[25] **S Caracciolo**, **A D Sokal**, **A Sportiello**, *Algebraic/combinatorial proofs of Cayley-type identities for derivatives of determinants and Pfaffians*, Adv. in Appl. Math. 50 (2013) 474–594 MR

[26] **S S Chern**, **R S Hamilton**, *On Riemannian metrics adapted to three-dimensional contact manifolds*, from "Workshop Bonn 1984" (F Hirzebruch, J Schwermer, S Suter, editors), Lecture Notes in Math. 1111, Springer (1985) 279–308 MR

[27] **H Donnelly**, *Heat equation and the volume of tubes*, Invent. Math. 29 (1975) 239–243 MR

[28] **D Faifman**, *An extension of Schäffer's dual girth conjecture to Grassmannians*, J. Differential Geom. 92 (2012) 201–220 MR

[29] **D Faifman**, *Crofton formulas and indefinite signature*, Geom. Funct. Anal. 27 (2017) 489–540 MR

[30] **J H G Fu**, **T Wannerer**, *Riemannian curvature measures*, Geom. Funct. Anal. 29 (2019) 343–381 MR

[31] **I M Gelfand**, **M M Smirnov**, *Lagrangians satisfying Crofton formulas, Radon transforms, and nonlocal differentials*, Adv. Math. 109 (1994) 188–227 MR

[32] **E D Gluskin**, **Y Ostrover**, *Asymptotic equivalence of symplectic capacities*, Comment. Math. Helv. 91 (2016) 131–144 MR

[33] **D Gourevitch**, **S Sahi**, **E Sayag**, *Invariant functionals on Speh representations*, Transform. Groups 20 (2015) 1023–1042 MR

[34] **V Guillemin**, **S Sternberg**, *Geometric asymptotics*, Mathematical Surveys 14, Amer. Math. Soc., Providence, RI (1977) MR

[35] **H Hadwiger**, *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*, Springer (1957) MR

[36] **L Hörmander**, *The analysis of linear partial differential operators, I: Distribution theory and Fourier analysis*, 2nd edition, Grundl. Math. Wissen. 256, Springer (1990) MR

[37] **D A Klain**, *Even valuations on convex bodies*, Trans. Amer. Math. Soc. 352 (2000) 71–93 MR

[38] **A Koldobsky**, **D Ryabogin**, **A Zvavitch**, *Projections of convex bodies and the Fourier transform*, Israel J. Math. 139 (2004) 361–380 MR

[39] **J A C Kolk**, **V S Varadarajan**, *On the transverse symbol of vectorial distributions and some applications to harmonic analysis*, Indag. Math. 7 (1996) 67–96 MR

[40] **M Ludwig**, **M Reitzner**, *A characterization of affine surface area*, Adv. Math. 147 (1999) 138–172 MR

[41] **M Ludwig**, **M Reitzner**, *A classification of $\mathrm{SL}(n)$ invariant valuations*, Ann. of Math. 172 (2010) 1219–1267 MR

[42] **P McMullen**, *Valuations and Euler-type relations on certain classes of convex polytopes*, Proc. London Math. Soc. 35 (1977) 113–135 MR

[43] **M Muro**, *Singular invariant hyperfunctions on the square matrix space and the alternating matrix space*, Nagoya Math. J. 169 (2003) 19–75 MR

[44] **Y Ostrover**, *When symplectic topology meets Banach space geometry*, from "Proceedings of the International Congress of Mathematicians" (S Y Jang, Y R Kim, D-W Lee, I Ye, editors), volume 2, Kyung Moon Sa, Seoul (2014) 959–981 MR

[45] **V Ovsienko**, *Vector fields in the presence of a contact structure*, Enseign. Math. 52 (2006) 215–229 MR

[46] **R Ponge**, *The tangent groupoid of a Heisenberg manifold*, Pacific J. Math. 227 (2006) 151–175 MR

[47] **R S Ponge**, *Heisenberg calculus and spectral theory of hypoelliptic operators on Heisenberg manifolds*, Mem. Amer. Math. Soc. 906, Amer. Math. Soc., Providence, RI (2008) MR

[48] **B L Reinhart**, *The second fundamental form of a plane field*, J. Differential Geom. 12 (1977) 619–627 MR

[49] **M Rumin**, *Formes différentielles sur les variétés de contact*, J. Differential Geom. 39 (1994) 281–330 MR

[50] **R Schneider**, *Convex bodies: the Brunn–Minkowski theory*, expanded edition, Encyclopedia of Mathematics and its Applications 151, Cambridge Univ. Press (2014) MR

[51] **G Solanes**, **T Wannerer**, *Integral geometry of exceptional spheres*, preprint (2017) arXiv To appear in J. Differential Geom.

[52] **H Tasaki**, *Generalization of Kähler angle and integral geometry in complex projective spaces*, from "Steps in differential geometry" (L Kozma, P T Nagy, L Tamássy, editors), Inst. Math. Inform., Debrecen (2001) 349–361 MR

[53] **C Viterbo**, *Metric and isoperimetric problems in symplectic geometry*, J. Amer. Math. Soc. 13 (2000) 411–431 MR

[54] **H Weyl**, *On the volume of tubes*, Amer. J. Math. 61 (1939) 461–472 MR

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