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Homotopy groups of the observer moduli space of Ricci positive metrics

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The observer moduli space of Riemannian metrics is the quotient of the space $\mathcal{R}(M)$ of all Riemannian metrics on a manifold M by the group of diffeomorphisms $\mathrm{Diff}_{x_0}(M)$ which fix both a basepoint x_0 and the tangent space at x_0. The group $\mathrm{Diff}_{x_0}(M)$ acts freely on $\mathcal{R}(M)$ provided that M is connected. This offers certain advantages over the classic moduli space, which is the quotient by the full diffeomorphism group. Results due to Botvinnik, Hanke, Schick and Walsh, and Hanke, Schick and Steimle have demonstrated that the higher homotopy groups of the observer moduli space $\mathcal{M}_{x_0}^{s>0}(M)$ of positive scalar curvature metrics are, in many cases, nontrivial. The aim in the current paper is to establish similar results for the moduli space $\mathcal{M}_{x_0}^{\mathrm{Ric}>0}(M)$ of metrics with positive Ricci curvature. In particular we show that for a given k, there are infinite-order elements in the homotopy group $\pi_{4k}\mathcal{M}_{x_0}^{\mathrm{Ric}>0}(S^n)$ provided the dimension n is odd and sufficiently large. In establishing this we make use of a gluing result of Perelman. We provide full details of the proof of this gluing theorem, which we believe have not appeared before in the literature. We also extend this to a family gluing theorem for Ricci positive manifolds.

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1 Introduction

1.1 Motivation and main result

In recent years, there have been great efforts made to better understand the topology of moduli spaces of Riemannian metrics of positive scalar curvature on a smooth compact (usually spin) manifold; see Botvinnik and Gilkey [2], Botvinnik, Hanke, Schick and Walsh [3], Carr [4] and Hanke, Schick and Steimle [13]. Apart from results of Kreck and Stolz [16], Wraith [23], and Dessai, Klaus and Tuschmann [6] concerning path-connectivity, we know very little about topology of the corresponding moduli spaces of positive Ricci curvature metrics. (In this context we should also mention work of Crowley, Schick and Steimle [5] on the space of Ricci positive metrics on certain manifolds.) Whether or not there is any nontriviality in the higher homotopy groups of such moduli spaces is still an open question. Here we study the topology of its closest relative, the *observer moduli space* $\mathcal{M}_{x_0}^{\mathrm{Ric}>0}(S^n)$ of positive Ricci curvature metrics on the sphere S^n.

We denote by ds_n^2 the standard round metric on S^n, and by $[ds_n^2]$ its orbit in the moduli space $\mathcal{M}_{x_0}^{\mathrm{Ric}>0}(S^n)$. Here is our main result:

Main Theorem *For any $m \in \mathbb{N}$, there is an integer $N(m)$ such that for all odd $n > N(m)$, the group $\pi_i(\mathcal{M}_{x_0}^{\mathrm{Ric}>0}(S^n), [ds_n^2]) \otimes \mathbb{Q}$ is nontrivial when $i = 4k$ and $k \le m$.*

We would like to emphasize that the observer moduli space is indeed the most tractable moduli space of metrics. Let $\mathcal{R}(M)$ be the space of all metrics on a compact closed manifold M, and $\mathrm{Diff}(M)$ be the group of diffeomorphisms which acts naturally on $\mathcal{R}(M)$ by pull-back. Even though the space $\mathcal{R}(M)$ is contractible, the moduli space of all metrics, ie the orbit space $\mathcal{R}(M)/\mathrm{Diff}(M)$, could be very complicated since some metrics have nontrivial isometry groups. Hence, in general, the action of $\mathrm{Diff}(M)$ on the space of metrics $\mathcal{R}(M)$ is far from being tractable. Following ideas from gauge theory, we fix an observer, ie a base point $x_0 \in M$ together with a frame at the tangent space $T_{x_0}M$. We then obtain the *observer moduli space* $\mathcal{M}_{x_0}(M) := \mathcal{R}(M)/\mathrm{Diff}_{x_0}(M)$, where the gauge group $\mathrm{Diff}_{x_0}(M)$ fixes such an observer. It is easy to see that the gauge group $\mathrm{Diff}_{x_0}(M)$ acts freely on the space of metrics provided M is a connected manifold. Then the observer moduli space $\mathcal{M}_{x_0}(M)$ is homotopy equivalent to the classifying space $\mathrm{BDiff}_{x_0}(M)$, and the corresponding observer moduli space $\mathcal{M}_{x_0}^{\mathrm{Ric}>0}(M)$ of positive Ricci curvature metrics maps naturally to $\mathcal{M}_{x_0}(M)$; see below for more details.

The proof of the Main Theorem is based on an analogous theorem by Botvinnik, Hanke, Schick and Walsh [3] for the observer moduli space of positive *scalar* curvature metrics. Both proofs rely heavily on work of Farrell and Hsiang [8], Goette [9] and Hatcher. Techniques for constructing families of metrics are also required. In the scalar curvature case, this means a family version of surgery technique of Gromov and Lawson [12], described in Walsh [22]. Due to the flexibility of the scalar curvature and the strength of the Gromov–Lawson construction, this technique permits the detection of nontriviality for manifolds besides the sphere. Unsurprisingly, the Ricci curvature case requires a more delicate construction, which is based on a gluing theorem of Perelman. As yet, we have not demonstrated nontriviality beyond the case of the sphere.

1.2 The observer moduli spaces of metrics

Let M be a smooth closed connected manifold of dimension n. We denote by $\mathcal{R}(M)$, the space of all Riemannian metrics on M equipped with the smooth Whitney topology. For a metric $g \in \mathcal{R}(M)$, we denote by s_g and Ric_g its scalar and Ricci curvatures. We then consider the subspaces

$$\mathcal{R}^{s>0}(M) \subset \mathcal{R}(M) \quad \text{and} \quad \mathcal{R}^{\mathrm{Ric}>0}(M) \subset \mathcal{R}(M)$$

of metrics with positive scalar and positive Ricci curvatures, respectively. Let $\mathrm{Diff}(M)$ be the group of diffeomorphisms on M. This group acts on the space of metrics by pull-back:

$$\mathrm{Diff}(M) \times \mathcal{R}(M) \to \mathcal{R}(M), \quad (\phi, g) \mapsto \phi^* g.$$

Recalling that M is connected, we fix a base point $x_0 \in M$ which plays the role of an *observer* in a sense which will become clear shortly. Let $\mathrm{Diff}_{x_0}(M) \subset \mathrm{Diff}(M)$ be the subgroup of diffeomorphisms $\phi\colon M \to M$ with $\phi(x_0) = x_0$ and such that the derivative $d\phi_{x_0}\colon T_{x_0}M \to T_{x_0}M$ is the identity. This is the *observer diffeomorphism group* of M based at x_0.

As we have mentioned, the group $\mathrm{Diff}_{x_0}(M)$ acts freely on the space of metrics $\mathcal{R}(M)$ provided M is a connected manifold; see [3, Lemma 1.2]. The orbit space $\mathcal{M}_{x_0}(M) := \mathcal{R}(M)/\mathrm{Diff}_{x_0}(M)$ is the *observer moduli space of metrics on* M. Since the space $\mathcal{R}(M)$ is contractible and the action of $\mathrm{Diff}_{x_0}(M)$ on $\mathcal{R}(M)$ is *proper* (see Ebin [7]), the observer moduli space $\mathcal{M}_{x_0}(M)$ is homotopy equivalent to the classifying space $\mathrm{BDiff}_{x_0}(M)$ of the group $\mathrm{Diff}_{x_0}(M)$. In particular, we have a $\mathrm{Diff}_{x_0}(M)$–principal bundle:

$$\mathrm{Diff}_{x_0}(M) \to \mathcal{R}(M) \to \mathcal{M}_{x_0}(M).$$

By restricting the action of $\mathrm{Diff}_{x_0}(M)$ to the appropriate subspaces, we obtain the *observer moduli spaces*

$$\mathcal{M}^{s>0}_{x_0}(M) := \mathcal{R}^{s>0}(M)/\mathrm{Diff}_{x_0}(M) \quad \text{and} \quad \mathcal{M}^{\mathrm{Ric}>0}_{x_0}(M) := \mathcal{R}^{\mathrm{Ric}>0}(M)/\mathrm{Diff}_{x_0}(M)$$

of positive scalar and *of positive Ricci curvature metrics*, respectively. The inclusions of spaces of metrics $\mathcal{R}^{\mathrm{Ric}>0}(M) \subset \mathcal{R}^{s>0}(M) \subset \mathcal{R}(M)$ then induce maps of principal $\mathrm{Diff}_{x_0}(M)$–bundles:

$$\begin{tikzcd} \mathcal{R}^{\mathrm{Ric}>0}(M) \arrow[r] \arrow[d] & \mathcal{R}^{s>0}(M) \arrow[r] \arrow[d] & \mathcal{R}(M) \arrow[d] \\ \mathcal{M}^{\mathrm{Ric}>0}_{x_0}(M) \arrow[r, "\iota_1"] & \mathcal{M}^{s>0}_{x_0}(M) \arrow[r, "\iota_0"] & \mathcal{M}_{x_0}(M) \end{tikzcd} \tag{1-1}$$

We write $\iota := \iota_0 \circ \iota_1 \colon \mathcal{M}^{\mathrm{Ric}>0}_{x_0}(M) \to \mathcal{M}_{x_0}(M)$. The fibre bundles (1-1) give rise to the following commutative diagram, where the horizontal lines are Serre fibrations:

$$\begin{tikzcd} \mathcal{R}^{\mathrm{Ric}>0}(M) \arrow[r] \arrow[d] & \mathcal{M}^{\mathrm{Ric}>0}_{x_0}(M) \arrow[r, "\iota"] \arrow[d, "\iota_1"] & \mathcal{M}_{x_0}(M) \arrow[d, "\mathrm{Id}"] \\ \mathcal{R}^{s>0}(M) \arrow[r] & \mathcal{M}^{s>0}_{x_0}(M) \arrow[r, "\iota_0"] & \mathcal{M}_{x_0}(M) \end{tikzcd} \tag{1-2}$$

Let us take a Ricci positive metric g_0 as basepoint for the space $\mathcal{R}(M)$. Consider the induced diagram of homotopy group homomorphisms below:

$$\begin{tikzcd} \pi_i(\mathcal{M}^{\mathrm{Ric}>0}_{x_0}(M), [g_0]) \arrow[r, "\iota_*"] \arrow[d, "\iota_{1*}"] & \pi_i(\mathcal{M}_{x_0}(M), [g_0]) \arrow[d, "\mathrm{Id}"] \\ \pi_i(\mathcal{M}^{s>0}_{x_0}(M), [g_0]) \arrow[r, "\iota_{0*}"] & \pi_i(\mathcal{M}_{x_0}(M), [g_0]) \end{tikzcd} \tag{1-3}$$

Because $\mathcal{M}_{x_0}(M)$ is a model for $\mathrm{BDiff}_{x_0}(M)$, an element in the homotopy group $\pi_i(\mathcal{M}_{x_0}(M), [g_0])$ can be represented by a smooth fibre bundle $E \to S^i$ with fibre M. Hence to show that such an element lies in the image of ι_{0*}, it is enough to show that there exists a metric on the total space E which restricts to a psc metric on every fibre; see [3]. Here our task is more difficult: we have to construct such a metric on E which is fibrewise Ricci positive, and the methods used involve geometric constructions which are quite different from the positive scalar curvature case. This is one of the reasons why we restrict our attention to the case when $M = S^n$. Next, we focus on the geometric properties of the moduli space $\mathcal{M}_{x_0}(M)$.

1.3 The universal fibre metric

As we have mentioned earlier, the observer moduli space $\mathcal{M}_{x_0}(M)$ is homotopy equivalent to the classifying space $\mathrm{BDiff}_{x_0}(M)$.

We say that a fibre bundle $E \to X$ with fibre M is a *smooth M–fibre bundle* if its structure group is a subgroup of $\mathrm{Diff}_{x_0}(M)$. Now we consider the universal principal bundle $\mathcal{R}(M) \to \mathcal{M}_{x_0}(M)$. Here the group $\mathrm{Diff}_{x_0}(M)$ acts freely on $\mathcal{R}(M)$, and the Borel construction gives the universal smooth M–fibre bundle $E(M) \to \mathcal{M}_{x_0}(M)$, where $E(M) := \mathcal{R}(M) \times_{\mathrm{Diff}_{x_0}(M)} M$. Recall that the space $\mathcal{R}(M) \times_{\mathrm{Diff}_{x_0}(M)} M$ is defined as the quotient of $\mathcal{R}(M) \times M$ by the action of $\mathrm{Diff}_{x_0}(M)$ given by $\phi.(h,x) = ((\phi^{-1})^* h, \phi(x))$, where $\phi \in \mathrm{Diff}_{x_0}(M)$, $h \in \mathcal{R}(M)$ and $x \in M$.

Given that X is a paracompact Hausdorff space, recall that the isomorphism classes of principal $\mathrm{Diff}_{x_0}(M)$–bundles over X are in one-to-one correspondence with homotopy classes $[X, \mathcal{M}_{x_0}(M)]$ of maps $X \to \mathcal{M}_{x_0}(M)$. In particular, given a map $f\colon X \to \mathcal{M}_{x_0}(M)$, we obtain a commutative diagram

$$\begin{array}{ccc} E_f & \longrightarrow & E(M) \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & \mathcal{M}_{x_0}(M) \end{array}$$

where the bundle $E_f \to X$ is the pull-back of the universal smooth M–fibre bundle by the map f.

There is, however, a more refined structure which we can associate to such a bundle. The total space $E(M) = \mathcal{R}(M) \times_{\mathrm{Diff}_{x_0}(M)} M$ admits a "universal fibre metric" which we will now define. We begin with an arbitrary point $[h,x] \in \mathcal{R}(M) \times_{\mathrm{Diff}_{x_0}(M)} M$. The fibre at this point is of course diffeomorphic to M. Let us now consider the tangent space to this fibre. Suppose $(h,x), (h',x') \in \mathcal{R}(M) \times M$ both represent the point $[h,x] \in \mathcal{R}(M) \times_{\mathrm{Diff}_{x_0}(M)} M$. Then the tangent spaces $T_x M$ and $T_{x'} M$ are isomorphically related by the derivative map ϕ_* of some diffeomorphism $\phi \in \mathrm{Diff}_{x_0} M$ which satisfies $\phi(x) = x'$ and $h' = (\phi^{-1})^* h$. Thus, the tangent space to the fibre through $[h,x]$ can be thought of as the isomorphic identification of all tangent spaces $T_{x'} M$ where $x' \in M$ lies in the orbit of x under the action of $\mathrm{Diff}_{x_0} M$. Suppose now that $[u]$ and $[v]$ denote tangent vectors to the fibre at $[h,x]$ represented by tangent vectors $u, v \in T_x M$. We specify an inner product to the tangent space to the fibre at $[h,x]$ by the formula

$$\langle [u], [v] \rangle_{[h,x]} = h_x(u, v),$$

where h_x is the restriction of the Riemannian metric h to the tangent space $T_x M$. It is an easy exercise to show that this is well defined and varies smoothly over $E(M)$; see Tuschmann and Wraith [21, page 61]. Notice that this does not give a Riemannian metric on $E(M)$ as we only specify the inner product on fibres.

Given a map $f\colon X \to \mathcal{M}_{x_0}(M)$, this universal fibre metric then pulls back to a continuous fibrewise family of Riemannian metrics on E_f. More precisely, each fibre of the bundle $E_f \to X$, already diffeomorphic to M, is now equipped with a Riemannian metric which depends continuously on X. Clearly, varying the map f by a homotopy alters the fibrewise metric structure of the bundle.

Suppose, on the other hand, we begin with a continuous fibrewise family of metrics on an M–fibre bundle over X. Notice that the choice of basepoint $x_0 \in M$ gives rise (by construction) to a section of the universal bundle $E(M)$, which in turn produces a section in any pull-back of $E(M)$. We can think of this as a natural family of "basepoints" in the fibres. Equally, a choice of frame at the basepoint gives rise to a corresponding family of frames spanning the tangent space to the fibre at each point along the section. If we now identify each fibre with a standard copy of M in any way provided that both the basepoints and their tangent frames are preserved, then by pulling back metrics, we obtain a well-defined and continuous map $f\colon X \to \mathcal{M}_{x_0}(M)$. To see that this is well defined, note that any two such identifications of a fibre with the standard copy of M differ by an element of $\mathrm{Diff}_{x_0}(M)$, and hence the resulting pull-back metrics represent the same point in the moduli space $\mathcal{M}_{x_0}(M)$. The continuity of f follows automatically from the continuity of the family of fibrewise metrics. Thus, we obtain a one-to-one correspondence between maps $X \to \mathcal{M}_{x_0}(M)$ and fibrewise metrics on the M–fibre bundle corresponding to this map.

Assuming X is the sphere S^i, we return to the homomorphism of homotopy groups

$$\iota_*\colon \pi_i(\mathcal{M}^{\mathrm{Ric}>0}_{x_0}(M), [g_0]) \to \pi_i(\mathcal{M}_{x_0}(M), [g_0])$$

induced by the inclusion $\iota\colon \mathcal{M}^{\mathrm{Ric}>0}_{x_0}(M) \hookrightarrow \mathcal{M}_{x_0}(M)$. Let $f\colon S^i \to \mathcal{M}_{x_0}(M)$ represent an element of $\pi_i(\mathcal{M}_{x_0}(M), [g_0])$. This element determines (and is determined by) an M–bundle $E_f \to S^i$ as above, together with a fibrewise family of metrics on E_f. Thus, it is possible to lift this element of $\pi_i(\mathcal{M}_{x_0}(M), [g_0])$ to an element of $\pi_i(\mathcal{M}^{\mathrm{Ric}>0}_{x_0}(M), [g_0])$, provided we can construct a fibrewise family of positive Ricci curvature metrics on E_f.

1.4 The work of Farrell and Hsiang

At this stage we have established that lifting an element of $\pi_i(\mathcal{M}_{x_0}(M), [g_0])$ to $\pi_i(\mathcal{M}_{x_0}^{\mathrm{Ric}>0}(M), [g_0])$ involves the construction of a family of fibrewise Ricci positive Riemannian metrics on some bundle over S^i. However, we have not yet discussed the particular elements in the homotopy groups of $\mathcal{M}_{x_0}(M)$ which we plan to lift. It is here that we recall a result of Farrell and Hsiang [8] which identifies the rational homotopy groups of $\mathrm{BDiff}_{x_0}(S^n)$ in a stable range, using algebraic K–theory and Waldhausen K–theory computations. Recalling that $\mathcal{M}_{x_0}(S^n)$ is homotopy equivalent to the classifying space $\mathrm{BDiff}_{x_0}(S^n)$, the result of these computations can be stated as follows.

Theorem 1 (Farrell and Hsiang [8]) *For any $m \in \mathbb{N}$, there is an integer $N(m)$ such that for all odd $n > N(m)$ and $i \le 4m$,*

$$\pi_i \mathcal{M}_{x_0}(S^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } n \text{ is odd and } i \equiv 0 \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for appropriate i, we now have lots of nontrivial groups $\pi_i(\mathcal{M}_{x_0}(S^n), [g_0]) \otimes \mathbb{Q}$. This also explains the hypotheses of the main theorem.

This leaves the question of which S^n–bundles over S^i $(i \equiv 0 \bmod 4)$ can represent the nontrivial elements of $\pi_i(\mathcal{M}_{x_0}(S^n), [g_0]) \otimes \mathbb{Q}$. It turns out that these elements can be represented by so-called "Hatcher bundles". In Section 3 we will provide a description of these, based on the work of Goette, Igusa and Williams [9; 11; 10; 15]. Our approach to the topological construction of Hatcher bundles is guided by the geometric constructions we must subsequently perform, namely the production of fibrewise Ricci positive metrics. These metric issues will be addressed in Section 4, and will involve a generalized version of a powerful gluing theorem due to Perelman. Perelman's theorem and our generalization of this is the subject of Theorem 2.

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2 Gluing manifolds and a theorem of Perelman

2.1 The gluing construction

The purpose of this section is to present a theorem of Perelman which allows for the construction of Ricci positive metrics by a metric gluing procedure on Ricci positive manifolds with compact isometric boundaries, subject to a boundary convexity condition. This result is the principal geometric tool used in achieving our goal of obtaining a fibrewise family of positive Ricci curvature metrics on the total space of a Hatcher bundle. Perelman's theorem is originally published in [19] and justified with a brief outline, omitting the details. Following this outline, we provide a comprehensive proof addressing all of these missing details. This thorough account is important for establishing the family version of Perelman's theorem, which appears at the end of this section.

Aside from establishing a family version, we were in part motivated by the prospect of promoting Perelman's theorem to a wider audience. On first encountering this very useful result, it seemed to us that it was not widely known. We have since learned that some of our work in this section overlaps with results of Menguy in [18]. Moreover, among experts in the construction of spaces with Ricci curvature bounds, Perelman's result (as well as many significant improvements by Menguy) is frequently cited. That said, we hope that those unfamiliar with it may find the details and intuition we provide helpful.

We begin with a brief review of the notion of gluing smooth manifolds, something we make extensive use of throughout the paper. Consider a pair of smooth n–dimensional manifolds, M_1 and M_2, each with compact nonempty boundary. We further assume that ∂M_1 and ∂M_2 are diffeomorphic via a diffeomorphism $\phi\colon \partial M_1 \to \partial M_2$. From this, we may form the adjunction space, $W = M_1 \cup_\phi M_2$, obtained as the quotient of $M_1 \sqcup M_2$ by identifying each $x \in \partial M_1$ with $\phi(x) \in \partial M_2$. In particular, the quotient map embeds both M_1 and M_2 into this space. For simplicity then, we identify M_1 and M_2 with their images in W and write $X = \partial M_1 = \partial M_2$. Consider collar neighbourhoods $\partial M_1 \times (-\delta, 0]$ and $\partial M_2 \times [0, \delta)$ about X for some small $\delta > 0$, for example determined by the normal coordinate from ∂M_1 and ∂M_2 with respect to some choice of metrics on M_1 and M_2. Denote by N the union of the images of these collar neighbourhoods in W. We then have a homeomorphism between $X \times (-\delta, \delta)$ and N given by

$$(x, r) \mapsto \begin{cases} (m_1, r) & \text{if } r \le 0, \\ (\phi(m_1), r) & \text{if } r \ge 0, \end{cases}$$

where $x \in X$ is the equivalence class $x = \{m_1, \phi(m_1)\}$ for some $m_1 \in \partial M_1$.

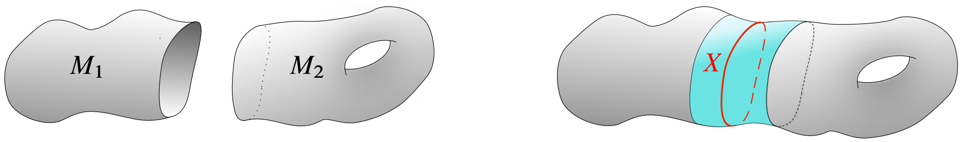

Figure 1: The manifolds with boundary M_1 and M_2 (left) along with the adjunction space W and tubular neighbourhood N of $X \subset W$ (right).

We can now use this to give N a differentiable structure, by pulling back the standard differentiable structure on $X \times (-\delta, \delta)$ via the inverse homeomorphism. Finally, we extend this differentiable structure over M_1 and M_2 to give a differentiable structure on W. Although there are many choices involved in this construction, leading to many possible differentiable structures, it is a well known fact that the diffeomorphism type of the resulting smooth manifold W is independent of these choices; see [14, Chapter 8, Section 2]. We now consider such a gluing in the Riemannian setting, equipping M_1 and M_2 with Riemannian metrics h_1 and h_2. Let us assume that the restrictions of these metrics to their respective boundaries are isometric via ϕ. More precisely, we assume

$$h_1|_{\partial M_1} = \phi^* h_2|_{\partial M_2}.$$

This automatically leads to a well-defined C^0–metric $h = h_1 \cup_\phi h_2$ on $M_1 \cup_\phi M_2$. Notice that this adjunction metric is smooth if and only if it is smooth in a collar neighbourhood of $X \subset M_1 \cup_\phi M_2$. In view of the adjunction space discussion above, this will be the case if the metric $h_1|_{\partial M_1 \times (-\delta, 0]}$ glues smoothly with $(\phi \times \mathrm{Id}_{[0,\delta)})^*(h_2|_{\partial M_2 \times [0,\delta)})$.

2.2 The theorem of Perelman

We will be interested in smoothing the above metric $h_1 \cup_\phi h_2$ on $M_1 \cup_\phi M_2$ within positive Ricci curvature in the case where h_1 and h_2 individually have positive Ricci curvature. This is not always possible. However, the following theorem of Perelman shows that under certain additional assumptions involving the second fundamental form of h_1 and h_2 at the boundary, such a smoothing can be performed.

Theorem 2 *Let (M_1, h_1) and (M_2, h_2) be a pair of Riemannian manifolds with positive Ricci curvature and compact nonempty boundaries, and let $\phi\colon (\partial M_1, h_1|_{\partial M_1}) \to (\partial M_2, h_2|_{\partial M_2})$ be an isometry of the boundaries. Suppose that the second fundamental forms II_1 and II_2 of ∂M_1 and ∂M_2 with respect to the inward normals satisfy the condition $\mathrm{II}_1 + \phi^* \mathrm{II}_2 > 0$. Then the C^0–metric $h = h_1 \cup_\phi h_2$ on the smooth* *manifold $M_1 \cup_\phi M_2$ can be replaced by a C^∞–metric with positive Ricci curvature, agreeing with h_1 and h_2 outside a neighbourhood of the glued boundaries.*

We will establish Perelman's theorem via a sequence of lemmas. The general setup is as follows. As above, we will denote by X the hypersurface of $M_1 \cup_\phi M_2$ along which M_1 and M_2 are joined, and assume that the normal parameter t through the hypersurface X gives rise to collar neighbourhoods $\partial M_2 \times [0,\delta)$ in M_2 and $\partial M_1 \times (-\delta, 0]$ in M_1 for some $\delta > 0$. Since we will be working exclusively in a collar neighbourhood of X, for convenience we can simply relabel the metric $(\phi \times \mathrm{Id}_{[0,\delta)})^*(h_2|_{\partial M_2 \times [0,\delta)})$ by h_2, assume that $\partial M_1 = \partial M_2$, and assume that ϕ is the identity map. Thus from now on we will write $h_1 \cup h_2$ for the C^0–metric in the theorem, and $M_1 \cup M_2$ for the manifold. The normal parameter t to X measures the distance from X with respect to $h = h_1 \cup h_2$, with X corresponding to $t = 0$. Observe that $M_1 \cup M_2$ has a smooth topological structure (though not a smooth metric structure) and that with respect to this structure t is smooth.

Choose a small parameter $\epsilon > 0$. (We will say more about an appropriate size for ϵ later.) Our next task is to write down a new metric on $X \times [-\epsilon, \epsilon]$ which joins with h_1 for $t < -\epsilon$ and h_2 for $t > \epsilon$ to give a C^1–metric on $M_1 \cup M_2$. This new metric will take the form $dt^2 + g(t)$. The metrics h_i for $i = 1, 2$ induce metrics on the hypersurfaces at constant distance t from X. We will denote these induced metrics by $h_i(t)$, and so in a neighbourhood of X we can write $h_i = dt^2 + h_i(t)$. We then set $g(t)$ to be the following cubic expression in t:

$$
\begin{aligned}
g(t) = {} & \frac{t+\epsilon}{2\epsilon} h_2(\epsilon) - \frac{t-\epsilon}{2\epsilon} h_1(-\epsilon) \\
& + \frac{(t-\epsilon)^2(t+\epsilon)}{4\epsilon^2} \Big[h_1'(-\epsilon) - \frac{1}{2\epsilon}[h_2(\epsilon) - h_1(-\epsilon)] \Big] \\
& + \frac{(t+\epsilon)^2(t-\epsilon)}{4\epsilon^2} \Big[h_2'(\epsilon) - \frac{1}{2\epsilon}[h_2(\epsilon) - h_1(-\epsilon)] \Big].
\end{aligned}
\tag{2-1}
$$

For convenience, we will assume from now on that the topological product structure in a neighbourhood of X extends over $X \times [-2\epsilon, 2\epsilon]$ and that the metrics h_i take the form $dt^2 + h_i(t)$ for $t \in [-2\epsilon, 0]$ when $i = 1$, and respectively for $t \in [0, 2\epsilon]$ when $i = 2$.

Lemma 3 *Assume that the metrics h_1 and h_2 satisfy the hypotheses in Theorem 2, and suppose $g(t)$ is as in (2-1). Then given any $\epsilon > 0$, the metric $\tilde{h}$ obtained from gluing h_1 to $dt^2 + g(t)$ at $t = -\epsilon$, and $dt^2 + g(t)$ to h_2 at $t = \epsilon$, is C^1 at $t = \pm\epsilon$, and smooth elsewhere.*

Proof First, we find the t–derivative of this metric. By a straightforward calculation,

$$g'(t) = \frac{1}{2\epsilon}[h_2(\epsilon)-h_1(-\epsilon)] + \frac{2(t^2-\epsilon^2)+(t-\epsilon)^2}{4\epsilon^2}\Big[h_1'(-\epsilon)-\frac{1}{2\epsilon}[h_2(\epsilon)-h_1(-\epsilon)]\Big] + \frac{2(t^2-\epsilon^2)+(t+\epsilon)^2}{4\epsilon^2}\Big[h_2'(\epsilon)-\frac{1}{2\epsilon}[h_2(\epsilon)-h_1(-\epsilon)]\Big]. \tag{2-2}$$

It is now an easy exercise to check that the metric g forms a C^1 join with the h_i at $t=\pm\epsilon$. (The metric g is of course smooth.) □

With a view to studying the curvature of $dt^2+g(t)$, our next task is to investigate $g''(t)$. An easy calculation shows that

$$g''(t) = \frac{1}{4\epsilon^2}(6t-2\epsilon)\Big[h_1'(-\epsilon)-\frac{1}{2\epsilon}[h_2(\epsilon)-h_1(-\epsilon)]\Big] + \frac{1}{4\epsilon^2}(6t+2\epsilon)\Big[h_2'(\epsilon)-\frac{1}{2\epsilon}[h_2(\epsilon)-h_1(-\epsilon)]\Big].$$

We will investigate the limiting behaviour of $g''(\pm\epsilon)$ as $\epsilon\to 0$. At $t=\epsilon$ we have

$$g''(\epsilon) = \frac{1}{\epsilon}\Big[h_1'(-\epsilon)+2h_2'(\epsilon)-\frac{3}{2}\frac{h_2(\epsilon)-h_1(-\epsilon)}{\epsilon}\Big]. \tag{2-3}$$

Consider the term $(h_2(\epsilon)-h_1(-\epsilon))/\epsilon$ in (2-3). As $\epsilon\to 0$ we see by l'Hôpital's rule that the value of the limit is $h_1'(0)+h_2'(0)$, where $h_1'(0)$ and $h_2'(0)$ are to be interpreted as one-sided derivatives. Clearly, the overall limit of the bracketed term in (2-3) is

$$h_1'(0)+2h_2'(0)-\tfrac{3}{2}(h_1'(0)+h_2'(0)) = \tfrac{1}{2}(h_2'(0)-h_1'(0)).$$

A similar calculation shows that the corresponding term in $\lim_{\epsilon\to 0} g''(-\epsilon)$ yields exactly the same expression. Thus we obtain

$$\lim_{\epsilon\to 0}\epsilon g''(\pm\epsilon) = \tfrac{1}{2}(h_2'(0)-h_1'(0)).$$

Finally, observe that $g''(t)$ has a linear dependence on t. Thus if $g''(-\epsilon)$ and $g''(\epsilon)$ have the same sign, then this sign persists for all $t\in[-\epsilon,\epsilon]$.[1]

Lemma 4 *Under the Perelman second fundamental form condition from Theorem 2, given any $A>0$, there exists $\epsilon_0=\epsilon_0(A,h_1,h_2)>0$ such that for all positive $\epsilon<\epsilon_0$,*

[1] Notice what we have used so far. For the C^1 cubic expression we require no assumptions. In order to obtain the limiting formula for the second derivative we only need that the original metric on the union be continuous at $t=0$.

the metric $g(t)$ defined in equation (2-1) satisfies

$$g''(t)(u,u) < -A|u|^2,$$

where u is any vector tangent to a hypersurface $t = \text{constant}$ for $t \in (-\epsilon,\epsilon)$.

Proof To begin with, let u be any fixed vector tangent to X at some point $x_0 \in X$. Extend u to a local vector field on X, and then further extend by parallel translation in the t direction to obtain a vector field in a neighbourhood of the line (t, x_0) for $t \in (-\epsilon,\epsilon)$. We will also denote this vector field by u.

Define the *normal curvature function* $k(u)$ by $k(u) = \langle \nabla_u \partial_t, u\rangle$. We could equally write $k(u) = \mathrm{II}(u,u)$, where II is the second fundamental form of the hypersurface $t = \text{constant}$ for $t \in (-\epsilon,\epsilon)$ with respect to the normal direction $-\partial/\partial t$.

Now observe that we can rearrange the definition of $k(u)$ to yield $k(u) = \frac{1}{2}(\partial/\partial t)\langle u,u\rangle$, which is just $\frac{1}{2}g'(t)(u,u)$. Differentiating with respect to t we obtain $k'(u) = \frac{1}{2}g''(t)(u,u)$.

Considering the difference of the normal curvatures for u across the ϵ–neighbourhood of X, we obtain

$$\frac{1}{2}\lim_{\epsilon\to 0}\big(g'(\epsilon)(u,u) - g'(-\epsilon)(u,u)\big) = \frac{1}{2}\big(h_2'(0)(u,u) - h_1'(0)(u,u)\big). \tag{2-4}$$

Bearing in mind that the second fundamental form hypothesis in Theorem 2 involves the inward normal directions for both M_1 and M_2, we define $k_i(u) = \mathrm{II}_i(u,u)$ for $i = 1, 2$ for u tangent to X, and note that

$$\begin{aligned} k_1(u) &= \lim_{t\to 0^-} k(u) = \frac{1}{2}\lim_{t\to 0^-} g'(t)(u,u) = \frac{1}{2}h_1'(0)(u,u), \\ k_2(u) &= -\lim_{t\to 0^+} k(u) = -\frac{1}{2}\lim_{t\to 0^+} g'(t)(u,u) = -\frac{1}{2}h_2'(0)(u,u). \end{aligned}$$

Thus both sides of (2-4) are equal to $-(k_1(u)+k_2(u))$. Now the Perelman second fundamental form condition is equivalent to demanding that $k_1(u) + k_2(u) > 0$, which means that both sides of (2-4) are negative, and hence $\lim_{\epsilon\to 0}\epsilon g''(\pm\epsilon) = \frac{1}{2}(h_2'(0) - h_1'(0)) < 0$. Therefore given any $A > 0$, since $g''(t)$ is linear in t, we can bound $g''(t)(u,u)$ above by $-A|u|^2$ by choosing ϵ sufficiently small. It is also clear that the upper bound on $\epsilon > 0$ for which the above upper bound on $g''(t)$ holds depends only on A, h_1 and h_2, as claimed. □

It follows immediately from Lemma 4 that $k'(u)$ can similarly be bounded above. The relevance of $k'(u)$ is that it can be rewritten in terms of the curvature tensor applied to u and $\partial_t := \partial/\partial t$, and we can use the arbitrarily large negative feature of $k'(u)$ to produce an arbitrarily large positive lower bound for $R(\partial_t, u, u, \partial_t)$.

Lemma 5 *Given any $B > 0$, there exists $\epsilon_1 = \epsilon_1(B, h_1, h_2) > 0$ such that for all positive $\epsilon < \epsilon_1$, the curvature tensor of the metric $dt^2 + g(t)$ satisfies*

$$R(\partial_t, u, u, \partial_t) > B|u|^2,$$

where u is any vector tangent to a hypersurface $t = \text{constant}$ for $t \in (-\epsilon, \epsilon)$.

Proof We have

$$\begin{aligned} k'(u) = \partial_t \langle \nabla_u \partial_t, u \rangle &= \langle \nabla_{\partial_t} \nabla_u \partial_t, u \rangle + \langle \nabla_u \partial_t, \nabla_{\partial_t} u \rangle \\ &= \langle \nabla_{\partial_t} \nabla_u \partial_t, u \rangle + |S(u)|^2, \end{aligned}$$

where $S(u)$ denotes the shape operator of the hypersurfaces given by constant values of t, and where we have used the fact that $\nabla_{\partial_t} u = \nabla_u \partial_t$ since $[\partial_t, u] \equiv 0$. On the other hand, we have

$$\begin{aligned} R(\partial_t, u, u, \partial_t) &= -R(\partial_t, u, \partial_t, u) \\ &= -[\langle \nabla_{\partial_t} \nabla_u \partial_t, u \rangle - \langle \nabla_u \nabla_{\partial_t} \partial_t, u \rangle] \\ &= -\langle \nabla_{\partial_t} \nabla_u \partial_t, u \rangle \end{aligned}$$

as $\nabla_{\partial_t} \partial_t \equiv 0$. Thus we conclude that

$$R(\partial_t, u, u, \partial_t) = -k'(u) + |S(u)|^2. \tag{2-5}$$

In particular, since $k'(u) = \frac{1}{2} g''(u, u)$, and $g''(t)(u, u) < -A|u|^2$ by Lemma 4 for any given $A > 0$ provided ϵ is sufficiently small, we conclude that $R(\partial_t, u, u, \partial_t)$ can be bounded below as claimed. □

Lemma 6 *For all $\epsilon > 0$ sufficiently small (depending on h_1 and h_2), we have* $\mathrm{Ric}_{dt^2 + g(t)} > 0$ *for $t \in (-\epsilon, \epsilon)$.*

Proof We begin by observing that any Ricci curvature expression must contain the large positive term from Lemma 5, and we therefore get positive Ricci curvature for the metric $dt^2 + g(t)$ provided we can show that all other curvature tensor expressions remain bounded as $\epsilon \to 0$. It is easily checked that this boundedness reduces to showing that $\| R(u_i, u_j) u_k \|$ is bounded above by some constant independent of ϵ for all vectors u_i, u_j and u_k tangent to X which are unit with respect to say $h_1(0) = h_2(0)$.

With the above curvature expression (2-5) in mind, consider the first and second derivatives of the metric $g(t)$ in directions orthogonal to t. The quantities $h_1(-\epsilon)$, $h_1'(-\epsilon)$, $h_2(\epsilon)$ and $h_2'(\epsilon)$ and their derivatives can clearly be bounded independent of ϵ as a consequence of the compactness of X. We also know that the other terms appearing in (2-1) involving ϵ,

$$\frac{t\pm\epsilon}{2\epsilon}, \quad \frac{1}{2\epsilon}(h_2(\epsilon)-h_1(-\epsilon)), \quad \frac{(t\pm\epsilon)^2(t\mp\epsilon)}{4\epsilon^2},$$

all remain bounded for $t\in[-\epsilon,\epsilon]$ as $\epsilon\to 0$. Therefore the first and second derivatives of $g(t)$ orthogonal to the t direction must stay bounded independent of ϵ.

We also claim that the first derivative of $g(t)$ with respect to t is bounded independent of ϵ. We showed in Lemma 4 that for ϵ sufficiently small the sign of $g''(t)(u,u)$ is negative, from which we see that the values of $g'(t)(u,u)$ must lie between those at $t=\pm\epsilon$, and hence are bounded independent of ϵ. We also notice that boundedness can then also be deduced for $g'(t)(u,v)$ via the polarization formula for inner products.

By the compactness of X, we see from (2-2) that the derivatives of $g'(t)$ in X directions remain bounded as $\epsilon\to 0$.

We conclude that the norm $\|R(u_i,u_j)u_k\|$ is bounded for all $t\in[-\epsilon,\epsilon]$ independent of ϵ provided the curvature $R(u_i,u_j)u_k$ does not depend on the second derivative of the metric with respect to t. Without loss of generality assume that u_i, u_j and u_k are coordinate vector fields for some coordinate system on X extended to a coordinate system in a neighbourhood of X by the parameter t. The relevant expression for the components of $R(u_i,u_j)u_k$ in terms of Christoffel symbols is

$$R^l_{ijk}=\partial_i\Gamma^l_{jk}-\partial_j\Gamma^l_{ik}+\sum_{m=1}^{n}(\Gamma^m_{jk}\Gamma^l_{im}-\Gamma^m_{ik}\Gamma^l_{jm}), \tag{2-6}$$

where l runs over all possible subscripts, including t. The Christoffel symbols in (2-6) have at most one derivative with respect to t. Consequently, we obtain the desired boundedness property of $\|R(u_i,u_j)u_k)\|$, since expressed in terms of metric components, R^l_{ijk} involves the inverse components g^{kl}, the second derivatives of g in X directions, and the first derivatives of $g'(t)$ in X directions. These derivatives are all bounded, as discussed above, and the inverse components are bounded since for very small ϵ, $g^{kl}\approx h_i^{kl}$, where the latter components are clearly bounded. Thus we can fix a small $\epsilon>0$ so that the metric $dt^2+g(t)$ has positive Ricci curvature. □

Let us fix once and for all an $\epsilon > 0$ in accordance with Lemma 6. Our next goal is to show how to effect a C^2–smoothing of the metric $\widetilde{h}$ from Lemma 3 in some τ–neighbourhoods of $t = \pm\epsilon$ with $\tau \ll \epsilon$ in such a way that we will be able to maintain positive Ricci curvature. We will create the desired C^2–metric by quintic interpolation, in exactly the same way that we created a C^1–metric using a cubic interpolation. By considering metric components, it will suffice to prove a C^2–smoothing result for a function $f\colon \mathbb{R} \to \mathbb{R}$ over an interval $[-\tau, \tau]$.

Lemma 7 *Consider a function $f\colon \mathbb{R} \to \mathbb{R}$ which is C^1 at $t = 0$ and smooth otherwise. Assume that the second derivatives of $f(t)$ remain bounded as $t \to 0^{\pm}$. Then given $\epsilon > 0$, for all sufficiently small $\tau > 0$ there exists a C^2–function $\widetilde{f}(t)$ such that $f(t) = \widetilde{f}(t)$ for all $t \notin (-\tau, \tau)$, $\|f(t) - \widetilde{f}(t)\|_{C^1} < \epsilon$, and for all $t \in [-\tau, \tau]$, $\min\{f''(-\tau), f''(\tau)\} \le \widetilde{f}''(t) \le \max\{f''(-\tau), f''(\tau)\}$.*

Proof The idea is to replace $f(t)$ for $t \in [-\tau, \tau]$ by a quintic polynomial $p(t)$ which will agree to second order with $f(t)$ at $t = \pm\tau$.

Let $p(t) = \sum_{n=0}^{5} c_n t^n$, and suppose that $f(\tau) = a_0$, $f(-\tau) = b_0$, $f'(\tau) = a_1$, $f'(-\tau) = b_1$, $f''(\tau) = a_2$ and $f''(-\tau) = b_2$. Assuming that $p^{(i)}(\pm\tau) = f^{(i)}(\pm\tau)$ for $i = 0, 1, 2$ yields a $(6{\times}6)$–linear system with the c_n as the unknowns and the a_i and b_i as coefficients. Solving this system shows that the polynomial $p(t)$ is uniquely determined by the above requirements, and is equal to

$$p(t) = \frac{\tau^2(a_2-b_2)-3\tau(a_1+b_1)+3(a_0-b_0)}{16\tau^5}t^5 - \frac{-\tau(a_2+b_2)+(a_1-b_1)}{16\tau^3}t^4 - \frac{\tau^2(a_2-b_2)-5\tau(a_1+b_1)+5(a_0-b_0)}{8\tau^3}t^3 + \frac{-\tau(a_2+b_2)+3(a_1-b_1)}{8\tau}t^2 + \frac{\tau^2(a_2-b_2)-7\tau(a_1+b_1)+15(a_0-b_0)}{16\tau}t + \frac{\tau^2(a_2+b_2)-5\tau(a_1-b_1)}{16} + \tfrac{1}{2}(a_0+b_0). \tag{2-7}$$

Consider next the effect on $p(t)$ (for $t \in [-\tau, \tau]$) of letting $\tau \to 0$. As this limit is approached, the term involving t^5 in the above expression for $p(t)$ approaches $\frac{3}{16}(a_0 - b_0)(t^5/\tau^5)$, the t^3 term approaches $-\frac{5}{8}(a_0 - b_0)(t^3/\tau^3)$, and the first order term in t contributes $\frac{15}{16}(a_0 - b_0)(t/\tau)$.

Recalling that $|t| \le |\tau|$, the limits of the t^5–, t^3–, and t–terms are bounded by

$$\tfrac{3}{16}|a_0 - b_0|, \quad \tfrac{5}{8}|a_0 - b_0| \quad \text{and} \quad \tfrac{15}{16}|a_0 - b_0|,$$

respectively. Since $|t| \leq |\tau|$, the terms of degrees four and two in t contribute nothing in the limit, and the zeroth-order term yields $\frac{1}{2}(a_0 + b_0)$. In our case we can say more, however. Clearly, the coefficients a_i and b_i are functions of τ, ie $a_i = a_i(\tau)$ and $b_i = b_i(\tau)$ for $i = 0, 1, 2$, and since $f(t)$ is assumed C^1 at $t = 0$ we see that $\lim_{\tau\to 0} a_j(\tau) = \lim_{\tau\to 0} b_j(\tau) = f^{(j)}(0)$ for $j = 0, 1$. Thus we conclude that for τ sufficiently small, the polynomial $p(t)$ can C^0–approximate the constant function with value $\frac{1}{2}(a_0 + b_0) = a_0 = b_0 = f(0)$ over the interval $[-\tau, \tau]$ to within any desired degree.

By applying the same analysis to $p'(t)$, we see that for τ sufficiently small, $p'(t)$ C^0–approximates the constant function with value $\frac{1}{2}(a_1 + b_1) = a_1 = b_1 = f'(0)$ over the interval $[-\tau, \tau]$ to within any desired degree; ie by choosing τ sufficiently small, $p(t)$ will C^1–approximate $f(t)$ over $[-\tau, \tau]$ to within any desired accuracy. In order to see this, we note first that

$$\begin{aligned} \text{(2-8)} \qquad \lim_{\tau\to 0} \frac{a_0(\tau) - b_0(\tau)}{\tau} &= \lim_{\tau\to 0} \frac{f(\tau) - f(-\tau)}{\tau} \\ &= \lim_{\tau\to 0} \frac{f(\tau) - f(0)}{\tau} + \lim_{\tau\to 0} \frac{f(-\tau) - f(0)}{-\tau} \\ &= 2f'(0). \end{aligned}$$

Substituting this into the expression for $p'(t)$ obtained by differentiating (2-7), the only term which makes a contribution in the limit as $\tau \to 0$ is the fifth term on the right-hand side, ie

$$\frac{\tau^2(a_2 - b_2) - 7\tau(a_1 + b_1) + 15(a_0 - b_0)}{16\tau}.$$

In the limit we therefore need to consider

$$-\tfrac{7}{16}(a_1 + b_1) + \tfrac{15}{16}(a_0 - b_0).$$

As $t \to 0$ this yields

$$-\tfrac{14}{16} f'(0) + \tfrac{30}{16} f'(0) = f'(0),$$

as claimed.

Finally, we must consider the behaviour of $p''(t)$. Arguments analogous to the above show that for τ sufficiently small, $p''(t)$ can be C^0–approximated over $[-\tau, \tau]$ to within any desired degree of accuracy by the cubic

$$\text{(2-9)} \qquad \frac{5}{4}(a_2 - b_2)\frac{t^3}{\tau^3} - \frac{3}{4}(a_2 - b_2)\frac{t}{\tau} + \frac{1}{2}(a_2 + b_2).$$

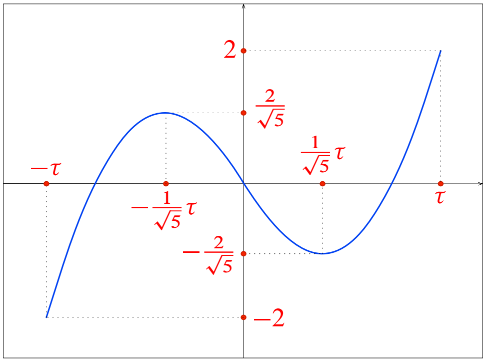

Figure 2: The graph of the function $q(t)$.

To understand the behaviour of this cubic, it clearly suffices to examine the function $q(t) = (5/\tau^3)t^3 - (3/\tau)t$ over $t \in [-\tau, \tau]$. An elementary calculation shows that the maximum and minimum values taken by $q(t)$ over $[-\tau, \tau]$ are $q(\tau) = 2$ and $q(-\tau) = -2$. This function is depicted in Figure 2. Thus the values of $q(t)$ for all other t in this interval lie between the endpoint values. It follows immediately that the same is true for $p''(t)$ with respect to its endpoint values. □

Proposition 8 *Let $\tilde{h}$ be the metric defined in Lemma 3, and over the neighbourhood $X \times [-2\epsilon, 2\epsilon]$ express this as $\tilde{h} = dt^2 + \tilde{h}(t)$, where $\tilde{h}(t)$ is the induced metric on the hypersurface $t = \text{constant}$. Suppose that $\epsilon > 0$ is chosen as in Lemma 6, so that $\mathrm{Ric}_{dt^2+g(t)} > 0$ for $t \in (-\epsilon, \epsilon)$. Then there exists $\tau > 0$ with $\tau \ll \epsilon$, and a C^2–metric $dt^2 + \tilde{g}(t)$ for $t \in [-2\epsilon, 2\epsilon]$ such that*

(i) $\mathrm{Ric}_{dt^2+\tilde{g}(t)} > 0$ *for* $t \in (-2\epsilon, 2\epsilon)$;

(ii) $\tilde{g}(t) = \tilde{h}(t)$ *for* $t \notin (\pm\epsilon - \tau, \pm\epsilon + \tau)$;

(iii) *the metrics $dt^2 + \tilde{g}(t)$ and $\tilde{h}$ are arbitrarily C^1–close on $X \times [-2\epsilon, 2\epsilon]$.*

Proof We choose a smooth coordinate system on X and extend in the obvious way to a coordinate system on $X \times [-2\epsilon, 2\epsilon]$. With respect to this system, we express the metrics $\tilde{h}(t)$ in terms of its components $\tilde{h}_{ij}(t)$, which are smooth when $t \neq \pm\epsilon$ and C^1–functions at $t = \pm\epsilon$. We then use the polynomial (2-7) to modify each function $\tilde{h}_{ij}(t)$ in τ–neighbourhoods of $t = \pm\epsilon$ to obtain functions $\tilde{g}_{ij}(t)$. In order to do this, however, we need to argue that the quantities $a_2(\tau)$ and $b_2(\tau)$ for each metric component (using the language of Lemma 7) remain bounded as $\tau \to 0$, as this was a hypothesis in Lemma 7. These quantities are second derivatives of the metric either side of $t=\pm\epsilon$, with ϵ now fixed. Thus the desired boundedness now follows automatically from the boundedness of the components of $h_1''(-\epsilon)$, $h_2''(\epsilon)$ and $g''(\pm\epsilon)$ for the given choice of $\epsilon>0$.

By choosing τ sufficiently small we can therefore bound the variation in the metric components $\tilde{g}_{ij}(t)$ and their first derivatives across $t\in[\pm\epsilon-\tau,\pm\epsilon+\tau]$ by an arbitrarily small constant, whereas the second derivatives vary between their values at the endpoints. As curvature is a C^2 phenomenon which depends linearly on the second derivatives of the metric, we can arrange for any open convex curvature condition satisfied by both "halves" of the C^1–metric (ie either side of $t=\pm\epsilon$) to continue to hold for the resulting C^2–metric, as τ can be chosen arbitrarily small. Since the positivity of the Ricci curvature is an open and convex condition we deduce that the our C^2–metric $dt^2+\tilde{g}(t)$ will have positive Ricci curvature for all τ sufficiently small. □

Proof of Theorem 2 Given Proposition 8, it remains to smooth the metric from C^2 to C^∞. By general smoothing theory for functions, we know that the set of C^2–functions on a smooth manifold is dense in the space of C^∞–functions (see for example [14, Theorem 2.6]). Thus we can make a C^2–arbitrarily small adjustment to our C^2–metric to render it smooth, and in so doing ensure the positivity of the Ricci curvature is preserved. This proves Theorem 2. □

Theorem 2 immediately gives us the following corollary, which will play a key role in Section 4.

Corollary 9 *The conclusion of Theorem 2 holds if the principal curvatures at both boundaries (with respect to the inward normals) are all positive.*

2.3 A family version of Perelman's theorem

We will also need a family version of Theorem 2, which allows us to perform simultaneous Ricci positive smoothings on the fibres of a bundle.

Theorem 10 *Let $\pi_i\colon E_i\to B$ for $i=1,2$ be smooth fibre bundles with fibre M_i and compact base B, where $\partial M_i\neq\varnothing$. Suppose that each of these bundles is equipped with a smoothly varying family of fibrewise Ricci positive metrics $\{h_i(b)\}_{b\in B}$ and that with respect to these metrics, there is a smoothly varying family of fibrewise isometries $\phi:=\{\phi_b\}_{b\in B}$ for the boundary bundles $\partial\pi_i\colon\partial E_i\to B$ (with fibre ∂M_i), that is, $\phi_b\colon\partial\pi_1^{-1}(b)\cong\partial\pi_2^{-1}(b)$ for each $b\in B$. Then provided the second fundamental* *forms of the fibre boundaries (with respect to inward normals) satisfy* $\mathrm{II}_1^b + \phi_b^* \mathrm{II}_2^b$ *for each* $b \in B$*, the fibrewise* C^0*–metric* $h := \{h_1(b) \cup_{\phi(b)} h_2(b)\}_{b \in B}$ *on* $E_1 \cup_\phi E_2$ *can be smoothed within fibrewise positive Ricci curvature in such a way that the resulting metric agrees with the original outside a neighbourhood of the glued boundaries.*

Proof The key observation is that in the proof of Theorem 2, the C^2–smoothing constructed only depends on the metrics together with two small positive parameters ϵ and τ. Now suppose we have a smooth variation of the metrics on M_1 and M_2, which nevertheless always satisfies the requirements of Theorem 2. It is clear that ϵ, the first chosen parameter in the construction which together with the given metrics determines the C^1–smoothing, can be chosen to vary continuously with the metric. Similarly the second parameter, τ, needed to construct the C^2–smoothing, can be chosen to vary continuously with the metrics and ϵ.

In the situation of the current theorem, it follows from the above observations and the compactness of B that we can make uniform choices for ϵ and τ which will work for all fibres in our bundles E_1 and E_2. Having made these choices, the C^2–metric smoothing performed after gluing each pair of fibres is then completely determined by the metrics on these fibres. Moreover, since this is a smoothing by polynomials, it follows trivially that the resulting metrics will vary smoothly from fibre to fibre.

Finally, the same argument as employed at the end of the proof of Theorem 2 shows that our fibrewise Ricci positive metric on $E_1 \cup_\phi E_2$ can be smoothed to class C^∞ within fibrewise positive Ricci curvature. (We could always extend our fibrewise C^2–metric to a global C^2–metric for which the intrinsic fibre metrics have positive Ricci curvature. This can then be globally smoothed by a C^2–arbitrarily small deformation, preserving the intrinsic positive Ricci curvature on the fibres, then restricting to the fibres yields the desired smooth fibrewise metric.) □

3 Hatcher bundles

3.1 The work of Goette and Igusa

The aim of this section is to review the construction and properties of certain smooth S^n–bundles over S^i which are known as “Hatcher bundles”. In short, a Hatcher bundle $E_\lambda \to S^i$ is a smooth S^n–bundle determined by an element $\lambda \in \ker J$, where $J\colon \pi_{i-1} O(p) \to \pi_{i-1+p} S^p$ is the J–homomorphism, and where $0 < p < n$. A

Hatcher bundle $E_\lambda \to S^i$ has structure group $\mathrm{Diff}_{x_0}(S^n)$ and thus is classified by some map $f_\lambda\colon S^i \to \mathrm{BDiff}_{x_0}(S^n)$. We then say that a Hatcher bundle $E_\lambda \to S^i$ represents the element $[f_\lambda] \in \pi_i \mathrm{BDiff}_{x_0}(S^n)$; below we make the identification $\mathrm{BDiff}_{x_0}(S^n) = \mathcal{M}_{x_0}(S^n)$.

In the introduction we stated a theorem of Farrell and Hsiang (Theorem 1) concerning the rational homotopy groups $\pi_i \mathcal{M}_{x_0}(S^n) \otimes \mathbb{Q}$. In fact, each element of those groups may be represented by a Hatcher bundle.

The following summarizes the results of [15, Theorem 6.5.5] and [9, Theorem 5.13]:

Theorem 11 *Suppose that n and k satisfy the hypotheses of Theorem 1 so that $\pi_{4k}\mathcal{M}_{x_0}(S^n) \otimes \mathbb{Q} \cong \mathbb{Q}$. Then, for each element $[f] \in \pi_{4k}\mathcal{M}_{x_0}(S^n) \otimes \mathbb{Q}$, there is an integer p with $0 < p < n$, and an element $\lambda \in \ker J$, where $J\colon \pi_{4k-1}O(p) \to \pi_{4k-1+p}S^p$ is the J–homomorphism, such that the Hatcher bundle E_λ represents the element $[f]$.*

Remark Theorem 6.5.5 from [15] shows that Hatcher bundles represent the above elements $[f] \in \pi_{4k}\mathcal{M}_{x_0}(S^n) \otimes \mathbb{Q}$. The nontriviality of these elements is detected by computing the *higher Franz–Reidemeister torsion* for the corresponding Hatcher bundles. Below we essentially use the construction from [9, Section 5], and [9, Theorem 5.13] shows the nontriviality of the same Hatcher bundles by computing a relevant *analytical torsion class* of those bundles.

We recall that the key feature of these bundles is that they are *exotic smooth S^n–bundles*, in the sense that each one is homeomorphic to, but not diffeomorphic to, the trivial bundle $S^i \times S^n \to S^i$. We will develop Goette's construction so as to provide the appropriate setting for our geometric arguments in the next section. An in-depth description of these bundles and their properties is given in [9], and we refer the reader to that paper for further details.

3.2 Preliminary constructions

Throughout this section, we will assume that n is odd and is sufficiently large for all of our purposes. The groups $\pi_i \mathcal{M}_{x_0}(S^n) \otimes \mathbb{Q}$ are trivial unless $i = 4k$ for appropriate k, and so we will consider only bundles which have base manifold S^{4k} and fibre S^n.

Let us begin with the trivial bundle $S^{4k} \times S^n \to S^{4k}$. By decomposing the fibre sphere S^n into a pair of northern and southern hemispherical discs, D^n_+ and D^n_-,

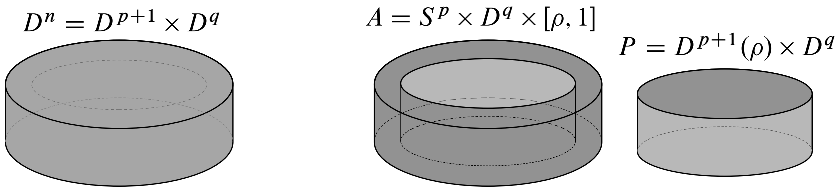

Figure 3: The decomposition of the fibre disc D^n into A and P.

we can decompose the entire bundle into a pair of disc bundles, $S^{4k} \times D^n_+ \to S^{4k}$ and $S^{4k} \times D^n_- \to S^{4k}$, glued together in the obvious way. Therefore the trivial bundle $S^{4k} \times S^n \to S^{4k}$ can be regarded as the double of the trivial disc bundle $S^{4k} \times D^n \to S^{4k}$. We will always assume that discs are closed unless otherwise stated.

To construct a Hatcher bundle, we will make certain adjustments to the trivial D^n–bundle over S^{4k} to obtain a smooth bundle which is homeomorphic to, but not diffeomorphic to, the trivial disc bundle. We will then form the double of this exotic D^n–bundle to obtain the desired exotic S^n–bundle over S^{4k}, which will represent a nontrivial element of $\pi_{4k}\mathcal{M}_{x_0}(S^n) \otimes \mathbb{Q}$.

We begin with the trivial disc bundle $S^{4k} \times D^n \to S^{4k}$. The fibre D^n decomposes as

$$D^n = D^{p+1} \times D^q = (D^{p+1}(\rho) \times D^q) \cup ((S^p \times [\rho, 1]) \times D^q),$$

where $p+q+1=n$, $\rho \in (0,1)$, and $D^{p+1}(\rho) \times D^q$ is a smaller version of the original disc product (with the D^{p+1} factor having radius ρ) surrounded by an annular region $(S^p \times [\rho, 1]) \times D^q$. The integers p and q may be assumed to be positive; in fact at various stages in the construction, it is necessary to allow both p and q to be large. It will be convenient for later considerations to reorder the factors and write the annular region as $S^p \times D^q \times [\rho, 1]$. Henceforth we will denote this by A and the remaining piece, $D^{p+1}(\rho) \times D^q$, by P.[2] Thus, as illustrated in Figure 3, we have

$$D^n = A \cup P.$$

The regions A and P share a common piece of boundary, $S^p \times D^q \times \{\rho\}$, and are glued together via the identity map on $S^p \times D^q$. The trivial bundle $S^{4k} \times D^n \to S^{4k}$, therefore, can be thought of as a union of subbundles $S^{4k} \times A \to S^{4k}$ and $S^{4k} \times P \to S^{4k}$, glued together in the obvious way.

[2]This smaller product of discs resembles an ice-hockey puck, hence the notation.

From now on we will write $A_y = \{y\} \times A$ and $P_y = \{y\} \times P$ to denote the fibres at $y \in S^{4k}$ of the respective subbundles $S^{4k} \times A \to S^{4k}$ and $S^{4k} \times P \to S^{4k}$. Below we will specify a smooth diffeomorphism

$$\Lambda_y\colon S^p \times D^q \to S^p \times D^q \tag{3-1}$$

over each $y \in S^{4k}$, where the domain is $(\partial D^{p+1}) \times D^q \subset \partial P_y$ and the target space is the product $S^p \times D^q \times \{\rho\} \subset \partial A_y$. The idea will be to replace the identity map on $S^p \times D^q$, which glues P_y to A_y to form $D^{p+1} \times D^q$, with the map Λ_y.

Before we can begin the construction we will need a further decomposition: that of the base manifold S^{4k} into northern and southern hemispherical discs

$$S^{4k} = D^{4k}_+ \cup D^{4k}_-.$$

Over the disc D^{4k}_- we take the trivial bundle $D^{4k}_- \times D^n \to D^{4k}_-$, that is, we define the map Λ_y to be the identity map on $S^p \times D^q$ for all $y \in D^{4k}_-$. We therefore need to specify the maps Λ_y for $y \in D^{4k}_+$ in order to describe the bundle over D^{4k}_+, and finally we need to show how to glue the two bundles together over $S^{4k-1} = \partial D^{4k}_- = \partial D^{4k}_+$.

Over the disc D^{4k}_+ we will actually work with a slightly different, though topologically equivalent, annulus A'_y, which we will define below. We will also work with diffeomorphisms Λ_y as above, however we must adjust the target space to lie in the boundary of A'_y. Gluing the P_y to A'_y creates a fibre bundle over D^{4k}_+ with fibres $P_y \cup_{\Lambda_y} A'_y$.

In order to make these constructions, let us first suppose we have a collection of embeddings

$$\bar{\lambda}_y\colon S^p \times D^q \to S^p \times D^q$$

for $y \in D^{4k}_+$ which vary smoothly with y. For any given $y \in D^{4k}_+$, the image of the embedding $\bar{\lambda}_y$, which we denote by $\operatorname{Im}\bar{\lambda}_y$, is schematically depicted in Figure 4. We will define the annulus A'_y by

$$A'_y = \{y\} \times \operatorname{Im}\bar{\lambda}_y \times [\rho, 1] \subset \{y\} \times S^p \times D^q \times [\rho, 1].$$

We will furthermore define the diffeomorphism Λ_y to be simply the map $\bar{\lambda}_y$ with target space $\operatorname{Im}\bar{\lambda}_y$. Thus we can glue P_y to A'_y using Λ_y, by identifying the points

$$(y, x) \in \{y\} \times (S^p \times D^q) \subset \partial P_y \quad \text{and} \quad (y, \bar{\lambda}_y(x), \rho) \in \{y\} \times \operatorname{Im}\bar{\lambda}_y \times \{\rho\} \subset \partial A'_y,$$

where $x \in S^p \times D^q$. The spaces P_y and A'_y (as a subset of $\{y\} \times S^p \times D^q \times [\rho, 1]$) are depicted in Figure 5. Applying this gluing fibrewise for all $y \in D^{4k}_+$ gives rise to

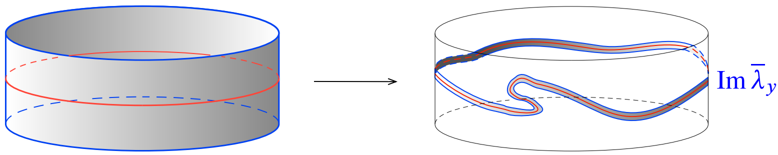

Figure 4: An embedding $\overline{\lambda}_y$ from $S^p \times D^q$ (left) into $S^p \times D^q$ (right).

the desired bundle over D^{4k}_+. For convenience we will denote the bundles over $D^{4k}_\pm$ by $\mathcal{E}_\pm \to D^{4k}_\pm$. We claim that for each $y \in D^{4k}_+$ we have a canonical diffeomorphism

$$D^n = D^{p+1} \times D^q = P_y \cup_{\mathrm{Id}} A_y \cong P_y \cup_{\Lambda_y} A'_y.$$

To this end, define a map $\phi_y\colon A_y \to A'_y$ by setting $\phi_y = \Lambda_y \times \mathrm{Id}_{[\rho,1]}$, where we are viewing $A_y = (S^p \times D^q) \times [\rho, 1]$ and $A'_y = \mathrm{Im}\,\overline{\lambda}_y \times [\rho, 1]$. Using this map we can define a further map $\Phi_y\colon P_y \cup_{\mathrm{Id}} A_y \to P_y \cup_{\Lambda_y} A'_y$ by

$$\Phi_y(z) = \begin{cases} z & \text{if } z \in P_y, \\ \phi_y(z) & \text{if } z \in A_y. \end{cases} \tag{3-2}$$

It is clear that Φ_y is a homeomorphism for each y. Following the discussion on adjunction spaces at the start of Theorem 2 (or see [14, Chapter 8, Section 2]), we see that with respect to the canonical differentiable structure on $P_y \cup_{\Lambda_y} A'_y$, the map Φ_y is actually a diffeomorphism. Thus the exotic structure which Hatcher bundles display does not occur at the level of individual fibres: it is a global bundle phenomenon.

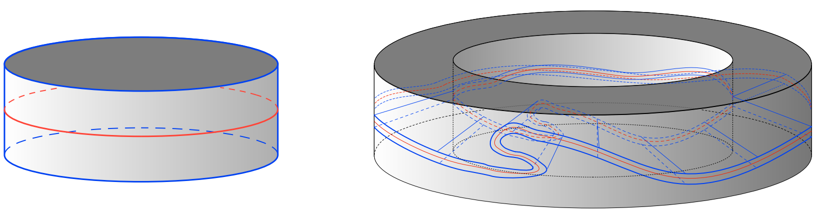

Figure 5: The spaces P_y (left) and A'_y as a subset of $S^p \times D^q \times [\rho, 1]$ (right).

3.3 Recollection of the J–homomorphism

Here we will follow [9] to construct the family of embeddings $\overline{\lambda}_y\colon S^p \times D^{q+1} \to S^p \times D^{q+1}$, for $y \in D^{4k}_+$, as above. This requires us to consider the J–homomorphism, which is a map

$$J\colon \pi_{4k-1} O(p) \to \pi_{4k-1+p} S^p,$$

where p is sufficiently large.[3] We can think of the J–homomorphism as follows. Consider a map $\lambda\colon S^{4k-1} \to O(p)$ determined by a choice of element $[\lambda] \in \pi_{4k-1}O(p)$; by the Whitney approximation theorem, without loss of generality we can assume that λ is smooth. This then determines a map $S^{4k-1} \times \mathbb{R}^p \to \mathbb{R}^p$, by sending a point (y,z) to the orthogonal transformation $\lambda(y) \in O(p)$ applied to $z \in \mathbb{R}^p$. Since $S^p = \mathbb{R}^p \cup \{x_0\}$, it is convenient to identify λ with a map $\lambda\colon S^{4k-1} \to \mathcal{C}(S^p, S^p)$, where $\mathcal{C}(X,Y)$ is the space of basepoint-preserving continuous maps $X \to Y$ with the compact–open topology. It will also be convenient to denote by $\lambda_y\colon S^p \to S^p$ the map λ evaluated at $y \in S^{4k-1}$. Passing to homotopy classes gives a map $\pi_{4k-1}O(p) \to \pi_{4k-1}\mathcal{C}(S^p,S^p)$, and composing this with an isomorphism $\pi_{4k-1}\mathcal{C}(S^p,S^p) \cong \pi_{4k-1+p}S^p$ then gives the J–homomorphism.

Recall that for p sufficiently large compared to k, the groups $\pi_{4k-1}O(p)$ and $\pi_{4k-1+p}S^p$ are independent of p and that $\pi_{4k-1+p}S^p$ is a finite group, while $\pi_{4k-1}O(p)$ is infinite cyclic.

Choose a map $\lambda\colon S^{4k-1} \to O(p)$ such that $[\lambda] \neq 0$ and $[\lambda] \in \ker J$. This means that λ extends to a map

$$\tilde{\lambda}\colon D^{4k}_+ \to \mathcal{C}(S^p, S^p),$$

where D^{4k}_+ is a disc of radius 1. We can assume that $\tilde{\lambda}$ restricted to the collar $S^{4k-1} \times \left(\frac{1}{2}, 1\right] \subset D^{4k}_+$ is independent of the $\left(\frac{1}{2}, 1\right]$–coordinate. For any $q \geq 4k$, we denote by $\iota\colon S^p \to S^p \times D^q$ the inclusion $\iota\colon x \mapsto (x, 0)$. The map λ and its extension $\tilde{\lambda}$ give a commutative diagram

$$\begin{tikzcd} S^{4k-1} \times S^p \arrow[r, "\lambda"] \arrow[d, "i \times \mathrm{Id}"] & S^p \arrow[r, "\iota"] \arrow[d, "\mathrm{Id}"] & S^p \times D^q \arrow[d, "\mathrm{Id}"] \\ D^{4k}_+ \times S^p \arrow[r, "\tilde{\lambda}"] & S^p \arrow[r, "\iota"] & S^p \times D^q \end{tikzcd} \tag{3-3}$$

where i is the inclusion of the boundary $S^{4k-1} \to D^{4k}_+$.

Lemma 12 *For sufficiently large q, we can approximate the map $\iota \circ \tilde{\lambda}$ by a smoothly varying family of smooth embeddings*

$$\hat{\lambda}_y\colon S^p \to S^p \times D^q, \quad y \in D^{4k}_+, \tag{3-4}$$

which retain the property that for $y \in S^{4k-1} \times \left(\frac{1}{2}, 1\right] \subset D^{4k}_+$, the maps $\hat{\lambda}_y$ agree with $\iota \circ \lambda_y$.

[3] Here we assume that the base point in S^p is the north pole.

Proof We begin by recalling that the map $\lambda\colon S^{4k-1} \to O(p)$ is assumed to be smooth. By applying the Whitney approximation theorem to $\widetilde{\lambda}$ viewed as a map $\widetilde{\lambda}\colon D^{4k}_+ \times S^p \to S^p$ (see for example [17, Theorem 6.26]), we see that by a C^0 arbitrarily small homotopy relative to a boundary neighbourhood in the domain, we can adjust this map to be smooth. Thus, without loss of generality, we may as well assume in the first place that $\widetilde{\lambda}\colon D^{4k}_+ \times S^p \to S^p$ is smooth.

To construct the embeddings $\widehat{\lambda}_y\colon S^p \to S^p \times D^q$, we first let $\eta\colon S^p \to D^q$ be an arbitrary embedding of the sphere into a ball $D^q \subset \mathbb{R}^q$ centred at the origin, and denote by $\epsilon\eta$ the composition consisting of η followed by a scaling of D^q onto itself by a factor of $\epsilon \geq 0$. (The embedding η clearly exists provided that $q \geq p+1$.) Next, we introduce a *function* $\epsilon\colon D^{4k}_+ \to \mathbb{R}$, which is identically zero in a small neighbourhood of the region in which $\widetilde{\lambda}$ is independent of the radial parameter, is strictly positive otherwise, and is everywhere smooth.

Finally, set $\widehat{\lambda}\colon D^{4k}_+ \times S^p \to S^p \times D^q$ to be the map

$$\widehat{\lambda}(y,x) \mapsto (\widetilde{\lambda}_y(x), \epsilon(y)\eta(x)).$$

It is now immediate that this restricts to give a smoothly varying family of smooth embeddings $\widehat{\lambda}_y\colon S^p \to S^p \times D^q$, by virtue of the fact that η is an embedding. Moreover, these embeddings clearly agree with $\iota \circ \widetilde{\lambda}_y$ for $y \in S^{4k-1} \times \left(\frac{1}{2}, 1\right]$, since ϵ vanishes in this region. $\square$

Denote by $N_y \to S^p$ the normal bundle of the embedding $\widehat{\lambda}_y$. Since both S^p and $S^p \times D^q$ have stably trivial tangent bundles, it follows that the normal bundle N_y is also stably trivial. In fact, by increasing q if necessary, we can assume that N_y is trivial for all y. Considering now all $y \in D^{4k}_+$ at once, we obtain a vector bundle $N \to D^{4k}_+ \times S^p$. Since $N_y \to S^p$ is trivial for each y, it follows that the bundle $N \to D^{4k}_+ \times S^p$ is also trivial. Then by fixing a trivialization of N and using the normal exponential map, we extend the family of embeddings (3-4) to the family of embeddings

$$\overline{\lambda}_y\colon S^p \times D^q \to S^p \times D^q, \quad y \in D^{4k}_+. \tag{3-5}$$

Lemma 13 [9, Proposition 5.4] *The family of embeddings* (3-5) *is smoothly isotopic to a family of embeddings $S^p \times D^q \to S^p \times D^q$ which restricts over $S^{4k-1} = \partial D^{4k}_+$ to give transformations*

$$(x,z) \mapsto \left(\lambda_y(x), (\lambda_y^{-1} \oplus \mathrm{Id}_{\mathbb{R}^{q-p}})(z)\right) \in S^p \times D^q, \quad y \in S^{4k-1}. \tag{3-6}$$

Remark The effect on the S^p factor in Lemma 13 is simply the original orthogonal transformation λ_y, and on the D^q factor we also have an orthogonal transformation, made up of the inverse transformation λ_y^{-1} on the first p coordinates, with the complementary space being fixed. In particular, since the map $\lambda\colon S^{4k-1} \to O(p)$ gives a continuous family of maps of this form, (3-6) is a product of rotations for all $y \in \partial D^{4k}_+$. Lemma 13 shows we can assume that the family of embeddings (3-5) satisfies the condition (3-6). We notice also that (3-6) implies that $\operatorname{Im}\bar{\lambda}_y$ coincides with $\{y\} \times S^p \times D^q \subset \partial A_y$. In particular, we can assume that

$$A'_y = A_y \quad \text{if } y \in D^{4k}_+ \text{ is near the boundary } \partial D^{4k}_+. \tag{3-7}$$

Recall that the embeddings $\bar{\lambda}_y$ give rise to the desired family of diffeomorphisms Λ_y, which by definition coincide with the maps $\bar{\lambda}_y$ when the target space is restricted to $\operatorname{Im}\bar{\lambda}_y$. This completes the construction of the bundle $\mathcal{E}_+ \to D^{4k}_+$.

We conclude this section with the result below, which follows from the proof of Lemma 12:

Corollary 14 *The diffeomorphisms $\Lambda_y\colon S^p \times D^q \to S^p \times D^q$ are determined by their restriction to an arbitrarily small neighbourhood of the sphere $S^p \times \{0\} \subset S^p \times D^q \subset \partial P_y$ and its image in $\partial A'_y$, for each $y \in D^{4k}_+$.*

Proof We merely have to observe that in the proof of Lemma 12 we can choose the embedding η so that its image is contained in an arbitrarily small ball about the origin in D^q, and we can choose the function ϵ to have an arbitrarily small upper bound. □

3.4 The Hatcher bundle E_λ

It remains to describe how the bundle $\mathcal{E}_+ \to D^{4k}_+$ is to be glued to the trivial disc bundle $\mathcal{E}_- \to D^{4k}_-$ along the boundary of the base discs $\partial D^{4k}_+ = \partial D^{4k}_- = S^{4k-1}$. Recall that each disc fibre is the union of an annulus and a "puck", and according to (3-7), we can assume the annulus parts A'_y of the fibres are equal to A_y near the boundary of D^{4k}_+.

We can therefore begin by gluing the annulus parts A_y and A'_y of the fibres at the boundary of $\mathcal{E}_+$ and $\mathcal{E}_-$ via the identity map. To glue the "puck" part of the fibres we observe that, according to Lemma 13, the maps $\Lambda_y\colon S^p \times D^q \to S^p \times D^q$ defined above are products of rotations for each y near ∂D^{4k}_+. Thus the Λ_y extend to diffeomorphisms $\widetilde{\Lambda}_y\colon P_y \to P_y$ for such y, using the rotations (3-6). We use the $\widetilde{\Lambda}_y$ to glue the puck subbundles of $\mathcal{E}_+$ to $\mathcal{E}_-$ for $y \in S^{4k-1}$, noting that this is consistent with the gluing of the annuli. The disc bundle over S^{4k} which results from this gluing we will denote by $\mathcal{E}_\lambda$, since it ultimately depends on our choice of $[\lambda] \in \ker J$. As noted at the start of this section, we can then double this disc bundle to produce the desired Hatcher bundle $E_\lambda := \mathcal{E}_\lambda \cup \mathcal{E}_\lambda$ over S^{4k}.

It can be shown that E_λ is bundle homeomorphic but not bundle diffeomorphic to the corresponding trivial bundle; see [9, Proposition 5.8, Theorem 5.13] and [10, Section 1].

4 The fibrewise Ricci positive metric construction

4.1 Foreword

In this section we will ultimately prove the Main Theorem. As discussed in Section 1, this reduces to showing that a Hatcher bundle admits a fibrewise Ricci positive metric. Our general strategy is to show the existence of fibrewise Ricci positive metrics on the Hatcher disc bundles constructed in the last section, and then use the family version of the Perelman gluing result, Theorem 10, to glue two copies of such a disc bundle together within Ricci positivity to create the desired object.

As we will see, the actual construction of the fibrewise Ricci positive metric on the Hatcher disc bundle (carried out in Section 4.5) is quite delicate, and one might wonder why such an elaborate construction is necessary. The issue is essentially due to the nonlinearity of the Hatcher bundle. Given a linear sphere bundle and a suitable horizontal distribution (ie a connection on the associated principal bundle), there is a standard procedure for constructing fibrewise metrics; see for example [1, Theorem 9.59]. In the first instance, this requires the existence of a fibre metric invariant under the action of the structural group. However, in the case of Hatcher bundles, all we can say is that the structural group is $\mathrm{Diff}_{x_0}(S^n)$. As this is noncompact, we cannot guarantee the existence of an invariant metric, and so another approach to metric construction is required.

One might also ask why our technique does not extend to the case of Ricci positive exotic spheres. The reason is that the families of Ricci positive metrics on S^n giving the nontrivial elements in the Main Theorem are constructed by doubling metrics on the disc D^n. To obtain an exotic sphere this way, we would have to glue two copies of D^n via an exotic diffeomorphism. This would require a boundary metric invariant under

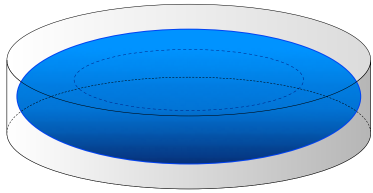

Figure 6: The solid ellipsoid contained in $D^m \times D^n$.

the diffeomorphism, and producing such metrics in this context is currently beyond the scope of our methods.

In order to perform the disc bundle gluing, we need to consider the second fundamental forms at the boundary of the disc fibres. There is an immediate problem, however: the discs were constructed as products $D^{p+1} \times D^q$. Thus, as written, each of these is a manifold with corners. Moreover, in order for these discs to be equipped with Ricci positive metrics, it is natural to consider product metrics which respect the topological product structure. The resulting boundary is not smooth, however, and we need a smooth boundary in order to apply the Perelman gluing technique.

In order to deal with this issue, our approach is to cut out a solid "ellipsoid" from within the product of discs; see Figure 6. This will be constructed to have a smooth boundary, with principal curvatures at the boundary (with respect to the inward normal) all positive. Thus, provided the ambient metric on the product of discs has positive Ricci curvature, we can glue two such ellipsoids together using the Perelman gluing technique. Therefore the main tasks in the next three subsections are respectively to construct the ambient metric, to construct the ellipsoid, and to prove the ellipsoid has the desired convexity properties at the boundary.

To avoid any confusion with indices, it is convenient to work with the product $D^m \times D^n$, where the roles of m and n are symmetric. We will specialize our formulas to the case $m = p+1$ and $n = q$ in Section 4.5, where we prove the Main Theorem.

4.2 The metric on $D^m \times D^n$

First, we will consider the following metric on $D^m \times D^n$:

$$h := ds^2 + \alpha^2(s)\, ds_{m-1}^2 + dt^2 + \beta^2(t)\, ds_{n-1}^2,$$

where s and t are the radial parameters in the discs D^m and D^n respectively. (In our later metric constructions we will use a slight variant of this metric.) Let us assume that the radii of the two discs are s_1 and t_1 respectively. We will impose the following conditions on the smooth warping functions α and β:

- α and β are odd in a small neighbourhood of $s = 0$ respectively $t = 0$ (or rather, one can extend α and β to negative values of s and t such that this extended function is smooth and odd), and in particular $\alpha(0) = \beta(0) = 0$.
- $\alpha'(0) = \beta'(0) = 1$.
- $\alpha' > 0$ and $\beta' > 0$ whenever s respectively t is positive.
- $\alpha''(s) < 0$ for all $s \in [0, s_1]$ and $\beta''(t) < 0$ for all $t \in [0, t_1]$.

It follows easily from the warped product formulas for Ricci curvature that these conditions ensure that the metric h has strictly positive Ricci curvature; see, for example, [1, Section 9J].

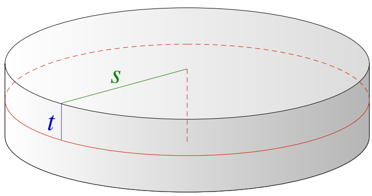

Figure 7: Coordinates on the space $X = D^m \times D^n$.

4.3 Specifying the ellipsoid

In order to construct the ellipsoid, we introduce a unit speed curve $\mu = \mu(r)$ into the st–plane. This curve will have the profile given in Figure 8.

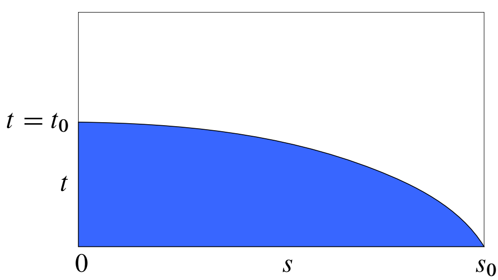

Figure 8: The curve μ which gives rise to the ellipsoid $\boldsymbol{E}$.

Notice that the illustrated curve separates the rectangle $[0, s_1] \times [0, t_1]$ into two regions, and suppose that the parameter r is such that $\mu(0) = (0, t_0)$ and $\mu(r_0) = (s_0, 0)$ for some $s_0 \in (0, s_1)$, $t_0 \in (0, t_1)$ and $r_0 > 0$. We will define the ellipsoid $\boldsymbol{E}$ to be the subset of $D^{p+1} \times D^q$ consisting of all elements whose s and t coordinates lie in the region on or below this curve.

We need to specify μ in more detail, and to this end we will write $\mu(r) = (\mu_s(r), \mu_t(r))$. Let us impose the following conditions on μ_s and μ_t:

(1) $\mu_s(0) = 0$, $\mu_s(r_0) = s_0$, $\mu_s'(0) = 1$, $\mu_s'(r_0) = 0$, $\mu_s''(r) < 0$ for all $r \in [0, r_0]$, and μ_s is odd in a neighbourhood of $r = 0$ and is even locally about the point $r = r_0$ (in the sense that there is a smooth extension such that $\mu_s(r_0 - \epsilon) = \mu_s(r_0 + \epsilon)$ for all sufficiently small $\epsilon > 0$).

(2) $\mu_t(0) = t_0$, $\mu_t(r_0) = 0$, $\mu_t'(0) = 0$, $\mu_t'(r_0) = -1$, $\mu_t''(r) < 0$ for all $r \in [0, r_0]$, and μ_t is even about the point $r = 0$ and odd in a neighbourhood of $r = r_0$.

Given a unit speed curve μ satisfying (1) and (2) above, we need to check that the resulting ellipsoid has a smooth boundary. First, it is clear from the smoothness of all the functions involved that this boundary will indeed be smooth everywhere except possibly when $r = 0$ or $r = r_0$. We must therefore check the corresponding "ends" of the ellipsoid for smoothness. With this in mind, we begin by observing that the metric induced by h on the boundary of the ellipsoid is

$$h_{\boldsymbol{E}} := dr^2 + \alpha^2(\mu_s(r))\, ds_{m-1}^2 + \beta^2(\mu_t(r))\, ds_{n-1}^2. \tag{4-1}$$

Given that $\mu_s(0) = 0$ and $\mu_t(r_0) = 0$ and that $\mu_s, \mu_t > 0$ otherwise, we see immediately from (4-1) that the metric $h_{\boldsymbol{E}}$ is a (not necessarily smooth) metric on a sphere of dimension $m + n - 1$. The boundary conditions which such a metric must satisfy in order to give a smooth sphere metric are well known (see for example [20, Section 1.4]); the scaling functions $\alpha(\mu_s(r))$ and $\beta(\mu_t(r))$ must obey the following rules:

(i) They must be everywhere nonnegative, with $\alpha(\mu_s(r)) = 0$ if and only if $r = 0$, and $\beta(\mu_t(r)) = 0$ if and only if $r = r_0$.

(ii) $\alpha(\mu_s(r))$ must be odd at $r = 0$ and even at $r = r_0$.

(iii) $\beta(\mu_t(r))$ must be even at $r = 0$ and odd at $r = r_0$.

(iv) The derivative of $\alpha(\mu_s(r))$ must take the value 1 at $r = 0$, and that of $\beta(\mu_t(r))$ must take the value -1 at $r = r_0$.

Property (i) follows immediately from the conditions imposed on α, β, μ_s and μ_t. For property (iii) we note that by definition $\mu_t(r)$ is odd at $r = r_0$ and $\beta(t)$ is odd at $t = 0$, and it follows trivially from this that the composition $\beta(\mu_t(r))$ is odd at $r = r_0$. For the evenness requirement it suffices to note that the composition of an even function followed by an arbitrary function is trivially even. Property (ii) follows by similar arguments. Finally, property (iv) follows by the chain rule since $\alpha'(0)\mu_s'(0) = 1$, and $\beta'(0)\mu_t'(r_0) = -1$.

In summary then, we have demonstrated how to choose a unit speed curve μ such that the resulting ellipsoid is smooth, and we will work with the same subset of $D^{p+1} \times D^q$ for each fibre of the Hatcher disc bundle when we construct the fibrewise metric later in this section.

4.4 Principal curvatures of the ellipsoid

The other issue we need to address in relation to the Perelman gluing of discs (or rather ellipsoids) is that of the second fundamental form at the boundary. As observed previously (Corollary 9), assuming the ellipsoid (or more generally $D^m \times D^n$) has positive Ricci curvature, then it is sufficient for our purposes that the principal curvatures all be positive with respect to the inward-pointing normal. It turns out, however, that the principal curvatures of the ellipsoid we have constructed are only nonnegative with respect to the ambient metric h, and in particular vanish at the points of the ellipsoid corresponding to $r = 0$ and $r = r_0$. To rectify this situation we work with the same ellipsoid, but a slightly modified metric on $D^m \times D^n$.

Let us define a metric g on $D^m \times D^n$ as follows:

$$g := \delta^2(t)\, ds^2 + \delta^2(t)\alpha^2(s)\, ds_{m-1}^2 + \gamma^2(s)\, dt^2 + \gamma^2(s)\beta^2(t)\, ds_{n-1}^2. \tag{4-2}$$

The new functions introduced here, $\delta(t)$ and $\gamma(s)$, are chosen so as to satisfy the following properties:

(a) $\delta'(t) \geq 0$ for all $t \in [0, t_1]$, $\delta'(t_0) > 0$ and $\delta(t) \equiv 1$ in a neighbourhood of $t = 0$.

(b) $\gamma'(s) \geq 0$ for all $s \in [0, s_1]$, $\gamma'(s_0) > 0$ and $\gamma(s) \equiv 1$ in a neighbourhood of $s = 0$.

We will see that the positivity of the derivatives of δ and γ at t_0, respectively s_0, is enough to give us strictly positive principal curvatures globally. Of course we must not forget that the metric g must have positive Ricci curvature. By the openness of the positivity condition we can choose δ and γ satisfying (a) and (b) above sufficiently close in a C^2–sense to the constant function with value 1 so that $\mathrm{Ric}(g) > 0$. We therefore add a third condition:

(c) δ and γ are such that $\mathrm{Ric}(g) > 0$, at least in some neighbourhood of $\boldsymbol{E}$.

Lemma 15 *The principal curvatures at the boundary of the ellipsoid $\boldsymbol{E}$ are all strictly positive with respect to the ambient metric g.*

Proof We work locally, and begin by fixing a point $x_1 = (s_1, a_1, t_1, b_1) \in \boldsymbol{E}$, where $a_1 \in S^{m-1}$ and $b_1 \in S^{n-1}$. About the points a_1 and b_1, introduce normal coordinate systems locally into S^{m-1} and S^{n-1}. Together with the s and t coordinates, these combine to give a local coordinate system in $D^m \times D^n$. With respect to these coordinates we can represent g by the block-diagonal matrix

$$g = \begin{bmatrix} \delta^2(t) & & & \\ & \gamma^2(s) & & \\ & & \delta^2(t)\alpha^2(s)A_{m-1} & \\ & & & \gamma^2(s)\beta^2(t)B_{n-1} \end{bmatrix},$$

where A_{m-1} and B_{n-1} represent ds^2_{n-1} and ds^2_{m-1} with respect to the chosen normal coordinate systems on the spheres. Note that at the points a_1 and b_1, the matrices A_{m-1} and B_{n-1} are both identity matrices and have vanishing first derivatives. Hence at the point x_1 we have $g_{ij} \neq 0$ if and only if $i = j$, $g^{ii} = 1/g_{ii}$, and the derivatives $g_{ij,k}$ equal 0 whenever k is a direction tangent to S^{m-1} or S^{n-1}. We will assume that all computations below are carried out at this point.

Using the formula

$$\Gamma^k_{ij} = \tfrac{1}{2} g^{kl}(g_{il,j} + g_{jl,i} - g_{ij,l}),$$

it is straightforward to compute the corresponding Christoffel symbols. The list below consists of precisely those Christoffel symbols which are nonzero. Beginning with the case when each of the indices i, j and k are s or t, we have

$$\Gamma^s_{st} = \Gamma^s_{ts} = \frac{\delta'(t)}{\delta(t)}, \quad \Gamma^s_{tt} = \frac{-\gamma'(s)\gamma(s)}{\delta^2(t)}, \quad \Gamma^t_{ss} = \frac{-\delta(t)\delta'(t)}{\gamma^2(s)}, \quad \Gamma^t_{st} = \Gamma^t_{ts} = \frac{\gamma'(s)}{\gamma(s)}.$$

Then, using the symbols a and b to represent any of the coordinate functions on S^{m-1} or S^{n-1} respectively, we list the remaining nonzero Christoffel symbols:

$$\begin{aligned} \Gamma^s_{aa} &= -\alpha(s)\alpha'(s), & \Gamma^a_{sa} &= \Gamma^a_{as} = \frac{\alpha'(s)}{\alpha(s)}, \\ \Gamma^t_{aa} &= \frac{-\delta(t)\delta'(t)\alpha^2(s)}{\gamma^2(s)}, & \Gamma^a_{ta} &= \Gamma^a_{at} = \frac{\delta'(t)}{\delta(t)}, \\ \Gamma^s_{bb} &= \frac{-\gamma(s)\gamma'(s)\beta^2(t)}{\delta^2(t)}, & \Gamma^b_{sb} &= \Gamma^b_{bs} = \frac{\gamma'(s)}{\gamma(s)}, \\ \Gamma^t_{bb} &= -\beta(t)\beta'(t), & \Gamma^b_{tb} &= \Gamma^b_{bt} = \frac{\beta'(t)}{\beta(t)}. \end{aligned}$$

From this we compute certain covariant derivatives involving coordinate vector fields, ∂_s, ∂_t, ∂_a and ∂_b, which will we will make use of shortly. In particular, we see that at the point x_1 we have

$$\nabla_{\partial_s}\partial_s = \frac{-\delta(t)\delta'(t)}{\gamma^2(s)}\partial_t, \quad \nabla_{\partial_t}\partial_t = \frac{-\gamma'(s)\gamma(s)}{\delta^2(t)}\partial_s,$$

$$\nabla_{\partial_s}\partial_t = \nabla_{\partial_t}\partial_s = \frac{\delta'(t)}{\delta(t)}\partial_s + \frac{\gamma'(s)}{\gamma(s)}\partial_t,$$

$$\nabla_{\partial_t}\partial_a = \nabla_{\partial_a}\partial_t = \frac{\delta'}{\delta}\partial_a, \qquad \nabla_{\partial_s}\partial_a = \nabla_{\partial_a}\partial_s = \frac{\alpha'}{\alpha}\partial_a,$$

$$\nabla_{\partial_t}\partial_b = \nabla_{\partial_b}\partial_t = \frac{\beta'}{\beta}\partial_b, \qquad \nabla_{\partial_s}\partial_b = \nabla_{\partial_b}\partial_s = \frac{\gamma'}{\gamma}\partial_b,$$

$$\nabla_{\partial_a}\partial_a = -\frac{\delta'\delta\alpha^2}{\gamma^2}\partial_t - \alpha'\alpha\partial_s, \quad \nabla_{\partial_b}\partial_b = -\frac{\gamma'\gamma\beta^2}{\delta^2}\partial_s - \beta'\beta\partial_t,$$

and $\nabla_{\partial_a}\partial_b = \nabla_{\partial_b}\partial_a = 0$.

We now compute second fundamental forms, and will break up the computation into directions tangent to S^{m-1} and S^{n-1}, and tangent to the curve μ. Notice that $\mu'(r)$ is everywhere tangent to the boundary of the ellipsoid, and this direction is orthogonal (with respect to g) to both S^{m-1} and S^{n-1}. Explicitly we have $T(r) := \mu'(r) = \mu_s'(r)\partial_s + \mu_t'(r)\partial_t$. It is easy to see that the normal vector to the ellipsoid lies in the st–plane. If we represent the *outward* normal as $N = c_s\partial_s + c_t\partial_t$ then it is clear that the coefficients c_s and c_t are functions of r. Moreover, it is evident from our choice of μ that $c_s(r_0) = 0$ and $c_t(0) = 0$ and that $c_s, c_t > 0$ otherwise.

The second fundamental form $\mathrm{II}(u, v)$ with respect to the inward normal $-N$ is given by $\mathrm{II}(u, v) = -g(\nabla_u v, N)$. Thus in order to show this is positive definite, it suffices to establish that the components of $\nabla_u u$ in the ∂_s and ∂_t directions are nonpositive, at least one of the coefficients is negative for all $r \in (0, r_0)$, at $r = 0$ (where $c_t = 0$) the coefficient of ∂_s is negative, and at $r = r_0$ (where $c_s = 0$) the coefficient of ∂_t is negative. (Of course if $u \in TS^{m-1}$ then we must automatically have $r > 0$ else this sphere is not defined, and similarly we need $r < r_0$ if $u \in TS^{n-1}$.)

Consider first $\partial_a \in TS^{m-1}$. From the covariant derivative expressions above we observe that the coefficient of ∂_s, namely $-\alpha'\alpha$, is nonpositive and strictly negative for all $r \in (0, r_0)$; however, it vanishes at $r = r_0$. (We have $r > 0$ in order for the vector ∂_a to make sense, as noted above.) The coefficient of ∂_t is $-\delta'\delta\alpha^2\gamma^{-2}$, which is clearly nonpositive, and negative at $r = r_0$ since $\delta'(t_0) > 0$ by definition. Thus we have $\mathrm{II}(\partial_a, \partial_a) > 0$ as required. Analogous arguments apply for $\mathrm{II}(\partial_b, \partial_b)$.

Finally, we investigate $\nabla_T T$. We have

$$\nabla_T T = \mu_s'(\partial_s \mu_s')\partial_s + {\mu_s'}^2 \nabla_{\partial_s}\partial_s + \mu_s'(\partial_s\mu_t')\partial_t + \mu_s'\mu_t'\nabla_{\partial_s}\partial_t + \mu_t'(\partial_t\mu_s')\partial_s \\ + \mu_t'\mu_s'\nabla_{\partial_t}\partial_s + \mu_t'(\partial_t\mu_t')\partial_t + {\mu_t'}^2\nabla_{\partial_t}\partial_t.$$

In order to simplify this expression, we note that by definition of μ, the coordinate functions $\mu_s(r)$ and $\mu_t(r)$ are one-to-one, and therefore invertible. Viewing s as a function of r along μ we clearly have $s(r) = \mu_s(r)$, and hence $r(s) = \mu_s^{-1}(s)$. Differentiating with respect to s then yields

$$\partial_s\mu_s'(r) = \partial_s\mu_s'(\mu_s^{-1}(s)) = \mu_s''(\mu_s^{-1}(s))\frac{1}{\mu_s'(\mu^{-1}(s))} = \frac{\mu_s''(r)}{\mu_s'(r)}.$$

Analogous computations give

$$\partial_s\mu_t' = \frac{\mu_t''}{\mu_s'}, \quad \partial_t\mu_s' = \frac{\mu_s''}{\mu_t'}, \quad \partial_t\mu_t' = \frac{\mu_t''}{\mu_t'}. \tag{4-3}$$

It follows immediately that

$$\mu_s'(\partial_s\mu_s') = \mu_s'', \quad \mu_s'(\partial_s\mu_t') = \mu_t'', \quad \mu_t'(\partial_t\mu_s') = \mu_s'', \quad \mu_t'(\partial_t\mu_t') = \mu_t''. \tag{4-4}$$

Notice that for the above calculations to be valid as stated, we must assume that $\mu_s', \mu_t' \neq 0$. This is fine precisely when $r \in (0, r_0)$. However, observe that the right-hand sides of the expressions (4-4) are defined for all $r \in [0, r_0]$, and we can infer from this that the limits as $r \to 0^+$ and $r \to r_0^-$ in (4-3) must be well behaved.

We can now use the above calculations to simplify the expression for $\nabla_T T$:

$$\nabla_T T = \left(2\mu_s'' - {\mu_s'}^2\frac{\delta'\delta}{\gamma^2} + 2\mu_s'\mu_t'\frac{\delta'}{\delta}\right)\partial_s + \left(2\mu_t'' - {\mu_t'}^2\frac{\gamma'\gamma}{\delta^2} + 2\mu_s'\mu_t'\frac{\gamma'}{\gamma}\right)\partial_t.$$

In each of the above brackets, notice that the terms are negative, nonpositive and nonpositive respectively. It follows that $\mathrm{II}(T, T) > 0$, and hence we conclude that II is positive definite as claimed. □

Let us summarize the above constructions:

Proposition 16 *There is a Ricci positive metric g on $D^m \times D^n$ and a codimension-zero solid ellipsoid $\boldsymbol{E} \subset D^m \times D^n$ such that $\partial\boldsymbol{E}$ is a smooth submanifold of $D^m \times D^n$ and the principal curvatures of $\partial\boldsymbol{E}$ (with respect to the inward-pointing normal) are all positive.*

4.5 Proof of the Main Theorem

Recall from Section 1 that to establish the theorem it suffices to construct a fibrewise Ricci positive metric on each Hatcher sphere bundle. In order to do this, we will begin by reconsidering the construction of the Hatcher disc bundle from Section 3.

Now we switch to the relevant notation, ie $m = p+1$ and $n = q$. For each point $y \in D^{4k}_+$ we have

$$D^{p+1} \times D^q = P_y \cup_{\mathrm{Id}} A_y \overset{\Phi_y}{\cong} P_y \cup_{\Lambda_y} A'_y,$$

where we refer the reader to Section 3 for the notation. The ellipsoid $\boldsymbol{E}$ is a subset of $D^{p+1} \times D^q$, and so for each $y \in D^{4k}_+$ there is an ellipsoid

$$\boldsymbol{E}_y := \Phi_y(\boldsymbol{E}) \subset P_y \cup_{\Lambda_y} A'_y.$$

Collectively, these ellipsoid fibres form a subbundle $\boldsymbol{\mathcal{E}}^{\mathrm{ell}}_+$ of $\mathcal{E}_+$. Pushing forward the metric g via Φ_y and restricting to $\boldsymbol{E}_y$ equips each $\boldsymbol{E}_y$ with a Ricci positive metric with positive principal curvatures (with respect to the inward normal) at the boundary. Moreover, as y varies across D^{4k}_+, we obtain in this way a smoothly varying family of fibre metrics on $\boldsymbol{\mathcal{E}}^{\mathrm{ell}}_+$.

We similarly form a product bundle $\boldsymbol{\mathcal{E}}^{\mathrm{ell}}_- \to D^{4k}_-$ with total space $D^{4k}_- \times \boldsymbol{E}$, and take the obvious fibrewise metric where each fibre $\mathbf{E}$ is equipped with the metric induced by g. For each fibre $\boldsymbol{E}_y \subset \boldsymbol{\mathcal{E}}^{\mathrm{ell}}_+$, notice that we have a decomposition

$$\boldsymbol{E}_y = (\boldsymbol{E}_y \cap P_y) \cup (\boldsymbol{E}_y \cap A'_y),$$

and similarly for the fibres of $\boldsymbol{\mathcal{E}}^{\mathrm{ell}}_-$.

In order to form the Hatcher disc bundle, we need to glue the bundles $\boldsymbol{\mathcal{E}}^{\mathrm{ell}}_+$ and $\boldsymbol{\mathcal{E}}^{\mathrm{ell}}_-$ along the boundaries of their base discs. The procedure for gluing the "full" disc bundles $\mathcal{E}_+$ and $\mathcal{E}_-$ is described at the end of Section 3. Recall that for each pair of fibres in $\mathcal{E}_+$ and $\mathcal{E}_-$ being identified, the annulus parts are identified via the identity map, but the inner "puck" regions are identified using diffeomorphisms $\widetilde{\Lambda}_y \colon P_y \to P_y$, which by Lemma 13 split as a product of rotations on the two disc factors. Before proceeding further, we note that these gluing maps restrict to give gluing maps between $\boldsymbol{\mathcal{E}}^{\mathrm{ell}}_+$ and $\boldsymbol{\mathcal{E}}^{\mathrm{ell}}_-$, since the annulus and puck parts of the respective ellipsoid bundles agree near the boundary of the base discs, and are invariant under rotation of the factors. Note further that by Corollary 14 in Section 3, we do not lose any gluing information by reducing the fibres from the original product of discs considered in Section 3 to the ellipsoids currently under consideration. Thus the bundle we will construct using $\mathcal{E}^{\mathrm{ell}}_{+}$ and $\mathcal{E}^{\mathrm{ell}}_{-}$ will be diffeomorphic to that formed from $\mathcal{E}_{+}$ and $\mathcal{E}_{-}$.

From a metric perspective, let us focus first on the puck subbundles within $\mathcal{E}^{\mathrm{ell}}_{+}$ and $\mathcal{E}^{\mathrm{ell}}_{-}$. As Φ_y is the identity mapping on P_y, the puck subbundle within $\mathcal{E}^{\mathrm{ell}}_{+}$ is just a product, with each fibre equipped with the restriction of g. Now the metric g displays rotational symmetry with respect to both disc factors, and so pulling back $g|_{P_y}$ via the map $\widetilde{\Lambda}_y$ results in a metric identical to $g|_{P_y}$. Since we have set things up so that the metrics near the boundaries of both $\mathcal{E}_{+}$ and $\mathcal{E}_{-}$ are independent of the radial parameter in the base, we see that gluing the puck subbundles of $\mathcal{E}^{\mathrm{ell}}_{+}$ and $\mathcal{E}^{\mathrm{ell}}_{-}$ along S^{4k-1} in this way yields a smooth fibrewise metric. (It is perhaps worth remarking that if we were trying to construct a submersion metric on the *whole* Hatcher disc bundle — as opposed to creating a mere fibrewise metric — then the twisting involved in gluing the bundles $\mathcal{E}_{+}$ and $\mathcal{E}_{-}$ would have nontrivial metric implications in directions transverse to the fibres.)

Turning our attention to the gluing of the annular regions, we similarly observe that the metric on the annuli close to the boundary of $\mathcal{E}_{+}$ is a push-forward via Φ_y of the rotationally symmetric metric $g|_{A_y}$. Although Φ_y acts nontrivially on the annuli, it nevertheless acts by rotation in both S^{m-1} and S^{n-1} directions for y close to ∂D^{4k}_{+}. Thus the pull-back metric on the annuli is identical to the original over the boundary of the base disk, and so gluing the annular part of $\mathcal{E}^{\mathrm{ell}}_{+}$ to $\mathcal{E}^{\mathrm{ell}}_{-}$ via the identity creates a smooth fibrewise metric in the annular region also.

In summary, we have a smooth fibrewise Ricci positive metric on the fibres of each of the ellipsoid subbundles $\mathcal{E}^{\mathrm{ell}}{}_{-}$ and $\mathcal{E}^{\mathrm{ell}}_{+}$, which glue to create a fibrewise Ricci positive metric on the ellipsoid subbundle of the Hatcher disc bundle $\mathcal{E}^{\mathrm{ell}} \subset \mathcal{E}$, with the principal curvatures at the boundary of each fibre being positive with respect to the inward normal.

Finally, we wish to glue two identical copies of the Hatcher disc bundle $\mathcal{E}^{\mathrm{ell}}$ equipped with the above fibrewise metric so as to construct the desired Hatcher sphere bundle. Metrically this is now possible using the family gluing result, Theorem 10, as a consequence of the positive principal curvatures at the boundary. We thus create a Hatcher sphere bundle with a smooth fibrewise Ricci positive metric, as required to establish the theorem. □

References

[1] **A L Besse**, *Einstein manifolds*, Ergeb. Math. Grenzgeb. 10, Springer (1987) MR

[2] **B Botvinnik**, **P B Gilkey**, *The eta invariant and metrics of positive scalar curvature*, Math. Ann. 302 (1995) 507–517 MR

[3] **B Botvinnik**, **B Hanke**, **T Schick**, **M Walsh**, *Homotopy groups of the moduli space of metrics of positive scalar curvature*, Geom. Topol. 14 (2010) 2047–2076 MR

[4] **R Carr**, *Construction of manifolds of positive scalar curvature*, Trans. Amer. Math. Soc. 307 (1988) 63–74 MR

[5] **D Crowley**, **T Schick**, **W Steimle**, *Harmonic spinors and metrics of positive curvature via the Gromoll filtration and Toda brackets*, J. Topol. 11 (2018) 1077–1099

[6] **A Dessai**, **S Klaus**, **W Tuschmann**, *Nonconnected moduli spaces of nonnegative sectional curvature metrics on simply connected manifolds*, Bull. Lond. Math. Soc. 50 (2018) 96–107 MR

[7] **D G Ebin**, *The manifold of Riemannian metrics*, from "Global analysis" (S-S Chern, S Smale, editors), Proc. Sympos. Pure Math. 15, Amer. Math. Soc., Providence, RI (1970) 11–40 MR

[8] **F T Farrell**, **W C Hsiang**, *On the rational homotopy groups of the diffeomorphism groups of discs, spheres and aspherical manifolds*, from "Algebraic and geometric topology, I" (R J Milgram, editor), Proc. Sympos. Pure Math. 32, Amer. Math. Soc., Providence, RI (1978) 325–337 MR

[9] **S Goette**, *Morse theory and higher torsion invariants, I*, preprint (2001) arXiv

[10] **S Goette**, **K Igusa**, *Exotic smooth structures on topological fiber bundles, II*, Trans. Amer. Math. Soc. 366 (2014) 791–832 MR

[11] **S Goette**, **K Igusa**, **B Williams**, *Exotic smooth structures on topological fiber bundles, I*, Trans. Amer. Math. Soc. 366 (2014) 749–790 MR

[12] **M Gromov**, **H B Lawson, Jr**, *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. 111 (1980) 423–434 MR

[13] **B Hanke**, **T Schick**, **W Steimle**, *The space of metrics of positive scalar curvature*, Publ. Math. Inst. Hautes Études Sci. 120 (2014) 335–367 MR

[14] **M W Hirsch**, *Differential topology*, Graduate Texts in Mathematics 33, Springer (1976) MR

[15] **K Igusa**, *Higher Franz–Reidemeister torsion*, AMS/IP Studies in Advanced Mathematics 31, Amer. Math. Soc., Providence, RI (2002) MR

[16] **M Kreck**, **S Stolz**, *Nonconnected moduli spaces of positive sectional curvature metrics*, J. Amer. Math. Soc. 6 (1993) 825–850 MR

[17] **J M Lee**, *Introduction to smooth manifolds*, 2nd edition, Graduate Texts in Mathematics 218, Springer (2013) MR

[18] **X Menguy**, *Noncollapsing examples with positive Ricci curvature and infinite topological type*, Geom. Funct. Anal. 10 (2000) 600–627 MR

[19] **G Perelman**, *Construction of manifolds of positive Ricci curvature with big volume and large Betti numbers*, from "Comparison geometry" (K Grove, P Petersen, editors), Math. Sci. Res. Inst. Publ. 30, Cambridge Univ. Press (1997) 157–163 MR

[20] **P Petersen**, *Riemannian geometry*, Graduate Texts in Mathematics 171, Springer (1998) MR

[21] **W Tuschmann**, **D J Wraith**, *Moduli spaces of Riemannian metrics*, Oberwolfach Seminars 46, Birkhäuser, Basel (2015) MR

[22] **M Walsh**, *Metrics of positive scalar curvature and generalised Morse functions, II*, Trans. Amer. Math. Soc. 366 (2014) 1–50 MR

[23] **D J Wraith**, *On the moduli space of positive Ricci curvature metrics on homotopy spheres*, Geom. Topol. 15 (2011) 1983–2015 MR

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