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Rationality, universal generation and the integral Hodge conjecture

MINGMIN SHEN

We use the universal generation of algebraic cycles to relate (stable) rationality to the integral Hodge conjecture. We show that the Chow group of 1–cycles on a cubic hypersurface is universally generated by lines. Applications are mainly in cubic hypersurfaces of low dimensions. For example, we show that if a generic cubic fourfold is stably rational then the Beauville–Bogomolov form on its variety of lines, viewed as an integral Hodge class on the self product of its variety of lines, is algebraic. In dimensions 3 and 5, we relate stable rationality with the geometry of the associated intermediate Jacobian.

14C25, 14C30, 14E08

1 Introduction

An algebraic variety X is *rational* if it contains an open subset that can be identified with an open subset of the projective space of the same dimension. It is called *stably rational* if the product of X and some projective space is rational. The rationality problem is to tell whether a given variety is (stably) rational or not. It is one of the most subtle problems in algebraic geometry.

We work over the field $\mathbb{C}$ of complex numbers unless otherwise stated. The Lüroth theorem and Castelnuovo's criterion settled the rationality problem in dimensions one and two. One breakthrough in dimension three was made by Clemens and Griffiths [5], where they showed that a smooth cubic threefold is not rational. Other important methods that appeared around the same time include Artin and Mumford [3] and Iskovskikh and Manin [13].

The (stable) rationality problem in dimension three is closely related to the geometry of the intermediate Jacobian.

Theorem 1.1 *Let X be a smooth projective variety of dimension three and let $(J^3(X), \Theta)$ be its intermediate Jacobian.*

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(1) (Voisin [25]) *If X is stably rational, then the minimal class $\Theta^{g-1}/(g-1)!$ is algebraic.*

(2) (Clemens and Griffiths [5]) *If X is rational, then the minimal class $\Theta^{g-1}/(g-1)!$ is algebraic and effective* (which is equivalent to that $J^3(X)$ is a Jacobian of curves).

The integral Hodge conjecture is the statement that every integral Hodge class is an *algebraic class*, namely the cohomology class of some integral algebraic cycle. It is known that the integral Hodge conjecture is false in general. The relation between the rationality problem and the integral Hodge conjecture is very mysterious. Theorem 1.1, especially the first statement, can be viewed as a beautiful connection between (stable) rationality and the integral Hodge conjecture. In this paper, we develop a method to achieve more such statements. The main applications will be given to cubic threefolds and cubic fourfolds. We first recall the definition of a decomposition of the diagonal.

Definition 1.2 (Voisin [24; 25]) Let X be a smooth projective variety of dimension d. We say that X admits a *Chow-theoretical decomposition of the diagonal* if

$$\Delta_X = X \times x + Z \quad \text{in } \mathrm{CH}_d(X \times X),$$

where $x \in X$ is a closed point on X and Z is an algebraic cycle supported on $D \times X$ for some divisor $D \subset X$. We say that X has a *cohomological decomposition of the diagonal* if the above equality holds in $\mathrm{H}^{2d}(X \times X, \mathbb{Z})$.

One important fact is that a stably rational variety always admits a Chow-theoretical (and hence a cohomological) decomposition of the diagonal. Voisin [25] used decomposition of the diagonal to show new examples of three-dimensional unirational varieties which are not stably rational. Then Colliot-Thélène and Pirutka [6] generalized this method to show that a very general quartic threefold is not stably rational. Along the same line of ideas, Totaro [18] showed that a very general hypersurface of degree in a certain range is not stably rational. Many new results of nonrationality were obtained in recent years by Hassett, Kresch, Pirutka and Tschinkel [9; 10; 11; 12] and Okada [15]. The smallest possible degree for a hypersurface to be irrational is three. In dimension three, it is not known whether there exists a smooth cubic threefold that is stably rational or that is not stably rational. When it comes to the case of smooth cubic fourfolds, we know there exist rational cubic fourfolds; see Beauville and Donagi [4], Hassett [8] and Addington, Hassett, Tschinkel and Várilly-Alvarado [1]. It is expected that a very general cubic fourfold is not rational. However, no single cubic fourfold has been proven irrational.

1.1 Main results

Let $X \subset \mathbb{P}^{d+1}_{\mathbb{C}}$ be a smooth cubic hypersurface of dimension $d \geq 3$ and let $F = F(X)$ be the variety of lines on X. It is known by Altman and Kleiman [2] that F is a smooth projective variety of dimension $2d-4$. Over F we have the universal family of lines

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] & X \\ F & \end{tikzcd}$$

Then we can view $P \times P$ as a correspondence from $F \times F$ to $X \times X$. Let $h \in \mathrm{CH}^1(X)$ be the class of a hyperplane section.

Theorem 1.3 *Assume that X admits a Chow-theoretical decomposition of the diagonal.*

(1) *If $d = 3$, then there exists a symmetric 1–cycle θ on $F \times F$ such that*

$$\Delta_X = X \times x + x \times X + \gamma \times h + h \times \gamma + (P \times P)_*\theta \quad \text{in } \mathrm{CH}^3(X \times X)$$

for some $\gamma \in \mathrm{CH}_1(X)$.

(2) *If $d = 4$, then there exists a symmetric 2–cycle θ on $F \times F$ such that*

$$\Delta_X = X \times x + x \times X + \Sigma + (P \times P)_*\theta \quad \text{in } \mathrm{CH}^4(X \times X),$$

where $\Sigma \in \mathrm{CH}^2(X) \otimes \mathrm{CH}^2(X)$ is a symmetric decomposable 4–cycle. Moreover, if X is very general, then Σ can be chosen to be zero.

When $d = 4$, the variety F is a hyperkähler fourfold that is deformation equivalent to the Hilbert scheme of two points on a $K3$–surface. The canonical Beauville–Bogomolov bilinear form

$$\mathfrak{B}\colon \mathrm{H}^2(F, \mathbb{Z}) \times \mathrm{H}^2(F, \mathbb{Z}) \to \mathbb{Z}$$

gives rise to an integral Hodge class $q_{\mathfrak{B}} \in \mathrm{H}^{12}(F \times F, \mathbb{Z})$. We have the corresponding statement at the level of cohomology as follows:

Theorem 1.4 *Let X be a smooth cubic hypersurface of dimension $d = 3$ or 4 and let F be the variety of lines on X. Then X admits a cohomological decomposition of the diagonal if and only if there exists a symmetric $(d-2)$–cycle θ on $F \times F$ such that*

$$[\theta] \cdot \hat{\alpha} \otimes \hat{\beta} = \langle \alpha, \beta \rangle_X$$

for all $\alpha, \beta \in \mathrm{H}^d(X, \mathbb{Z})_{\mathrm{tr}}$, *where* $\hat{\alpha} := P^*\alpha$. *If* $d = 4$ *and* X *is very general, then the above condition is also equivalent to the fact that the Beauville–Bogomolov form* $q_{\mathfrak{B}}$ *is algebraic.*

In the case of $d = 3$, the above result has the following interesting application:

Corollary 1.5 *Let* X *be a smooth cubic threefold and let* $(J^3(X), \Theta)$ *be its intermediate Jacobian. If* X *admits a decomposition of the diagonal, then the following statements hold*:

(1) *The minimal class of* $J^3(X)$ *is algebraic and supported on a divisor of cohomology class* 3Θ.

(2) *The double of the minimal class of* $J^3(X)$ *is represented by a symmetric* (*with respect to the multiplication-by-*(-1) *morphism of* $J^3(X)$) 1*–cycle supported on a theta divisor.*

In the case of cubic fivefolds, our method can also relate rationality with the geometry of the intermediate Jacobian. The price we pay here is that we have to consider the rationality of X and F simultaneously.

Theorem 1.6 *Let* X *be a smooth cubic fivefold and let* F *be its variety of lines. If both* X *and* F *admit a Chow-theoretical decomposition of the diagonal*, *then the intermediate Jacobian* $J^5(X)$ *is a direct summand of the Jacobian of a* (*possibly reducible*) *curve* (*without respecting the principal polarizations*).

1.2 Universal generation

The main ingredient in the proof of the above results is the universal generation of the Chow group of 1–cycles on a cubic hypersurface by lines. This universal generation works over a general base field. Let Y and Z be smooth projective varieties defined over a field K. A cycle $\gamma \in \mathrm{CH}^r(Z_{K(Y)})$ is universally generating if a spreading $\Gamma \in \mathrm{CH}^r(Y \times Z)$ induces a universally surjective homomorphism $\Gamma_*\colon \mathrm{CH}_0(Y) \to \mathrm{CH}^r(Z)$. This means

$$(\Gamma_L)_*\colon \mathrm{CH}_0(Y_L) \to \mathrm{CH}^r(Z_L)$$

is surjective for all field extensions $L \supset K$.

Theorem 1.7 *Let $X \subset \mathbb{P}^{d+1}_K$ be a smooth cubic hypersurface of dimension $d \geq 3$ and let F be its variety of lines. Assume that $\mathrm{CH}_0(F)$ contains an element of degree one. Then the universal line $P \subset F \times X$ restricts to a universally generating 1–cycle $P|_{\eta_F} \in \mathrm{CH}_1(X_{K(F)})$. Namely, $P_*\colon \mathrm{CH}_0(F) \to \mathrm{CH}_1(X)$ is universally surjective.*

O Benoist pointed out that the condition that F admits a 0–cycle of degree one cannot be removed since the above universal generation fails when X is the universal cubic hypersurface over the generic point of the moduli space.

1.3 Convention and notation

Let X be a smooth projective variety of dimension d.

- $\mathrm{H}^p(X)$ denotes $\mathrm{H}^p(X, \mathbb{Z})$ modulo torsion.
- If $\alpha_i \in \mathrm{H}^{k_i}(X)$, $1 \leq i \leq r$, such that $\sum k_i = 2d$ then $\alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_r$ denotes the intersection number, namely the class $\alpha_1 \cup \dots \cup \alpha_r$ evaluated against the fundamental class $[X]$. In the special case of the middle cohomology, we write

$$\langle -,- \rangle_X\colon \mathrm{H}^d(X) \times \mathrm{H}^d(X) \to \mathbb{Z}, \quad (\alpha, \beta) \mapsto \alpha \cdot \beta.$$

- If Λ is a Hodge structure, then we use $\mathrm{Hdg}(\Lambda)$ to denote the Hodge classes in Λ. In the special case $\Lambda = \mathrm{H}^{2i}(X)$, we use $\mathrm{Hdg}^{2i}(X)$ to denote the Hodge classes in $\mathrm{H}^{2i}(X)$. The transcendental cohomology $\mathrm{H}^p(X)_{\mathrm{tr}}$ is the group of all elements $\alpha \in \mathrm{H}^p(X)$ such that

$$\alpha \cdot \beta = 0 \quad \text{for all } \beta \in \mathrm{Hdg}^{2d-p}(X).$$

  We use $\mathrm{H}^{2i}(X)_{\mathrm{alg}} \subseteq \mathrm{Hdg}^{2i}(X)$ to be the subgroup of algebraic classes. The same notation is defined for $\mathbb{Z}$ and $\mathbb{Q}$ coefficients.

- When X is given an ample class $h \in \mathrm{H}^2(X, \mathbb{Z})$, we use $\mathrm{H}^p(X)_{\mathrm{prim}}$ to denote the associated primitive cohomology.
- For any $\alpha \in \mathrm{H}^i(X)$ and $\beta \in \mathrm{H}^j(Y)$, we use $\alpha \otimes \beta$ denote the element in $\mathrm{H}^{i+j}(X \times Y)$ which is obtained via the Künneth decomposition. Similarly, for $\alpha \in \mathrm{CH}^i(X)$ and $\beta \in \mathrm{CH}^j(X)$, we use $\alpha \otimes \beta$ to denote the decomposable cycle $p_1^*\alpha \cdot p_2^*\beta \in \mathrm{CH}^{i+j}(X \times Y)$.
- η_X denotes the generic point of X.

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2 On a filtration on cohomology

In this section, we study a filtration on cohomology given by Definition 2.1. If a variety X is rational, then X can be obtained from the projective space $\mathbb{P}^d$ by a successive blow-ups and blow-downs, with centers of dimension at most $d-2$. Hence the cohomology of X comes from those centers. In this case the filtration corresponds to the dimensions of the centers.

In the first subsection, we give some general facts about the filtration including the behavior under a smooth blow-up (Proposition 2.4). In the second subsection, we treat an important feature of the filtration, namely that some algebraic cycle of X over a function field gives rise to a bilinear pairing on certain piece of the filtration.

Let X be a smooth projective variety of dimension d. Recall that we use $\mathrm{H}^p(-)$ to denote the integral cohomology $\mathrm{H}^p(-,\mathbb{Z})$ modulo torsion.

Definition 2.1 We define an increasing filtration on the middle cohomology $\mathrm{H}^d(X,\mathbb{Z})$ by

$$F^i\mathrm{H}^d(X,\mathbb{Z}) := \bigcap_{(Y,\Gamma)} \ker\{[\Gamma]^*\colon \mathrm{H}^d(X,\mathbb{Z}) \to \mathrm{H}^*(Y,\mathbb{Z})\}, \quad i \geq 1, \tag{1}$$

where (Y,Γ) runs through all smooth projective varieties Y of dimension at most $d-i$ and all correspondences $\Gamma \in \mathrm{CH}_*(Y \times X)$. We can similarly define the corresponding notion on $\mathrm{H}^d(X)$ and $\mathrm{H}^d(X,\mathbb{Q})$.

The following lemma shows that we can actually put some extra restriction on the dimension of Γ in the above definition:

Lemma 2.2 *The filtration $F^i\mathrm{H}^d(X,\mathbb{Z})$ can also be defined as in (1), where (Y,Γ) runs through all smooth projective varieties Y of dimension at most $d-i$ and all correspondences $\Gamma \in \mathrm{CH}_l(Y \times X)$ with $l \leq d-1$.*

Proof Let $F'^i\mathrm{H}^d(X,\mathbb{Z})$ be the filtration as defined in the lemma. Then it is clear that $F^i \subseteq F'^i$. Let $\alpha \in F'^i\mathrm{H}^d(X,\mathbb{Z})$ and let Y be a smooth projective variety of dimension at most $d-i$ and $\Gamma \in \mathrm{CH}_l(Y \times X)$. We need to show that $\beta := [\Gamma]^*\alpha = 0$ in $\mathrm{H}^{d-2(l-d_Y)}(Y,\mathbb{Z})$. For this, we may further assume that Γ is represented by an irreducible subvariety of $Y \times X$ which dominates Y. Indeed, if this case is proved and we have a case where Γ maps to a proper closed subvariety $j\colon Z \hookrightarrow Y$, then the action of Γ factors through a resolution Z' of Z. To be more precise, $[\Gamma]^* = j'_* \circ [\Gamma']^*$, where Γ' is a correspondence between Z' and X which dominates Z' and $j'\colon Z' \to X$ is the resolution morphism $Z' \to Z$ followed by the inclusion j. By assumption, we have $[\Gamma']^*\alpha = 0$ and it follows that $[\Gamma]^*\alpha = 0$.

Now we assume that Γ dominates Y and hence $l \geq d_Y$. If $l \leq d-1$, then by the definition of F'^i we have $[\Gamma]^*\alpha = 0$. Assume that $l \geq d$. Note that $d' := d-2(l-d_Y) = d_Y-(l-d)-(l-d_Y) \leq d_Y$. The equality holds only when $l = d = d_Y$, which forces $i = 0$. Hence we have $d' < d_Y$. Thus we can take a general complete intersection $Y' \subset Y$ of very ample divisors in Y such that $\dim Y' = d'$. The Lefschetz hyperplane theorem implies that $\beta = 0$ if and only if $\beta|_{Y'} = 0$. But we have $\beta|_{Y'} = [\Gamma']^*\alpha$, where $\Gamma' = \Gamma|_{Y' \times X} \in \mathrm{CH}_{l'}(Y' \times X)$ with $l' = l-(d_Y-d') = d-(l-d_Y) \leq d$. If this is a strict inequality, then by assumption, we have $\beta|_{Y'} = 0$ and hence $\beta = 0$. Otherwise, we have $l = d_Y \geq d$, which again forces $i = 0$. □

From the above lemma, we see that $F^1\mathrm{H}^d(X,\mathbb{Z})$ consists of all $\alpha \in \mathrm{H}^d(X,\mathbb{Z})$ such that $f^*\alpha = 0$ in $\mathrm{H}^d(Y,\mathbb{Z})$ for all morphisms $f\colon Y \to X$ where $\dim Y \leq d-1$. Let $\alpha \in \mathrm{H}^{d,0}(X)$ be the class represented by a global holomorphic d–form, then $f^*\alpha = 0$ for all $f\colon Y \to X$ with $\dim Y \leq d-1$, since $\mathrm{H}^{d,0}(Y) = 0$ for dimension reasons.

2.1 The filtration under a blow-up

In this subsection we take X to be a smooth projective variety of dimension d.

Proposition 2.3 *Let X' be another smooth projective variety of dimension d and $\Gamma \in \mathrm{CH}_d(X' \times X)$, then $[\Gamma]$ induces*

$$[\Gamma]^*\colon F^k\mathrm{H}^d(X,\mathbb{Z}) \to F^k\mathrm{H}^d(X',\mathbb{Z}).$$

In particular, for any morphism $f\colon X' \to X$, the homomorphism f^* on middle cohomology respects the filtration.

Proof This can be checked directly using composition of correspondences. □

Proposition 2.4 *Let $\rho\colon \widetilde{X}\to X$ be a blow-up along a smooth center $Y\subset X$ of codimension r. Then*

$$\rho^*\colon F^i\mathrm{H}^d(X,\mathbb{Z})\to F^i\mathrm{H}^d(\widetilde{X},\mathbb{Z})$$

is an isomorphism for all $i\le r$. In particular, the groups $F^1\mathrm{H}^d(X,\mathbb{Z})$ and $F^2\mathrm{H}^d(X,\mathbb{Z})$ are birational invariants, and they vanish if X is rational.

Proof Consider the blow-up diagram

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "\pi"'] & \widetilde{X} \arrow[d, "\rho"] \\ Y \arrow[r, "i_Y"] & X \end{tikzcd}$$

where $E\cong\mathbb{P}(\mathcal{N}_{Y/X})$ is the exceptional divisor. Then it is known that

$$\begin{aligned} \mathrm{H}^d(X,\mathbb{Z})\oplus\Bigl(\bigoplus_{l=1}^{r-1}\mathrm{H}^{d-2l}(Y,\mathbb{Z})\Bigr)&\cong\mathrm{H}^d(\widetilde{X},\mathbb{Z}),\\ (\alpha,\beta_1,\dots,\beta_{r-1})&\mapsto\rho^*\alpha+\sum_{l=1}^{r-1}j_*(\xi^{l-1}\cup\pi^*\beta_l), \end{aligned} \tag{2}$$

where $\xi\in\mathrm{H}^2(E,\mathbb{Z})$ is the class of the relative $\mathcal{O}(1)$–bundle on E. Since $\rho_*\rho^*=\mathrm{id}$, we have

$$\rho^*\colon F^i\mathrm{H}^d(X,\mathbb{Z})\hookrightarrow F^i\mathrm{H}^d(\widetilde{X},\mathbb{Z}). \tag{3}$$

Now let

$$\tilde{\alpha}=\rho^*\alpha+\sum_{l=1}^{r-1}j_*(\xi^{l-1}\cup\pi^*\beta_l)$$

be an element in $F^i\mathrm{H}^d(\widetilde{X},\mathbb{Z})$. Then let

$$\Gamma_{i'}:=(\pi,j)_*\xi^{i'-1}\in\mathrm{CH}_{d-i'}(Y\times\widetilde{X}),\quad 1\le i'\le r-1,$$

where $(\pi,j)\colon E\to Y\times X$ is the natural morphism. Since $\dim Y=d-r\le d-i$, we have

$$0=\Gamma_1^*\tilde{\alpha}=\pi_*j^*\tilde{\alpha}=-\beta_{r-1}.$$

Then we apply Γ_2^* to $\tilde{\alpha}$ and get $\beta_{r-2}=0$. By induction, we get $\beta_l=0$ for all $l=1,\dots,r-1$. To show that $\alpha\in F^i\mathrm{H}^d(X,\mathbb{Z})$, pick an arbitrary variety Z of dimension $d-i$ and a correspondence $\Gamma \in \mathrm{CH}_*(Z \times X)$. Then

$$\Gamma^*\alpha = \Gamma^*\rho_*\widetilde{\alpha} = ({}^t\rho \circ \Gamma)^*\widetilde{\alpha} = 0$$

and hence $\alpha \in F^i\mathrm{H}^d(X,\mathbb{Z})$. It follows that ρ^* in equation (3) is also surjective. □

Proposition 2.5 *Let X be a smooth cubic hypersurface of dimension $d = 3$ or 4. Then*

$$F^1\mathrm{H}^d(X,\mathbb{Z}) = F^2\mathrm{H}^d(X,\mathbb{Z}) = 0, \quad F^3\mathrm{H}^d(X,\mathbb{Z}) = \mathrm{H}^d(X,\mathbb{Z})_{\mathrm{tr}}.$$

Proof We first note that the middle cohomology of X is torsion-free. Let F be the variety of lines on X. Let $l \subset X$ be a general line on X and let $D_l \subseteq F$ be the variety of lines that meet l. Then it is known that D_l is smooth (see [5, Lemma 10.5] for $d=3$ and [21, Section 3, Lemme 1] for $d=4$). The associated Abel–Jacobi homomorphism

$$\Phi_l\colon \mathrm{H}^d(X,\mathbb{Z}) \to \mathrm{H}^{d-2}(D_l,\mathbb{Z})$$

is injective. This can be seen from the intersection property that

$$\Phi_l(\alpha)\cdot\Phi_l(\beta) = [D_l]\cdot\hat{\alpha}\cdot\hat{\beta} = -2\langle\alpha,\beta\rangle_X$$

for all $\alpha,\beta \in \mathrm{H}^d(X,\mathbb{Z})_{\mathrm{prim}}$; see [5; 17]. As a consequence, we see that

$$F^2\mathrm{H}^d(X,\mathbb{Z}) = F^1\mathrm{H}^d(X,\mathbb{Z}) = 0.$$

Let $\alpha \in F^3\mathrm{H}^d(X,\mathbb{Z})$. If $d=3$, then we have $\mathrm{H}^5(X,\mathbb{Z}) = 0$ and $\mathrm{H}^7(X,\mathbb{Z}) = 0$. As a consequence, we have $F^3\mathrm{H}^3(X,\mathbb{Z}) = \mathrm{H}^3(X,\mathbb{Z})_{\mathrm{tr}} = \mathrm{H}^3(X,\mathbb{Z})$. If $d=4$, then for all curves C and correspondences $\Sigma \in \mathrm{CH}_l(C \times X)$ for $l \le 3$, we have $\Sigma^*\alpha = 0$ in $\mathrm{H}^{6-2l}(C,\mathbb{Z})$. This condition is nontrivial only when $l=2$ or 3. In either case, it is equivalent to $[Z]\cdot\alpha = 0$ for some surface $Z \subset X$. So we have $F^3\mathrm{H}^4(X,\mathbb{Z}) = \mathrm{H}^4(X,\mathbb{Z})_{\mathrm{tr}}$. □

2.2 Bilinear form associated to a cycle over a function field

Definition–Lemma 2.6 *Let K be the function field of a variety of dimension $d-2r$. For any $\gamma \in \mathrm{CH}_r(X_K)$ we can define a $(-1)^d$–symmetric bilinear form*

$$\langle -,-\rangle_\gamma\colon F^{2r+1}\mathrm{H}^d(X) \times F^{2r+1}\mathrm{H}^d(X) \to \mathbb{Z}$$

by $(\alpha,\beta) \mapsto \langle\Gamma^*\alpha,\Gamma^*\beta\rangle_Z$, where Z is a model of K and $\Gamma \subset Z \times X$ is a spreading of γ.

Proof The above bilinear form is independent of the choices made. Indeed, a different choice of Z and Γ gives rise to an action on $\mathrm{H}^d(X)$ that differs by the action of a correspondence that factors through a variety of dimension at most $d-2r-1$. Hence the difference action is zero on $F^{2r+1}\mathrm{H}^d(X)$ by definition. □

Proposition 2.7 *If $\gamma_1,\gamma_2\in\mathrm{CH}_r(X_K)$ such that $\gamma_1-\gamma_2$ is torsion in $\mathrm{CH}_r(X_K)$, then $\langle -,-\rangle_{\gamma_1}=\langle -,-\rangle_{\gamma_2}$.*

Proof By definition $n(\gamma_1-\gamma_2)=0$ in $\mathrm{CH}_r(X_K)$. If we take some spreading $\Gamma_i\in\mathrm{CH}_{d-r}(Z\times X)$ of γ_i, we see that $n(\Gamma_1-\Gamma_2)$ is supported over a proper closed subset of Z. Thus the action of $n(\Gamma_1-\Gamma_2)$ factors through varieties of dimension at most $d-2r-1$. Hence, by definition, we have

$$[n(\Gamma_1-\Gamma_2)]^*\alpha=0 \quad \text{for all } \alpha\in F^{2r+1}\mathrm{H}^d(X).$$

It follows that $\Gamma_1^*\alpha=\Gamma_2^*\alpha$ since $\mathrm{H}^d(X)$, namely $\mathrm{H}^d(X,\mathbb{Z})$ modulo torsion, is torsion-free. □

3 Universal generation

In this section we give the definition of universal generation of algebraic cycles and discuss its basic properties. Then we discuss how universal generation is related to the decomposition of the diagonal.

We will eventually work over $\mathbb{C}$, but for the definitions, we assume that X is a smooth projective variety of dimension d over an arbitrary field k. Let Z be a smooth projective variety with function field $K=k(Z)$. For any cycle $\gamma\in\mathrm{CH}_r(X_K)$, we can define

$$\gamma_*\colon \mathrm{CH}_0(Z)\to\mathrm{CH}_r(X), \quad \tau\mapsto\Gamma_*\tau, \tag{4}$$

where $\Gamma\in\mathrm{CH}_{d_Z+r}(Z\times X)$ is a spreading of γ. Namely, $\Gamma|_{\eta_Z\times X}=\gamma$ in $\mathrm{CH}_r(X_K)$. If Γ' is another spreading of γ, then $\Gamma-\Gamma'$ is supported on $D\times X$ for some divisor D of Z. It follows that $\Gamma_*\tau=\Gamma'_*\tau$ for all $\tau\in\mathrm{CH}_0(Z)$. Thus the homomorphism (4) only depends on the class γ. Furthermore, for every field extension $L\supset k$, we have the induced homomorphism

$$(\gamma_L)_*\colon \mathrm{CH}_0(Z_L)\to\mathrm{CH}_r(X_L), \quad \tau\mapsto(\Gamma_L)_*\tau. \tag{5}$$

The above construction can be generalized to the following situation. Let Z be the disjoint union of smooth projective varieties Z_i with function field K_i and $\gamma = \sum_i \gamma_i$, where $\gamma_i \in \mathrm{CH}_r(X_{K_i})$. Then we can again define

$$(\gamma_L)_*\colon \bigoplus_i \mathrm{CH}_0((Z_i)_L) \to \mathrm{CH}_r(X_L). \tag{6}$$

Definition 3.1 The cycle

$$\gamma = \sum_{i=1}^{n} \gamma_i \in \bigoplus_{i=1}^{n} \mathrm{CH}_r(X_{K_i})$$

is *universally generating* if the natural homomorphism (6) is surjective for all field extensions $L \supset k$. We say that $\mathrm{CH}_r(X)$ is *universally trivial* if the natural homomorphism $\mathrm{CH}_r(X) \to \mathrm{CH}_r(X_L)$ is an isomorphism for all field extensions $L \supset k$.

For a d–dimensional variety X with a k–point x, the universal triviality of $\mathrm{CH}_0(X)$ is equivalent to the existence of a Chow-theoretic decomposition of the diagonal of the form $\Delta_X = X \times x + \Gamma$ in $\mathrm{CH}_d(X \times X)$, where Γ is supported on $D \times X$ for some divisor $D \subset X$. Indeed, the existence of a decomposition of the diagonal implies that $\mathrm{CH}_0(X)$ is universally generated by the point x. Conversely, if $\mathrm{CH}_0(X)$ is universally trivial in the above sense, then there exists $\gamma \in \mathrm{CH}_0(X)$ such that

$$\delta_X := \Delta_X|_{\eta_X \times X} = \gamma \quad \text{in } \mathrm{CH}_0(X_K),$$

where $K = k(X)$ is the function field of X. This implies that

$$\Delta_X = X \times \gamma + \Gamma \quad \text{in } \mathrm{CH}_d(X \times X),$$

where Γ is supported on $D \times X$ for some divisor $D \subset X$. We apply the above correspondence to $x \in \mathrm{CH}_0(X)$ and get

$$x = (\Delta_X)_* x = (X \times \gamma)_* x = \gamma.$$

Hence we have $\Delta_X = X \times x + \Gamma$ in $\mathrm{CH}_d(X \times X)$, which is a decomposition of the diagonal. Thus the universal triviality of CH_0 as in Definition 3.1 agrees with the usual definition; see for example [26].

3.1 Universal generation from decomposition of the diagonal

Proposition 3.2 (Voisin [26]) *Let X be a smooth projective variety of dimension d over $\mathbb{C}$. Then X admits a Chow-theoretical decomposition of the diagonal if and only if the following condition holds*:

($*$) *There exist smooth projective varieties Z_i of dimension $d-2$, correspondences $\Gamma_i \in \mathrm{CH}^{d-1}(Z_i \times X)$ and integers $n_i \in \mathbb{Z}$ for $i = 1, 2, \dots, r$ such that*

$$\Delta_X = \sum_{i=1}^{r} n_i \Gamma_i \circ {}^t\Gamma_i + X \times x + x \times X \quad \textit{in } \mathrm{CH}^d(X \times X),$$

where $x \in X$ is a closed point.

Proof The proof is the same as that of [26, Theorem 3.1], with homological equivalence replaced by rational equivalence. We only sketch the main steps here. It is clear that the condition ($*$) is a special form of Chow-theoretical decomposition of the diagonal. For the converse we assume that X has a Chow-theoretical decomposition of the diagonal

$$\Delta_X - X \times x = Z \quad \text{in } \mathrm{CH}_d(X \times X),$$

where Z is supported on $D \times X$ for some divisor $D \subset X$. We may replace X by a blow-up and assume that $D = \bigcup D_i$ is a global normal crossing divisor. Let $k_i \colon D_i \to X$ be the inclusion map. Then there exist $\Gamma_i' \in \mathrm{CH}_d(D_i \times X)$ such that

$$\Delta_X - X \times x = \sum (k_i, \mathrm{Id}_X)_* \Gamma_i' = \sum \Gamma_i' \circ k_i^*.$$

Composing the above equation with its transpose, we get

$$\Delta_X - X \times x - x \times X = (\Delta_X - X \times x) \circ (\Delta_X - x \times X) = \sum_{i,j} \Gamma_i' \circ k_i^* \circ k_{j,*} \circ {}^t\Gamma_j'.$$

For each i, assume that $k_i^* D_i = \sum_l n_{i,l} Z_{i,l}'$, where $Z_{i,l}' \subset D_i$ are irreducible divisors. Let $Z_{i,l}$ be a resolution of $Z_{i,l}'$ and let $\Gamma_{i,l} \in \mathrm{CH}_{d-1}(Z_{i,l} \times X)$ be the restriction of Γ_i' to $Z_{i,l}$; then we have

$$\Gamma_i' \circ k_i^* \circ k_{i,*} \circ {}^t\Gamma_i' = \sum_l n_{i,l} \Gamma_{i,l} \circ {}^t\Gamma_{i,l}.$$

For $i \neq j$, we let $Z_{\{i,j\}}$ be the intersection of D_i and D_j. Let $\Gamma_{i,j}' \in \mathrm{CH}_{d-1}(Z_{\{i,j\}} \times X)$ be the restriction of Γ_i' to $Z_{\{i,j\}}$. Thus we have

$$\Gamma_i' \circ k_i^* \circ k_{j,*} \circ {}^t\Gamma_j' = \Gamma_{i,j}' \circ {}^t\Gamma_{j,i}'.$$

It follows that

$$\begin{aligned}\Gamma_i' \circ k_i^* \circ k_{j,*} \circ {}^t\Gamma_j' + \Gamma_j' \circ k_j^* \circ k_{i,*} \circ {}^t\Gamma_i' \\ = (\Gamma_{i,j}' + \Gamma_{j,i}') \circ ({}^t\Gamma_{i,j}' + {}^t\Gamma_{j,i}') - \Gamma_{i,j}' \circ {}^t\Gamma_{i,j}' - \Gamma_{j,i}' \circ {}^t\Gamma_{j,i}'.\end{aligned}$$

Hence $\Delta_X - X \times x - x \times X$ is of the given form. □

The above result of Voisin is sufficient for our main application to cubic threefolds and cubic fourfolds. However, there is a more general version which we need for a later application to cubic fivefolds.

Proposition 3.3 *Let X be a smooth projective variety of dimension d over $\mathbb{C}$. Let $r \geq 1$ be an integer such that $2r \leq d$. Assume that $\mathrm{CH}_i(X)$ is universally trivial for $i = 0, 1, \ldots, r-1$. Then we have the higher decomposition of the diagonal*

$$\Delta_X = (X \times x + x \times X) + \Sigma + \Gamma \quad \text{in } \mathrm{CH}_d(X \times X),$$

which satisfies the following conditions:

(1) *The cycle*

$$\Sigma \in \bigoplus_{i<r} (\mathrm{CH}^i(X) \otimes \mathrm{CH}_i(X) \oplus \mathrm{CH}_i(X) \otimes \mathrm{CH}^i(X))$$

is a decomposable symmetric d–cycle, namely a linear combination of cycles of the form $\gamma \otimes \gamma' + \gamma' \otimes \gamma$.

(2) *There are smooth projective varieties Z_i of dimension $d-2r$, correspondences $\Gamma_i \in \mathrm{CH}_{d-r}(Z_i \times X)$ and integers n_i such that*

$$\Gamma = \sum_i n_i \Gamma_i \circ \sigma_i \circ {}^t\Gamma_i \quad \text{in } \mathrm{CH}_d(X \times X),$$

where $\sigma_i \colon Z_i \to Z_i$ is a (possibly trivial) involution. If $r = 1$, then all σ_i can be taken to be the identity.

The proof requires the following lemma concerning symmetric cycles:

Lemma 3.4 *Let X be a smooth projective variety. The following statements are true:*

(i) *Let Γ_1 and Γ_2 be two symmetric cycles on $X \times X$; then $\Gamma_1 \cdot \Gamma_2$ is represented by a symmetric cycle.*

(ii) *Let Z be another smooth projective varieties and let $\Gamma \in \mathrm{CH}^p(Z \times X)$ be a correspondence. If a self-correspondence $\Sigma \in \mathrm{CH}^q(X \times X)$ is represented by a symmetric cycle, then the self-correspondence ${}^t\Gamma \circ \Sigma \circ \Gamma \in \mathrm{CH}^*(Z \times Z)$ is also represented by a symmetric cycle.*

Proof Let $d = \dim X$ and let $\rho_X \colon \widetilde{X \times X} \to X \times X$ be the blow-up along the diagonal. Let $\sigma_X \colon \widetilde{X \times X} \to X^{[2]}$ be a ramified degree 2 cover of the Hilbert square of X. Let $\mu_X = \sigma_{X,*}\rho_X^* \in \mathrm{CH}^{2d}(X \times X \times X^{[2]})$ be the associated correspondence. Let $\iota_X\colon X \to X \times X$ be the diagonal embedding. Our first observation is that a cycle class $\gamma \in \mathrm{CH}_r(X \times X)$ is represented by a symmetric cycle if and only if $\gamma = \iota_{X,*}\alpha + \mu_X^*\beta$ for some $\alpha \in \mathrm{CH}_r(X)$ and $\beta \in \mathrm{CH}_r(X^{[2]})$. This can be seen as follows. If $\gamma = \iota_{X,*}\alpha + \mu_X^*\beta$ then it is clear that γ is represented by a symmetric cycle. Conversely, assume that γ is represented by a symmetric cycle Γ on $X \times X$. Let Γ_0 be the restriction of Γ to $X \times X \setminus \Delta_X$. Then there exists a cycle Γ_0' on $X^{(2)} \setminus \Delta_X$ whose pullback to $X \times X \setminus \Delta_X$ is Γ_0. Note that $X^{(2)} \setminus \Delta_X \subset X^{[2]}$ is canonically an open subvariety. Let $\overline{\Gamma_0'}$ be the closure of Γ_0' in $X^{[2]}$. Let $\beta \in \mathrm{CH}_r(X^{[2]})$ be the class of $\overline{\Gamma_0'}$. By construction γ and $\mu_X^*\beta$ agrees on $X \times X \setminus \Delta_X$. Hence, by the localization sequence, there exists $\alpha \in \mathrm{CH}_r(X)$ such that $\gamma = \iota_{X,*}\alpha + \mu_X^*\beta$.

To prove (i), we write $\Gamma_i = \iota_{X,*}\alpha_i + \mu_X^*\beta_i$ for $i = 1, 2$. Then we have

$$\begin{aligned}
\Gamma_1 \cdot \Gamma_2 &= (\iota_{X,*}\alpha_1 + \mu_X^*\beta_1) \cdot (\iota_{X,*}\alpha_2 + \mu_X^*\beta_2) \\
&= \mu_X^*\beta_1 \cdot \mu_X^*\beta_2 && \mod \iota_{X,*}\mathrm{CH}_*(X) \\
&= \rho_{X,*}\sigma_X^*\beta_1 \cdot \rho_{X,*}\sigma_X^*\beta_2 && \mod \iota_{X,*}\mathrm{CH}_*(X) \\
&= \rho_{X,*}(\sigma_X^*\beta_1 \cdot \rho_X^*\rho_{X,*}\sigma_X^*\beta_2) && \mod \iota_{X,*}\mathrm{CH}_*(X) \\
&= \rho_{X,*}(\sigma_X^*\beta_1 \cdot (\sigma_X^*\beta_2 + \tau)) && \mod \iota_{X,*}\mathrm{CH}_*(X),
\end{aligned}$$

where τ is supported on the exceptional divisor $E_X \subset \widetilde{X \times X}$ of the blow-up ρ_X. Thus $\rho_{X,*}(\tau \cdot \sigma_X^*\beta_1)$ is supported on Δ_X. As a consequence, we have

$$\Gamma_1 \cdot \Gamma_2 = \mu_X^*(\beta_1 \cdot \beta_2) + \iota_{X,*}\alpha$$

for some $\alpha \in \mathrm{CH}_*(X)$. It follows that $\Gamma_1 \cdot \Gamma_2$ is represented by a symmetric cycle.

Now we prove (ii). By definition,

$${}^t\Gamma \circ \Sigma \circ \Gamma = (p_{Z \times Z})_*((p_{X \times X})^*\Sigma \cdot (\Gamma \times \Gamma)).$$

Note that $(p_{X \times X})^*\Sigma$ is a symmetric cycle on $Z \times X \times Z \times X$ and so is $\Gamma \times \Gamma$. We apply (i) and conclude that ${}^t\Gamma \circ \Sigma \circ \Gamma$ is the pushforward of a symmetric cycle on $Z \times X \times Z \times X$, which is again symmetric. □

Proof of Proposition 3.3 Since $\mathrm{CH}_0(X)$ is universally trivial, we have a decomposition of the diagonal

$$\Delta_X = X \times x + \Gamma' \quad \text{in } \mathrm{CH}_d(X \times X),$$

where Γ' is supported on $D \times X$ for some divisor D. We assume that $D = \bigcup_j D_j$. Then the restriction of Γ' to the generic point of D_j is a 1–cycle on $X_{\mathbb{C}(D_j)}$. By the universal triviality of $\mathrm{CH}_1(X)$, there exist a 1–cycle $\gamma_j \in \mathrm{CH}_1(X)$ such that $D_j \otimes \gamma_j$ agrees with Γ' over the generic point of D_j. Thus we have

$$\Delta_X = X \times x + \Gamma_1' + \Gamma'' \quad \text{in } \mathrm{CH}_d(X \times X),$$

where $\Gamma_1' = \sum_j D_j \otimes \gamma_j$ and Γ'' is supported on $Y \times X$, where $Y \subset X$ is a closed subset of codimension 2. By repeating the above argument, we eventually get

$$\Delta_X = X \times x + \Gamma_1' + \dots + \Gamma_{r-1}' + \Gamma''' \quad \text{in } \mathrm{CH}_d(X \times X),$$

where $\Gamma_i' \in \mathrm{CH}^i(X) \otimes \mathrm{CH}_i(X)$ for $1 \le i \le r-1$ and Γ''' is supported on $Z \times X$ for some closed subset $Z \subset X$ of codimension r. Then we carry out the symmetrization argument $\Delta_X = \Delta_X \circ {}^t\Delta_X$. We note that $\Gamma_i' \circ (-)$ (resp. $(-) \circ {}^t\Gamma_i'$) is again decomposable and contained in $\mathrm{CH}^i(X) \otimes \mathrm{CH}_i(X)$ (resp. in $\mathrm{CH}_i(X) \otimes \mathrm{CH}^i(X)$). Then we carry out a similar argument as before to show that $\Gamma''' \circ {}^t\Gamma'''$ is of the required form. First, there exist smooth projective varieties D_i' of dimension $d-r$, correspondences $\Gamma_i''' \in \mathrm{CH}_d(D_i' \times X)$ and morphisms $f_i \colon D_i' \to X$ such that

$$\Gamma''' = \sum_i (f_i, \mathrm{Id}_X)_* \Gamma_i''' = \sum_i \Gamma_i''' \circ f_i^*.$$

Then we have

$$\Gamma''' \circ {}^t\Gamma''' = \sum_{i,j} \Gamma_i''' \circ f_i^* \circ f_{j,*} \circ {}^t\Gamma_j'''.$$

We write down the cycles $f_i^* \circ f_{j,*} \in \mathrm{CH}_{d-2r}(D_i' \times D_j')$ explicitly. Then the terms of the above sum can be grouped into the following types:

Term type 1 $\Gamma_1 \circ {}^t\Gamma_2 + \Gamma_2 \circ {}^t\Gamma_1$, where $\Gamma_1, \Gamma_2 \in \mathrm{CH}_{d-r}(Z \times X)$ for some smooth projective variety Z of dimension $d-2r$. Such terms appear in

$$\Gamma_i''' \circ f_i^* \circ f_{j,*} \circ {}^t\Gamma_j''' + \Gamma_j''' \circ f_j^* \circ f_{i,*} \circ {}^t\Gamma_i''',$$

where $i < j$. Such a type 1 term can be written as

$$(\Gamma_1 + \Gamma_2) \circ {}^t(\Gamma_1 + \Gamma_2) - \Gamma_1 \circ {}^t\Gamma_1 - \Gamma_2 \circ {}^t\Gamma_2.$$

Thus such a term can be written as the required form.

We still need to deal with the terms $\Gamma_i''' \circ f_i^* \circ f_{i,*} \circ {}^t\Gamma_i'''$. By Lemma 3.4, such a term can also be written as $\Gamma_i''' \circ \Sigma_i \circ {}^t\Gamma_i'''$, where Σ_i is a symmetric cycle of dimension

$d-2r$ on $D_i' \times D_i'$. Then Σ_i is a linear combination of cycles of the following forms: (a) $Z + {}^tZ$, where $Z \subset D_i' \times D_i'$ is an irreducible subvariety of dimension $d-2r$; (b) a subvariety $Z \subset D_i' \times D_i'$ of dimension $d-2r$, which is contained in the diagonal $\Delta_{D_i'}$; (c) an irreducible subvariety $Z \subset D_i' \times D_i'$, which is not supported on the diagonal but invariant under the involution of switching the two factors of $D_i' \times D_i'$. After resolution of the singularities of Z if necessary, we see the following: (a) produces terms of type 1; (b) produces terms for the form $\Gamma \circ {}^t\Gamma$, which satisfies the required form; (c) produces a new term of type 2, as follows:

Term type 2 $\Gamma \circ \sigma \circ {}^t\Gamma$, where $\Gamma \in \mathrm{CH}_{d-r}(Z \times X)$ for some smooth projective variety Z of dimension $d-2r$ and σ is a nontrivial involution of Z (induced by the involution of $D_i' \times D_i'$). □

Remark 3.5 In Proposition 3.2, only diagonalized terms $\Gamma_i \circ {}^t\Gamma_i$ appear since we are allowed to blow up X to make the D_i to be normal crossing. However, blow-up only preserves universal triviality of CH_0 and hence is not allowed in Proposition 3.3. Thus the image of D_i in X can fail to be normal and that produces the terms $\Gamma_i \circ \sigma_i \circ {}^t\Gamma_i$ that are not diagonalized.

Corollary 3.6 *Let X be a smooth projective variety of dimension d over $\mathbb{C}$ such that $\mathrm{CH}_i(X)$ is universally trivial for all $i = 0, 1, \dots, r-1$. Then the following are true:*

(i) *There exist smooth projective varieties Z_i of dimension $d-2r$, cycles $\Gamma_i \in \mathrm{CH}_{d-r}(Z_i \times X)$ and integers $n_i \in \mathbb{Z}$ such that $\Gamma = \sum n_i \Gamma_i$ induces a universally surjective homomorphism $\bigoplus_i \mathrm{CH}_0(Z_i) \to \mathrm{CH}_r(X)$ and*

$$\sum n_i \langle \Gamma_i^* \alpha, \sigma_i^* \Gamma_i^* \beta \rangle_{Z_i} = \langle \alpha, \beta \rangle_X$$

for all $\alpha, \beta \in \mathrm{H}^d(X, \mathbb{Z})$, where $\sigma_i \colon Z_i \to Z_i$ is either the identity map or an involution.

(ii) $\mathrm{H}^{p,q}(X) = 0$ *for all $p \neq q$ and $\min\{p,q\} < r$.*

(iii) $F^i \mathrm{H}^d(X, \mathbb{Z}) = 0$ *for all $i = 0, 1, \dots, 2r$.*

Proof Let L/k be an arbitrary field extension. Let $\gamma \in \mathrm{CH}_r(X_L)$; then

$$\gamma = (\Delta_X)_* \gamma = \Sigma_* \gamma + \Gamma_* \gamma = \Gamma_* \gamma.$$

Here all the correspondences are understood to be their base changes to L; we also use the fact $\Sigma_*\gamma = 0$, which is a consequence of

$$\Sigma \in \bigoplus_{i<r} (\mathrm{CH}^i(X) \otimes \mathrm{CH}_i(X) \oplus \mathrm{CH}_i(X) \otimes \mathrm{CH}^i(X)) \quad \text{and} \quad \gamma \in \mathrm{CH}_r(X_L).$$

Since $\Gamma_*\gamma$ factors through $\sum \mathrm{CH}_0((Z_i)_L)$ the universal surjection in statement (i) follows immediately. Let $\alpha, \beta \in \mathrm{H}^d(X, \mathbb{Z})$. Then we have

$$\langle \alpha, \beta \rangle_X = [\Delta_X] \cup (\alpha \otimes \beta) = [X \times x + x \times X + \Sigma + \Gamma] \cup (\alpha \otimes \beta) = [\Gamma] \cup (\alpha \otimes \beta).$$

Note that here $\Sigma \cup (\alpha \otimes \beta) = 0$ since the algebraic cycle Σ is of the special form as above. Indeed, if $\Sigma = \Sigma_1 \otimes \Sigma_2 \in \mathrm{CH}^i(X) \otimes \mathrm{CH}_i(X)$, then $\Sigma \cup (\alpha \otimes \beta) = \langle [\Sigma_1], \alpha \rangle_X \langle [\Sigma_2], \beta \rangle_X = 0$ since $2i < d$. At the same time, we have

$$[\Gamma] \cup (\alpha \otimes \beta) = \Big[\sum n_i \Gamma_i \circ \sigma_i \circ {}^t\Gamma_i \Big] \cup (\alpha \otimes \beta) = \sum n_i \langle \Gamma_i^* \alpha, \sigma_i^* \Gamma_i^* \beta \rangle_{Z_i}.$$

Let $\omega \in \mathrm{H}^{p,q}(X)$. Then we have

$$\omega = (\Delta_X)_* \omega = (x \times X + X \times x + \Sigma)_* \omega + \Gamma_* \omega,$$

where the first term vanishes whenever $p \neq q$. If $\max\{p, q\} < r$, then the second term also vanishes since it factors through $\mathrm{H}^{p-r, q-r}(Z_i)$. Statement (ii) follows. Let $\alpha \in F^i \mathrm{H}^d(X, \mathbb{Z})$, where $i \leq 2r$. Since Δ_X factors through varieties of dimension at most $d - 2r$, it follows that $\alpha = (\Delta_X)^* \alpha = 0$. Thus (iii) is proved. □

Remark 3.7 The statement (ii) holds under the weaker assumption that $\mathrm{CH}_i(X)_{\mathbb{Q}}$ is universally trivial for $i = 0, 1, \dots, r-1$.

Theorem 3.8 *Let X be a smooth projective variety of dimension d over $\mathbb{C}$ such that $\mathrm{CH}_i(X)$ is universally trivial for all $i = 0, 1, \dots, r-1$, where r is a positive integer with $2r \leq d$. Let F be a smooth projective variety and let $\gamma \in \mathrm{CH}_r(X_{\mathbb{C}(F)})$ be a universally generating r–cycle. Then there exists a symmetric algebraic cycle $\theta \in \mathrm{CH}_{d-2r}(F \times F)$ such that*

$$[\theta] \cdot (\hat{\alpha} \otimes \hat{\beta}) = \langle \alpha, \beta \rangle_X$$

for all $\alpha, \beta \in F^{2r+1} \mathrm{H}^d(X, \mathbb{Z})$, where $\hat{\alpha} := \Gamma^* \alpha$ for some spreading $\Gamma \in \mathrm{CH}^r(F \times X)$ of γ.

Proof By Corollary 3.6, there exist smooth projective varieties Z_i of dimension $d-2r$, cycles $\Gamma_i \in \mathrm{CH}_{d-r}(Z_i \times X)$ and integers n_i such that

$$\sum n_i \langle \Gamma_i^* \alpha, \sigma_i^* \Gamma_i^* \beta \rangle_{Z_i} = \langle \alpha, \beta \rangle_X$$

for all $\alpha, \beta \in \mathrm{H}^d(X, \mathbb{Z})$. Let $\gamma_i := \Gamma_i|_{\eta_{Z_i}} \in \mathrm{CH}_r(X_{K_i})$, where $K_i = \mathbb{C}(Z_i)$. For each i, by the universal generating property of Γ, there exists $\tau_i \in \mathrm{CH}_0(F_{K_i})$ such that $\Gamma_* \tau_i = \gamma_i$. Let $T_i \in \mathrm{CH}_{d-2r}(Z_i \times F)$ be a spreading of τ_i and set

$$\theta = \sum n_i T_i \circ \sigma_i \circ {}^t T_i \in \mathrm{CH}_{d-2r}(F \times F).$$

Note that the condition $\Gamma_* \tau_i = \gamma_i$ means that $\Gamma_i' := \Gamma \circ T_i$ and Γ_i agrees over η_{Z_i}. Hence $\Gamma_i - \Gamma_i'$ factors through a divisor of Z_i. It follows that $\Gamma_i^* \alpha = \Gamma_i'^* \alpha$ for all $\alpha \in F^{2r+1} \mathrm{H}^d(X, \mathbb{Z})$. Thus, for $\alpha, \beta \in F^{2r+1} H^d(X, \mathbb{Z})$, we have

$$\begin{aligned}
[\theta] \cdot (\hat{\alpha} \otimes \hat{\beta}) &= \sum n_i [T_i \circ \sigma_i \circ {}^t T_i] \cup (\Gamma^* \alpha \otimes \Gamma^* \beta) \\
&= \sum n_i \langle T_i^* \Gamma^* \alpha, \sigma_i^* T_i^* \Gamma^* \beta \rangle_{Z_i} \\
&= \sum n_i \langle \Gamma_i'^* \alpha, \sigma_i^* \Gamma_i'^* \beta \rangle_{Z_i} \\
&= \sum n_i \langle {\Gamma_i}^* \alpha, \sigma_i^* {\Gamma_i}^* \beta \rangle_{Z_i} \\
&= \langle \alpha, \beta \rangle_X .
\end{aligned}$$

This finishes the proof. □

4 Universal generation of 1–cycles on cubic hypersurfaces

In this section we show that the Chow group of 1–cycles on a smooth cubic hypersurface is universally generated by lines. Let K be an arbitrary base field of any characteristic. Let $X \subseteq \mathbb{P}_K^{d+1}$ be a smooth cubic hypersurface of dimension d over K. Let $F = F(X)$ be the variety of lines on X. It is known by [2] that F/K is smooth of dimension $2d-4$. Let

$$\begin{array}{ccc} P & \xrightarrow{q} & X \\ {\scriptstyle p}\big\downarrow & & \\ F & & \end{array}$$

be the universal line.

Theorem 4.1 *Let K be an arbitrary field and X/K a smooth cubic hypersurface of dimension $d \geq 3$. Then the following are true*:

(i) *If $\gamma \in \mathrm{CH}_1(X)$, then $3\gamma \in q_* p^* \mathrm{CH}_0(F) + \mathbb{Z} h^{d-1}$.*

(ii) *If* $\mathrm{CH}_1(X)$ *contains an element of degree not divisible by* 3, *then*

$$\mathrm{CH}_1(X) = q_* p^* \mathrm{CH}_0(F) + \mathbb{Z}h^{d-1}.$$

(iii) *If* $\mathrm{CH}_0(F)$ *contains an element of degree* 1, *then*

$$P_* = q_* p^* \colon \mathrm{CH}_0(F) \to \mathrm{CH}_1(X)$$

is universally surjective.

The key ingredient one needs to prove the above universal generation is the following relations among 1–cycles on X:

Proposition 4.2 *Let* $\gamma_1, \gamma_2 \in \mathrm{CH}_1(X)$ *be* 1*–cycles of degree* e_1 *and* e_2, *respectively. Let* $h \in \mathrm{CH}^1(X)$ *be the class of a hyperplane section.*

(i) *There exists a* 0*–cycle* $\mathfrak{a} \in \mathrm{CH}_0(F)$ *such that*

$$(2e_1 - 3)\gamma_1 + q_* p^* \mathfrak{a} = a h^{d-1} \quad \text{in } \mathrm{CH}_1(X)$$

for some integer a. *If* γ_1 *is represented by a geometrically irreducible curve* C *in general position, then* $\mathfrak{a}$ *can be taken to be all the lines that meet* C *in two points.*

(ii) *We have*

$$2e_2\gamma_1 + 2e_1\gamma_2 + q_* p^* \mathfrak{a}' = 3e_1e_2h^{d-1} \quad \text{in } \mathrm{CH}_1(X),$$

where $\mathfrak{a}' = p_* q^* \gamma_1 \cdot p_* q^* \gamma_2 \in \mathrm{CH}_0(F)$ *is a* 0*–cycle of degree* $5e_1e_2$.

(iii) *Let* $\xi \in \mathrm{CH}^r(X)$ *with* $r < d-1$. *Then*

$$2e_1\xi + q_* p^* \mathfrak{a}'' = b h^r \quad \text{in } \mathrm{CH}^r(X),$$

where $\mathfrak{a}'' = p_* q^* \gamma_1 \cdot p_* q^* \xi \in \mathrm{CH}^{d+r-3}(F)$ *and* $b \in \mathbb{Z}$.

We now prove Theorem 4.1 assuming Proposition 4.2.

Proof of Theorem 4.1 Apply Proposition 4.2(i) to $\gamma_1 = \gamma$; we get $(2e-3)\gamma \in q_* p^* \mathrm{CH}_0(F) + \mathbb{Z}h^{d-1}$. Apply Proposition 4.2(ii) to the case of $\gamma_1 = \gamma$ and $\gamma_2 = h^{d-1}$; we get $6\gamma \in q_* p^* \mathrm{CH}_0(F) + \mathbb{Z}h^{d-1}$. Note that the greatest common divisor of 6 and $2e-3$ is either 3 or 1. We see that $3\gamma \in q_* p^* \mathrm{CH}_0(F) + \mathbb{Z}h^{d-1}$. This proves (i).

If $e = \deg(\gamma)$ is not divisible by 3, then the greatest common divisor of 6 and $2e-3$ is 1. Hence $\gamma \in q_* p^* \mathrm{CH}_0(F) + \mathbb{Z}h^{d-1}$. Now let $\gamma' \in \mathrm{CH}_1(X)$ be an arbitrary

1–cycle. We apply Proposition 4.2(ii) to γ and γ' and get

$$2e\gamma' + 2e'\gamma \in q_*p^*\mathrm{CH}_0(F) + \mathbb{Z}h^{d-1}.$$

Since $\gamma \in q_*p^*\mathrm{CH}_0(F) + \mathbb{Z}h^{d-1}$, we conclude that $2e\gamma' \in q_*p^*\mathrm{CH}_0(F) + \mathbb{Z}h^{d-1}$. We already know that $3\gamma' \in q_*p^*\mathrm{CH}_0(F) + \mathbb{Z}h^{d-1}$. By assumption, the greatest common divisor of 3 and $2e$ is 1. We conclude $\gamma' \in q_*p^*\mathrm{CH}_0(F) + \mathbb{Z}h^{d-1}$, which establishes (ii).

Since our base field K is arbitrary, for (iii) it suffices to show that $q_*p^*\colon \mathrm{CH}_0(F) \to \mathrm{CH}_1(X)$ is surjective. Let $\mathfrak{a}_0 \in \mathrm{CH}_0(F)$ be an element of degree 1. Take $\mathfrak{l} = q_*p^*\mathfrak{a}_0 \in \mathrm{CH}_1(X)$, which is of degree 1. By (ii), we only need to show that h^{d-1} is contained in $q_*p^*\mathrm{CH}_0(F)$. Apply Proposition 4.2(ii) to $\gamma_1 = \gamma_2 = h^{d-1}$; we get

$$6h^{d-1} + 6h^{d-1} + q_*p^*\mathfrak{a}_1 = 27h^{d-1} \quad \text{with } \mathfrak{a}_1 \in \mathrm{CH}_0(F).$$

It follows that $15h^{d-1} = q_*p^*\mathfrak{a}_1$. We apply the proposition again to h^{d-1} and $\mathfrak{l}$ and get

$$2h^{d-1} + 6\mathfrak{l} + q_*p^*\mathfrak{a}_2 = 9h^{d-1} \quad \text{with } \mathfrak{a}_2 \in \mathrm{CH}_0(F).$$

Thus $7h^{d-1} = q_*p^*(\mathfrak{a}_2 + 6\mathfrak{a}_0)$. Thus we conclude that $h^{d-1} \in q_*p^*\mathrm{CH}_0(F)$. This proves (iii). □

The proof of Proposition 4.2 over an algebraically closed field is given in [16; 17]. In this section, we develop a universal version of the techniques in [16] which leads to a proof of Proposition 4.2. We would like to study the geometry of $X^{[2]}$, the Hilbert scheme of two points on X. Let $\delta \in \mathrm{CH}^1(X^{[2]})$ be the "half boundary". Namely, 2δ is the class of the boundary divisor parametrizing nonreduced length-2 subschemes.

Following Galkin and Shinder [7], we define a rational map

$$\Phi\colon X^{[2]} \dashrightarrow P_X := \mathbb{P}(\mathcal{T}_{\mathbb{P}_K^{d+1}}|_X)$$

as follows. Let $x, y \in X$ be two distinct points and they determine a line $L_{x,y} \subset \mathbb{P}_K^{d+1}$. If $L_{x,y}$ is not contained in X, then $L_{x,y}$ intersects X in a third point $z \in X$. Then $\Phi([x,y])$ is the point represented by the 1–dimensional subspace $\mathcal{T}_{L_{x,y},z}$ in $\mathbb{P}(\mathcal{T}_{\mathbb{P}^{d+1},z})$. Let $P_F^{[2]} \subset X^{[2]}$ be the relative Hilbert scheme of P/F. Let $p'\colon P_F^{[2]} \to F$ be the structure morphism. Then it is a fact that the indeterminacy of Φ can be resolved by blowing up $X^{[2]}$ along $P_F^{[2]}$. This is proved in Voisin [26, Proposition 2.9] for the case $K = \mathbb{C}$ and the same proof works over an arbitrary field K. The resulting morphism $\widetilde{\Phi}\colon \widetilde{X^{[2]}} \to P_X$ is the blow-up of P_X along $P \subset P_X$. The inclusion

$i_1\colon P \hookrightarrow P_X$ sends a point $(x \in l)$ to the direction $\mathcal{T}_{l,x}$ in $\mathcal{T}_{X,x}$. The two blow-ups $\tau\colon \widetilde{X^{[2]}} \to X^{[2]}$ and $\widetilde{\Phi}\colon \widetilde{X^{[2]}} \to P_X$ share the same exceptional divisor $E \subset \widetilde{X^{[2]}}$. To summarize, we have the commutative diagram

$$\begin{tikzcd}
E \arrow[rr, "\pi_1"] \arrow[dr, "j"] \arrow[dd, "\pi_2"'] & & P \arrow[d, "i_1"'] \\
& \widetilde{X^{[2]}} \arrow[r, "\widetilde{\Phi}"] \arrow[d, "\tau"'] & P_X \\
P_F^{[2]} \arrow[r, "i_2"] & X^{[2]} \arrow[ur, dashed, "\Phi"'] &
\end{tikzcd}$$

We also have the natural identification

$$E = P \times_F P_F^{[2]}$$

and the morphisms π_1 and π_2 are the two projections.

There is a double cover $\sigma\colon \widetilde{X \times X} \to X^{[2]}$, where $\widetilde{X \times X}$ is the blow-up of $X \times X$ along the diagonal. There is also a morphism

$$\Psi\colon \widetilde{X \times X} \to P_X, \quad (x, y) \mapsto [\mathcal{T}_{L_{x,y},x}].$$

Note that the composition $\widetilde{X \times X} \to P_X \to X$ is the blow-up morphism ρ followed by the projection onto the first factor. The above morphisms form the commutative diagram

$$\begin{tikzcd}
X^{[2]} & \widetilde{X \times X} \arrow[l, "\sigma"'] \arrow[r, "\Psi"] \arrow[d, "\rho"'] & P_X \arrow[d, "\pi"] \\
& X \times X \arrow[r, "p_1"] & X
\end{tikzcd}$$

Given algebraic cycles $\alpha, \beta \in \mathrm{CH}_*(X)$, we write

$$\alpha \mathbin{\widehat{\otimes}} \beta := \sigma_* \rho^* (\alpha \times \beta) \in \mathrm{CH}_*(X^{[2]}).$$

Two distinct points $x, y \in X$ determine a line $L_{x,y}$ in $\mathbb{P}^{d+1}$. This defines a morphism

$$\varphi\colon X^{[2]} \to G(2, d+2),$$

where $G(2, d+2)$ is the Grassmannian of rank two subspaces of K^{d+2}. Together with the previous morphisms, we have a commutative diagram, with all squares being fiber products,

$$
\begin{tikzcd}
\widetilde{X \times X} \cup \widetilde{X^{[2]}} \arrow[rr] \arrow[d] & & P_X \arrow[r, "\pi"] \arrow[d, "i'"] & X \arrow[d, "i"] \\
Q \arrow[rr, "\varphi'"] \arrow[d, "f"] & & G(1,2,d+2) \arrow[r, "\tilde{\pi}"] \arrow[d, "\tilde{f}"] & \mathbb{P}_K^{d+1} \\
X^{[2]} \arrow[rr, "\varphi"] & & G(2,d+2) &
\end{tikzcd}
\tag{7}
$$

Let $h \in \mathrm{CH}^1(\mathbb{P}_K^{d+1})$ be the class of a hyperplane section. We still use $h \in \mathrm{CH}^1(X)$ to denote the restriction of h to X and let $h_Q \in \mathrm{CH}^1(Q)$ be the pullback of h to Q via the natural morphism $Q \to \mathbb{P}_K^{d+1}$.

Lemma 4.3 *We have*

$$E = -h_Q|_{\widetilde{X^{[2]}}} - \tau^*(2h \mathbin{\widehat{\otimes}} 1 - 3\delta)$$

in $\mathrm{CH}^1(\widetilde{X^{[2]}})$.

Proof Let $\mathscr{E} \subset \mathcal{O}_G^{d+2}$ be the tautological rank-2 subbundle on $G := G(2,d+2)$. Note that $Q \cong \mathbb{P}(\mathscr{E}|_{X^{[2]}})$ is a $\mathbb{P}^1$–bundle over $X^{[2]}$. The projective bundle formula for Chow groups implies that, as divisors on Q,

$$\widetilde{X^{[2]}} = h_Q + f^*\mathfrak{a} \quad \text{in } \mathrm{CH}^1(Q)$$

for some $\mathfrak{a} \in \mathrm{CH}^1(X^{[2]})$. We also have the short exact sequence (derived from the Euler sequence)

$$0 \to \mathcal{O}(h_Q) \otimes \frac{f^*(\mathscr{E}|_{X^{[2]}})}{\mathcal{O}(-h_Q)} \to \mathcal{T}_Q \to f^*\mathcal{T}_{X^{[2]}} \to 0.$$

As a consequence, we have

$$K_Q = -c_1(\mathcal{T}_Q) = f^*K_{X^{[2]}} - 2h_Q - f^*c_1(\mathscr{E}|_{X^{[2]}}).$$

The canonical divisor of $\widetilde{X^{[2]}}$ can be computed by the adjunction formula as

$$K_{\widetilde{X^{[2]}}} = (K_Q + \widetilde{X^{[2]}})|_{\widetilde{X^{[2]}}} = \tau^*K_{X^{[2]}} - h_Q|_{\widetilde{X^{[2]}}} - \tau^*c_1(\mathscr{E}|_{X^{[2]}}) + \tau^*\mathfrak{a}.$$

Since $\widetilde{X^{[2]}}$ is the blow-up of $X^{[2]}$, we also have

$$K_{\widetilde{X^{[2]}}} = \tau^*K_{X^{[2]}} + E.$$

Comparing the above two expressions, we get

$$E = -h_Q|_{\widetilde{X^{[2]}}} - \tau^* c_1(\mathscr{E}|_{X^{[2]}}) + \tau^*\mathfrak{a} \quad \text{in } \mathrm{CH}^1(\widetilde{X^{[2]}}). \tag{8}$$

To determine the value of $\mathfrak{a}$, we apply τ_* to the above equation and get

$$0 = -f_*(h_Q \cdot \widetilde{X^{[2]}}) - c_1(\mathscr{E}|_{X^{[2]}}) + \mathfrak{a} \tag{9}$$

in $\mathrm{CH}^1(X^{[2]})$. The first term on the right-hand side can be computed as follows:

$$\begin{aligned} f_*(h_Q \cdot \widetilde{X^{[2]}}) &= f_*(h_Q \cdot (\widetilde{X^{[2]}} + \widetilde{X \times X})) - f_*(h_Q \cdot \widetilde{X \times X}) \\ &= f_*(h_Q \cdot 3h_Q) - \sigma_*\rho^*(h \otimes 1) \\ &= 3f_* h_Q^2 - h \mathbin{\widehat{\otimes}} 1. \end{aligned}$$

Note that $Q = \mathbb{P}(\mathscr{E}|_{X^{[2]}})$ is a $\mathbb{P}^1$–bundle over $X^{[2]}$ and h_Q is the class of the associated relative $\mathcal{O}(1)$–bundle. Thus we have the equation

$$h_Q^2 + f^*c_1(\mathscr{E}|_{X^{[2]}}) \cdot h_Q + f^*c_2(\mathscr{E}|_{X^{[2]}}) = 0$$

and it follows that

$$f_* h_Q^2 = -c_1(\mathscr{E}|_{X^{[2]}}).$$

This combined with equation (9) gives

$$\mathfrak{a} = -2c_1(\mathscr{E}|_{X^{[2]}}) - h \mathbin{\widehat{\otimes}} 1.$$

We plug this into (8) and get

$$E = -h_Q|_{\widetilde{X^{[2]}}} - \tau^*(h \mathbin{\widehat{\otimes}} 1) - 3\tau^* c_1(\mathscr{E}|_{X^{[2]}}) \quad \text{in } \mathrm{CH}^1(\widetilde{X^{[2]}}). \tag{10}$$

We still need to compute $c_1(\mathscr{E}|_{X^{[2]}})$. For simplicity, let $V \cong K^{d+2}$ be a $(d+2)$–dimensional vector space with a fixed identification $\mathbb{P}_K^{d+1} = \mathbb{P}(V)$. For any coherent sheaf $\mathscr{F}$ on X, we define

$$\mathscr{F}^{[2]} := \sigma_*\rho^* p_1^* \mathscr{F}.$$

If $\mathscr{F}$ is locally free of rank r, then $\mathscr{F}^{[2]}$ is locally free of rank $2r$. The inclusion $\mathcal{O}_X(-1) \hookrightarrow V \otimes_K \mathcal{O}_X$ induces an inclusion

$$\mathcal{O}(-1)^{[2]} \hookrightarrow V \otimes \mathcal{O}_X^{[2]} = V \otimes \sigma_* \mathcal{O}_{\widetilde{X \times X}}. \tag{11}$$

Recall that $\delta \in \mathrm{CH}^1(X^{[2]})$ is the "half boundary", which fits into the short exact sequence

$$0 \to \mathcal{O}_{X^{[2]}} \to \sigma_* \mathcal{O}_{\widetilde{X \times X}} \to \mathcal{O}_{X^{[2]}}(-\delta) \to 0. \tag{12}$$

Applying $V \otimes_K -$ to the above sequence, we get

$$0 \to V \otimes \mathcal{O}_{X^{[2]}} \to V \otimes \sigma_* \mathcal{O}_{\widetilde{X \times X}} \to V \otimes \mathcal{O}_{X^{[2]}}(-\delta) \to 0. \tag{13}$$

Pulling it back to $\widetilde{X \times X}$, we get

$$0 \to V \otimes \mathcal{O}_{\widetilde{X \times X}} \to V \otimes \sigma^* \sigma_* \mathcal{O}_{\widetilde{X \times X}} \to V \otimes \mathcal{O}_{\widetilde{X \times X}}(-D) \to 0, \tag{14}$$

where $D \subset \widetilde{X \times X}$ is the exceptional divisor of the diagonal blow-up $\rho\colon \widetilde{X \times X} \to X \times X$. The short exact sequence (14) restricted to $\widetilde{X \times X} \setminus D$ becomes

$$0 \to V \to V \oplus V \to V \to 0,$$

where the first map is $v \mapsto (v, v)$ and the second map is $(v, v') \mapsto v - v'$. Consider the restriction of the second map of (14) to the canonical rank two subbundle $\rho^*(p_1^* \mathcal{O}_X(-1) \oplus p_2^* \mathcal{O}_X(-1)) \hookrightarrow (V \oplus V) \otimes \mathcal{O}_{\widetilde{X \times X}}$ and we get

$$\mu\colon \rho^*(p_1^* \mathcal{O}_X(-1) \oplus p_2^* \mathcal{O}_X(-1)) \to V \otimes \mathcal{O}_{\widetilde{X \times X}}(-D).$$

Let x and y be two distinct points of X. Then the image of μ at the point (x, y) is simply $V_x + V_y \subset V$, where $V_x \subset V$ (resp. $V_y \subset V$) is the one-dimensional subspace associated to $x \in \mathbb{P}(V)$ (resp. y). Thus $V_x + V_y \subset V$ is the two-dimensional subspace associated to the line $L_{x,y}$ passing through x and y. Thus $\varphi^* \mathscr{E}$ should be the unique rank two subbundle of $V \otimes \mathcal{O}_{X^{[2]}}$ which pulls back to $\rho^*(p_1^* \mathcal{O}_X(-1) \oplus p_2^* \mathcal{O}_X(-1))$ on $\widetilde{X \times X} \setminus D$.

To find $\varphi^* \mathscr{E}$, we combine (13) with the inclusion (11) and get a rank two subbundle

$$\mathcal{O}(-1)^{[2]} \otimes \mathcal{O}(\delta) \to V \otimes \mathcal{O}_{X^{[2]}}.$$

Note that

$$\sigma^* \mathcal{O}(-1)^{[2]} = \sigma^* \sigma_* \rho^* p_1^* \mathcal{O}_X(-1) = \rho^*(p_1^* \mathcal{O}_X(-1) \oplus p_2^* \mathcal{O}_X(-1)).$$

Thus the rank 2 subbundle of $V \otimes \mathcal{O}_{X^{[2]}}$ constructed above satisfies the required condition and hence

$$\varphi^* \mathscr{E} \cong \mathcal{O}(-1)^{[2]} \otimes \mathcal{O}(\delta).$$

The Grothendieck–Riemann–Roch formula gives

$$c_1(\mathcal{O}(-1)^{[2]}) = -h \mathbin{\widehat{\otimes}} 1 - \delta.$$

The Grothendieck–Riemann–Roch formula uses rational coefficients, but this is fine here since $\mathrm{CH}^1(X^{[2]})$ is torsion-free. Another way to get the above formula is to apply the exact functor $(-)^{[2]}$ to the short exact sequence $0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_H \to 0$ and then take first Chern classes. It follows from the above equation that

$$c_1(\mathscr{E}|_{X^{[2]}}) = c_1(\varphi^*\mathscr{E}) = -h \mathbin{\widehat{\otimes}} 1 + \delta.$$

We combine this with equation (10) and prove the lemma. □

Lemma 4.4 *Let* $\Gamma \in \mathrm{CH}^r(X^{[2]})$.

(i) *The following equation holds in* $\mathrm{CH}^{r+1}(P_X)$:

$$\widetilde{\Phi}_*(E \cdot \tau^*\Gamma) = (i_1)_* p^*\gamma,$$

where $\gamma = p'_*(i_2)^*\Gamma$ *in* $\mathrm{CH}^{r-2}(F)$. *In particular, we have*

$$\pi_*\widetilde{\Phi}_*(E \cdot \tau^*\Gamma) = q_* p^*\gamma \quad \text{in } \mathrm{CH}^{r-3}(X).$$

(ii) *We have*

$$\widetilde{\Phi}_*\tau^*\Gamma = (i')^* \tilde{f}^* \varphi_*\Gamma - \Psi_*\sigma^*\Gamma \quad \text{in } \mathrm{CH}^r(P_X).$$

Proof The statement (i) follows from a straightforward calculation as follows:

$$\widetilde{\Phi}_*(E \cdot \tau^*\Gamma) = \widetilde{\Phi}_* j_* j^* \tau^*\Gamma = (i_1)_*(\pi_1)_*\pi_2^* i_2^*\Gamma = (i_1)_* p^* p'_* i_2^*\Gamma.$$

We also have

$$\widetilde{\Phi}_*\tau^*\Gamma = (\widetilde{\Phi}_*\tau^*\Gamma + \Psi_*\sigma^*\Gamma) - \Psi_*\sigma^*\Gamma = (i')^* \tilde{f}^* \varphi_*\Gamma - \Psi_*\sigma^*\Gamma.$$

This proves (ii). □

Lemma 4.5 *Let* $\{\widetilde{\Gamma}_1, \dots, \widetilde{\Gamma}_n, \widetilde{\Sigma}_1, \dots, \widetilde{\Sigma}_m\}$ *be a set of distinct irreducible reduced curves in* X. *Let* $\widetilde{\Gamma} = \sum n_k \widetilde{\Gamma}_k$ *and* $\widetilde{\Sigma} = \sum m_k \widetilde{\Sigma}_k$ *be two algebraic cycles of dimension one in* X. *Let* e *(resp.* e'*) be the degree of* $\widetilde{\Gamma}$ *(resp.* $\widetilde{\Sigma}$*) as a* 1*–cycle on* X.

(i) *There exists an algebraic cycle* Γ *on* $X^{[2]}$ *such that*

$$\begin{aligned} \pi_*\Psi_*\sigma^*\Gamma &= 0, \\ \pi_*\Psi_*\sigma^*(h \mathbin{\widehat{\otimes}} 1 \cdot \Gamma) &= e\widetilde{\Gamma}, \\ \pi_*\Psi_*\sigma^*(\delta \cdot \Gamma) &= \sum n_k^2 \widetilde{\Gamma}_k. \end{aligned}$$

(ii) *There exists an algebraic cycle Γ' on $X^{[2]}$ such that*

$$\pi_*\Psi_*\sigma^*\Gamma' = 0,$$
$$\pi_*\Psi_*\sigma^*(h \mathbin{\widehat{\otimes}} 1 \cdot \Gamma') = e'\widetilde{\Gamma} + e\widetilde{\Sigma},$$
$$\pi_*\Psi_*\sigma^*(\delta \cdot \Gamma') = 0.$$

(iii) *Let $\widetilde{\Xi}$ be an algebraic cycle of codimension $r < d-1$ on X. Then there exists an algebraic cycle Γ'' on $X^{[2]}$ such that*

$$\pi_*\Psi_*\sigma^*\Gamma'' = 0,$$
$$\pi_*\Psi_*\sigma^*(h \mathbin{\widehat{\otimes}} 1 \cdot \Gamma'') = e\widetilde{\Xi},$$
$$\pi_*\Psi_*\sigma^*(\delta \cdot \Gamma'') = 0.$$

Proof Let $\widetilde{\Gamma}_{kl} := \widetilde{\Gamma}_k \times \widetilde{\Gamma}_l \subset X \times X$. Let $\Gamma_{kl} \subset \widetilde{X \times X}$ be the strict transform of $\widetilde{\Gamma}_{kl}$. Note that $\Gamma_{kl} + \Gamma_{lk} = \sigma^*\sigma_*\Gamma_{kl}$ for all $k < l$. The cycle Γ_{ll} maps with degree two onto its image. By abuse of notation we use $\Gamma_l^{[2]} \subset X^{[2]}$ to denote this image. Then we have $\Gamma_{ll} = \sigma^*\Gamma_l^{[2]}$. Then the algebraic cycle $\Gamma = \sum_{k<l} n_k n_l \sigma_*\Gamma_{kl} + \sum n_l^2 \Gamma_l^{[2]}$ on $X^{[2]}$ satisfies $\sigma^*\Gamma = \sum n_k n_l \Gamma_{kl}$. Thus

$$\pi_*\Psi_*\sigma^*\Gamma = \sum_{k,l} n_k n_l (p_1)_*\rho_*\Gamma_{kl} = 0$$

and

$$\pi_*\Psi_*\sigma^*(\Gamma \cdot h \mathbin{\widehat{\otimes}} 1) = \sum n_k n_l (p_1)_*\rho_*(\Gamma_{kl} \cdot \rho^*(h \otimes 1 + 1 \otimes h)) = \sum n_k n_l e_k \widetilde{\Gamma}_l = e\widetilde{\Gamma},$$

where $e_k = \deg(\widetilde{\Gamma}_k|_{X_{\eta_B}})$ and $e = \sum n_k e_k$. The remaining equation holds because

$$\pi_*\Psi_*(\Gamma_{kl} \cdot \sigma^*\delta) = \begin{cases} 0 & \text{if } k \neq l, \\ \widetilde{\Gamma}_k & \text{if } k = l. \end{cases}$$

This proves (i). Statements (ii) and (iii) are proved similarly. For example, Γ' is obtained as the image of the strict transform of $\widetilde{\Gamma} \times \widetilde{\Sigma}$ and Γ'' is obtained as the image of the strict transform of $\widetilde{\Gamma} \times \widetilde{\Xi}$. □

Proof of Proposition 4.2 Using the moving lemma, we may assume that γ_1 and γ_2 do not meet each other. This allows us to apply Lemma 4.5 to the situation of $\widetilde{\Gamma} = \gamma_1$ and $\widetilde{\Sigma} = \gamma_2$. Thus we get a 2–cycle Γ' on $X^{[2]}$ as in Lemma 4.5(ii). We apply $\pi_*\widetilde{\Phi}_*((-) \cdot \tau^*\Gamma')$ to the equality in Lemma 4.3 and get

$$\pi_*\widetilde{\Phi}_*(E \cdot \tau^*\Gamma') = -\pi_*\widetilde{\Phi}_*(h_Q|_{\widetilde{X^{[2]}}} \cdot \tau^*\Gamma') - \pi_*\widetilde{\Phi}_*\tau^*((2h \mathbin{\widehat{\otimes}} 1 - 3\delta) \cdot \Gamma'). \tag{15}$$

For the last term, we apply Lemma 4.4(ii) and have

$$\begin{aligned}\pi_*\widetilde{\Phi}_*\tau^*((2h\mathbin{\widehat{\otimes}}1-3\delta)\cdot\Gamma') &= \pi_*(i')^*\widetilde{f}^*\alpha'+\pi_*\Psi_*((2h\mathbin{\widehat{\otimes}}1-3\delta)\cdot\Gamma')\\ &= i^*\widetilde{\pi}_*\widetilde{f}^*\alpha'+2e_2\gamma_1+2e_1\gamma_2,\end{aligned}$$

where $\alpha'=\varphi_*((2h\mathbin{\widehat{\otimes}}1-3\delta)\cdot\Gamma')$ is a 1–cycle on $G(2,d+2)$. Similarly, we also have

$$\pi_*\widetilde{\Phi}_*(h_Q|_{\widetilde{X^{[2]}}}\cdot\tau^*\Gamma')=h\cdot\pi_*\widetilde{\Phi}_*(\tau^*\Gamma')=h\cdot\pi_*(i')^*\widetilde{f}^*\alpha''-\pi_*\Psi_*\sigma^*\Gamma'=h\cdot i^*\widetilde{\pi}_*\widetilde{f}^*\alpha'',$$

where $\alpha''=\varphi_*\Gamma'$ is a 2–cycle on $G(2,d+2)$. We apply Lemma 4.4(i) to the left-hand side of (15) and get

$$\pi_*\widetilde{\Phi}_*(E\cdot\tau^*\Gamma')=q_*p^*\mathfrak{a}',\quad \mathfrak{a}'=p'_*(i_2)^*\Gamma'.$$

Combining the above equations, we get

$$2e_2\gamma_1+2e_1\gamma_2+q_*p^*\mathfrak{a}'=-i^*(\widetilde{\pi}_*\widetilde{f}^*(\alpha')+h\cdot\widetilde{\pi}_*\widetilde{f}^*(\alpha'')).$$

The right-hand side of this equation is a 2–cycle on $\mathbb{P}_K^{d+1}$ restricted to X and hence it must be a multiple of h^{d-1}. We track the definition of $\mathfrak{a}'$ and get

$$\mathfrak{a}'=p'_*i_2^*\Gamma'=p'_*i_2^*(\gamma_1\mathbin{\widehat{\otimes}}\gamma_2)=p_*q^*\gamma_1\cdot p_*q^*\gamma_2.$$

Then Proposition 4.2(ii) follows easily after comparing the degree of each side.

The statement of Proposition 4.2(iii) is proved similarly.

To prove Proposition 4.2(i), we first reduce it to the case when γ_1 is represented by a single irreducible curve. Assume that (i) is true for γ_1 and γ_1'. Namely,

$$\begin{aligned}(2e_1-3)\gamma_1&\in q_*p^*\mathrm{CH}_0(F)+\mathbb{Z}h^{d-1},\\ (2e_1'-3)\gamma_1'&\in q_*p^*\mathrm{CH}_0(F)+\mathbb{Z}h^{d-1},\end{aligned}$$

where $e_1=\deg(\gamma_1)$ and $e_1'=\deg(\gamma_1')$. Then clearly (i) is also true for $-\gamma_1$. Now we show that (i) also holds for $\gamma_1+\gamma_1'$. By (ii), we know that

$$2e_1'\gamma_1+2e_1\gamma_1'\in q_*p^*\mathrm{CH}_0(F)+\mathbb{Z}h^{d-1}.$$

Thus we get

$$\begin{aligned}(2(e_1+e_1')-3)(\gamma_1+\gamma_1')&=(2e_1-3)\gamma_1+(2e_1'-3)\gamma_1'+(2e_1'\gamma_1+2e_1\gamma_1')\\ &\in q_*p^*\mathrm{CH}_0(F)+\mathbb{Z}h^{d-1}.\end{aligned}$$

Now we assume that γ_1 is represented by a single irreducible curve $\widetilde{\Gamma} \subset X$. Thus we get a 2–cycle Γ on $X^{[2]}$ as in Lemma 4.5(i). We apply $\pi_*\widetilde{\Phi}_*((-)\cdot\tau^*\Gamma)$ to the equality in Lemma 4.3 and get

(16) $$\pi_*\widetilde{\Phi}_*(E\cdot\tau^*\Gamma) = -\pi_*\widetilde{\Phi}_*(h_Q|_{\widetilde{X^{[2]}}}\cdot\tau^*\Gamma) - \pi_*\widetilde{\Phi}_*\tau^*((2h\mathbin{\hat{\otimes}}1-3\delta)\cdot\Gamma).$$

Then we apply Lemma 4.4(ii) to the last term and get

$$\begin{aligned}\pi_*\widetilde{\Phi}_*\tau^*((2h\mathbin{\hat{\otimes}}1-3\delta)\cdot\Gamma) &= i^*\tilde{\pi}_*\tilde{f}^*\beta' - \pi_*\Psi_*\sigma^*((2h\mathbin{\hat{\otimes}}1-3\delta)\cdot\Gamma)\\ &= i^*\tilde{\pi}_*\tilde{f}^*\beta' - (2e_1-3)\gamma_1,\end{aligned}$$

where $\beta' = \varphi_*((2h\mathbin{\hat{\otimes}}1-3\delta)\cdot\Gamma)$. The other term on the right-hand side of (16) can be dealt with by

$$\pi_*\widetilde{\Phi}_*(h_Q|_{\widetilde{X^{[2]}}}\cdot\tau^*\Gamma) = h\cdot i^*\tilde{\pi}_*\widetilde{F}^*\beta'' - h\cdot\pi_*\Psi_*\sigma^*\Gamma = h\cdot i^*\tilde{\pi}_*\widetilde{F}^*\beta'',$$

where $\beta'' = \varphi_*\Gamma$. Apply Lemma 4.4(i) to the left-hand side of (16); we get

$$\pi_*\widetilde{\Phi}_*(E\cdot\tau^*\Gamma) = q_*p^*\mathfrak{a} \quad \text{with } \mathfrak{a} = p'_*i_2^*\Gamma \in \mathrm{CH}_0(F).$$

The above three displayed equations combined with equation (16) gives

$$(2e_1-3)\gamma_1 \in q_*p^*\mathrm{CH}_0(F) + \mathbb{Z}h^{d-1}.$$

This finishes the proof. □

5 Cubic hypersurfaces of small dimensions

This section is devoted to applications of the universal generation result of the previous section. We relate the rationality problem of a cubic hypersurface of small dimension to the geometry of its variety of lines.

5.1 A special decomposition of the diagonal

In this subsection we fix a smooth cubic hypersurface $X \subset \mathbb{P}^{d+1}_{\mathbb{C}}$ of dimension $d=3$ or 4. Let $F=F(X)$ be the variety of lines on X and $P \subset F\times X$ the universal line. Let $h \in \mathrm{CH}^1(X)$ be the class of a hyperplane section.

Theorem 5.1 *Assume that X admits a Chow-theoretical decomposition of the diagonal. Then the following holds*:

(i) *If $d = 3$, then there exists a symmetric 1–cycle $\theta \in \mathrm{CH}_1(F \times F)$ such that*

$$\Delta_X = x \times X + X \times x + (\gamma \times h + h \times \gamma) + (P \times P)_*\theta \quad \text{in } \mathrm{CH}^3(X \times X),$$

where $\gamma \in \mathrm{CH}_1(X)$.

(ii) *If $d = 4$, then there exists a symmetric 2–cycle $\theta \in \mathrm{CH}_2(F \times F)$ such that*

$$\Delta_X = x \times X + X \times x + \Sigma + (P \times P)_*\theta \quad \text{in } \mathrm{CH}^4(X \times X),$$

where $\Sigma \in \mathrm{CH}_2(X) \otimes \mathrm{CH}_2(X)$ is a symmetric decomposable cycle. Moreover, Σ can be chosen to be zero if $\mathrm{Hdg}^4(X) = \mathbb{Z}h^2$.

Proof We will see that the construction of θ as in Theorem 3.8 suffices in the case of $d = 3$. However, it is insufficient in the case of $d = 4$ since it produces correspondences factoring through curves. Using an argument of Voisin [26], we can show that these problematic terms can be absorbed into the terms $(P \times P)_*\theta$ and Σ. We make this more precise:

Assume that $d = 3$. Then, by Proposition 3.2, there exist curves Z_i, correspondences $\Gamma_i \in \mathrm{CH}^2(Z_i \times X)$ and integers n_i such that

$$\Delta_X = x \times X + X \times x + \sum n_i \Gamma_i \circ {}^t\Gamma_i \quad \text{in } \mathrm{CH}^3(X \times X).$$

As in the proof of Theorem 3.8, we use the universal generation of $\mathrm{CH}_1(X)$ by lines to get $T_i \in \mathrm{CH}^2(Z_i \times F)$ such that $\Gamma_i' := P \circ T_i \in \mathrm{CH}^2(Z_i \times X)$ agrees with Γ_i over the generic point of Z_i. It follows that $\Gamma_i \circ {}^t\Gamma_i - \Gamma_i' \circ {}^t\Gamma_i'$ is a decomposable cycle, ie of the form $\gamma_i \otimes h + h \otimes \gamma_i$ for some $\gamma_i \in \mathrm{CH}_1(X)$. Indeed, $\Gamma_i = \Gamma_i' + \Delta_i$, where Δ_i factors through points. Thus $\Gamma_i \circ {}^t\Gamma_i - \Gamma_i' \circ {}^t\Gamma_i'$ also factors through points and hence is decomposable. As a consequence, we have

$$\begin{aligned}
\Delta_X &= x \times X + X \times x + \sum n_i \Gamma_i \circ {}^t\Gamma_i \\
&= x \times X + X \times x + \sum n_i \Gamma_i' \circ {}^t\Gamma_i' + \sum n_i(\gamma_i \otimes h + h \otimes \gamma_i) \\
&= x \times X + X \times x + (P \times P)_*\Big(\sum n_i T_i \circ {}^tT_i\Big) + \gamma \otimes h + h \otimes \gamma,
\end{aligned}$$

where

$$\gamma = \sum n_i \gamma_i.$$

Statement (i) follows by taking $\theta = \sum n_i T_i \circ {}^tT_i$.

Assume $d=4$; then we can obtain surfaces Z_i and correspondences $T_i\in \mathrm{CH}_3(Z_i\times X)$ similarly. Then we conclude that the cohomology class of

$$\Delta_X - x\times X - X\times x - (P\times P)_*\theta$$

is decomposable. Since the integral Hodge conjecture holds for X (see Voisin [23]), we know that

$$\Gamma := \Delta_X - x\times X - X\times x - (P\times P)_*\theta - \gamma\otimes h - h\otimes\gamma - \Sigma = 0 \quad \text{in } \mathrm{H}^4(X\times X,\mathbb{Z}) \tag{17}$$

for some $\gamma\in\mathrm{CH}_1(X)$ and some symmetric cycle $\Sigma\in\mathrm{CH}_2(X)\otimes\mathrm{CH}_2(X)$. Since $\mathrm{CH}_2(F)\to\mathrm{CH}^1(X)$ is surjective and there exists $\tau\in\mathrm{CH}_2(F)$ such that $P_*\tau=h$, the term $\gamma\otimes h+h\otimes\gamma$ can be absorbed into the term $(P\times P)_*\theta$ and hence we can assume that $\gamma=0$ in equation (17). There also exists $\tau'\in\mathrm{CH}_1(F)$ such that $P_*\tau'=h^2$ in cohomology. Thus we can assume that Σ does not appear in equation (17) if $\mathrm{Hdg}^4(X)=\mathbb{Z}h^2$. By Proposition 5.3(ii) below, we can modify θ by a homologically trivial cycle supported on the diagonal of $F\times F$ and assume that Γ is algebraically trivial. Now, by [20; 22], we know that $\Gamma^{\circ N}=0$ in $\mathrm{CH}^4(X\times X)$ for some sufficiently large integer N. In the expansion of this equation, any term involving Σ is again decomposable of the same form and any power of $(P\times P)_*\theta$ is again of this form. After modifying θ and Σ, we get the equation

$$\Delta_X = x\times X + X\times x + \Sigma + (P\times P)_*\theta \quad \text{in } \mathrm{CH}^4(X\times X).$$

This finishes the proof. □

Remark 5.2 When $d=3$, the term $\gamma\otimes h+h\otimes\gamma$ cannot be absorbed into θ. This is because the homomorphism $q_*p^*\colon \mathrm{H}^2(F,\mathbb{Z})\to\mathrm{H}^2(X,\mathbb{Z})$ is not surjective. Given any 0–cycle $\mathfrak{o}_F\in\mathrm{CH}_0(F)$ of degree 1, we set $\mathfrak{l}=q_*p^*\mathfrak{o}_F\in\mathrm{CH}_1(X)$. Then the cycle γ can always be chosen to be a multiple of $\mathfrak{l}$. This can be seen as follows. For any $\gamma'\in\mathrm{CH}_1(X)$ with $\deg(\gamma')=0$, there exists $\gamma''\in\mathrm{CH}_1(X)$ such that $\gamma'=5\gamma''$. Thus $\gamma'\otimes h=\gamma''\otimes 5h$ is contained in $(P\times P)_*(\mathrm{CH}_0(F)\otimes\mathrm{CH}_1(F))$. Thus $\gamma\otimes h-\deg(\gamma)\mathfrak{l}\otimes h$ can be absorbed into $(P\times P)_*\theta$.

When $d=4$, it is known that $q_*p^*\colon \mathrm{H}^6(F,\mathbb{Z})\to\mathrm{H}^4(X,\mathbb{Z})$ is an isomorphism. If we assume that the integral Hodge conjecture holds for 1–cycles on F, then the term Σ can always be absorbed into θ. Note that this assumption has recently been established by Mongardi and Ottem [14].

Proposition 5.3 *Let X be a smooth cubic hypersurface of dimension $d=3$ or 4. Let Γ be a symmetric cycle on $X\times X$ such that $[\Delta_X]=[\Gamma]$ in $\mathrm{H}^d(X\times X,\mathbb{Z})$.*

(i) *If* $d=3$, *then*

$$\Delta_X - \Gamma = 0 \quad \textit{in } \mathrm{CH}_d(X\times X)/\mathrm{alg}.$$

(ii) *If* $d=4$, *then there exists a homologically trivial cycle* $\sum_i a_i S_i \in \mathrm{CH}_2(F)$ *such that*

$$\Delta_X - \Gamma - \sum a_i P_i \times_{S_i} P_i = 0 \quad \textit{in } \mathrm{CH}_4(X\times X)/\mathrm{alg},$$

where $P_i = P|_{S_i}$ *and* $P_i \times_{S_i} P_i$ *is understood to be its image in* $X\times X$.

Proof Recall that $h\in \mathrm{CH}^1(X)$ is the class of a hyperplane section. Let $l\subset X$ be a line. We note that, by Totaro [19], all cohomology groups involved in the proof are torsion-free and hence $\mathrm{H}^*(-)$ should be understood to be $\mathrm{H}^*(-,\mathbb{Z})$. By [26, Corollary 2.4], there exists a d–cycle Γ' on $X^{[2]}$ such that

$$\mu^*\Gamma' = \Delta_X - \Gamma$$

as algebraic cycles, where $\mu\in \mathrm{CH}_{2d}((X\times X)\times X^{[2]})$ is the correspondence defined by the closure of the graph of the rational map $X^2 \dashrightarrow X^{[2]}$. As is explained in the proof of [26, Proposition 2.6], we can require that

$$[\Gamma'] = 0 \quad \text{in } \mathrm{H}^{2d}(X^{[2]}).$$

If $d=3$, then Lemma 5.4(i) applies. If $d=4$, then by Lemma 5.4(ii) we can find homologically trivial $\sum_i a_i S_i \in \mathrm{CH}_2(F)$ such that $\Delta_X - \Gamma - \sum_i a_i P_i \times_{S_i} P_i$ is algebraically trivial. □

Lemma 5.4 *Let* $\Gamma' \in \mathrm{CH}^d(X^{[2]})_{\mathrm{hom}}$.

(i) *If* $d=3$, *then* Γ' *is algebraically trivial.*

(ii) *If* $d=4$, *then there exist surfaces* $S_i\subset F$ *and integers* a_i *such that* $\sum a_i[S_i]=0$ *in* $\mathrm{H}^4(F,\mathbb{Z})$ *and*

$$\mu^*\Gamma' = \sum a_i P_i \times_{S_i} P_i \quad \textit{in } \mathrm{CH}^d(X\times X)/\mathrm{alg},$$

where P_i *is the universal line* P *restricted to* S_i.

Proof This is a consequence of the explicit resolution $\widetilde{\Phi}$ of the birational map Φ between $X^{[2]}$ and $P_X := \mathbb{P}(\mathcal{T}_{\mathbb{P}^{d+1}}|_X)$; see the previous section and [26, Proposition 2.9].

Recall that we have the commutative diagram

$$\begin{tikzcd}
E \arrow[rr, "\pi_1"] \arrow[dr, "j"] \arrow[dd, "\pi_2"'] & & P_1 \arrow[d, "i_1"] \\
& \widetilde{X^{[2]}} \arrow[r, "\widetilde{\Phi}"] \arrow[d, "\tau"] & P_X \\
P_2 \arrow[r, "i_2"] & X^{[2]} &
\end{tikzcd}$$

Here $P_2 = P^{[2]}/F$ is the relative Hilbert scheme of two points on the universal line P/F and $P_1 = P$. The morphism τ is the blow-up of $X^{[2]}$ along P_2 and $\widetilde{\Phi}$ is the blow-up of P_X along P_1. The two blow-up morphisms share the same exceptional divisor

$$E = P_1 \times_F P_2.$$

Note that $\eta_1\colon P_1 \to F$ is a $\mathbb{P}^1$–bundle over F and $\eta_2\colon P_2 \to F$ is a $\mathbb{P}^2$–bundle over F. Let $\xi_i \in \mathrm{CH}^1(P_i)$ for $i = 1, 2$ be the first Chern classes of the relative $\mathcal{O}(1)$–bundles. By abuse of notation, we still use ξ_i to denote its pullback to E. By the blow-up formula, we know that

$$\mathrm{CH}_d(\widetilde{X^{[2]}}) = \widetilde{\Phi}^*\mathrm{CH}_d(P_X) \oplus j_*\pi_1^*\mathrm{CH}_{d-2}(P_1) \oplus j_*(\xi_2 \cdot \pi_1^*\mathrm{CH}_{d-1}(P_1)).$$

Thus

$$\tau^*\Gamma' = \widetilde{\Phi}^*\Gamma'_0 + j_*\pi_1^*\Gamma'_1 + j_*(\xi_2\pi_1^*\Gamma'_2), \tag{18}$$

where

$$\Gamma'_0 \in \mathrm{CH}_d(P_X)_{\mathrm{hom}}, \quad \Gamma'_1 \in \mathrm{CH}_{d-2}(P_1)_{\mathrm{hom}}, \quad \Gamma'_2 \in \mathrm{CH}_{d-1}(P_1)_{\mathrm{hom}}.$$

By the projective bundle formula, we have

$$\begin{aligned}
\Gamma'_1 &= \eta_1^*\Gamma'_{1,0} + \xi_1 \cdot \eta_1^*\Gamma'_{1,1} \quad \text{with } \Gamma'_{1,0} \in \mathrm{CH}_{d-3}(F)_{\mathrm{hom}} \text{ and } \Gamma'_{1,1} \in \mathrm{CH}_{d-2}(F)_{\mathrm{hom}};\\
\Gamma'_2 &= \eta_1^*\Gamma'_{2,0} + \xi_1 \cdot \eta_1^*\Gamma'_{2,1} \quad \text{with } \Gamma'_{2,0} \in \mathrm{CH}_{d-2}(F)_{\mathrm{hom}} \text{ and } \Gamma'_{2,1} \in \mathrm{CH}_{d-1}(F)_{\mathrm{hom}}.
\end{aligned}$$

Note that

$$\tau_* j_*(\pi_1^*\eta_1^*\Gamma'_{1,0}) = i_{2,*}\pi_{2,*}(\pi_1^*\eta_1^*\Gamma'_{1,0}) = 0,$$
$$\tau_* j_*(\xi_2 \cdot \pi_1^*\eta_1^*\Gamma'_{2,0}) = i_{2,*}\pi_{2,*}(\xi_2 \cdot \pi_1^*\eta_1^*\Gamma'_{2,0}) = 0.$$

Applying σ_* to equation (18), we get

$$\begin{aligned}\Gamma' &= \Phi^*\Gamma'_0 + i_{2,*}\pi_{2,*}\pi_1^*(\xi_1\cdot\eta_1^*\Gamma'_{1,1}) + i_{2,*}\pi_{2,*}(\xi_2\cdot\pi_1^*(\xi_1\cdot\eta_1^*\Gamma'_{2,1}))\\ &= \Phi^*\Gamma'_0 + i_{2,*}\eta_2^*\Gamma'_{1,1} + i_{2,*}(\xi_2\cdot\eta_2^*\Gamma'_{2,1}).\end{aligned}$$

Now we follow the argument in the proof of [26, Theorem 1.1]. The first fact is that the algebraic equivalence is the same as the homological equivalence on P_X. Thus we see that Γ'_0 is algebraically equivalent to zero. When $d=3$ or 4, the cycle $\Gamma'_{2,1}$ is either of codimension 0 or of codimension 1. Thus we always have that $\Gamma'_{2,1}$ is algebraically equivalent to zero. When $d=3$ we have $\dim F=2$ and $\Gamma'_{1,1}\in \mathrm{CH}^1(F)_{\mathrm{hom}}$. So $\Gamma'_{1,1}$ is also algebraically equivalent to zero in this case. Statement (i) follows.

Assume $d=4$ and hence $\dim F=4$. Thus $\Gamma'_{1,1}\in\mathrm{CH}_2(F)_{\mathrm{hom}}$. We can write

$$\Gamma'_{1,1}=\sum_i a_iS_i,$$

where $S_i\subset F$ are surfaces. Then an explicit computation gives

$$\mu^*(i_{2,*}\eta_2^*S_i)=P_i\times_{S_i}P_i$$

as cycles. Hence the lemma is proved. □

5.2 Algebraicity of the Beauville–Bogomolov form

Let X be a smooth cubic fourfold and let F be its variety of lines. It is known that F is a hyperkähler variety. By Beauville and Donagi [4], we know that $\alpha\mapsto\hat\alpha:=p_*q^*\alpha$ gives an isomorphism between $\mathrm{H}^4(X,\mathbb{Z})$ and $\mathrm{H}^2(F,\mathbb{Z})$, and the Beauville–Bogomolov bilinear form on $\mathrm{H}^2(F,\mathbb{Z})$ is given by

$$\mathfrak{B}(\hat\alpha,\hat\beta)=\langle\alpha,h^2\rangle_X\langle\beta,h^2\rangle_X-\langle\alpha,\beta\rangle_X$$

for all $\alpha,\beta\in\mathrm{H}^4(X,\mathbb{Z})$. Let α_i for $i=1,\dots,23$ be an integral basis of $\mathrm{H}^4(X,\mathbb{Z})$. Then $\{\hat\alpha_i\}$ form an integral basis of $\mathrm{H}^2(F,\mathbb{Z})$ and let $\{\hat\alpha_i^\vee\}$ be the dual basis of $\mathrm{H}^6(F,\mathbb{Z})$. Then the Beauville–Bogomolov form corresponds to the canonical integral Hodge class

$$q_{\mathfrak{B}}=\sum_{i,j=1}^{23} b_{ij}\hat\alpha_i^\vee\otimes\hat\alpha_j^\vee\in\mathrm{H}^{12}(F\times F,\mathbb{Z}),$$

where $b_{ij}=\mathfrak{B}(\hat\alpha_i,\hat\alpha_j)$.

Proposition 5.5 *Let X be a smooth cubic fourfold and F be its variety of lines as above. If $q_{\mathfrak{B}} \in \mathrm{H}^{12}(F \times F, \mathbb{Z})$ is algebraic then X is universally CH_0–trivial. The converse is true if the integral Hodge conjecture holds for 1–cycles on F* (eg if $\mathrm{Hdg}^4(X)$ is generated by h^2).[1]

Proof Assume that $q_{\mathfrak{B}}$ is algebraic. Since $[\Delta_X]+(P\times P)_*q_{\mathfrak{B}}$ pairs to zero with $\alpha\otimes\beta$ for all $\alpha,\beta\in \mathrm{H}^4(X,\mathbb{Z})_{\mathrm{prim}}$, we know that $[\Delta_X]+(P\times P)_*q_{\mathfrak{B}}$ is decomposable. Thus X admits a cohomological decomposition of the diagonal, which implies universal CH_0–triviality by Voisin [26]. Assume that $\mathrm{CH}_0(X)$ is universally trivial. Then we get the cycle $\theta \in \mathrm{CH}_2(F\times F)$ as in Theorem 5.1. The Hodge class $q_{\mathfrak{B}}+[\theta]$ is decomposable and hence algebraic by the assumption that the integral Hodge conjecture holds for 1–cycles on F. □

5.3 The minimal class on the intermediate Jacobian of a cubic threefold

Let X be a smooth cubic threefold and let F be the surface of lines on X. Let $J^3(X)$ be the intermediate Jacobian of X. It is known that the Abel–Jacobi map

$$\phi\colon F\to J^3(X)$$

induces an isomorphism

$$\phi^*\colon \mathrm{H}^1(J^3(X),\mathbb{Z})\to \mathrm{H}^1(F,\mathbb{Z}).$$

There is a natural identification $\mathrm{H}^1(J^3(X),\mathbb{Z})\cong \mathrm{H}^3(X,\mathbb{Z})$, under which we have

$$\phi^*\alpha=\hat{\alpha}=p_*q^*\alpha \tag{19}$$

for all $\alpha\in \mathrm{H}^1(J^3(X),\mathbb{Z})=\mathrm{H}^3(X,\mathbb{Z})$. By taking the difference or the sum, we have the morphisms

$$\begin{aligned}\phi_+\colon F\times F\to J^3(X), &\quad (u,v)\mapsto \phi(x)+\phi(y),\\ \phi_-\colon F\times F\to J^3(X), &\quad (u,v)\mapsto \phi(x)-\phi(y).\end{aligned}$$

By [5, Theorem 13.4] and its proof, the image of ϕ_- is a theta divisor of $J^3(X)$ and ϕ_- has degree 6 onto its image.

Lemma 5.6 *The image of ϕ_+ is a divisor D_+ of cohomological class 3Θ and ϕ_+ has degree 2 onto its image.*

[1] The integral Hodge conjecture for 1–cycles on F has recently been established by Mongardi and Ottem [14].

Proof We first give a description of the generic behavior of the degree 6 morphism ϕ_-. Let l_1 and l_2 be a general pair of lines on X. Then l_1 and l_2 span a linear $\mathbb{P}^3$ and its intersection with X is a smooth cubic surface Σ containing l_1 and l_2. Realize Σ as a blow-up of $\mathbb{P}^2$ in 6 points $\{P_1, P_2, \dots, P_6\}$ with E_i for $i = 1, 2, \dots, 6$ being the exceptional curves. This can be chosen in such a way that $E_1 = l_1$ and $E_2 = l_2$. Let $C_i \subset \Sigma$ be the strict transform of the conic in $\mathbb{P}^2$ that passes all the 6 points except P_i. Let L_{ij} for $1 \le i < j \le 6$ be the strict transform of the line in $\mathbb{P}^2$ passing through the points P_i and P_j. Then E_i, C_j and $L_{i'j'}$ are the 27 lines on Σ. It is clear that, in $\mathrm{CH}^1(\Sigma)$, we have

$$E_1 - E_2 = C_1 - C_2 = L_{23} - L_{13} = L_{24} - L_{14} = L_{25} - L_{15} = L_{26} - L_{16}.$$

It follows that $(\phi_-)^{-1}\phi_-([l_1],[l_2])$ consists of the 6 points

$$\begin{gathered}([E_1],[E_2]), \quad ([C_1],[C_2]), \quad ([L_{23}],[L_{13}]), \\ ([L_{24}],[L_{14}]), \quad ([L_{25}],[L_{15}]), \quad ([L_{26}],[L_{16}]).\end{gathered} \tag{20}$$

Now let l'_1 and l'_2 another general pair of lines on X. Assume that l''_1 and l''_2 be a different pair of lines such that $\phi_+([l'_1],[l'_2]) = \phi_+([l''_1],[l''_2])$. Then we have $\phi_-([l'_1],[l''_2]) = \phi_-([l''_1],[l'_2])$ and hence $([l'_1],[l''_2])$ and $([l''_1],[l'_2])$ form two distinct points from the list (20). This implies that l'_1 meets l'_2 (and that l''_1 meets l''_2), which is a contradiction since l'_1 and l'_2 are general. This implies that ϕ_+ is of degree 2 onto its image.

The lemma follows from the fact that $(\phi_+)_*[F \times F] = (\phi_-)_*[F \times F] = 6\Theta$ in $\mathrm{H}^2(J^3(X),\mathbb{Z})$. This fact can be seen as follows. It is known that $\phi_*[F] = \Theta^3/3!$ in $\mathrm{H}^6(J^3(X),\mathbb{Z})$. It follows that $[-1]_*\phi_*[F] = \phi_*[F]$ in $\mathrm{H}^6(J^3(X),\mathbb{Z})$. Then we have

$$\begin{aligned}(\phi_-)_*[F \times F] &= (\mu_+)_*(\mathrm{Id} \times [-1])_*(\phi \times \phi)_*[F \times F] \\ &= (\mu_+)_*(\phi_*[F] \otimes [-1]_*\phi_*[F]) \\ &= (\mu_+)_*(\phi_*[F] \otimes \phi_*[F]) \\ &= (\mu_+)_*[F \times F],\end{aligned}$$

where $\mu_+\colon J^3(X) \times J^3(X) \to J^3(X)$ is the addition morphism. □

Proposition 5.7 *If* $\mathrm{CH}_0(X)$ *is universally trivial, then the following holds:*

(i) *The minimal class of* $J^3(X)$ *is algebraic and supported on the divisor* $D_+ \subset J^3(X)$ *of cohomology class* 3Θ.

(ii) *The double of the minimal class is represented by a symmetric* 1*–cycle supported on a theta divisor of* $J^3(X)$.

Proof The morphism ϕ_+ factors as

$$F \times F \xrightarrow{(\phi,\phi)} J^3(X) \times J^3(X) \xrightarrow{\mu_+} J^3(X),$$

where $\mu_+(x,y) = x+y$ is the summation morphism. Thus for any $\alpha \in \mathrm{H}^1(J^3(X),\mathbb{Z})$, we have

$$(\phi_+)^*\alpha = (\phi,\phi)^*(\mu_+)^*\alpha = (\phi,\phi)^*(\alpha\otimes 1 + 1\otimes\alpha) = \phi^*\alpha\otimes 1 + 1\otimes\phi^*\alpha.$$

Similarly, we also have $(\phi_-)^*\alpha = \phi^*\alpha\otimes 1 - 1\otimes\phi^*\alpha$. Let $\theta\in \mathrm{CH}_1(F\times F)$ be the symmetric cycle as in Theorem 5.1. Then for all $\alpha,\beta\in\mathrm{H}^1(J^3(X),\mathbb{Z})$ we have

$$\begin{aligned}(\phi_+)_*[\theta]\cup\alpha\cup\beta &= [\theta]\cup(\phi_+)^*\alpha\cup(\phi_+)^*\beta\\ &= [\theta]\cup(\phi^*\alpha\otimes 1 + 1\otimes\phi^*\alpha)\cup(\phi^*\beta\otimes 1 + 1\otimes\beta)\\ &= [\theta]\cup((\hat\alpha\cup\hat\beta)\otimes 1 + \hat\alpha\otimes\hat\beta - \hat\beta\otimes\hat\alpha + 1\otimes(\hat\alpha\cup\hat\beta))\\ &= 2\phi_*[\theta_1]\cup\alpha\cup\beta + 2\langle\alpha,\beta\rangle_X,\end{aligned}$$

where $\theta_1 = (\mathrm{pr}_1)_*\theta\in\mathrm{CH}_1(F)$. The same computation shows that

$$(\phi_+)_*[\theta_1\otimes\mathfrak{o}]\cup\alpha\cup\beta = (\phi_+)_*[\mathfrak{o}\otimes\theta_1]\cup\alpha\cup\beta = \phi_*[\theta_1]\cup\alpha\cup\beta,$$

where $\mathfrak{o}\in J^3(X)$ is the zero element. Take $\tilde\theta = \theta - \theta_1\otimes\mathfrak{o} - \mathfrak{o}\otimes\theta_1$; then

$$(\phi_+)_*[\tilde\theta]\cup\alpha\cup\beta = 2\langle\alpha,\beta\rangle_X.$$

Since $\tilde\theta$ is again a symmetric cycle, we know that $(\phi_+)_*\tilde\theta = 2\eta$ for some $\eta\in\mathrm{CH}_1(J^3(X))$ supported on D_+. Thus $[\eta]\cup\alpha\cup\beta = \langle\alpha,\beta\rangle_X$ and hence $-[\eta]$ is the minimal class on $J^3(X)$. We carry out the same computation for ϕ_- and see that

$$(\phi_-)_*[\tilde\theta]\cup\alpha\cup\beta = -2\langle\alpha,\beta\rangle_X.$$

Thus the cohomology class of $(\phi_-)_*\tilde\theta$ is twice the minimal class. Furthermore it is symmetric (with respect to multiplication by -1 on $J^3(X)$) and supported on the image of ϕ_-, which is a theta divisor. □

5.4 Cubic fivefolds

Let $X\subset\mathbb{P}^6$ be a cubic fivefold and let F be its variety of lines. Let $J^5(X)$ be the intermediate Jacobian of X, which happens to a be a principally polarized abelian variety.

Proposition 5.8 *If both $\mathrm{CH}_0(X)$ and $\mathrm{CH}_0(F)$ are universally trivial, then there exist finitely many curves C_i together with a splitting surjective homomorphism $\bigoplus_i J(C_i)\to J^5(X)$.*

Proof Since $\mathrm{CH}_0(F) \to \mathrm{CH}_1(X)$ is universally surjective, we know that $\mathrm{CH}_1(X)$ is universally trivial. By Corollary 3.6, the universal triviality of both $\mathrm{CH}_0(X)$ and $\mathrm{CH}_1(X)$ implies the existence of curves C_i, correspondences $\Gamma_i \in \mathrm{CH}^3(C_i \times X)$ and integers n_i such that

$$\sum n_i \langle \Gamma_i^* \alpha, \sigma_i^* \Gamma_i^* \beta \rangle_{C_i} = \langle \alpha, \beta \rangle_X \tag{21}$$

for all $\alpha, \beta \in F^5\mathrm{H}^5(X, \mathbb{Z}) = \mathrm{H}^5(X, \mathbb{Z})$, where $\sigma_i \colon C_i \to C_i$ is either the identity map or an involution. Each cycle Γ_i defines the associated Abel–Jacobi map

$$\phi_i \colon J(C_i) \to J^5(X).$$

Combining them together, we have the surjective homomorphism

$$\phi \colon \bigoplus_i J(C_i) \to J^5(X).$$

Then the equation (21) implies that ϕ has a section given by $\sum \sigma_i^\vee \circ \phi_i^\vee$. □

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KdV Institute for Mathematics, University of Amsterdam
Amsterdam, Netherlands

m.shen@uva.nl

Proposed: Dan Abramovich
Seconded: Richard P Thomas, Jim Bryan

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