On strongly quasiconvex subgroups

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We develop a theory of *strongly quasiconvex subgroups* of an arbitrary finitely generated group. Strong quasiconvexity generalizes quasiconvexity in hyperbolic groups and is preserved under quasi-isometry. We show that strongly quasiconvex subgroups are also more reflective of the ambient group's geometry than the stable subgroups defined by Durham and Taylor, while still having many properties analogous to those of quasiconvex subgroups of hyperbolic groups. We characterize strongly quasiconvex subgroups in terms of the lower relative divergence of ambient groups with respect to them.

We also study strong quasiconvexity and stability in relatively hyperbolic groups, right-angled Coxeter groups, and right-angled Artin groups. We give complete descriptions of strong quasiconvexity and stability in relatively hyperbolic groups and we characterize strongly quasiconvex special subgroups and stable special subgroups of two-dimensional right-angled Coxeter groups. In the case of right-angled Artin groups, we prove that the two notions of strong quasiconvexity and stability are equivalent when the right-angled Artin group is one-ended and the subgroups have infinite index. We also characterize nontrivial strongly quasiconvex subgroups of infinite index (ie nontrivial stable subgroups) in right-angled Artin groups by quadratic lower relative divergence, expanding the work of Koberda, Mangahas, and Taylor on characterizing purely loxodromic subgroups of right-angled Artin groups.

1 Introduction

One method to understand the structure of a group G is to investigate the subgroups of G. To utilize this method in the study of the geometry of the word metric on finitely generated groups, one needs to investigate subgroups which reflect the geometry of G and are invariant under choices of finite generating set for G. Quasiconvex subgroups of hyperbolic groups are a primary example of such subgroups. A quasiconvex subgroup of a hyperbolic group is also hyperbolic. Further, the invariance of quasiconvexity under quasi-isometries between hyperbolic spaces ensures that the quasiconvexity of a subgroup is independent of choice of finite generating set for the ambient group.

Unfortunately, quasiconvexity is not as useful for arbitrary finitely generated groups. In nonhyperbolic spaces, quasiconvexity is not preserved under quasi-isometry. Therefore the quasiconvex subgroups of a nonhyperbolic group G will depend upon the choice of finite generating set for G. In [17], Durham and Taylor introduce a strong notion of quasiconvexity in finitely generated groups, called *stability*, which is preserved under quasi-isometry, and agrees with quasiconvexity when the ambient group is hyperbolic. Moreover, stable subgroups of mapping class groups are precisely the *convex cocompact subgroups* defined by Farb and Mosher in [18] and such subgroups of mapping class groups are a primary motivation for the concept of stable subgroups of arbitrary finitely generated groups (see [17]). Stable subgroups have many properties similar to quasiconvex subgroups of hyperbolic groups and the study of stable subgroups has received much attention in recent years (see Koberda, Mangahas, and Taylor [26], Aougab, Durham, and Taylor [3], Antolín, Mj, Sisto, and Taylor [2], Abbott, Durham, and Behrstock [1], and Cordes and Hume [11]).

However, a stable subgroup of a finitely generated group is always hyperbolic regardless of the geometry of the ambient group (see [17]). Thus, the geometry of a stable subgroup does not completely reflect that of the ambient group. Therefore, we introduce another concept of quasiconvexity, called *strong quasiconvexity*, which is strong enough to be preserved under quasi-isometry and relaxed enough to capture the geometry of ambient groups.

Definition 1.1 Let G be a finitely generated group and H a subgroup of G. We say H is *strongly quasiconvex* in G if for every $K \geq 1$ and $C \geq 0$ there is some $M = M(K, C)$ such that every (K, C)–quasigeodesic in G with endpoints on H is contained in the M–neighborhood of H.

Outside the hyperbolic setting, there are many strongly quasiconvex subgroups that are not stable. For example, nonhyperbolic peripheral subgroups of a relatively hyperbolic group and nonhyperbolic hyperbolically embedded subgroups of a finitely generated group are nonstable strongly quasiconvex subgroups (see Druțu and Sapir [16] and Sisto [32]). In Section 7, we also provide many examples of nonstable strongly quasiconvex subgroups in right-angled Coxeter groups that are not relatively hyperbolic.

However, there is a strong connection between strong quasiconvexity and stability. More precisely, we prove that a subgroup is stable if and only if it is strongly quasiconvex and hyperbolic (see Theorem 4.8). As a result, our study of strongly quasiconvex subgroups

yields new results for stable subgroups of relatively hyperbolic groups, right-angled Coxeter groups, and right-angled Artin groups (see Corollaries 1.10 and 1.12 and Theorem 1.16).

In this paper, we prove that strongly quasiconvex subgroups of any finitely generated group have many of the same properties as quasiconvex subgroups in hyperbolic groups. We also characterize a strongly quasiconvex subgroup via the completely superlinear lower relative divergence of the ambient group with respect to the subgroup. Finally, we study strongly quasiconvex subgroups of relatively hyperbolic groups, right-angled Artin groups, and right-angled Coxeter groups.

The concept of strongly quasiconvex subgroups was also introduced independently in [19] by Genevois under the name *Morse subgroups*. In that paper, Genevois characterizes strongly quasiconvex subgroups in cubulable groups and then he uses strongly quasiconvex subgroups to give a characterization of hyperbolically embedded subgroups in the same group collection. He also independently studies strongly quasiconvex subgroups in right-angled Artin groups and right-angled Coxeter groups.

1.1 Some properties of strongly quasiconvex subgroups

A quasiconvex subgroup of a hyperbolic group is always finitely generated, undistorted, and has finite index in its commensurator (see Kapovich and Short [24; 31]). Further, the intersection between two quasiconvex subgroups of a hyperbolic group is also a quasiconvex subgroup (see Short [31]) and any collection of quasiconvex subgroups of a hyperbolic group has finite height, finite width, and bounded packing (see Gitik, Mitra, Rips, and Sageev [20] and Hruska and Wise [23]). In this paper, we prove that strongly quasiconvex subgroups of arbitrary finitely generated groups have analogous properties.

Theorem 1.2 *Let G be an arbitrary finitely generated group. Then:*

(1) *If H is a strongly quasiconvex subgroup of G, then H is a finitely generated, undistorted subgroup and H has finite index in its commensurator.*

(2) *If H_1 and H_2 are arbitrary strongly quasiconvex subgroups of G, then $H_1 \cap H_2$ is strongly quasiconvex in G, H_1, and H_2.*

(3) *If $\mathcal{H} = \{H_1, \ldots, H_\ell\}$ is a finite collection of strongly quasiconvex subgroups of G, then $\mathcal{H}$ has finite height, finite width, and bounded packing. (We refer the reader to Definitions 4.13 and 4.17 for the concepts of finite height, finite width, and bounded packing of a finite collection of subgroups.)*

In [2], Antolín, Mj, Sisto, and Taylor prove that if $\mathcal{H}$ is a finite collection of stable subgroups of a finitely generated group, then $\mathcal{H}$ has finite height, finite width, and bounded packing. The above theorem strengthens their work to strongly quasiconvex subgroups. Combining Theorem 1.2 with the work of Sageev [29; 30] and Hruska and Wise [23], we have the following as an immediate corollary:

Corollary 1.3 *Suppose H is a strongly quasiconvex codimension 1 subgroup of a finitely generated group G. Then the corresponding* CAT(0) *cube complex is finite-dimensional.*

In the hyperbolic setting, the inclusion map $H \hookrightarrow G$ of a quasiconvex subgroup H into G induces a topological embedding of the Gromov boundary of H into the Gromov boundary of G. The image of this embedding is also the limit set ΛH of H in the Gromov boundary of G. Moreover, if H_1 and H_2 are two quasiconvex subgroups of a hyperbolic group G, then the limit set $\Lambda(H_1 \cap H_2)$ is the intersection $\Lambda H_1 \cap \Lambda H_2$ (see Gromov [21]).

In order to prove analogies of the above properties for strongly quasiconvex subgroups, we need a quasi-isometry invariant boundary for any finitely generated group. In [10], Cordes introduced the *Morse boundary* of proper geodesic metric spaces which generalizes the contracting boundary of CAT(0) spaces (see Charney and Sultan [9]). Critically, the Morse boundary is a quasi-isometry invariant and it agrees with the Gromov boundary on hyperbolic spaces (see [10]). In this paper, we prove strongly quasiconvex subgroups interact with the Morse boundary similarly to how quasiconvex subgroups of hyperbolic groups interact with the Gromov boundary.

Theorem 1.4 *Let G be a finitely generated group. Then:*

(1) *If H is a strongly quasiconvex subgroup of G, then the inclusion $i\colon H \hookrightarrow G$ induces a topological embedding $\hat{i}\colon \partial_M H \to \partial_M G$ of the Morse boundary of H into the Morse boundary of G such that $\hat{i}(\partial_M H) = \Lambda H$, where ΛH is the limit set of H in the Morse boundary $\partial_M G$ of G.*

(2) *If H_1 and H_2 are strongly quasiconvex subgroups of G, then $H_1 \cap H_2$ is strongly quasiconvex in G and $\Lambda(H_1 \cap H_2) = \Lambda H_1 \cap \Lambda H_2$.*

1.2 Strong quasiconvexity and lower relative divergence

Lower relative divergence was introduced by the author in [33] to study the geometric connection between a geodesic space and its subspaces. Lower relative divergence

generalizes the lower divergence of quasigeodesics studied by Charney and Sultan [9] and Arzhantseva, Cashen, Gruber, and Hume [4] to any subset of a geodesic space. Roughly speaking, relative divergence measures the distance distortion of the complement of the r–neighborhood of a subspace in the whole space when r increases. Since lower relative divergence is preserved under quasi-isometry (see Proposition 4.9 in [33]), we can define the lower relative divergence of a pair of groups (G, H), where G is finitely generated and $H \leq G$. We refer the reader to Section 2 for precise definitions.

In [33] the lower relative divergence of a hyperbolic group with respect to a quasiconvex subgroup is at least exponential. For strongly quasiconvex subgroups of arbitrary finitely generated groups, the lower relative divergence must always be completely superlinear.

Theorem 1.5 *Let G be a finitely generated group and H an infinite subgroup of G. Then H is strongly quasiconvex in G if and only if the lower relative divergence of G with respect to H is completely superlinear.*

Theorem 1.5 is actually a corollary of the following more general statement about the lower relative divergence of geodesic spaces with respect to Morse subsets.

Theorem 1.6 *Let X be a geodesic space and A a subset of X of infinite diameter. Then A is Morse in X if and only if the lower relative divergence of X with respect to A is completely superlinear.*

In [35], the author proves that the lower relative divergence of a nonhyperbolic group with respect to a stable subgroup can be any polynomial. In this paper, we prove that the same result holds for nonstable strongly quasiconvex subgroups.

Theorem 1.7 *For each $d \geq 2$ there is a nonhyperbolic group G_d together with a nonstable strongly quasiconvex subgroup whose lower relative divergence is a polynomial of degree d.*

The above theorem shows that the lower relative divergences with respect to strongly quasiconvex subgroups are potentially diverse. Since lower relative divergence is a quasi-isometry invariant, Theorem 1.7 suggests that classifying strongly quasiconvex subgroups using lower relative divergence would produce a rich source of new quasi-isometry invariants.

Theorem 1.6 is a generalization of the work in [9] and [4], in which the authors characterize Morse quasigeodesics as those which have completely superlinear lower

divergence. In fact, Charney and Sultan showed in the setting of CAT(0) spaces that Morse quasigeodesics are actually characterized by quadratic lower divergence. Thus Theorem 1.6 suggests the following interesting question:

Question 1.8 Is the lower relative divergence of a CAT(0) group with respect to an infinite strongly quasiconvex subgroup at least quadratic?

1.3 Strong quasiconvexity and stability in relatively hyperbolic groups

In this paper, we fully describe strongly quasiconvex subgroups and stable subgroups of relatively hyperbolic groups. We remark that strongly quasiconvex subgroups and stable subgroups of relatively hyperbolic groups are distinct from the relatively quasiconvex subgroups defined by Dahmani [12] and Osin [27]. We first come up with the description of strongly quasiconvex subgroups of relatively hyperbolic groups.

Theorem 1.9 *Let $(G, \mathbb{P})$ be a finitely generated relatively hyperbolic group and H a finitely generated undistorted subgroup of G. Then the following are equivalent:*

(1) *The subgroup H is strongly quasiconvex in G.*

(2) *The subgroup $H \cap P^g$ is strongly quasiconvex in P^g for each conjugate P^g of a peripheral subgroup in $\mathbb{P}$.*

(3) *The subgroup $H \cap P^g$ is strongly quasiconvex in G for each conjugate P^g of a peripheral subgroup in $\mathbb{P}$.*

In [3], Aougab, Durham, and Taylor characterize stability in relatively hyperbolic groups whose peripheral subgroups are one-ended and have linear divergence. In this paper, we give a complete characterization of stable subgroups of arbitrary relatively hyperbolic groups.

Corollary 1.10 *Let $(G, \mathbb{P})$ be a finitely generated relatively hyperbolic group and H a finitely generated undistorted subgroup of G. Then the following are equivalent:*

(1) *The subgroup H is stable in G.*

(2) *The subgroup $H \cap P^g$ is stable in P^g for each conjugate P^g of a peripheral subgroup in $\mathbb{P}$.*

(3) *The subgroup $H \cap P^g$ is stable in G for each conjugate P^g of a peripheral subgroup in $\mathbb{P}$.*

1.4 Strong quasiconvexity, stability and the Morse boundary of right-angled Coxeter groups

In this paper, we establish a characterization of strongly quasiconvex special subgroups of two-dimensional right-angled Coxeter groups. Independent work of Genevois has recently partially expanded this characterization to special subgroups of right-angled Coxeter groups of arbitrary dimension (see Proposition 4.9 in [19]).

Theorem 1.11 *Let Γ be a simplicial, triangle-free graph with vertex set S, and let K be a subgroup of G_Γ generated by some subset S_1 of S. Then the following conditions are equivalent:*

(1) *The subgroup K is strongly quasiconvex in G_Γ.*

(2) *If S_1 contains two nonadjacent vertices of an induced 4–cycle σ, then S_1 contains all vertices of σ.*

(3) *Either $|K| < \infty$ or the lower relative divergence of G_Γ with respect to K is at least quadratic.*

An immediate corollary of the above theorem is a characterization of stable special subgroups of two-dimensional right-angled Coxeter groups.

Corollary 1.12 *Let Γ be a simplicial, triangle-free graph with vertex set S, and let K be a subgroup of G_Γ generated by some subset S_1 of S. Then the following conditions are equivalent:*

(1) *The subgroup K is stable in G_Γ.*

(2) *The set S_1 does not contain a pair of nonadjacent vertices of an induced 4–cycle in Γ.*

Behrstock [5] produced the first example of a $\mathcal{CFS}$ right-angled Coxeter group whose Morse boundary was not totally disconnected. Using Corollary 1.12 and Theorem 1.4, we establish a sufficient condition for the Morse boundary of a two-dimensional right-angled Coxeter group to be not totally disconnected. This provides an alternative proof that the Morse boundary of the example in [5] is not totally disconnected:

Corollary 1.13 *If the simplicial, triangle-free graph Γ contains an induced loop σ of length greater than 4 such that the vertex set of σ does not contain a pair of nonadjacent vertices of an induced 4–cycle in Γ, then the Morse boundary of the right-angled Coxeter group G_Γ is not totally disconnected.*

Corollary 1.13 inspires the following conjecture on the connectedness of the Morse boundary of right-angled Coxeter groups.

Conjecture 1.14 *The Morse boundary of a right-angled Coxeter group G_Γ is not totally disconnected if and only if the defining graph Γ contains an induced loop σ of length greater than 4 such that the vertex set of σ does not contain a pair of nonadjacent vertices of an induced 4–cycle in Γ.*

1.5 Strong quasiconvexity, stability, and lower relative divergence in right-angled Artin groups

As discussed above, strong quasiconvexity and stability are equivalent in the hyperbolic setting, but different in general. It is natural to ask about the existence of a nonhyperbolic ambient group in which the two notions of strongly quasiconvex subgroup of infinite index and stable subgroup are equivalent:

Question 1.15 Is there any nonhyperbolic ambient group setting in which the two notions of strongly quasiconvex subgroup of infinite index and stable subgroup are equivalent? In other words, is there any nonhyperbolic ambient group setting in which all strongly quasiconvex subgroups of infinite index are hyperbolic?

The following theorem answers the above question:

Theorem 1.16 *Let Γ be a simplicial, finite, connected graph such that Γ does not decompose as a nontrivial join. Let H be a nontrivial, infinite-index subgroup of the right-angled Artin group A_Γ. Then the following are equivalent:*

(1) *H is strongly quasiconvex.*

(2) *H is stable.*

(3) *The lower relative divergence of A_Γ with respect to H is quadratic.*

(4) *The lower relative divergence of A_Γ with respect to H is completely superlinear.*

In [26], Koberda, Mangahas, and Taylor give several characterizations of purely loxodromic subgroups in right-angled Artin groups. One of these is the equivalence between purely loxodromic subgroups and stable subgroups. The above theorem builds on the work of Koberda, Mangahas, and Taylor in [26] to provide several new characterizations of purely loxodromic subgroups in one-ended right-angled Artin groups.

Corollary 1.17 *Let Γ be a simplicial, finite, connected graph such that Γ does not decompose as a nontrivial join. Let H be a nontrivial, finitely generated, infinite-index subgroup of the right-angled Artin group A_Γ. Then the following are equivalent:*

(1) *H is purely loxodromic.*

(2) *H is strongly quasiconvex.*

(3) *The lower relative divergence of A_Γ with respect to H is quadratic.*

(4) *The lower relative divergence of A_Γ with respect to H is completely superlinear.*

Genevois provides an alternative proof for the equivalence among strongly quasiconvex, stable, and purely loxodromic subgroups of one-ended right-angled Artin groups in Theorem B.1 of [19]. However, Corollary 1.17 also provides a classification of the possible lower relative divergences for any infinite-index subgroup of a one-ended right-angled Artin group. In [25], Kim has recently proved a similar characterization for convex cocompact subgroups of mapping class groups.

1.6 Some questions

Hierarchically hyperbolic groups were recently introduced by Behrstock, Hagen, and Sisto [7] to provide a uniform framework in which to study many important families of groups, including mapping class groups, right-angled Coxeter groups, most 3–manifold groups, and right-angled Artin groups. In [1], Abbott, Behrstock, and Durham prove a characterization of stability in hierarchically hyperbolic groups. We expect a similar characterization of strong quasiconvexity in hierarchically hyperbolic groups is possible.

Question 1.18 Characterize strong quasiconvexity in hierarchically hyperbolic groups.

We also hope there is a connection between strong quasiconvexity and stability in some hierarchically hyperbolic groups which is analogous to the one demonstrated in Theorem 1.16 for right-angled Artin groups.

Question 1.19 What are conditions on a hierarchically hyperbolic group to ensure that all strongly quasiconvex subgroups of infinite index are stable?

Since quasiconvex subgroups of hyperbolic groups are hyperbolic, it is natural to ask about the existence of hierarchically hyperbolic structures on a strongly quasiconvex subgroup of a hierarchically hyperbolic group.

Question 1.20 Is a strongly quasiconvex subgroup of a hierarchically hyperbolic group hierarchically hyperbolic?

Moving on from hierarchically hyperbolic groups, we wonder about further connections between strongly quasiconvex subgroups and hyperbolically embedded subgroups. In particular, Dahmani, Guirardel, and Osin [13] and Sisto [32] prove that a hyperbolically embedded subgroup is almost malnormal and strongly quasiconvex, and Genevois [19] shows that the converse holds in the case of cubulable groups. We ask about other conditions of a group where one can obtain a converse statement.

Question 1.21 Under what other conditions of a finitely generated group are all almost malnormal, strongly quasiconvex subgroups hyperbolically embedded?

In [28], Osin characterizes acylindrically hyperbolic groups using hyperbolically embedded subgroups. Therefore, the answer to the above question may give some connection between the two notions of strongly quasiconvex subgroups and acylindrically hyperbolic groups.

Acknowledgements

2 Preliminaries

In this section, we review some well-known concepts in geometric group theory: geodesic spaces, geodesics, quasigeodesics, quasi-isometries, quasi-isometric embeddings, and Morse subsets. We also discuss some recently developed concepts: lower relative divergence, geodesic divergence, and geodesic lower divergence.

Definition 2.1 Let (X, d_X) and (Y, d_Y) be two metric spaces. A map Φ from X to Y is a (K, L)–*quasi-isometric embedding* if for all x_1 and x_2 in X the inequality

$$\frac{1}{K} d_X(x_1, x_2) - L \leq d_Y(\Phi(x_1), \Phi(x_2)) \leq K d_X(x_1, x_2) + L$$

holds. If, in addition, $N_L(\Phi(X)) = Y$, then Φ is called a (K, L)–*quasi-isometry*. Two spaces X and Y are quasi-isometric if there is a (K, L)–quasi-isometry from X to Y.

The special case of a quasi-isometric embedding where the domain is a connected interval in $\mathbb{R}$ (possibly all of $\mathbb{R}$) is called a (K, L)–*quasigeodesic*. A *geodesic* is a $(1, 0)$–quasigeodesic. The metric space X is a *geodesic space* if any pair of points in X can be joined by a geodesic segment.

Definition 2.2 Let X be a geodesic space and A a subset of X. Let r be any positive number.

(1) $N_r(A) = \{x \in X \mid d_X(x, A) < r\}$.

(2) $\partial N_r(A) = \{x \in X \mid d_X(x, A) = r\}$.

(3) $C_r(A) = X - N_r(A)$.

(4)   Let $d_{r,A}$ be the induced length metric on the complement of the r–neighborhood of A in X. If the subspace A is clear from context, we can use the notation d_r instead of using $d_{r,A}$.

Definition 2.3 A subset A of a geodesic metric space X is *Morse* if for every $K \geq 1$ and $C \geq 0$ there is some $M = M(K, C)$ such that every (K, C)–quasigeodesic with endpoints on A is contained in the M–neighborhood of A. We call the function M a *Morse gauge*.

Before we define the concepts of lower relative divergence, geodesic divergence, and geodesic lower divergence, we need to build the notions of domination and equivalence.

Definition 2.4 Let $\mathcal{M}$ be the collection of all functions from $[0, \infty)$ to $[0, \infty]$. Let f and g be arbitrary elements of $\mathcal{M}$. The function f is *dominated* by the function g, denoted by $f \preceq g$, if there are positive constants A, B, C, and D such that $f(x) \leq Ag(Bx) + Cx$ for all $x > D$. Two functions f and g are *equivalent*, denoted by $f \sim g$, if $f \preceq g$ and $g \preceq f$. The function f is *strictly dominated* by the function g, denoted by $f \prec g$, if f is dominated by g and they are not equivalent.

We say a function f in $\mathcal{M}$ is *completely superlinear* if for every choice of $C > 0$ the collection of $x \in [0, \infty)$ such that $f(x) \leq Cx$ is bounded.

Remark 2.5 The relations $\preceq$ and $\prec$ are transitive. The relation $\sim$ is an equivalence relation on the set $\mathcal{M}$. If $f \preceq g$ in $\mathcal{M}$ and f is completely superlinear, then g is also completely superlinear. Therefore, if $f \sim g$ in $\mathcal{M}$ and one of them is completely superlinear, then the other is also completely superlinear.

It is clear that two polynomial functions of degrees 0 and 1 are always equivalent. Now we let f and g be two polynomial functions of degree at least 1 in the family $\mathcal{M}$.

We observe that f is dominated by g if and only if the degree of f is less than or equal to the degree of g. Also, the two functions f and g are equivalent if and only if they have the same degree. All exponential functions of the form a^{bx+c}, where $a > 1$ and $b > 0$, are equivalent. Therefore, a function f in $\mathcal{M}$ is *linear, quadratic* or *exponential* if f is respectively equivalent to any polynomial with degree one or two or any function of the form a^{bx+c}, where $a > 1$ and $b > 0$.

Definition 2.6 Let $\{\delta_\rho^n\}$ and $\{\delta_\rho'^n\}$ be two families of functions of $\mathcal{M}$, indexed over $\rho \in (0, 1]$ and positive integers $n \geq 2$. The family $\{\delta_\rho^n\}$ is *dominated* by the family $\{\delta_\rho'^n\}$, denoted by $\{\delta_\rho^n\} \preceq \{\delta_\rho'^n\}$, if there exists a constant $L \in (0, 1]$ and a positive integer M such that $\delta_{L\rho}^n \preceq \delta_\rho'^{Mn}$. Two families $\{\delta_\rho^n\}$ and $\{\delta_\rho'^n\}$ are *equivalent*, denoted by $\{\delta_\rho^n\} \sim \{\delta_\rho'^n\}$, if $\{\delta_\rho^n\} \preceq \{\delta_\rho'^n\}$ and $\{\delta_\rho'^n\} \preceq \{\delta_\rho^n\}$.

We say a family $\{\delta_\rho^n\}$ of functions in $\mathcal{M}$ is *completely superlinear* if there is some $n_0 \geq 3$ such that $\delta_\rho^{n_0}$ is completely superlinear for all $\rho \in (0, 1]$.

Remark 2.7 A family $\{\delta_\rho^n\}$ is dominated by (or dominates) a function f in $\mathcal{M}$ if $\{\delta_\rho^n\}$ is dominated by (or dominates) the family $\{\delta_\rho'^n\}$, where $\delta_\rho'^n = f$ for all ρ and n. The equivalence between a family $\{\delta_\rho^n\}$ and a function f in $\mathcal{M}$ can be defined similarly. Thus, a family $\{\delta_\rho^n\}$ is linear, quadratic, exponential, etc if $\{\delta_\rho^n\}$ is equivalent to the function f where f is linear, quadratic, exponential, etc.

If $\{\delta_\rho^n\} \preceq \{\delta_\rho'^n\}$ and $\{\delta_\rho^n\}$ is completely superlinear, then $\{\delta_\rho'^n\}$ is also completely superlinear. Therefore, if $\{\delta_\rho^n\} \sim \{\delta_\rho'^n\}$ and one of them is completely superlinear, then the other is also completely superlinear.

We now review the concept of lower relative divergence, which was originally introduced by the author in [33].

Definition 2.8 (lower relative divergence in spaces) Let X be a geodesic space and A a subset of X of infinite diameter. For each $\rho \in (0, 1]$ and positive integer $n \geq 2$, we define the function $\sigma_\rho^n \colon [0, \infty) \to [0, \infty]$ as follows:

For each positive r, if there is no pair of $x_1, x_2 \in \partial N_r(A)$ such that $d_r(x_1, x_2) < \infty$ and $d(x_1, x_2) \geq nr$, we define $\sigma_\rho^n(r) = \infty$. Otherwise, we define $\sigma_\rho^n(r) = \inf d_{\rho r}(x_1, x_2)$, where the infimum is taken over all $x_1, x_2 \in \partial N_r(A)$ such that $d_r(x_1, x_2) < \infty$ and $d(x_1, x_2) \geq nr$.

The family of functions $\{\sigma_\rho^n\}$ is the *lower relative divergence* of X with respect A, denoted by $\mathrm{div}(X, A)$.

Convention 2.9 Let X be a geodesic space and A a subset of X of infinite diameter. Let $\{\sigma_\rho^n\}$ be the lower relative divergence of X with respect to A. Assume that $N_{r_0}(A) = X$ for some r_0. Therefore, $\partial N_r(A) = \varnothing$ for each $r > r_0$. This implies for each $\rho \in (0, 1]$ and positive integer $n \geq 2$ that $\sigma_\rho^n(r) = \infty$ for $r > r_0$.

Definition 2.10 (lower relative divergence in groups) Let G be a finitely generated group and H its subgroup. We define the *lower relative divergence* of G with respect to H, denoted by $\mathrm{div}(G, H)$, to be the lower relative divergence of the Cayley graph $\Gamma(G, S)$ with respect to H for some (any) finite generating set S.

Remark 2.11 The subgroup H in the above definition is not required to be finitely generated. Moreover, the lower relative divergence is a pair quasi-isometry invariant (see Proposition 4.9 in [33]). This implies that the lower relative divergence on a finitely generated group does not depend on the choice of finite generating sets of the ambient group.

We now define geodesic divergence and geodesic lower divergence.

Definition 2.12 (geodesic divergence) The *divergence* of two geodesic rays α and β with the same initial point x_0 in a geodesic space X, denoted by $\mathrm{Div}_{\alpha,\beta}$, is a function $g \colon (0, \infty) \to (0, \infty]$ defined as follows. For each positive r, if there is no path outside the open ball with radius r about x_0 connecting $\alpha(r)$ and $\beta(r)$, we define $g(r) = \infty$. Otherwise, we define $g(r)$ to be the infimum on the lengths of all paths outside the open ball with radius r about x_0 connecting $\alpha(r)$ and $\beta(r)$.

The *divergence of a bi-infinite geodesic* γ, denoted by Div_γ, is the divergence of the two geodesic rays obtained from γ with initial point $\gamma(0)$.

Definition 2.13 (geodesic lower divergence) Let α be a bi-infinite geodesic. For any $r > 0$ and $t \in \mathbb{R}$, if there is no path from $\alpha(t - r)$ to $\alpha(t + r)$ that lies outside the open ball of radius r about $\alpha(t)$, we define $\rho_\alpha(r, t) = \infty$. Otherwise, we let $\rho_\alpha(r, t)$ denote the infimum of the lengths of all paths from $\alpha(t - r)$ to $\alpha(t + r)$ that lie outside the open ball of radius r about $\alpha(t)$. Define the *lower divergence* of α to be the growth rate of the following function:

$$\mathrm{ldiv}_\alpha(r) = \inf_{t \in \mathbb{R}} \rho_\alpha(r, t)$$

Remark 2.14 The geodesic divergence is a classical notion related to curvature and the geodesic lower divergence was originally introduced by Charney and Sultan in [9]. In general, these concepts measure different aspects of a bi-infinite geodesic in a geodesic

space. More precisely, let α be a bi-infinite geodesic in a geodesic space X. It is not hard to see that $\mathrm{ldiv}_\alpha \leq \mathrm{Div}_\alpha$ and the two functions are not the same in general. However, $\mathrm{ldiv}_\alpha \sim \mathrm{Div}_\alpha$ if α is a periodic geodesic (ie there is an isometry g of X such that $g\alpha = \alpha$).

Also, the lower divergence ldiv_α of α in X is equivalent to the lower relative divergence $\mathrm{div}(X, \alpha)$ of X with respect to α. The proof of this fact is similar to the proof of Proposition 6.6 in [33] and we leave it to the reader.

3 Characterizing Morse subsets via lower relative divergence

In most parts of this section, we characterize Morse subsets with infinite diameter in terms of the lower divergence related to them. This result will be used to characterize strongly quasiconvex subgroups, which are defined in Section 4. We also study the behavior of geodesic rays in a finite neighborhood of some Morse subsets and this result will be applied to study the height of strongly quasiconvex subgroups later.

Proposition 3.1 (completely superlinear lower relative divergence implies Morse) *Let X be a geodesic space and A a subset of X of infinite diameter. If the lower relative divergence $\{\sigma_\rho^n\}$ of X with respect to A is completely superlinear, then A is Morse in X.*

Proof Let $n_0 \geq 3$ and $\rho_0 \in (0, 1]$ such that $\sigma_{\rho_0}^{n_0}$ is a completely superlinear function. Let $\gamma: [a, b] \to X$ be an arbitrary (K, L)–quasigeodesic in X connecting two points in A. By Lemma 1.11 of [8, Chapter III.H], we can find a continuous quasigeodesic $c: [a, b] \to X$ such that:

(1) $c(a) = \gamma(a)$ and $c(b) = \gamma(b)$.

(2) $\ell(c|_{[t,t']}) \leq K_1 d(c(t), c(t')) + L_1$, where $K_1 \geq 1$ and $L_1 \geq 0$ only depend on K and L.

(3) The Hausdorff distance between γ and c is less than C, where C only depends on K and L.

Therefore, it is sufficient to prove that c lies in some M–neighborhood of A, where M only depends on K_1, L_1, n_0, and the function $\sigma_{\rho_0}^{n_0}$. Let $t_0 \in [a, b]$ such that $c(t_0)$ is the farthest point from A, and let $m = d(c(t_0), A)$. If $m \leq 2L_1$, then c obviously lies in some M–neighborhood of A, where M only depends on L_1. Therefore, we can

assume that $m > 2L_1$. We claim that if $m > 2L_1$, then $\sigma_{\rho 0}^{n_0}(m/(2K_1 n_0)) \le (4K_1 + 2)m$. Let $m_1 = m/(2K_1 n_0)$, $t_1 \in [a, t_0]$, and $t_2 \in [t_0, b]$ such that $c(t_1)$ and $c(t_2)$ both lie in $\partial N_{m_1}(A)$ and $c[t_1, t_2]$ lies outside $N_{m_1}(A)$. We consider two cases:

Case 1 $$\ell(c|_{[t_1, t_2]}) \le 4K_1 m.$$

Because $d(c(t_0), A) = m$ and $d(c(t_1), A) = m/(2K_1 n_0)$, the length $\ell(c|_{[t_1, t_0]})$ is bounded below by $(1 - 1/(2K_1 n_0))m$. Similarly, the length $\ell(c|_{[t_0, t_2]})$ is bounded below by $(1 - 1/(2K_1 n_0))m$. Therefore, the length $\ell(c|_{[t_1, t_2]})$ is bounded below by $(2 - 1/(K_1 n_0))m$. This implies that

$$
\begin{aligned}
d(c(t_1), c(t_2)) &\ge \frac{1}{K_1}\ell(c|_{[t_1, t_2]}) - \frac{L_1}{K_1} \\
&\ge \frac{1}{K_1}\left(2 - \frac{1}{K_1 n_0}\right)m - \frac{L_1}{K_1} \\
&\ge \frac{m}{K_1} + \left(1 - \frac{1}{K_1 n_0}\right)\left(\frac{m}{K_1}\right) - \frac{L_1}{K_1} \\
&\ge \frac{m}{K_1} + \frac{m}{2K_1} - \frac{L_1}{K_1} \ge \frac{m}{K_1} \ge n_0\left(\frac{m}{2K_1 n_0}\right).
\end{aligned}
$$

Also, the path $c([t_1, t_2])$ lies outside the $(m/(2K_1 n_0))$–neighborhood of A. Therefore,

$$\sigma_{\rho 0}^{n_0}\left(\frac{m}{2K_1 n_0}\right) \le \ell(c|_{[t_1, t_2]}) \le 4K_1 m.$$

Case 2 $$\ell(c|_{[t_1, t_2]}) > 4K_1 m.$$

We can choose t_3 and t_4 in $[t_1, t_2]$ (with $t_3 < t_4$) such that $\ell(c|_{[t_3, t_4]}) = 4K_1 m$. Since $m_1 \le d(c(t_3), A) \le m$, we can choose a point u in $\partial N_{m_1}(A)$ such that the geodesic α_1 connecting u and $c(t_3)$ lies outside $N_{m_1}(A)$ and its length is bounded above by m. Similarly, we can choose a point v in $\partial N_{m_1}(A)$ such that the geodesic α_2 connecting v and $c(t_4)$ lies outside $N_{m_1}(A)$ and its length is bounded above by m. Let $\beta = \alpha_1 \bigcup c[t_3, t_4] \bigcup \alpha_2$. Then β is a path outside $N_{m_1}(A)$ that connects two points u and v on $\partial N_{m_1}(A)$. Moreover, the length of β is bounded above by $(4K_1 + 2)m$ by construction. Also,

$$
\begin{aligned}
d(u, v) &\ge d(c(t_3), c(t_4)) - 2m \ge \frac{1}{K_1}\ell(c|_{[t_3, t_4]}) - \frac{L_1}{K_1} - 2m \\
&\ge 4m - \frac{L_1}{K_1} - 2m \ge 2m - \frac{L_1}{K_1} \ge m \ge n_0\left(\frac{m}{2K_1 n_0}\right).
\end{aligned}
$$

Therefore,

$$\sigma_{\rho_0}^{n_0}\left(\frac{m}{2K_1 n_0}\right) \leq \ell(\beta) \leq (4K_1 + 2)m.$$

Since $\sigma_{\rho_0}^{n_0}$ is a completely superlinear function, there is an upper bound on m depending on K_1 and n_0. Therefore, c lies in some M–neighborhood of A, where M only depends on K_1, L_1, n_0, and the function $\sigma_{\rho_0}^{n_0}$.   □

The following two lemmas will be used to compute the lower relative divergence of geodesic spaces with respect to their Morse subsets. Most techniques for the proof of these lemmas were studied in [4].

Lemma 3.2 *Suppose γ is a concatenation of two $(L, 0)$–quasigeodesics γ_1 and γ_2, where γ_1, γ_2, and γ are all parametrized by arc length. Let $C > 1$ and $r > 0$ such that the distance between the two endpoints a and c of γ is at least r and the length of γ is at most Cr. For each $\rho \in (0, 1]$ and $L' > L + C + C/\rho + 1$, there exists an $(L', 0)$–quasigeodesic α parametrized by arc length connecting the two points a and c such that the image of α lies in the ρr–neighborhood of the image of γ and the length of α is bounded above by the length of γ.*

Proof For each path β parametrized by arc length we will abuse notation by identifying β with its image. Moreover, for each pair of points u and v on β we denote by $d_\beta(u, v)$ the length of the subpath of β connecting u and v. If γ is an $(L', 0)$–quasigeodesic, then $\alpha = \gamma$ is a desired quasigeodesic. We now assume that γ is not an $(L', 0)$–quasigeodesic.

Since $d(a, c) \geq r \geq d_\gamma(a, c)/C > d_\gamma(a, c)/L'$, there is a maximal subsegment $[x, y]_\gamma$ of γ such that $d(x, y) = d_\gamma(x, y)/L'$ by the continuity of γ. It is obvious that x and y cannot both lie in the same path γ_1 or γ_2. Therefore, we can assume $x \in \gamma_1$ and $y \in \gamma_2$. Let $[x, y]$ be a geodesic connecting x and y and $\alpha = [a, x]_{\gamma_1} \cup [x, y] \cup [y, c]_{\gamma_2}$. We parametrized α by arc length and we will prove that α is an $(L', 0)$–quasigeodesic.

Let u and v be two arbitrary points in α. If u and v both lie in the same one of the segments $[a, x]_{\gamma_1}$, $[x, y]$, or $[y, c]_{\gamma_2}$, then $d_\alpha(u, v) \leq L'd(u, v)$. We now consider the remaining three cases.

Case 1 One of the points u and v lies in $[a, x]_{\gamma_1}$ and the other lies in $[x, y]$. We can assume that u lies in $[a, x]_{\gamma_1}$ and v lies in $[x, y]$. We observe that

$$d(u, y) > d_\gamma(u, y)/L'$$

by the maximality of $[x, y]_\gamma$ and the continuity of γ. Therefore,

$$
d(u, v) \geq d(u, y) - d(v, y) \geq \frac{d_\gamma(u, y)}{L'} - d(v, y)
$$

$$
= \frac{d_\gamma(u, x) + d_\gamma(x, y)}{L'} - d(v, y) = \frac{d_\gamma(u, x)}{L'} + d(x, y) - d(v, y)
$$

$$
= \frac{d_\gamma(u, x)}{L'} + d(x, v) \geq \frac{d_\gamma(u, x) + d(x, v)}{L'} = \frac{d_\alpha(u, v)}{L'}
$$

or $d_\alpha(u, v) \leq L'd(u, v)$.

Case 2 One of the points u and v lies in $[y, c]_{\gamma_2}$ and the other lies in $[x, y]$. Using an argument symmetric to that in Case 1, we can prove $d_\alpha(u, v) \leq L'd(u, v)$.

Case 3 One of the points u and v lies in $[a, x]_{\gamma_1}$ and the other lies in $[y, c]_{\gamma_2}$. We can assume that u lies in $[a, x]_{\gamma_1}$ and v lies in $[y, c]_{\gamma_2}$. We observe again that $d(u, v) > d_\gamma(u, v)/L'$ by the maximality of $[x, y]_\gamma$ and the continuity of γ. Also, it is not hard to see $d_\alpha(u, v) \leq d_\gamma(u, v)$. This implies that $d(u, v) > d_\alpha(u, v)/L'$ or $d_\alpha(u, v) \leq L'd(u, v)$.

Therefore, α is an $(L', 0)$–quasigeodesic parametrized by arc length connecting a and c. It is not hard to see the length of α is less than or equal to the length of γ. In order to see that the path α lies completely inside the ρr–neighborhood of γ, we observe that

$$
d(x, y) = \frac{d_\gamma(x, y)}{L'} < \frac{Cr}{C/\rho} = \rho r. \qquad \square
$$

Lemma 3.3 *For each $C > 1$ and $\rho \in (0, 1]$ there is a constant $L = L(C, \rho) \geq 1$ such that the following holds. Let r be an arbitrary positive number and γ a continuous path with length less than Cr. Assume the distance between the two endpoints x and y of γ is at least r. Then there is an $(L, 0)$–quasigeodesic α connecting the two points x and y such that the image of α lies in the ρr–neighborhood of γ and the length of α is less than or equal to the length of γ.*

Proof Let $x_0 = x$, and let x_1 be the last point on γ such that $d(x_0, x_1) = \rho r/4$. Similarly, we define $x_i = y$ if $d(x_{i-1}, y) < \rho r/2$ or to be the last point on γ such that $d(x_{i-1}, x_i) = \rho r/4$. We observe that, by construction, $d(x_i, x_j) \geq \rho r/4$ if $i \neq j$ and $y = x_n$ for some $n \leq 4C/\rho$. Let α_1 be a concatenation of geodesics $[x_0, x_1][x_1, x_2] \cdots [x_{n-1}, x_n]$. Then α_1 connects the two points x and y, the length of α_1 is less than or equal to the length of γ, and α_1 lies completely inside the $(\rho r/2)$–neighborhood of γ.

Using Lemma 3.2 for each integer $1 \le i \le \lfloor n/2 \rfloor$, there are $(L_2, 0)$–quasigeodesics (where L_2 only depends on C and ρ) from $x_{2(i-1)}$ to x_{2i} such that the concatenation α_2 of these with the subpath $[x_{2\lfloor n/2 \rfloor}, x_n]_{\alpha_1}$ of α_1 satisfies the following conditions:

(1) α_2 lies completely inside the $(\rho r/4)$–neighborhood of α_1.

(2)   The length of α_2 is less than or equal to the length of α_1.

Repeat this process at most d times ($d \le n$) until we get an $(L_d, 0)$–quasigeodesic α_d (where L_d only depends on C and ρ) connecting the two points x and y satisfying the following conditions:

(1) α_d lies completely inside the $(\rho r/2^d)$–neighborhood of α_{d-1}.

(2)   The length of α_d is less than or equal to the length of α_{d-1}.

Therefore, the length of the $(L_d, 0)$–quasigeodesic α_d is less than the length of γ. Also,

$$\frac{\rho r}{2} + \frac{\rho r}{4} + \cdots + \frac{\rho r}{2^d} < \rho r.$$

This implies that α_d lies in the ρr–neighborhood of γ. Therefore, $L = L_d$ is the desired number only depending on C and ρ. $\square$

Proposition 3.4 (Morse implies completely superlinear lower relative divergence) *Let X be a geodesic space and A a subset of X of infinite diameter. If A is Morse in X, then the lower relative divergence $\{\sigma^n_\rho\}$ of X with respect to A is completely superlinear.*

Proof   We will prove the stronger result that σ^n_ρ is completely superlinear for all $n \ge 5$ and $\rho \in (0, 1]$. Let μ be the Morse gauge of A and assume for purposes of contradiction that $\sigma^{n_0}_{\rho_0}$ is not completely superlinear for some $n_0 \ge 5$ and $\rho_0 \in (0, 1]$. Then there is a $C > 0$ for which there is an unbounded sequence of numbers r_m and paths γ_m satisfying:

(1)   Each path γ_m lies outsides the $(\rho_0 r_m)$–neighborhood of A and its endpoints x_m and y_m lie in $\partial N_{r_m}(A)$.

(2) $d(x_m, y_m) \ge n_0 r_m$ and $\ell(\gamma_m) \le C r_m$.

For each m, let γ'_m be a geodesic of length less than $2r_m$ that connects x_m and some point u_m in A. Similarly, let γ''_m be a geodesic of length less than $2r_m$ that connects y_m and some point v_m in A. Let $\bar{\gamma}_m = \gamma'_m \cup \gamma_m \cup \gamma''_m$. Then $\bar{\gamma}_m$ is a

path of length at most $(C+4)r_m$ that connects two points u_m and v_m on A with $d(u_m, v_m) \geq (n_0 - 4)r_m \geq r_m$.

By Lemma 3.3, for each m there is an $(L, 0)$–quasigeodesic α_m (here L does not depend on m) connecting two points u_m and v_m satisfying:

(1) $\ell(\alpha_m) \leq \ell(\overline{\gamma}_m) \leq (C+4)r_m$.

(2) α_m lies in the $(\rho_0 r_m/2)$–neighborhood of $\overline{\gamma}_m$.

By the triangle inequality, each $(\rho_0 r_m/2)$–neighborhood of γ'_m is contained in the open ball $B(x_m, 2r_m + \rho_0 r_m/2)$ and each $(\rho_0 r_m/2)$–neighborhood of γ''_m is contained in the open ball $B(y_m, 2r_m + \rho_0 r_m/2)$. Since $d(x_m, y_m) \geq n_0 r_m \geq 5r_m$, the intersection between the two open balls $B(x_m, 2r_m + \rho_0 r_m/2)$ and $B(y_m, 2r_m + \rho_0 r_m/2)$ is empty. Therefore, the $(\rho_0 r_m/2)$–neighborhoods of γ'_m and γ''_m have an empty intersection. Since α_m is connected and α_m intersects both $(\rho_0 r_m/2)$–neighborhoods of γ'_m and γ''_m, it cannot lie completely inside the $(\rho_0 r_m/2)$–neighborhood of $\gamma'_m \cup \gamma''_m$. This implies that there is a point s_m on α_m that lies in the $(\rho_0 r_m/2)$–neighborhood of γ_m. Since each path γ_m lies outside the $(\rho_0 r_m)$–neighborhood of A, the point s_m must lie outside the $(\rho_0 r_m/2)$–neighborhood of A.

Recall that each α_m is an $(L, 0)$–quasi-geodesic with endpoints on A and that A is Morse in X with Morse gauge μ. Each path γ_m must lie completely inside the $\mu(L, 0)$–neighborhood of A. In particular, each point s_m must lie inside the $\mu(L, 0)$–neighborhood of A. Therefore, $\mu(L, 0) \geq \rho_0 r_m/2$ for all m, which contradicts the choice of sequence r_m. Therefore, σ_ρ^n is completely superlinear for all $n \geq 5$ and $\rho \in (0, 1]$. $\qquad\square$

The following theorem is deduced from Propositions 3.1 and 3.4.

Theorem 3.5 (characterization of Morse subsets) *Let X be a geodesic space and A a subset of X of infinite diameter. Then A is Morse in X if and only if the lower relative divergence of X with respect to A is completely superlinear.*

We now study the behavior of geodesic rays in a finite neighborhood of some Morse subset. We first start with a technical lemma that basically amounts to "quasigeodesic approximation" of certain concatenations of geodesics.

Lemma 3.6 *Let $\gamma = \gamma_1 \gamma_2 \gamma_3$ be the concatenation of three geodesics γ_1, γ_2, and γ_3. If there is a number $r > 0$ such that $\ell(\gamma_1) < r$, $\ell(\gamma_2) = 20r$, and $\ell(\gamma_3) < r$, then there is a $(5, 0)$–quasigeodesic α with the same endpoints as γ and $\alpha \cap \gamma_2 \neq \varnothing$.*

Proof For each path β parametrized by arc length we will abuse notation by identifying β with its image. Moreover, for each pair of points u and v on β we denote by $d_\beta(u, v)$ the length of the subpath of β connecting u and v. Let a and b be the endpoints of γ_1, let b and c be the endpoints of γ_2, and let c and d be the endpoints of γ_3. We claim that there are points $x_1 \in \gamma_1$ and $y_1 \in \gamma_2$ such that $d(x_1, y_1) \leq 5r$ and the concatenation $\alpha_1 = [a, x_1]_{\gamma_1} \cup [x_1, y_1] \cup [y_1, c]_{\gamma_2}$ is a $(5, 0)$–quasigeodesic, where $[x_1, y_1]$ is a geodesic connecting x_1 and y_1.

If the concatenation $\beta = \gamma_1 \gamma_2$ is a $(5, 0)$–quasigeodesic, then we can choose $x_1 = y_1 = b$ and $\alpha_1 = \beta$. We now assume that β is not a $(5, 0)$–quasigeodesic. Since

$$d(a, c) \geq d(b, c) - d(a, b) \geq 19r \quad \text{and} \quad d_\beta(a, c) \leq d(a, b) + d(b, c) \leq 21r,$$

$d(a, c) > \frac{1}{5} d_\beta(a, c)$. Therefore, there is a maximal subsegment $[x_1, y_1]_\beta$ of β such that $d(x_1, y_1) = \frac{1}{5} d_\beta(x_1, y_1)$, by the continuity of β. It is obvious that x_1 and y_1 cannot both lie in the same path γ_1 or γ_2. Therefore, we can assume $x_1 \in \gamma_1$ and $y_1 \in \gamma_2$. Obviously, $d(x_1, y_1) = \frac{1}{5} d_\beta(x_1, y_1) \leq \frac{1}{5} \ell(\beta) \leq 5r$. Moreover, α_1 is a $(5, 0)$–quasigeodesic by using the same argument as in Lemma 3.2.

Similarly, there are points $x_2 \in \gamma_3$ and $y_2 \in \gamma_2$ such that $d(x_2, y_2) \leq 5r$ and the concatenation $\alpha_2 = [b, y_2]_{\gamma_2} \cup [y_2, x_2] \cup [x_2, d]_{\gamma_3}$ is a $(5, 0)$–quasigeodesic, where $[y_2, x_2]$ is a geodesic connecting y_2 and x_2. Let

$$\alpha = [a, x_1]_{\gamma_1} \cup [x_1, y_1] \cup [y_1, y_2]_{\gamma_2} \cup [y_2, x_2] \cup [x_2, d]_{\gamma_3}.$$

Then $\alpha \subset \alpha_1 \cup \alpha_2$ and $\alpha \cap \gamma_2 \neq \varnothing$. We now prove that α is a $(5, 0)$–quasigeodesic. Let u and v be two arbitrary points in α. We consider several cases.

If two points u and v both lie in α_1, then $d_{\alpha_1}(u, v) \leq 5d(u, v)$ because α_1 is a $(5, 0)$–quasigeodesic. Also, $d_\alpha(u, v) = d_{\alpha_1}(u, v)$ by the construction of α and α_1. This implies that $d_\alpha(u, v) \leq 5d(u, v)$. We also obtain a similar inequality if two points u and v both lie in α_2. In the remaining case, u and v do not lie in the same path α_1 or α_2. Therefore, one of two points u and v must lie in $[a, x_1]_{\gamma_1} \cup [x_1, y_1]$ (say u) and the remaining point must lie in $[y_2, x_2] \cup [y_2, d]_{\gamma_3}$ (say v). Since the length of $[a, x_1]_{\gamma_1} \cup [x_1, y_1]$ is less than $6r$, $d(a, u) \leq 6r$. Similarly, $d(d, v) \leq 6r$. Therefore,

$$\begin{aligned} d(u, v) &\geq d(a, d) - d(a, u) - d(d, v) \\ &\geq d(b, c) - d(a, b) - d(c, d) - 12r \\ &\geq 20r - r - r - 12r \geq 6r. \end{aligned}$$

Also, $d_\alpha(u, v) \leq \ell(\alpha) \leq \ell(\gamma) \leq 22r$. Therefore, $d_\alpha(u, v) \leq 5d(u, v)$. Therefore, α is a $(5, 0)$–quasigeodesic. □

Proposition 3.7 Let X be a geodesic space and A a Morse subset of X with Morse gauge μ. Then there is a constant $D > 0$ such that the following holds. If $\gamma: [0, \infty) \to X$ is a geodesic ray that lies in some finite neighborhood of A, then there is a $C > 0$ such that $\gamma|_{[C, \infty)}$ lies in the D–neighborhood of A.

Proof Assume that γ lies in the r–neighborhood of A for some $r > 0$. For each $i \geq 1$ let $\gamma_i = \gamma|_{[20(i-1)r, 20ir]}$. Then each γ_i is a geodesic of length $20r$. Since γ lies in the r–neighborhood of A, for each $i \geq 0$ there is a geodesic β_i of length less than r that connects $\gamma(20ir)$ and A. By Lemma 3.6 for each $i \geq 1$ there is a $(5, 0)$–quasigeodesic α_i with the same endpoints as the concatenation $\beta_{i-1}\gamma_i\beta_i$ such that $\alpha_i \cap \gamma_i \neq \varnothing$.

Since each α_i is a $(5, 0)$–quasigeodesic with endpoints in A, each α_i lies entirely inside the $\mu(5, 0)$–neighborhood of A. We recall that $\alpha_i \cap \gamma_i \neq \varnothing$. Therefore, there is a $t_i \in [20(i-1)r, 20ir]$ such that $\gamma(t_i)$ lies in the $\mu(5, 0)$–neighborhood of A. Therefore, for each $i \geq 1$ each geodesic $\gamma|_{[t_i, t_{i+1}]}$ has endpoints in the $\mu(5, 0)$–neighborhood of A. This implies that $\gamma|_{[t_i, t_{i+1}]}$ lies in some D–neighborhood of A, where D only depends on μ. Set $C = t_1$. Then $\gamma|_{[C, \infty)}$ lies in the D–neighborhood of A obviously. □

4 Strong quasiconvexity, stability, and lower relative divergence

In this section, we propose the concept of strongly quasiconvex subgroups. We show the connection between strongly quasiconvex subgroups and stable subgroups and we characterize these subgroups using lower relative divergence. We also study some basic properties of strongly quasiconvex subgroups that are analogous to properties of quasiconvex subgroups in hyperbolic groups.

Definition 4.1 Let $\Phi: A \to X$ be a quasi-isometric embedding between geodesic metric spaces. We say A is *strongly quasiconvex* in X if the image $\Phi(A)$ is Morse in X. We say A is *stable* in X if for any $K \geq 1$ and $L \geq 0$ there is an $R = R(K, L) \geq 0$ such that if α and β are two (K, L)–quasigeodesics with the same endpoints in $\Phi(A)$, then the Hausdorff distance between α and β is less than R.

Note that when we say A is strongly quasiconvex (stable) in X we mean that A is strongly quasiconvex (stable) in X with respect to a particular quasi-isometric embedding $\Phi \colon A \to X$. Such a quasi-isometric embedding will always be clear from context, for example an undistorted subgroup H of a finitely generated group G. The following proposition provides some basic facts about strong quasiconvexity.

Proposition 4.2 *Suppose A, B, and X are geodesic metric spaces and $B \xrightarrow{f} A \xrightarrow{g} X$ are quasi-isometric embeddings.*

(1) *If B is strongly quasiconvex in X via $g \circ f$, then B is strongly quasiconvex in A via f.*

(2) *If B is strongly quasiconvex in A via f and A is strongly quasiconvex in X via g, then B is strongly quasiconvex in X via $g \circ f$.*

Proof We first prove statement (1). More precisely, we are going to prove that $f(B)$ is Morse in A. For each $K \geq 1$ and $L \geq 0$ let γ be an arbitrary (K, L)–quasigeodesic in A that connects two points in $f(B)$. Then $g(\gamma)$ is a (K', L')–quasigeodesic in X that connects two points in $g(f(B))$, where K and L' depend only on K, L, and the quasi-isometric embedding g. Since B is strongly quasiconvex in X via $g \circ f$, $g(\gamma)$ lies in some M–neighborhood of $g(f(B))$, where M depends only on K' and L'. Again, g is a quasi-isometric embedding. Therefore, γ lies in some M'–neighborhood of $f(B)$, where M' depends only on K, L, f, and the quasi-isometric embedding g. This implies that B is strongly quasiconvex in A via f.

We now prove statement (2). More precisely, we are going to prove that $(g \circ f)(B)$ is Morse in X. For each $K \geq 1$ and $L \geq 0$ let α be an arbitrary (K, L)–quasigeodesic in X that connects two points in $g(f(B))$. Since $g(f(B)) \subset g(A)$ and $g(A)$ is a Morse subset in X, we know that α lies in some M–neighborhood of $g(A)$, where M depends only on K and L. Also, $g \colon A \to X$ is a quasi-isometric embedding. Then there is a (K_1, L_1)–quasigeodesic β in A that connects two points in $f(B)$ such that the Hausdorff distance between $g(\beta)$ and α is bounded above by C, where K_1, L_1, and C depend only on K, L, M, and the quasi-isometric embedding g. Also, $f(B)$ is a Morse subset of A. Then β lies in some M_1–neighborhood $f(B)$, where M_1 depends only on K_1 and L_1. Again, $g \colon A \to X$ is a quasi-isometric embedding. Then $g(\beta)$ lies in some M_2–neighborhood of $g(f(B))$, where M_2 depends only on M_1 and the map g. Therefore, α lies in the $(M_2 + C)$–neighborhood of $g(f(B))$. Therefore, B is strongly quasiconvex in X via $g \circ f$. □

The following proposition gives the exact relationship between strongly quasiconvex and stable subspaces.

Proposition 4.3 *Let $\Phi\colon A \to X$ be a quasi-isometric embedding between geodesic metric spaces. Then A is stable in X if and only if A is strongly quasiconvex and hyperbolic.*

Proof One direction of the above proposition is deduced from Remark 3.1 and Lemma 3.3 in [17]. We now prove that if A is strongly quasiconvex and hyperbolic, then A is stable in X. For any $K \geq 1$ and $L \geq 0$, let α and β be two (K, L)–quasigeodesics with the same endpoints in $\Phi(A)$. Since $\Phi(A)$ is Morse in X, there is a constant $C \geq 0$ not depending on α and β such that α and β both lie in the C–neighborhood of $\Phi(A)$. It is an easy exercise that there are constants $K_1 \geq 1$, $L_1 \geq 0$, and $D \geq 0$ not depending on α and β and two (K_1, L_1)–quasigeodesics α_1 and β_1 with the same endpoints in A such that the Hausdorff distance between $\Phi(\alpha_1)$ and α and the Hausdorff distance between $\Phi(\beta_1)$ and β are both bounded above by D. Since A is a hyperbolic space and Φ is a quasi-isometric embedding, the Hausdorff distance between $\Phi(\alpha_1)$ and $\Phi(\beta_1)$ is bounded above by some constant D_1 which does not depend on α and β. Therefore, the Hausdorff distance between α and β is bounded above by $D_1 + 2D$. Therefore, A is stable in X. $\qquad\square$

We now define the concepts of strongly quasiconvex subgroups and stable subgroups.

Definition 4.4 Let G be a finitely generated group and S an arbitrary finite generating set of G. Let H be a finitely generated subgroup of G, and let T be an arbitrary finite generating set of H. The subgroup H is *undistorted* in G if the natural inclusion $i\colon H \to G$ induces a quasi-isometric embedding from the Cayley graph $\Gamma(H, T)$ into the Cayley graph $\Gamma(G, S)$. We say H is *stable* in G if $\Gamma(H, T)$ is stable in $\Gamma(G, S)$.

We remark that stable subgroups were proved to be independent of the choice of finite generating sets (see Section 3 in [17]).

Definition 4.5 Let G be a finitely generated group and H a subgroup of G. We say H is *quasiconvex* in G with respect to some finite generating set S of G if there exists some $C > 0$ such that every geodesic in the Cayley graph $\Gamma(G, S)$ that connects a pair of points in H lies inside the C–neighborhood of H. We say H is *strongly quasiconvex* in G if H is a Morse subset in the Cayley graph $\Gamma(G, S)$ for some (any) finite generating set S.

Remark 4.6 If H is a quasiconvex subgroup of a group G with respect to some finite generating set S, then H is also finitely generated and undistorted in G (see Lemma 3.5 of [8, Chapter III.Γ]). However, we emphasize that the concept of quasiconvex subgroups depends on the choice of finite generating set of the ambient group.

It is clear that if H is a Morse subset in the Cayley graph $\Gamma(G, S)$ with some finite generating set S, then H is also a quasiconvex subgroup of G with respect to S. In particular, H is finitely generated and undistorted in G. Therefore, the strong quasiconvexity of a subgroup does not depend on the choice of finite generating sets, by statement (1) in Proposition 4.2. Moreover, if a finitely generated group G acts properly and cocompactly in some space, then H is a strongly quasiconvex (stable) subgroup of G if and only if H is strongly quasiconvex (stable) in X via some (any) orbit map restricted to H.

The following theorem is a direct consequence of Theorem 3.5

Theorem 4.7 (characterizing strongly quasiconvex subgroups) *Let G be a finitely generated group and H an infinite subgroup of G. Then H is strongly quasiconvex in G if and only if the lower relative divergence of G with respect to H is completely superlinear.*

We also obtain several characterizations of stable subgroups by the following theorem, whose proof can be deduced from Proposition 4.3 and Theorem 4.7.

Theorem 4.8 (characterizing stable subgroups) *Let G be a finitely generated group and H an infinite subgroup of G. Then the following are equivalent:*

(1) *H is stable in G.*

(2) *H is hyperbolic and strongly quasiconvex in G.*

(3) *H is hyperbolic and the lower relative divergence of G with respect to H is completely superlinear.*

We now study some basic results about strongly quasiconvex subgroups related to subgroup inclusion and subgroup intersection.

Lemma 4.9 [22, Proposition 9.4] *Let G be a group with a finite generating set S. Suppose xH and yK are arbitrary left cosets of subgroups of G. For each constant L there is a constant $L' = L'(G, S, xH, yK, L)$ such that in the metric space (G, d_S),*

$$N_L(xH) \cap N_L(yK) \subset N_{L'}(xHx^{-1} \cap yKy^{-1}).$$

Proposition 4.10 *Let G be a finitely generated group, A a subgroup of G, and B a subgroup of A. Then:*

(1) *If A is finitely generated and undistorted in G, and B is strongly quasiconvex in G, then B is strongly quasiconvex in A.*

(2) *If B is strongly quasiconvex in A, and A is strongly quasiconvex in G, then B is strongly quasiconvex in G.*

Proof We note that if the subgroup A is undistorted in G, then B is undistorted in A if and only if B is undistorted in G. Therefore, the above proposition is a direct result of Proposition 4.2. $\square$

Statement (1) in Proposition 4.10 can be strengthened by the following proposition.

Proposition 4.11 *Let G be a finitely generated group and A an undistorted subgroup of G. If H is a strongly quasiconvex subgroup of G, then $H_1 = H \cap A$ is a strongly quasiconvex subgroup of A. In particular, H_1 is finitely generated and undistorted in A.*

Proof Fix finite generating sets S and T for G and A respectively. Let $f\colon \Gamma(A, T) \to \Gamma(G, S)$ be a quasi-isometric embedding which is an extension of the inclusion $A \hookrightarrow G$. We will prove that H_1 is a Morse subset of $\Gamma(A, T)$. For each $K \geq 1$ and $L \geq 0$, let α be an arbitrary (K, L)–quasigeodesic in $\Gamma(A, T)$ that connects two points in H_1. Since $f\colon \Gamma(A, T) \to \Gamma(G, S)$ is a quasi-isometric embedding which is an extension of the inclusion $A \hookrightarrow G$, $f(\alpha)$ is a (K', L')–quasigeodesic in $\Gamma(G, S)$ that connects two points in H_1, and $f(\alpha)$ lies in some D–neighborhood of A in $\Gamma(G, S)$, where K', L', and D depend only on K, L, and the map f. Since H is a Morse subset in $\Gamma(G, S)$, $f(\alpha)$ also lies in some D_1–neighborhood of H in $\Gamma(G, S)$, where D_1 depends only on K' and L'. By Lemma 4.9, there is an $M = M(G, S, A, H, D, D_1)$ such that $f(\alpha)$ lies in the M–neighborhood of $H_1 = H \cap A$ in $\Gamma(G, S)$. Again, $f\colon \Gamma(A, T) \to \Gamma(G, S)$ is a quasi-isometric embedding which is an extension of the inclusion $A \hookrightarrow G$. Then α lies in some M'–neighborhood of H_1 in $\Gamma(A, T)$, where M' depends only on M and the map f. $\square$

The following proposition is a direct result of Propositions 4.10 and 4.11.

Proposition 4.12 *Let G be a finitely generated group and H_1 and H_2 strongly quasiconvex subgroups of G. Then $H_1 \cap H_2$ is strongly quasiconvex in H_1, H_2, and G.*

Quasiconvex subgroups in hyperbolic groups have the well-known property of finite height, and therefore, they have finite index in their commensurators. We now show that strongly quasiconvex subgroups in general also have these properties.

Definition 4.13 Let G be a group and H a subgroup.

(1) Conjugates $g_1 H g_1^{-1}, \ldots, g_k H g_k^{-1}$ are called *essentially distinct* if the cosets $g_1 H, \ldots, g_k H$ are distinct.

(2) H has height at most n in G if the intersection of any $(n+1)$ essentially distinct conjugates is finite. The least n for which this is satisfied is called the height of H in G.

(3) The *width* of H is the maximal cardinality of the set
$$\{g_i H : |g_i H g_i^{-1} \cap g_j H g_j^{-1}| = \infty\},$$
where $\{g_i H\}$ ranges over all collections of distinct cosets.

Similarly, given a finite collection $\mathcal{H} = \{H_1, \ldots, H_l\}$:

(1) Conjugates $g_1 H_{\sigma(1)} g_1^{-1}, \ldots, g_k H_{\sigma(k)} g_k^{-1}$ are called *essentially distinct* if the cosets $g_1 H_{\sigma(1)}, \ldots, g_k H_{\sigma(k)}$ are distinct.

(2) The finite collection $\mathcal{H}$ of subgroups of G has height at most n if the intersection of any $(n+1)$ essentially distinct conjugates is finite. The least n for which this is satisfied is called the height of $\mathcal{H}$ in G.

(3) The *width* of $\mathcal{H}$ is the maximal cardinality of the set
$$\{g_{\sigma(i)} H_{\sigma(i)} : |g_{\sigma(i)} H_{\sigma(i)} g_{\sigma(i)}^{-1} \cap g_{\sigma(j)} H_{\sigma(j)} g_{\sigma(j)}^{-1}| = \infty\},$$
where $\{g_{\sigma(i)} H_{\sigma(i)}\}$ ranges over all collections of distinct cosets.

Definition 4.14 Let G be a group and H a subgroup. The *commensurator* of H in G, denoted by $\mathrm{Comm}_G(H)$, is defined as
$$\mathrm{Comm}_G(H) = \{g \in G \mid [H : H \cap gHg^{-1}] < \infty \text{ and } [gHg^{-1} : H \cap gHg^{-1}] < \infty\}.$$

Theorem 4.15 Let $\mathcal{H} = \{H_1, \ldots, H_\ell\}$ be a finite collection of strongly quasiconvex subgroups of a finitely generated group G. Then $\mathcal{H}$ has finite height.

Proof By Proposition 3.7 there is a constant $D > 0$ such that the following holds. If $\gamma \colon [0, \infty) \to \Gamma(G, S)$ is a geodesic ray in the Cayley graph $\Gamma(G, S)$ that lies in some finite neighborhood of H_i, then there is a $C > 0$ such that $\gamma|_{[C,\infty)}$ lies in the D–neighborhood of H_i. Let $g_1 H_{\sigma(1)} g_1^{-1}, \ldots, g_k H_{\sigma(k)} g_k^{-1}$ be essential distinct conjugates with infinite intersection $\bigcap g_i H_{\sigma(i)} g_i^{-1}$.

Since the Cayley graph $\Gamma(G, S)$ is proper and $\bigcap g_i H_{\sigma(i)} g_i^{-1}$ is an infinite strongly quasiconvex subgroup, there is a geodesic ray $\gamma \colon [0, \infty) \to \Gamma(G, S)$ in the Cayley graph $\Gamma(G, S)$ that lies in some finite neighborhood of $\bigcap g_i H_{\sigma(i)} g_i^{-1}$. Therefore, for each i the ray γ lies in some finite neighborhood of $g_i H_{\sigma(i)}$. This implies that the ray $g_i^{-1} \gamma$ lies in some finite neighborhood of $H_{\sigma(i)}$. Thus, there is some $C_i > 0$ such that $g_i^{-1} \gamma|_{[C_i,\infty)}$ lies in the D–neighborhood of $H_{\sigma(i)}$. In other words, $\gamma|_{[C_i,\infty)}$ lies in the D–neighborhood of $g_i H_{\sigma(i)}$. Therefore, all left cosets $g_i H_{\sigma(i)}$ intersect some ball $B(a, D)$. But there is a uniform bound N on the number of cosets of subgroups in $\mathcal{H}$ intersecting any metric ball of radius D. Thus, the collection $\mathcal{H}$ has finite height. $\square$

Corollary 4.16 *If H is an infinite strongly quasiconvex subgroup of a finitely generated group G, then H has finite index in its commensurator $\mathrm{Comm}_G(H)$.*

We now prove that a finite collection of strongly quasiconvex subgroups has bounded packing and finite width. We first recall the concept of bounded packing subgroups in [23].

Definition 4.17 Let G be a finitely generated group and Γ a Cayley graph with respect to a finite generating set. A subgroup H has *bounded packing* in G if, for all $D \geq 0$, there exists $N \in \mathbb{N}$ such that for any collection of N distinct cosets gH in G, at least two are separated by a distance of at least D.

Similarly, a finite collection $\mathcal{H} = \{H_1, \ldots, H_\ell\}$ of subgroups of G has *bounded packing* if, for all $D \geq 0$, there exist $N \in \mathbb{N}$ such that for any collection of N distinct cosets gH_i with $H_i \in \mathcal{H}$, at least two are separated by a distance of at least D.

Lemma 4.18 *Let (X, d) be a geodesic space. Let A, B, and C be Morse subsets of X with Morse gauge μ. For each $D > 0$ there is a number D' depending on D and μ such that if A, B, and C are pairwise D–close, then the intersection $N_{D'}(A) \cap N_{D'}(B) \cap N_{D'}(C)$ is nonempty.*

Proof Let

$$x \in N_D(A) \cap N_D(B), \quad y \in N_D(B) \cap N_D(C), \quad \text{and} \quad z \in N_D(C) \cap N_D(A).$$

Let u be a point in a geodesic $[y, z]$ connecting y and z such that $d(x, u) = d(x, [y, z])$. Let γ_1 be the concatenation of two geodesics $[x, u]$ and $[u, y]$. Similarly, let γ_2 be the concatenation of two geodesics $[x, u]$ and $[u, z]$. Then γ_1 and γ_2 are both $(3, 0)$–quasigeodesics. The proof of this claim is quite elementary and the reader can also see the proof of this claim in the proof of Lemma 2.2 in [10]. Since γ_1, γ_2, and $[y, z]$ are all $(3, 0)$–quasigeodesics and A, B, and C are all μ–Morse subsets of X, there is a number D' depending on D and μ such that

$$\gamma_1 \subset N_{D'}(B), \quad \gamma_2 \subset N_{D'}(A), \quad \text{and} \quad [y, z] \subset N_{D'}(C).$$

In particular, u belongs to the intersection $N_{D'}(A) \cap N_{D'}(B) \cap N_{D'}(C)$. Therefore, this intersection is nonempty.   □

Lemma 4.19 [23, Lemma 4.2] *Suppose $H \leq G$ has height $0 < n < \infty$ in G. Choose $g \in G$ so that $gH \neq H$, and let $K = H \cap gHg^{-1}$. Then K has height less than n in H.*

Theorem 4.20 *Let H be a strongly quasiconvex subgroup of a finitely generated group G. Then H has bounded packing in G.*

The proof of the above theorem follows the same line of argument as Theorem 4.8 in [23]. At some point in the proof of Theorem 4.8 in [23], Hruska and Wise need to use the thinness of a triangle in a hyperbolic space. However, due to the lack of hyperbolicity in the group G and subgroup H in the above theorem, we need to use Lemma 4.18 instead.

Proof Fix finite generating sets S and T for G and H respectively. By Theorem 4.15, we know that the height of H in G is finite. We will prove the theorem by induction on the height. If the height of H in G is zero, then H is a finite group. Therefore, H has bounded packing by Corollary 2.6 in [23]. We now assume by induction that the theorem holds for every finitely generated group G_0 and strongly quasiconvex subgroup H_0 with the height of H_0 in G_0 less than the height of H in G.

Let $\mathcal{H}$ be a set of left cosets gH which are pairwise D–close. We are going to prove that the cardinality of $\mathcal{H}$ is bounded by a number depending on D. Translating $\mathcal{H}$ if

necessary, assume that $H \in \mathcal{H}$. Observe that if $d(gH, H) < D$ then $gH = hxH$ for some $h \in H$ and $x \in B(1, D)$. It follows that the left cosets gH intersecting $N_D(H)$ lie in at most $|B(1, D)|$ distinct H–orbits. Thus it suffices to bound the number of elements of $\mathcal{H}$ in the orbit $H(gH)$ for each fixed $g \notin H$.

If we let $K = H \cap gHg^{-1}$, then Lemma 4.19 shows that the height of K in H is less than the height of H in G. Since K is a strongly quasiconvex subgroup H, the inductive hypothesis applied to $K \leq H$ gives for each D' a number $M' = M'(D')$ such that any collection of M' distinct cosets hK in H contains a pair separated by a d_T–distance at least D'. Furthermore, the proof of Lemma 4.2 in [23] shows that there is a well-defined map $hgH \to hK$ taking left cosets of H in the orbit of gH to left cosets of K. A similar argument shows that this map is bijective.

In order to complete the proof, we will show that D–closeness of distinct cosets h_1gH and h_2gH in (G, d_S) implies D'–closeness of the corresponding cosets h_1K and h_2K in (H, d_T), for some D' depending on D. The proof of this step in Theorem 4.8 in [23] requires the thinness of a triangle. However, we need to use Lemma 4.18 for our situation. In fact, there is a D_1 depending on D such that the intersection $N_{D_1}(H) \cap N_{D_1}(h_1gH) \cap N_{D_1}(h_2gH)$ contains an element u. By Lemma 4.9, there is a number D_2 depending on D_1 but independent of the choice of $h_i \in H$ such that

$$N_{D_1}(H) \cap N_{D_1}(h_igH) \subset N_{D_2}(h_iK).$$

Therefore, $u \in N_{D_2}(h_1K) \cap N_{D_2}(h_2K)$. In other words, $d_S(h_1K, h_2K) < 2D_2$. Since H is an undistorted subgroup of G, we can conclude that $d_T(h_1K, h_2K) < D'$ for some D' depending on D, as desired. $\square$

Theorem 4.21 *Let $\mathcal{H} = \{H_1, \ldots, H_\ell\}$ be a finite collection of strongly quasiconvex subgroups of a finitely generated group G. Then $\mathcal{H}$ has bounded packing in G. Further, $\mathcal{H}$ has finite width.*

Proof First, let $\mathcal{M}$ be a collection of cosets of subgroups in $\mathcal{H}$ which are pairwise D–close. By Theorem 4.20, for each $i = 1, 2, \ldots, \ell$, there is an $N_i \geq 0$ such that the number of cosets of H_i in $\mathcal{M}$ is at most N_i. Therefore, the number of elements of $\mathcal{M}$ is at most ΣN_i. Thus, $\mathcal{H}$ has bounded packing.

Arguing as in the proof of Theorem 4.15, we see that any two conjugates of subgroups in $\mathcal{H}$ with infinite intersection have cosets uniformly close together. Therefore, $\mathcal{H}$ has finite width. $\square$

5 Morse boundaries and strong quasiconvexity

We recall basic properties of a quasiconvex subgroup in a hyperbolic group: The subgroup is also a hyperbolic group and the subgroup inclusion induces a topological embedding of the Gromov boundary of the subgroup into the Gromov boundary of the ambient group. Moreover, the image of this topological embedding is identical to the limit set of the subgroup inside the ambient group. Also, the intersection of two quasiconvex subgroups is again a quasiconvex subgroup and the limit set of the intersection of two quasiconvex subgroups is equal to the intersection of the limit sets of the two subgroups. In this section, we prove some analogous properties for strongly quasiconvex subgroups in general.

We first review the concept of Morse boundary in [10].

Definition 5.1 Let $\mathcal{M}$ be the set of all Morse gauges. We put a partial ordering on $\mathcal{M}$ such that, for two Morse gauges $N, N' \in \mathcal{M}$, we say $N \leq N'$ if and only if $N(K, L) \leq N'(K, L)$ for all K and L.

Definition 5.2 Let X be a proper geodesic space. The *Morse boundary* of X with basepoint p, denoted by $\partial_M X_p$, is defined to be the set of all equivalence classes of Morse geodesic rays in X with initial point p, where two rays $\alpha, \alpha' \colon [0, \infty) \to X$ are equivalent if there exists a constant K such that $d_X(\alpha(t), \alpha'(t)) < K$ for all $t > 0$. We denote the equivalence class of a ray α in $\partial_M X_p$ by $[\alpha]$.

On $\partial_M X_p$, we could build a topology as follows:

Consider the subset of the Morse boundary

$\partial_M^N X_p = \{x \mid \text{the class } x \text{ contains at least one } N\text{–Morse geodesic ray } \alpha \text{ with } \alpha(0) = p\}.$

We define convergence in $\partial_M^N X_p$ by $x_n \to x$ as $n \to \infty$ if and only if there exists N–Morse geodesic rays α_n with $\alpha_n(0) = p$ and $[\alpha_n] = x_n$ such that every subsequence of α_n contains a subsequence that converges uniformly on compact sets to a geodesic ray α with $[\alpha] = x$. The closed subsets F in $\partial_M^N X_p$ are those satisfying the condition

$$[\{x_n\} \subset F \text{ and } x_n \to x] \implies x \in F.$$

We equip the Morse boundary $\partial_M X_p$ with the direct limit topology

$$\partial_M X_p = \varinjlim_{\mathcal{M}} \partial_M^N X_p.$$

Let A be a subset of X with basepoint p. The *limit set* ΛA of A in $\partial_M X_p$ is the set of all points c in $\partial_M X_p$ represented by rays based at p that lie in some finite neighborhood of A.

Remark 5.3 The direct limit topology on $\partial_M X_p$ is independent of the basepoint p (see Proposition 3.5 in [10]). Therefore, we can assume the basepoint is fixed, suppress it from the notation and write $\partial_M X$. Moreover, the Morse boundary is a quasi-isometry invariant (see Proposition 3.7 in [10]). Therefore, we define the *Morse boundary* of a finitely generated group G, denoted by $\partial_M G$, as the Morse boundary of its Cayley graph. We also define the *limit set* of a subgroup H of G in $\partial_M G$ accordingly.

We define an *action* of G on $\partial_M G$ as follows. For each element g in G and $[\alpha]$ in $\partial_M G$, we define $g[\alpha] = [\beta]$, where α and β are two rays at the basepoint in some Cayley graph of G such that the Hausdorff distance between $g\alpha$ and β is finite.

Definition 5.4 Let X and Y be proper geodesic metric spaces, $p \in X$, and $p' \in Y$. We say that $f\colon \partial_M X_p \to \partial_M Y_{p'}$ is *Morse-preserving* if given an N in $\mathcal{M}$ there exists an N' in $\mathcal{M}$ such that f injectively maps $\partial_M^N X_p$ to $\partial_M^{N'} Y_{p'}$.

The following proposition shows that if a quasi-isometric embedding defines a strongly quasiconvex subspace then it induces a Morse-preserving map. This proposition is a key lemma for the proof of the fact that inclusion map of a strongly quasiconvex subgroup into a finitely generated group induces a topological embedding on Morse boundaries.

Proposition 5.5 *Let X and A be two proper geodesic spaces and A be strongly quasiconvex in X via the map f. Then for each Morse gauge N there is another Morse gauge N' such that for every N–Morse geodesic ray $\gamma\colon [0, \infty) \to A$ there is an N'–Morse geodesic ray with basepoint $f(\gamma(0))$ in X bounded Hausdorff distance from $f(\gamma)$ (ie f induces a Morse-preserving map).*

Proof Since f is a quasi-isometric embedding, $f(\gamma)$ is a (K, L)–quasigeodesic, where K and L only depend on the quasi-isometric embedding constants of f. Also, $f(\gamma)$ is Morse with Morse gauge N_1 depending only on the Morse gauge N and the map f (see the proof of statement (2) of Proposition 4.2). Let (y_n) be a sequence of points in $f(\gamma)$ such that $d(f(\gamma(0)), y_n))$ approaches infinity as n approaches infinity. For each n let α_n be a geodesic segment connecting the basepoint $f(\gamma(0))$ and y_n. Since $f(\gamma)$ is a (K, L)–Morse quasigeodesic with Morse gauge N_1, each geodesic segment α_n lies in some D_1–neighborhood of $f(\gamma)$, where D_1 only depends

on the Morse gauge N_1. Since X is a proper geodesic space, some subsequence of (α_n) converges to a ray α based at $f(\gamma(0))$. Moreover, the Hausdorff distance between α and $f(\gamma)$ is bounded above by some constant D depending only on K, L, and the Morse gauge N_1. This implies that D only depends on the Morse gauge N and the map f. Since the Hausdorff distance between $f(\gamma)$ and α is D and $f(\gamma)$ is N_1–Morse, the ray α is N'–Morse, where N' only depends on D and the Morse gauge N_1. Therefore, N' only depends on the Morse gauge N and the map f. $\square$

Theorem 5.6 *Let X and A be two proper geodesic spaces and A be strongly quasiconvex in X via the map f. Then f induces a topological embedding $\partial_M f\colon \partial_M A \to \partial_M X$ such that $\partial_M f(\partial_M A) = \Lambda f(A)$.*

Proof The fact that f induces a topological embedding $\partial_M f\colon \partial_M A \to \partial_M X$ is a direct result of Proposition 5.5 via Proposition 4.2 in [10]. However, we need to check that $\partial_M f(\partial_M A) = \Lambda f(A)$.

We first recall the construction of the map $\partial_M f$ from Proposition 4.2 in [10]. Fix a point p in A and let $q = f(p)$. By Proposition 5.5, for each Morse gauge N there is another Morse gauge N' such that for every N–Morse geodesic ray α based at p there is an N'–Morse geodesic ray β with basepoint q in X bounded Hausdorff distance from $f(\alpha)$. We define $\partial_M f([\alpha]) = [\beta]$, and then $\partial_M f\colon \partial_M A \to \partial_M X$ is a topological embedding (see the proof of Proposition 4.2 in [10]). Obviously, $\partial_M f(\partial_M A) \subset \Lambda f(A)$ by construction. We now prove the opposite inclusion.

Let c be an arbitrary element in $\Lambda f(A)$. There is a Morse geodesic ray β_1 based at q in X such that $c = [\beta_1]$ and β_1 lies in some finite neighborhood of $f(A)$. Since f is a quasi-isometric embedding, there is a quasigeodesic α_1 based at p in A such that the Hausdorff distance between $f(\alpha_1)$ and β_1 is finite. Therefore, $f(\alpha_1)$ is also a Morse quasigeodesic. This implies that α_1 is also a Morse quasigeodesic in A by statement (1) of Proposition 4.2. Also, A is a proper geodesic space. Then by a similar argument as in the proof of Proposition 5.5 there is a Morse geodesic ray α based at p in X such that the Hausdorff distance between α and α_1 is finite. Thus, the Hausdorff distance between $f(\alpha)$ and $f(\alpha_1)$ is also finite. This implies that the Hausdorff distance between $f(\alpha)$ and β_1 is finite. By the construction of $\partial_M f$ we have $\partial_M f([\alpha]) = [\beta]$ for some geodesic ray β based at q such that the Hausdorff distance between $f(\alpha)$ and β is finite. Therefore, the Hausdorff distance between β and β_1 is finite. This implies that $c = [\beta_1] = [\beta] = \partial_M f([\alpha]) \in \partial_M f(\partial_M A)$. Therefore, $\partial_M f(\partial_M A) = \Lambda f(A)$. $\square$

We now state the main theorem of this section.

Theorem 5.7 *Let G be a finitely generated group. Then:*

(1) *If H is a finitely generated strongly quasiconvex subgroup of G, then the inclusion $i\colon H \hookrightarrow G$ induces a topological embedding $\hat{i}\colon \partial_M H \to \partial_M G$ such that $\hat{i}(\partial_M H) = \Lambda H$.*

(2) *If H_1 and H_2 are finitely generated strongly quasiconvex subgroups of G, then $H_1 \cap H_2$ is strongly quasiconvex in G and $\Lambda(H_1 \cap H_2) = \Lambda H_1 \cap \Lambda H_2$.*

Proof Statement (1) is a direct result of Theorem 5.6. Therefore, we only need to prove statement (2). The fact that $H_1 \cap H_2$ is strongly quasiconvex in G is a result of Proposition 4.12. Since $H_1 \cap H_2 \subset H_1$, $\Lambda(H_1 \cap H_2) \subset \Lambda H_1$. Similarly, $\Lambda(H_1 \cap H_2) \subset \Lambda H_2$. Therefore, $\Lambda(H_1 \cap H_2) \subset \Lambda H_1 \cap \Lambda H_2$. We now let c be an arbitrary element in $\Lambda H_1 \cap \Lambda H_2$. Then there is a Morse geodesic ray γ based at e in some Cayley graph $\Gamma(G, S)$ such that $[\gamma] = c$ and γ lies in some finite neighborhoods of H_1 and H_2. Therefore, γ also lies in a finite neighborhood of $H_1 \cap H_2$ by Lemma 4.9. This implies that c is also an element in $\Lambda(H_1 \cap H_2)$. $\qquad\square$

6 Strong quasiconvexity and stability in relatively hyperbolic groups

In this section, we investigate strongly quasiconvex subgroups and stable subgroups in relatively hyperbolic groups. We first recall the concepts of coned-off Cayley graphs and relative hyperbolic groups.

Definition 6.1 Given a finitely generated group G with Cayley graph $\Gamma(G, S)$ that is equipped with the path metric and a finite collection $\mathbb{P}$ of subgroups of G, one can construct the *coned-off Cayley graph* $\widehat{\Gamma}(G, S, \mathbb{P})$ as follows: For each left coset gP, where $P \in \mathbb{P}$, add a vertex v_{gP}, called a *peripheral vertex*, to the Cayley graph $\Gamma(G, S)$ and for each element x of gP, add an edge $e(x, gP)$ of length $\frac{1}{2}$ from x to the vertex v_{gP}. This results in a metric space that may not be proper (ie closed balls need not be compact).

Definition 6.2 (relatively hyperbolic group) A finitely generated group G is *hyperbolic relative to a finite collection $\mathbb{P}$ of subgroups of G* if the coned-off Cayley graph

is δ–hyperbolic and *fine* (ie for each positive number n, each edge of the coned-off Cayley graph is contained in only finitely many circuits of length n).

Each group $P \in \mathbb{P}$ is a *peripheral subgroup* and its left cosets are *peripheral left cosets* and we denote the collection of all peripheral left cosets by Π.

We now review some known results on relatively hyperbolic subgroups including undistorted subgroups and peripheral subgroups in relatively hyperbolic groups and relations between quasigeodesics in Cayley graphs and coned-off Cayley graphs.

Theorem 6.3 [22, Theorem 1.5 and Theorem 9.1] *Let $(G, \mathbb{P})$ be a finitely generated relatively hyperbolic group and let H be a finitely generated undistorted subgroup of G. Let S be some (any) finite generating set for G. Then:*

(1) *There is a constant $A = A(S)$ such that for each geodesic c in the coned-off Cayley graph $\hat{\Gamma}(G, S, \mathbb{P})$ connecting two points of H, every G–vertex of c lies within the A–neighborhood of H with respect to the metric d_S.*

(2) *All the subgroups of H that are of the form $H \cap P^g$, where $g \in G$, $P \in \mathbb{P}$, and $|H \cap P^g| = \infty$, lie in only finitely many conjugacy classes in H. Furthermore, if $\mathbb{O}$ is a set of representatives of these conjugacy classes then $(H, \mathbb{O})$ is relatively hyperbolic.*

Theorem 6.4 [16, Corollary 1.14] *A relatively hyperbolic group $(G, \mathbb{P})$ is hyperbolic if each peripheral subgroup in $\mathbb{P}$ is hyperbolic.*

Lemma 6.5 [22, Lemma 8.8] *Let $(G, \mathbb{P})$ be a finitely generated relatively hyperbolic group with a finite generating set S. For each $K \geq 1$ and $L \geq 0$ there is an $A = A(K, L) > 0$ such that the following holds. Let α be a (K, L)–quasigeodesic in $\Gamma(G, S)$ and c a geodesic in $\hat{\Gamma}(G, S, \mathbb{P})$ with the same endpoints in G. Then each G–vertex of c lies in the A–neighborhood of some vertex of α with respect to the metric d_S.*

Lemma 6.6 [16, Lemma 4.15] *Let $(G, \mathbb{P})$ be a finitely generated relatively hyperbolic group. Then each conjugate P^g of a peripheral subgroup in $\mathbb{P}$ is strongly quasiconvex in G. Moreover, each D–neighborhood of a peripheral left coset is M–Morse in the Cayley graph of G, where the Morse gauge M only depends on D and the choice of finite generating set of G.*

The following theorem provides characterizations of strongly quasiconvex subgroups in relatively hyperbolic groups. This is the main theorem of this section.

Theorem 6.7 *Let* $(G, \mathbb{P})$ *be a finitely generated relatively hyperbolic group and* H *a finitely generated undistorted subgroup of* G. *Then the following are equivalent:*

1. *The subgroup* H *is strongly quasiconvex in* G.

2. *The subgroup* $H \cap P^g$ *is strongly quasiconvex in* P^g *for each conjugate* P^g *of a peripheral subgroup in* $\mathbb{P}$.

3. *The subgroup* $H \cap P^g$ *is strongly quasiconvex in* G *for each conjugate* P^g *of a peripheral subgroup in* $\mathbb{P}$.

Proof The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are direct results from Propositions 4.10 and 4.12 and Lemma 6.6. We now prove the implication $(3) \Rightarrow (1)$. Let S be a finite generating set of G, and let $A_0 = A_0(S)$ be the constant in statement (1) of Theorem 6.3.

Let $K \geq 1$ and $L \geq 0$ be arbitrary. Let $A = A(K, L)$ be the constant in Lemma 6.5. By Lemma 4.9 there is a constant $A_1 > 0$ such that for each $t \in B(e, A_0) \cap G$ and $P \in \mathbb{P}$,

$$N_A(tP) \cap N_{A+A_0}(H) \subset N_{A_1}(tPt^{-1} \cap H).$$

Let μ be the maximum of all Morse gauge functions of all sets having the form $N_{A_1}(tPt^{-1} \cap H)$, where $|t|_S < A_0$ and $P \in \mathbb{P}$.

Let

$$D = K(2A + 1) + L + \mu(K, L) + A_0 + A + A_1 + 1.$$

Let $\alpha \colon [a, b] \to \Gamma(G, S)$ be an arbitrary (K, L)–quasigeodesic in $\Gamma(G, S)$ that connects two points h_1 and h_2 in H. By Lemma 1.11 of [8, Chapter III.H], we can assume that α is continuous and

$$\ell(\alpha|_{[t,t']}) \leq K d_S(\alpha(t), \alpha(t')) + L.$$

We will show α lies in the D–neighborhood of H. Let c be a geodesic in $\hat{\Gamma}(G, S, \mathbb{P})$ that connects two points h_1 and h_2, and let $h_1 = s_0, s_1, \ldots, s_n = h_2$ be all the G–vertices of c. By the choice of A_0, all the G–vertices s_i lie in the A_0–neighborhood of H. Also by the choice of A, there is a $t_i \in [a, b]$ such that $d_S(s_i, \alpha(t_i)) < A$. We consider $t_0 = s_0$ and $t_n = s_n$. Then each point $\alpha(t_i)$ lies in the $(A_0 + A)$–neighborhood of H. It is sufficient to show that each $\alpha(I_i)$ lies in the D–neighborhood of H, where I_i is a closed subinterval of $[a, b]$ with endpoints t_{i-1} and t_i. We consider two cases:

Case 1 Two G–vertices s_{i-1} and s_i are adjacent in $\Gamma(G, S)$. Then the distance between the two points $\alpha(t_{i-1})$ and $\alpha(t_i)$ with respect to the metric d_S is bounded above by $2A + 1$. Therefore,

$$\ell(\alpha|_{I_i}) \leq K d_S(\alpha(t_{i-1}), \alpha(t_i)) + L \leq K(2A + 1) + L.$$

Also, $\alpha(t_i)$ lies in the $(A + A_0)$–neighborhood of H. Therefore, $\alpha(I_i)$ lies in the D–neighborhood of H by the choice of D.

Case 2 Two G–vertices s_{i-1} and s_i are not adjacent in $\Gamma(G, S)$. Thus, s_{i-1} and s_i both lie in the same peripheral left coset gP. Since $d_S(gP, H) \leq d_S(s_i, H) < A_0$, $gP = htP$ for some $h \in H$ and $|t|_S < A_0$. Observe that the endpoints of $\alpha(I_i)$ both lie in $N_A(htP) \cap N_{A+A_0}(H)$. Therefore, the endpoints of $h^{-1}\alpha(I_i)$ both lie in $N_A(tP) \cap N_{A+A_0}(H) \subset N_{A_1}(tPt^{-1} \cap H)$. Since $h^{-1}\alpha$ is also a (K, L)–quasigeodesic, $h^{-1}\alpha(I_i)$ lies entirely in the $\mu(K, L)$–neighborhood of the set $N_{A_1}(tPt^{-1} \cap H)$. Therefore, $h^{-1}\alpha(I_i)$ lies in the $(\mu(K, L) + A_1)$–neighborhood of the set H. Translating by h, we see that $\alpha(I_i)$ also lies in the $(\mu(K, L) + A_1)$–neighborhood of the set H. By the choice of D, $\alpha(I_i)$ lies in the D–neighborhood of the set H.

Therefore, the subgroup H is strongly quasiconvex in G. □

Two characterizations of stable subgroups in relatively hyperbolic groups follow immediately.

Corollary 6.8 *Let* $(G, \mathbb{P})$ *be a finitely generated relatively hyperbolic group and* H *a finitely generated undistorted subgroup of* G. *Then the following are equivalent:*

(1) *The subgroup* H *is stable in* G.

(2) *The subgroup* $H \cap P^g$ *is stable in* P^g *for each conjugate* P^g *of a peripheral subgroup in* $\mathbb{P}$.

(3) *The subgroup* $H \cap P^g$ *is stable in* G *for each conjugate* P^g *of a peripheral subgroup in* $\mathbb{P}$.

Proof We can easily see that (1) implies (2) and (1) implies (3). In fact, if the subgroup H is stable in G, then H is strongly quasiconvex in G, and H is hyperbolic (see Theorem 4.8). Therefore, $H \cap P^g$ is strongly quasiconvex in H, P^g, and G by Proposition 4.12 and Lemma 6.6. In particular, $H \cap P^g$ is hyperbolic because H is a hyperbolic group. Therefore, $H \cap P^g$ is stable in P^g and G by Theorem 4.8. The equivalence between (2) and (3) can also be seen easily from Theorems 4.8 and 6.7.

We now prove that (3) implies (1). By the hypothesis, $H \cap P^g$ is strongly quasiconvex

in G for each conjugate P^g of a peripheral subgroup in $\mathbb{P}$. Therefore, H is strongly quasiconvex in G by Theorem 6.7. Also, $H \cap P^g$ is hyperbolic for each conjugate P^g of a peripheral subgroup in $\mathbb{P}$. Therefore, the subgroup H is also hyperbolic by statement (2) of Theorem 6.3 and Theorem 6.4. Therefore, the subgroup H is stable in G by Theorem 4.8. □

7 Strong quasiconvexity, stability in right-angled Coxeter groups and related lower relative divergence

In this section, we investigate strongly quasiconvex subgroups and stable subgroups in right-angled Coxeter groups. More precisely, we characterize strongly quasiconvex (stable) special subgroups in two-dimensional right-angled Coxeter groups. Also, for each integer $d \geq 2$, we construct a right-angled Coxeter group together with a nonstable strongly quasiconvex subgroup whose lower relative divergence is exactly polynomial of degree d. These are also the first examples of groups together with nonhyperbolic subgroups whose lower relative divergences are polynomial of degree d ($d \geq 2$).

7.1 Some background in right-angled Coxeter groups

This subsection provides background on right-angled Coxeter groups.

Definition 7.1 Given a finite, simplicial graph Γ, the associated *right-angled Coxeter group* G_Γ has generating set S the vertices of Γ, and relations $s^2 = 1$ for all s in S and $st = ts$ whenever s and t are adjacent vertices.

Let S_1 be a subset of S. The subgroup of G_Γ generated by S_1 is a right-angled Coxeter group G_{Γ_1}, where Γ_1 is the induced subgraph of Γ with vertex set S_1 (ie Γ_1 is the union of all edges of Γ with both endpoints in S_1). The subgroup G_{Γ_1} is called a *special subgroup* of G_Γ. Any of its conjugates is called a *parabolic subgroup* of G_Γ.

Definition 7.2 Given a finite, simplicial graph Γ, the associated *Davis complex* Σ_Γ is a cube complex constructed as follows. For every k–clique $T \subset \Gamma$, the special subgroup G_T is isomorphic to the direct product of k copies of Z_2. Hence, the Cayley graph of G_T is isomorphic to the 1–skeleton of a k–cube. The Davis complex Σ_Γ has 1–skeleton the Cayley graph of G_Γ, where edges are given unit length. Additionally, for each k–clique $T \subset \Gamma$ and coset gG_T, we glue a unit k–cube to $gG_T \subset \Sigma_\Gamma$. The Davis complex Σ_Γ is a CAT(0) cube complex and the group G_Γ acts properly and cocompactly on the Davis complex Σ_Γ (see [15]).

Remark 7.3 (see [15]) Let Γ be a finite simplicial graph with vertex set S and Γ_1 the induced subgraph of Γ generated by some subset T of S. Then the Davis complex Σ_{Γ_1} embeds isometrically in the Davis complex Σ_Γ. Moreover, the Cayley graph $\Sigma_{\Gamma_1}^{(1)}$ of the group G_{Γ_1} embeds isometrically in the Cayley graph $\Sigma_\Gamma^{(1)}$ of the group G_Γ.

We also remark that a Davis complex Σ_Γ is a two-dimensional cube complex if and only if the defining graph Γ is triangle-free and contains at least one edge.

Definition 7.4 Let Σ be a CAT(0) cube complex. In Σ, we consider the equivalence relation on the set of midcubes of cubes generated by the rule that two midcubes are related if they share a face. A *hyperplane H* is the union of the midcubes in a single equivalence class. We define the *support* of a hyperplane H, denoted by $N(H)$, to be the union of cubes which contain midcubes of H.

Remark 7.5 Let Γ be a finite simplicial graph with vertex set S and Σ_Γ the associated Davis complex. Each hyperplane in Σ_Γ separates Σ_Γ into two convex sets. It follows that the distance between a pair of vertices with respect to the metric d_S equals the number of hyperplanes in Σ_Γ separating those vertices.

For a generator v, let e_v denote the edge from the basepoint 1 to the vertex v. Any edge in Σ_Γ determines a unique hyperplane, namely the hyperplane containing the midpoint of that edge. Denote by H_v the hyperplane containing the midpoint of e_v.

For a cube in Σ_Γ, all of the parallel edges are labeled by the same generator v. It follows that all of the edges crossing a hyperplane H have the same label v, and we say this a hyperplane of *type v*. Obviously, if two hyperplanes of types v_1 and v_2 cross, then v_1 and v_2 commute. Since G_Γ acts transitively on edges labeled v, a hyperplane is of type v if and only if it is a translate of the standard hyperplane H_v. Obviously, the star subgroup $G_{\mathrm{st}(v)}$ is the stabilizer of the hyperplane H_v, and $G_{\mathrm{st}(v)}$ can also be viewed as the vertices of the support $N(H_v)$ of H_v. Therefore, the subgroup $g G_{\mathrm{st}(v)} g^{-1}$ is the stabilizer of the hyperplane $g H_v$, and $g G_{\mathrm{st}(v)}$ can also be viewed as the vertices of the support $N(g H_v)$ of $g H_v$.

7.2 Strongly quasiconvex special subgroup and stable special subgroup characterizations in right-angled Coxeter groups

In this subsection, we establish characterizations of strongly quasiconvex special subgroups and stable special subgroups in two-dimensional right-angled Coxeter groups in terms of defining graphs. As a consequence, we give a criterion on simplicial, triangle-

free graphs that guarantees that the associated right-angled Coxeter groups possess Morse boundaries which are not totally disconnected.

We first develop some results in right-angled Coxeter groups that help establish the characterizations of strongly quasiconvex (stable) special subgroups in two-dimensional right-angled Coxeter groups.

Lemma 7.6 *Let* Γ *be a simplicial graph with vertex set* S*. Let* K *be a special subgroup of* G_Γ *generated by some subset* S_1 *of* S *and* Γ_1 *the subgraph of* Γ *induced by* S_1*. For each* $g \in G_\Gamma$*,* γ *is a shortest path in* $\Sigma_\Gamma^{(1)}$ *that connects* g *to some point in* K *if and only if* γ *is a geodesic in* $\Sigma_\Gamma^{(1)}$ *and each hyperplane crossed by some edge of* γ *does not intersect the Cayley graph* $\Sigma_{\Gamma_1}^{(1)}$ *of* K*.*

Proof We first assume γ is a shortest path in $\Sigma_\Gamma^{(1)}$ that connects g to some point h in K. Obviously, γ must be a geodesic in $\Sigma_\Gamma^{(1)}$ connecting h and g. Let n be the length of γ. Then the geodesic γ along the direction from h to g is the concatenation $e_1 e_2 \cdots e_n$ of n edges, where each edge e_i is labeled by some vertex s_i in S. Therefore, $g = h(s_1 s_2 \cdots s_n)$.

Assume for purposes of contradiction that there is a hyperplane H_i crossed by the edge e_i that intersects $\Sigma_{\Gamma_1}^{(1)}$. We choose the smallest number i with that property. In particular, $s_i \in S_1 \subset K$. We claim that for each $j < i$ the hyperplane H_j crossed by the edge e_j of γ intersects H_i. Indeed, each hyperplane H_j (for $j < i$) does not intersect $\Sigma_{\Gamma_1}^{(1)}$ by the choice of i. Moreover, since $\Sigma_{\Gamma_1}^{(1)}$ is connected there is a path α in $\Sigma_{\Gamma_1}^{(1)}$ that connects h and a point u in $H_i \cap \Sigma_{\Gamma_1}^{(1)}$. Let β be the path in H_i that connects u to the midpoint of e_i. Then each hyperplane H_j (for $j < i$) must intersect the concatenation $\alpha\beta$. However, each hyperplane H_j (for $j < i$) does not intersect α because it does not intersect $\Sigma_{\Gamma_1}^{(1)}$. This implies that each hyperplane H_j (for $j < i$) intersects H_i. Therefore, s_j commutes with s_i for each $j < i$. Thus,

$$g = h(s_1 s_2 \cdots s_{i-1} s_i s_{i+1} \cdots s_n) = (hs_i)(s_1 s_2 \cdots s_{i-1} s_{i+1} \cdots s_n).$$

This equality shows there is path of length $n - 1$ that connects g to the point $hs_i \in K$. This is a contradiction. Therefore, each hyperplane crossed by some edge of γ does not intersect the Cayley graph $\Sigma_{\Gamma_1}^{(1)}$.

We now assume that γ is a geodesic in $\Sigma_\Gamma^{(1)}$ and each hyperplane crossed by some edge of γ does not intersect the Cayley graph $\Sigma_{\Gamma_1}^{(1)}$. Let γ' be an arbitrary path in $\Sigma_\Gamma^{(1)}$ that connects g to some point h_1 in K. Because $\Sigma_{\Gamma_1}^{(1)}$ is connected, there is a path α' in $\Sigma_{\Gamma_1}^{(1)}$

that connects h and h_1. Therefore, hyperplanes crossed by edges of γ intersect the concatenation $\alpha'\gamma'$. Also, these hyperplanes do not intersect α' since they do not intersect $\Sigma^{(1)}_{\Gamma_1}$. Therefore, all these hyperplanes must intersect γ'. This implies $\ell(\gamma) \leq \ell(\gamma')$. Therefore, γ is a shortest path in $\Sigma^{(1)}_{\Gamma}$ that connects g to some point in K.   □

Lemma 7.7 *Let Γ be a simplicial graph with vertex set S. Let K be a special subgroup of G_Γ generated by some subset S_1 of S. Let σ be an induced 4–cycle with two pairs of nonadjacent vertices (a_1, a_2) and (b_1, b_2) such that a_1 and a_2 both lie in S_1 but b_1 does not. Then $d_S((b_1b_2)^n, K) = 2n$ and $d_S((b_1b_2)^n b_1, K) = 2n+1$.*

Proof We will only prove that $d_S((b_1b_2)^n, K) = 2n$. The argument for the remaining equality is identical. Let Γ_1 the subgraph of Γ induced by S_1. Let $g = (b_1b_2)^n$ and choose the path γ in $\Sigma^{(1)}_{\Gamma}$ connecting the identity element e and g that reads the word $(b_1b_2)^n$. Since the two vertices b_1 and b_2 are not adjacent, the path γ is a geodesic in $\Sigma^{(1)}_{\Gamma}$ by Remark 7.3. Let H_i be the hyperplane crossed by the ith edge of γ in the direction from e to g. Since the hyperplane H_1 is labeled by $b_1 \notin S_1$, H_1 does not intersect $\Sigma^{(1)}_{\Gamma_1}$. Also, the geodesic γ is labeled by the noncommuting elements b_1 and b_2 alternately. This implies that H_i does not intersect H_j for $i \neq j$. In particular, H_i does not intersect H_1 for $i \neq 1$. This implies that H_i also does not intersect $\Sigma^{(1)}_{\Gamma_1}$ for $i \neq 1$. By Lemma 7.6, the geodesic γ is a shortest path that connects $(b_1b_2)^n$ and some point in K. Therefore, $d_S((b_1b_2)^n, K) = \ell(\gamma) = 2n$. Using a similar argument we also obtain the equality $d_S((b_1b_2)^n b_1, K) = 2n+1$. □

Proposition 7.8 *Let Γ be a simplicial graph with vertex set S. Let K be a special subgroup of G_Γ generated by some subset S_1 of S. Assume that there is an induced 4–cycle σ with two pairs of nonadjacent vertices (a_1, a_2) and (b_1, b_2) such that a_1 and a_2 both lie in S_1 but b_1 does not. Then K is not strongly quasiconvex in G_Γ.*

Proof By Theorem 4.7, it is sufficient to prove that the lower relative divergence of G_Γ with respect to K is not completely superlinear. Let $\{\sigma^n_\rho\}$ be the lower relative divergence of $\Sigma^{(1)}_{\Gamma}$ with respect to K. We claim that for each $n \geq 2$ and $\rho \in (0, 1]$,

$$\sigma^n_\rho(r) \leq (4n+2)r \quad \text{for each } r > 1.$$

Indeed, for each $r > 1$ choose an integer m in $[r, 2r]$. If $m = 2k$ for some integer k, we choose $x = (b_1b_2)^k$. Otherwise, $m = 2k+1$ for some integer k and we choose $x = (b_1b_2)^k b_1$. Then $d_S(x, K) = m$ by Lemma 7.7. We choose $y = (a_1a_2)^{mn}x$. Then

$$d_S(y, K) = d_S((a_1a_2)^{mn}x, K) = d_S(x, (a_1a_2)^{-mn}K) = d_S(x, K) = m.$$

Since x commutes with both a_1 and a_2, $y = x(a_1a_2)^{mn}$. This implies that there is a geodesic γ connecting x and y, and γ traces the word $(a_1a_2)^{mn}$. Obviously, $d_S(x, y) = \ell(\gamma) = 2mn$.

By construction, each vertex of γ is an element of the form $x(a_1a_2)^i$ or $x((a_1a_2)^ia_1)$ in G_Γ. Again, x commutes with both a_1 and a_2. Therefore,

$$d_S(x(a_1a_2)^i, K) = d_S((a_1a_2)^i x, K) = d_S(x, (a_1a_2)^{-i} K) = d_S(x, K) = m > r$$

and $d_S(x((a_1a_2)^ia_1), K) = m > r$ similarly. Therefore, the path γ lies outside the r–neighborhood of K.

Since $r \leq d_S(x, K) = m \leq 2r$, there is a path γ_1 in $\Sigma_\Gamma^{(1)}$ connecting x and some point u in $\partial N_r(K)$ such that γ_1 lies outside the r–neighborhood of K and its length is bounded above by r. In particular, $d_S(x, u) \leq r$. Similarly, there is a path γ_2 in $\Sigma_\Gamma^{(1)}$ connecting y and some point v in $\partial N_r(K)$ such that γ_2 lies outside the r–neighborhood of K and its length is bounded above by r. In particular, $d_S(x, u) \leq r$. Therefore,

$$d_S(u, v) \geq d_S(x, y) - d_S(x, u) - d_S(y, v) \geq 2mn - r - r \geq (2n - 2)r \geq nr.$$

We observe that $\overline{\gamma} = \gamma_1 \cup \gamma \cup \gamma_2$ is the path outside the r–neighborhood of K that connects the two points u and v in $\partial N_r(K)$. Also,

$$\ell(\overline{\gamma}) = \ell(\gamma_1) + \ell(\gamma) + \ell(\gamma_2) \leq r + 2mn + r \leq (4n + 2)r.$$

Therefore, $\sigma_\rho^n(r) \leq (4n + 2)r$ for each $r > 1$. This implies that the lower relative divergence of G_Γ with respect to K is not completely superlinear. Therefore, K is not strongly quasiconvex in G_Γ by Theorem 4.7. $\square$

Proposition 7.9 *Let Γ be a simplicial, triangle-free graph with vertex set S, and let K be a subgroup of G_Γ generated by some subset S_1 of S. We assume that if S_1 contains two nonadjacent vertices of an induced 4–cycle σ, then S_1 contains all vertices of σ. If K is an infinite subgroup of G_Γ, then the lower relative divergence of G_Γ with respect to K is at least quadratic.*

Proof Let $\{\sigma_\rho^n\}$ be the lower relative divergence of $\Sigma_\Gamma^{(1)}$ with respect to K. We claim that for each $n \geq 3$ and $\rho \in (0, 1]$,

$$\sigma_\rho^n(r) \geq (r - 1)(\rho r - 1) \quad \text{for each } r > 0.$$

In fact, if $\sigma_\rho^n(r) = \infty$, then the above inequality is true obviously. Otherwise, let γ be an arbitrary path outside $N_{\rho r}(K)$ that connects two points x and y in $\partial N_r(K)$ such

that $d_S(x, y) \geq nr$. Let γ_1' be a shortest geodesic in $\Sigma_\Gamma^{(1)}$ that connects x and some point k_1 in K. Similarly, let γ_2' be a shortest geodesic in $\Sigma_\Gamma^{(1)}$ that connects y and some point k_2 in K. Then $d_S(x, k_1) = d_S(y, k_2) = r$ obviously. This implies that

$$d_S(k_1, k_2) \geq d_S(x, y) - d_S(x, k_1) - d_S(y, k_2) \geq (n-2)r \geq r.$$

Let Γ_1 the subgraph of Γ induced by S_1. We connect k_1 and k_2 by a geodesic $\alpha = e_1 e_2 \cdots e_m$ of length m in the Cayley graph $\Sigma_{\Gamma_1}^{(1)}$ of K. Obviously $m \geq r$ and each edge e_i of α is labeled by some vertex s_i in S_1. Let H_i be the hyperplane crossed by the edge e_i of α. Then each H_i must intersect $\gamma_1' \cup \gamma \cup \gamma_2'$. However, H_i does not intersect $\gamma_1' \cup \gamma_2'$ by Lemma 7.6. Therefore, each hyperplane H_i must intersect some edge f_i of γ.

Let $h_0, h_1, h_2, \ldots, h_m$ be the vertices of α such that h_{i-1} and h_i are endpoints of e_i. For $1 \leq i \leq m$, let g_i be the endpoint of the edge f_i of γ that lies in the component of $\Sigma_\Gamma - H_i$ containing h_i. Let α_i be a the geodesic in the support of H_i that connects h_i and g_i. By the convexity of the graph $\Sigma_{\Gamma_1}^{(1)}$ in the graph $\Sigma_\Gamma^{(1)}$ each α_i is the concatenation of two geodesics β_i and β_i' such that β_i lies entirely in $\Sigma_{\Gamma_1}^{(1)}$, and β_i' intersects $\Sigma_{\Gamma_1}^{(1)}$ only at the common endpoint $h_i' \in K$ of β_i and β_i'.

We set $m_i = \ell(\beta_i')$ for $1 \leq i \leq m$. Then $m_i \geq \rho r$ since g_i lies outside the ρr–neighborhood of K. We assume that $\beta_i' = e_1^i e_2^i \cdots e_{m_i}^i$. Then each edge e_j^i of β_i' is labeled by an element in $\mathrm{lk}(s_i)$ and the first edge e_1^i is labeled by a vertex not in S_1. We claim that for each $1 \leq i \leq m-1$ there are at least $(m_i - 1)$ hyperplanes crossed by β_i' intersecting the subpath γ_i of γ connecting g_i and g_{i+1}. Since Γ is triangle-free, no pair of different vertices in $\mathrm{lk}(s_i)$ are adjacent. This implies that no pairs of hyperplanes crossed by two different edges of β_i' intersect. Since the first edge e_1^i of β_i' is labeled by a vertex not in S_1, the hyperplane crossed by e_1^i does not intersect $\Sigma_{\Gamma_1}^{(1)}$. Therefore, no subsequent hyperplane crossed by β_i' intersects $\Sigma_{\Gamma_1}^{(1)}$ either. We now consider the possibility that a hyperplane crossed by β_i' intersects β_{i+1}'.

If the hyperplane $H_{e_1^i}$ crossed by the first edge e_1^i of β_i' does not intersect β_{i+1}', no subsequent hyperplane crossed by β_i' intersects β_{i+1}' either (otherwise, some hyperplane crossed by an edge of β_i' intersects the hyperplane $H_{e_1^i}$, which is a contradiction). Therefore, all m_i hyperplanes crossed by β_i' intersect the subpath γ_i of γ. If the hyperplane crossed by e_1^i intersects β_{i+1}', then the edge e_1^i is labeled by a vertex a in $\mathrm{lk}(s_i) \cap \mathrm{lk}(s_{i+1})$. Since Γ is triangle-free, two consecutive vertices s_i and s_{i+1} are never adjacent. We claim that the hyperplane crossed by the second edge e_2^i of β_i' does not intersect β_{i+1}'. Otherwise, the second edge e_2^i of β_i' is labeled by a vertex b other

than a in $\mathrm{lk}(s_i) \cap \mathrm{lk}(s_{i+1})$. Therefore, the four points a, b, s_i, and s_{i+1} are all vertices of an induced 4–cycle σ in Γ. Moreover, two nonadjacent vertices s_i and s_{i+1} both lie in S_1 but a does not. This is a contradiction. Therefore, the hyperplane crossed by the second edge e_2^i of β_i' does not intersect β_{i+1}'. This also implies that the hyperplanes crossed by edges e_j^i (where $2 \le j \le m$) of β_i' do not intersect β_{i+1}'. Thus, all these hyperplanes must intersect the subpath γ_i of γ. Therefore, there are exactly $(m_i - 1)$ hyperplanes crossed by β_i' intersecting the subpath γ_i of γ.

For $1 \le i \le m - 1$, the length of the subpath γ_i of γ is at least $(m_i - 1)$ because there are at least $(m_i - 1)$ hyperplanes crossed by β_i' intersecting the subpath γ_i. Thus

$$\ell(\gamma) \ge \sum_{i=1}^{m-1} \ell(\gamma_i) \ge (m-1)(m_i - 1) \ge (r-1)(\rho r - 1).$$

Therefore, $\sigma_\rho^n(r) \ge (r-1)(\rho r - 1)$ for each $r > 0$. This implies that the lower relative divergence of G_Γ with respect to K is at least quadratic. $\square$

The following theorem establishes characterizations of strongly quasiconvex subgroups in two-dimensional right-angled Coxeter groups. This theorem is a direct result of Propositions 7.8 and 7.9 and Theorem 4.7. This is also the main theorem of this section.

Theorem 7.10 *Let Γ be a simplicial, triangle-free graph with vertex set S, and let K be a subgroup of G_Γ generated by some subset S_1 of S. Then the following conditions are equivalent:*

(1) *The subgroup K is strongly quasiconvex in G_Γ.*

(2) *If S_1 contains two nonadjacent vertices of an induced 4–cycle σ, then S_1 contains all vertices of σ.*

(3) *Either $|K| < \infty$ or the lower relative divergence of G_Γ with respect to K is at least quadratic.*

Question 7.11 Is the lower relative divergence of a CAT(0) group with respect to an infinite strongly quasiconvex subgroup at least quadratic?

The following corollary is a direct result of Theorem 7.10. The corollary establishes a characterization of stable subgroups in two-dimensional right-angled Coxeter groups.

Corollary 7.12 *Let Γ be a simplicial, triangle-free graph with vertex set S, and let K be a subgroup of G_Γ generated by some subset S_1 of S. Then the following conditions are equivalent:*

(1)   *The subgroup K is stable in G_Γ.*

(2)   *The set S_1 does not contain a pair of nonadjacent vertices of an induced 4–cycle in Γ.*

Proof   Assume that S_1 does not contain any pair of nonadjacent vertices of an induced 4–cycle of Γ. Then the subgroup K is hyperbolic (see Corollary 12.6.3 in [15]). Also, the subgroup K is strongly quasiconvex in G_Γ by Theorem 7.10. Therefore, K is stable subgroup by Theorem 4.8.

We now assume that S_1 contains a pair of nonadjacent vertices of an induced 4–cycle σ of Γ. We will prove that K is not stable in G_Γ. If K is not strongly quasiconvex in G_Γ, then K is not stable in G_Γ by Theorem 4.8. Otherwise, S_1 contains all vertices of the induced 4–cycle σ by Theorem 7.10. This implies that K is not hyperbolic by Corollary 12.6.3 in [15]. Thus, K is not stable in G_Γ.   □

Question 7.13   Can we characterize all strongly quasiconvex (stable) subgroups of right-angled Coxeter groups using defining graphs?

We guess that the recent work of Abbott, Behrstock, and Durham [1] can help us characterize all stable subgroups of right-angled Coxeter groups using defining graphs. However, characterizing all strongly quasiconvex subgroups of right-angled Coxeter groups using defining graphs is still a difficult question.

The following corollary gives a criterion on two-dimensional right-angled Coxeter groups to possess Morse boundaries which are not totally disconnected. We remark that the first example of a right-angled Coxeter group whose Morse boundary is not totally disconnected was constructed by Behrstock in [5]. The following corollary generalizes his example in some sense.

Corollary 7.14   *If the simplicial, triangle-free graph Γ contains an induced loop σ of length greater than 4 such that the vertex set of σ does not contain a pair of nonadjacent vertices of an induced 4–cycle in Γ, then the Morse boundary of the right-angled Coxeter group G_Γ is not totally disconnected.*

Proof   Since the vertex set S_1 of the loop σ satisfies the hypothesis of Corollary 7.12, the subgroup G_σ generated by S_1 is a stable subgroup in G_Γ. This implies that the Morse boundary of G_σ is topologically embedded in the Morse boundary of G_Γ, by Theorem 5.6. Also, G_σ is a two-dimensional hyperbolic orbifold group. Therefore, the Morse boundary of G_σ is also the Gromov boundary of G_σ, which is a topological

circle. This implies that the Morse boundary of G_Γ also contains a circle, and therefore, it is not totally disconnected. $\qquad\square$

Conjecture 7.15 *The Morse boundary of a right-angled Coxeter group G_Γ is not totally disconnected if and only if the defining graph Γ contains an induced loop σ of length greater than 4 such that the vertex set of σ does not contain a pair of nonadjacent vertices of an induced 4–cycle in Γ.*

7.3 Higher relative lower divergence in right-angled Coxeter groups

In this subsection, we construct right-angled Coxeter groups together with nonstable strongly quasiconvex subgroups whose lower relative divergences are arbitrary polynomials of degrees at least 2. We remark that the author in [35] also constructed various examples of right-angled Coxeter groups whose lower relative divergences with respect to some subgroups are arbitrary polynomials of degrees at least 2. However, all subgroups in those examples are stable and in this subsection we work on cases of nonstable subgroups.

We first establish a connection between the lower relative divergence of certain pairs of right-angled Coxeter groups and the divergence of certain geodesics in their Davis complexes. This connection is a key ingredient for the main examples in this subsection.

Lemma 7.16 *Let Γ be a simplicial graph with vertex set S, and let u and v be two nonadjacent vertices of Γ. Let Ω be a new graph obtained by coning off the two points u and v of Γ with a new vertex t, and let $\overline{S} = S \cup \{t\}$. Let x and y be endpoints of a path γ in $\Sigma_\Omega^{(1)} - N_1(G_\Gamma)$. If g_1 and g_2 are two elements in G_Γ such that $d_{\overline{S}}(x, G_\Gamma) = d_{\overline{S}}(x, g_1)$ and $d_{\overline{S}}(y, G_\Gamma) = d_{\overline{S}}(y, g_2)$, then $g_1^{-1}g_2$ lies in the subgroup I generated by u and v.*

Proof We can assume that x and y both lie in G_Ω. Let γ_1 be a geodesic in $\Sigma_\Omega^{(1)}$ connecting x and g_1, and let γ_2 be a geodesic in $\Sigma_\Omega^{(1)}$ connecting y and g_2. We can assume that γ_1 along the direction from g_1 to x is the concatenation $\gamma_1 = e_1 e_2 \cdots e_m$ of edges of $\Sigma_\Omega^{(1)}$. Similarly, γ_2 along the direction from g_2 to y is the concatenation $\gamma_2 = f_1 f_2 \cdots f_n$ of edges of $\Sigma_\Omega^{(1)}$. We claim that the two edges e_1 and f_1 are both crossed by the hyperplane which is labeled by t.

We first recall that G_Γ is the subgroup generated by S and $\overline{S} - S = \{t\}$. Also, γ_1 is a shortest geodesic that connects x and G_Γ. Then the first edge e_1 of γ_1 is labeled by t and we denote by H the hyperplane crossed by e_1. By Remark 7.5 the vertex set

of the support of H is $g_1 G_{\mathrm{st}(t)}$. Also, each element in $g_1 G_{\mathrm{st}(t)}$ has distance 0 or 1 from G_Γ. This implies that $H \cap \Sigma_\Omega^{(1)}$ lies in the 1–neighborhood of G_Γ. In particular, H does not intersect γ. Also, H does not intersect the Cayley graph $\Sigma_\Gamma^{(1)}$ of G_Γ. Thus, H intersects the path γ_2. Moreover, H must intersect the first edge f_1 of γ_2 because $H \cap \Sigma_\Omega^{(1)}$ lies in the 1–neighborhood of G_Γ. This fact implies that g_1 and g_2 are both vertices of the support of H. In other words, g_1 and g_2 lie in the same left coset of $G_{\mathrm{st}(t)}$ or $g_1^{-1} g_2$ lies in $G_{\mathrm{st}(t)}$. Also, $g_1^{-1} g_2$ lies in G_Γ, and $G_\Gamma \cap G_{\mathrm{st}(t)} = I$, where I is the subgroup generated by u and v. Thus, $g_1^{-1} g_2$ lies in the subgroup I. $\square$

Lemma 7.17 *Let Γ be a simplicial graph with vertex set S, and let u and v be two nonadjacent vertices of Γ. Let Ω be a new graph obtained by coning off the two points u and v of Γ with a new vertex t, and let $\overline{S} = S \cup \{t\}$. Let I be the subgroup generated by u and v. Then $d_{\overline{S}}(tx, G_\Gamma) = d_S(x, I) + 1$ for each x in $\Sigma_\Gamma^{(1)}$.*

Proof We assume that x lies in G_Γ and let $n = d_S(x, I)$. Let γ be a geodesic of length n in $\Sigma_\Gamma^{(1)}$ that connects x to some point h in I. We can express $\gamma = e_1 e_2 \cdots e_n$ as a concatenation of edges in $\Sigma_\Gamma^{(1)}$, where each edge e_i is labeled by some element $s_i \in S$. Then $x = h(s_1 s_2 \cdots s_n)$. Let $\gamma_1 = t\gamma$. Then γ_1 is a geodesic that connects th and tx. Moreover, we can express $\gamma_1 = f_1 f_2 \cdots f_n$, where each edge f_i is the translation of the edge e_i of γ by t. In particular, each edge f_i is also labeled by s_i. Since $th = ht$, there is an edge e_1' labeled by t with endpoints h and th. We construct $\overline{\gamma} = e_1' \cup \gamma_1$ and we claim that $\overline{\gamma}$ is a shortest geodesic that connects tx and G_Γ.

We first prove that $\overline{\gamma}$ is a geodesic in $\Sigma_\Omega^{(1)}$. Since the inclusion $\Sigma_\Gamma^{(1)} \hookrightarrow \Sigma_\Omega^{(1)}$ is an isometric embedding, γ is also a geodesic in $\Sigma_\Omega^{(1)}$. Thus, $\gamma_1 = t\gamma$ is a geodesic in $\Sigma_\Omega^{(1)}$. Since γ_1 is labeled by elements in S and the first edge e_1' of $\overline{\gamma}$ is labeled by $t \notin S$, the hyperplane crossed by e_1' does not intersect γ_1. Thus, $\overline{\gamma}$ is a geodesic in $\Sigma_\Omega^{(1)}$.

We now prove that $\overline{\gamma}$ is a shortest geodesic that connects tx and G_Γ. By [Lemma 7.6](), it is sufficient to prove that each hyperplane crossed by some edge of $\overline{\gamma}$ does not intersect the Cayley graph $\Sigma_\Gamma^{(1)}$ of G_Γ. Since G_Γ is the subgroup generated by S and the hyperplane H crossed by the first edge e_1' of $\overline{\gamma}$ is labeled by t, H does not intersect $\Sigma_\Gamma^{(1)}$. Assume for purposes of contradiction that a hyperplane H_i crossed by some edge f_i of $\overline{\gamma}$ intersects $\Sigma_\Gamma^{(1)}$. We choose i to be the smallest number with that property. In particular, H_i must intersect H, and therefore, s_i commutes with t. This implies that $s_i = u$ or $s_i = v$. Therefore, s_i is an element in I.

By the choice of i, each hyperplane H_j crossed by the edge f_j (where $j < i$) does not intersect $\Sigma_\Gamma^{(1)}$. Therefore, each H_j must intersect H_i. This implies that s_i commutes

with all s_j for $j < i$. Therefore,

$$x = h(s_1 s_2 \cdots s_{i-1} s_i s_{i+1} \cdots s_n) = (h s_i)(s_1 s_2 \cdots s_{i-1} s_{i+1} \cdots s_n).$$

This implies that there is a path in $\Sigma_\Gamma^{(1)}$ with length $n-1$ that connects x to the element $h s_i \in I$, which is a contradiction. Thus, each hyperplane crossed by some edge of $\overline{\gamma}$ does not intersect the Cayley graph $\Sigma_\Gamma^{(1)}$ of G_Γ. Therefore, the path $\overline{\gamma}$ is a shortest geodesic that connects tx and G_Γ. This implies that

$$d_{\overline{S}}(tx, G_\Gamma) = \ell(\overline{\gamma}) = \ell(\gamma_1) + 1 = \ell(\gamma) + 1 = d_S(x, I) + 1. \qquad \square$$

Proposition 7.18 *Let Γ be a simplicial graph with vertex set S, and let u and v be two nonadjacent vertices of Γ. Let Ω be a new graph obtained by coning off the two points u and v of Γ with a new vertex t, and let $\overline{S} = S \cup \{t\}$. Let α be the bi-infinite geodesic containing the identity e labeled by u and v alternately. Then*

$$\mathrm{Div}_\alpha^{\Sigma_\Omega^{(1)}} \preceq \mathrm{div}(G_\Omega, G_\Gamma) \preceq \mathrm{Div}_\alpha^{\Sigma_\Gamma^{(1)}},$$

where $\mathrm{Div}_\alpha^{\Sigma_\Omega^{(1)}}$ and $\mathrm{Div}_\alpha^{\Sigma_\Gamma^{(1)}}$ are the geodesic divergences of α in $\Sigma_\Omega^{(1)}$ and $\Sigma_\Gamma^{(1)}$ respectively.

Proof Let I be the subgroup generated by u and v. Then the geodesic α is the Cayley graph of I in $\Sigma_\Omega^{(1)}$ and $\Sigma_\Gamma^{(1)}$. Therefore, the lower relative divergences of $\Sigma_\Omega^{(1)}$ and $\Sigma_\Omega^{(1)}$ with respect to the path α are equivalent to the lower relative divergences of these spaces with respect to I. Also, by Remark 2.14, the lower relative divergence of a geodesic space with respect to a periodic bi-infinite geodesic is equivalent to the divergence of the geodesic in the space. Therefore, it is sufficient to prove

$$\mathrm{div}(\Sigma_\Omega^{(1)}, I) \preceq \mathrm{div}(\Sigma_\Omega^{(1)}, G_\Gamma) \preceq \mathrm{div}(\Sigma_\Gamma^{(1)}, I).$$

Let $\{\sigma_\rho^n\}$ be the lower relative divergence of $\Sigma_\Omega^{(1)}$ with respect to G_Γ, let $\{\overline{\sigma_\rho^n}\}$ be the lower relative divergence of $\Sigma_\Omega^{(1)}$ with respect to I, and let $\{\overline{\overline{\sigma_\rho^n}}\}$ be the lower relative divergence of $\Sigma_\Gamma^{(1)}$ with respect to I. We will prove that for each $n \geq 2$ and $\rho \in (0, 1]$

$$\overline{\sigma_\rho^n}(r) \leq \sigma_\rho^n(r) \leq \overline{\overline{\sigma_\rho^{2n}}}(r) + 2r \quad \text{for each } r > 1.$$

We first prove the left inequality. If $\sigma_\rho^n(r) = \infty$, the left inequality is true obviously. Otherwise, let x and y be two arbitrary points in $\partial N_r(G_\Gamma)$ such that there is a path outside $N_r(G_\Gamma)$ in $\Sigma_\Omega^{(1)}$ that connects x and y and $d_{\overline{S}}(x, y) \geq nr$. Let g_1 and g_2 be two elements in G_Γ such that $d_{\overline{S}}(x, G_\Gamma) = d_{\overline{S}}(x, g_1)$ and $d_{\overline{S}}(y, G_\Gamma) = d_{\overline{S}}(y, g_2)$. Then g_1 and g_2 both lie in the same left coset gI of the subgroup I by Lemma 7.16.

Since the element g in G_Ω acts isometrically on $\Sigma_\Omega^{(1)}$,

$$\mathrm{div}(\Sigma_\Omega^{(1)}, gI) = \mathrm{div}(\Sigma_\Omega^{(1)}, I) = \{\overline{\sigma_\rho^n}\}.$$

It is obvious that

$$d_{\overline{S}}(x, gI) = d_{\overline{S}}(x, G_\Gamma) = r \quad \text{and} \quad d_{\overline{S}}(y, gI) = d_{\overline{S}}(y, G_\Gamma) = r.$$

Also, any path that lies outside some s–neighborhood of G_Γ also lies outside the s–neighborhood of gI because gI is a subset of G_Γ. Therefore, $\overline{\sigma_\rho^n}(r) \le \sigma_\rho^n(r)$. This implies that $\mathrm{div}(\Sigma_\Omega^{(1)}, I) \preceq \mathrm{div}(\Sigma_\Omega^{(1)}, G_\Gamma)$.

We now prove the second inequality, $\sigma_\rho^n(r) \le \overline{\overline{\sigma_\rho^{2n}}}(r) + 2r$ for each $r > 1$. In the case that $\overline{\overline{\sigma_\rho^{2n}}}(r) = \infty$, the inequality is true obviously. Otherwise, let x_1 and y_1 be two arbitrary points in $\partial N_r(I) \subset \Sigma_\Gamma^{(1)}$ such that there is a path outside $N_r(I)$ connecting x_1 and y_1 and $d_S(x_1, y_1) \ge (2n)r$. Since the inclusion $\Sigma_\Gamma^{(1)} \hookrightarrow \Sigma_\Omega^{(1)}$ is an isometric embedding, $d_{\overline{S}}(x_1, y_1) = d_S(x_1, y_1) \ge (2n)r$.

Let β be an arbitrary path outside $N_{\rho r}(I)$ in $\Sigma_\Gamma^{(1)}$ connecting x_1 and y_1. Then the path $\beta_1 = t\beta$ connects the two points tx_1 and ty_1. By [Lemma 7.17](), β_1 lies outside the $(\rho r + 1)$–neighborhood of G_Γ in $\Sigma_\Omega^{(1)}$, and

$$d_{\overline{S}}(tx_1, G_\Gamma) = d_S(x_1, I) + 1 = r + 1 \quad \text{and} \quad d_{\overline{S}}(ty_1, G_\Gamma) = d_S(y_1, I) + 1 = r + 1.$$

This implies there is a geodesic β_2 with length 1 that lies outside the r–neighborhood of G_Γ in $\Sigma_\Omega^{(1)}$ and β_2 connects tx_1 with some point $x_2 \in N_r(G_\Gamma)$. Similarly, there is a geodesic β_3 with length 1 that lies outside the r–neighborhood of G_Γ in $\Sigma_\Omega^{(1)}$ and β_3 connects ty_1 with some point $y_2 \in N_r(G_\Gamma)$. Moreover,

$$\begin{aligned}
d_{\overline{S}}(x_2, y_2) &\ge d_{\overline{S}}(tx_1, ty_1) - d_{\overline{S}}(tx_1, x_2) - d_{\overline{S}}(ty_1, y_2) \\
&\ge d_{\overline{S}}(x_1, y_1) - d_{\overline{S}}(tx_1, x_2) - d_{\overline{S}}(ty_1, y_2) \\
&\ge 2nr - 1 - 1 \ge nr.
\end{aligned}$$

Also, $\overline{\beta} = \beta_2 \cup \beta_1 \cup \beta_3$ is the path outside the ρr–neighborhood of G_Γ connecting x_2 and y_2. Therefore,

$$\sigma_\rho^n(r) \le \ell(\overline{\beta}) \le \ell(\beta_2) + \ell(\beta_1) + \ell(\beta_3) \le 1 + \ell(\beta) + 1 \le \ell(\beta) + 2r.$$

This implies $\sigma_\rho^n(r) \le \overline{\overline{\sigma_\rho^{2n}}}(r) + 2r$. Therefore, $\mathrm{div}(\Sigma_\Omega^{(1)}, G_\Gamma) \preceq \mathrm{div}(\Sigma_\Gamma^{(1)}, I)$. Thus,

$$\mathrm{Div}_\alpha^{\Sigma_\Omega^{(1)}} \preceq \mathrm{div}(G_\Omega, G_\Gamma) \preceq \mathrm{Div}_\alpha^{\Sigma_\Gamma^{(1)}}. \qquad \square$$

We are now ready to construct the main examples for this subsection and we start with defining graphs. More precisely, for each integer $d \ge 2$, a graph Ω_d and its

subgraph Γ_d are constructed as in Figure 1. We remark that the graph Γ_d was originally constructed by Dani and Thomas [14] to study divergence of right-angled Coxeter groups and the graph Ω_d is a variation of Γ_d. However, we are going to use these pairs of graphs to construct corresponding pairs of right-angled Coxeter groups with desired lower relative divergence.

Proposition 7.19 [34, Proposition 3.19] *For each $d \geq 2$, let Γ_d be the graph in Figure 1. Let α_d be a bi-infinite geodesic containing e and labeled by $\cdots a_d b_d a_d b_d \cdots$. Then the divergence of α_d in $\Sigma_{\Gamma_d}^{(1)}$ is equivalent to a polynomial of degree d.*

We remark that the above proposition was proved in [34] but the proof was based mostly on the work of Dani and Thomas in [14]. Similarly, we also obtain an analogous proposition as follows.

Proposition 7.20 *For each $d \geq 2$, let Ω_d be the graph in Figure 1. Let α_d be a bi-infinite geodesic containing e and labeled by $\cdots a_d b_d a_d b_d \cdots$. Then the divergence of α_d in $\Sigma_{\Omega_d}^{(1)}$ is equivalent to a polynomial of degree d.*

Proof We observe that Γ_d is an induced subgraph of Ω_d. Therefore, the graph $\Sigma_{\Gamma_d}^{(1)}$ embeds isometrically in the graph $\Sigma_{\Omega_d}^{(1)}$. This implies that for each $r > 0$ any path in $\Sigma_{\Gamma_d}^{(1)}$ that stays outside the ball centered at the identity element e with radius r in $\Sigma_{\Gamma_d}^{(1)}$ also stays outside the ball with the same center and radius in $\Sigma_{\Omega_d}^{(1)}$. Therefore, the divergence of α_d in $\Sigma_{\Omega_d}^{(1)}$ is dominated by the divergence of α_d in $\Sigma_{\Gamma_d}^{(1)}$. This implies that the divergence of α_d in $\Sigma_{\Omega_d}^{(1)}$ is dominated by r^d by Proposition 7.19.

We now prove that the divergence of α_d in $\Sigma_{\Omega_d}^{(1)}$ dominates r^d. We observe that Ω_d is an induced subgraph of Γ_{d+2}. Using the same proof as that in the first paragraph, the divergence of α_d in $\Sigma_{\Omega_d}^{(1)}$ dominates the divergence of α_d in $\Sigma_{\Gamma_{d+2}}^{(1)}$. Therefore, it is sufficient to prove that the divergence of α_d in $\Sigma_{\Gamma_{d+2}}^{(1)}$ dominates r^d. In fact, for each $r > 0$, let γ be an arbitrary path connecting $\alpha_d(-r)$ and $\alpha_d(r)$ that lies outside the ball centered at e with radius r. Let α be the ray obtained from α_d containing $\alpha_d(r)$ with initial point e. Then α lies in the support of a hyperplane crossed by an edge labeled by b_{d+1}. Since the hyperplane H crossed by the first edge of α intersects γ, there is a geodesic β in the support of H connecting e and some point x in γ. Let γ_1 be the subpath of γ connecting x and $\alpha_d(r)$. Then the length of γ_1 is at least $(1/2^{d(d+1)})r^d$ by the proof of Proposition 5.3 in [14]. Therefore, the length of γ is also at least $(1/2^{d(d+1)})r^d$. This implies that the divergence of α_d in $\Sigma_{\Gamma_{d+2}}^{(1)}$ dominates r^d. Therefore, the divergence of α_d in $\Sigma_{\Omega_d}^{(1)}$ dominates r^d. $\square$
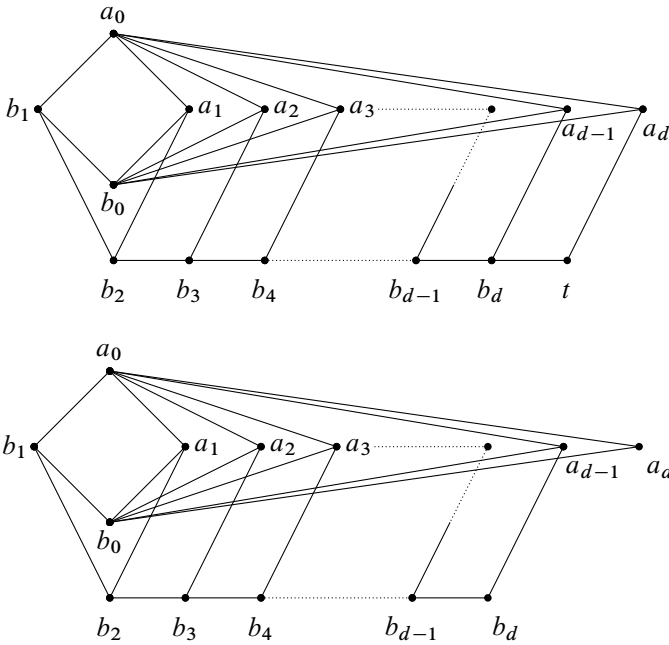
Figure 1: The graph Ω_d (top) is constructed from the graph Γ_d (bottom) by coning off two nonadjacent vertices a_d and b_d by a new vertex t.

We now state the main theorem of this subsection.

Theorem 7.21 For each $d \geq 2$, we construct a graph Ω_d and subgraph Γ_d as in Figure 1. Then G_{Γ_d} is a nonstable strongly quasiconvex subgroup of G_{Ω_d} and the lower relative divergence of G_{Ω_d} with respect to G_{Γ_d} is equivalent to a polynomial of degree d.

Proof By Propositions 7.18, 7.19, and 7.20, the lower relative divergence of G_{Ω_d} with respect to G_{Γ_d} is equivalent to a polynomial of degree d. In particular, G_{Γ_d} is a strongly quasiconvex subgroup of G_{Ω_d} by Theorem 4.7. However, the subgroup G_{Γ_d} is not hyperbolic because the graph Γ_d contains an induced 4–cycle. This implies that G_{Γ_d} is not a stable subgroup of G_{Ω_d}. $\qquad\square$

8 Strong quasiconvexity, stability, and lower relative divergence in right-angled Artin groups

In this section, we prove that the two notions of strong quasiconvexity and stability are equivalent in the right-angled Artin group A_Γ (except for the case of finite-index

subgroups). We also characterize nontrivial strongly quasiconvex subgroups of infinite index in A_Γ (ie nontrivial stable subgroups in A_Γ) by quadratic lower relative divergence. These results strengthen the work of Koberda, Mangahas, and Taylor [26] on characterizing purely loxodromic subgroups of right-angled Artin groups.

8.1 Some background in right-angled Artin groups

Definition 8.1 Given a finite simplicial graph Γ, the associated *right-angled Artin group* A_Γ has generating set S the vertices of Γ, and relations $st = ts$ whenever s and t are adjacent vertices.

Let S_1 be a subset of S. The subgroup of A_Γ generated by S_1 is a right-angled Artin group A_{Γ_1}, where Γ_1 is the induced subgraph of Γ with vertex set S_1 (ie Γ_1 is the union of all edges of Γ with both endpoints in S_1). The subgroup A_{Γ_1} is called a *special subgroup* of A_Γ.

A *reduced word* for a group element g in A_Γ is a minimal-length word in the free group $F(S)$ representing g.

Definition 8.2 Let Γ_1 and Γ_2 be two graphs. The *join* of Γ_1 and Γ_2 is a graph obtained by connecting every vertex of Γ_1 to every vertex of Γ_2 by an edge.

Let J be an induced subgraph of Γ which decomposes as a nontrivial join. We call A_J a *join subgroup* of A_Γ. A reduced word w in A_Γ is called a *join word* if w represents an element in some join subgroup. If β is a subword of w, we will say that β is a *join subword* of w when β is itself a join word.

Definition 8.3 Let Γ be a simplicial, finite, connected graph such that Γ does not decompose as a nontrivial join. A group element g in A_Γ is *loxodromic* if g is not conjugate into a join subgroup. If every nontrivial group element in a subgroup H of A_Γ is loxodromic, then H is *purely loxodromic*.

Definition 8.4 Let Γ be a finite simplicial graph with vertex set S. Let T be a torus of dimension $|S|$ with edges labeled by the elements of S. Let X_Γ denote the subcomplex of T consisting of all faces whose edge labels span a complete subgraph in Γ (or equivalently, mutually commute in A_Γ). We call X_Γ the *Salvetti complex*.

Remark 8.5 The fundamental group of X_Γ is A_Γ. The universal cover $\widetilde{X}_\Gamma$ of X_Γ is a CAT(0) cube complex with a free, cocompact action of A_Γ. Obviously, the 1–skeleton $\widetilde{X}_\Gamma^{(1)}$ of $\widetilde{X}_\Gamma$ is the Cayley graph of A_Γ with respect to the generating set S. Consequently, reduced words in A_Γ correspond to geodesics in $\widetilde{X}_\Gamma^{(1)}$, which we call

combinatorial geodesics. We refer to distance in $\widetilde{X}_\Gamma^{(1)}$ as *combinatorial distance*. If Γ_1 is an induced subgraph of a Γ, then there is a natural isometric embedding of the 1–skeleton $\widetilde{X}_{\Gamma_1}^{(1)}$ of $\widetilde{X}_{\Gamma_1}$ into the 1–skeleton $\widetilde{X}_\Gamma^{(1)}$ of $\widetilde{X}_\Gamma$.

Theorem 8.6 [26, Theorem 1.1, Theorem 5.2, and Corollary 6.2] *Let Γ be a simplicial, finite, connected graph such that Γ does not decompose as a nontrivial join. Let H be a finitely generated subgroup of A_Γ. Then the following are equivalent:*

(1) *H is a purely loxodromic subgroup.*

(2) *H is stable.*

(3) *There exists a positive number $N = N(H)$ such that for any reduced word w representing $h \in H$, and any join subword w' of w, we have $\ell(w') \leq N$.*

The following corollary is a direct consequence of the above theorem.

Corollary 8.7 *Let Γ be a simplicial, finite, connected graph with vertex set S such that Γ does not decompose as a nontrivial join. Let H be a nontrivial finitely generated purely loxodromic subgroup. Then there is a positive constant M such that every geodesic in the Cayley graph $\widetilde{X}_\Gamma^{(1)}$ connecting points in H lies in the M–neighborhood of H.*

The properties of hyperplanes of $\widetilde{X}_\Gamma$ in the following remark were observed in [6] by Behrstock and Charney.

Remark 8.8 Each hyperplane in $\widetilde{X}_\Gamma$ separates $\widetilde{X}_\Gamma$ into two convex sets. It follows that the combinatorial distance between a pair of vertices equals the number of hyperplanes in $\widetilde{X}_\Gamma$ separating those vertices.

For a generator v, let e_v denote the edge from the basepoint 1 to the vertex v. Any edge in $\widetilde{X}_\Gamma$ determines a unique hyperplane, namely the hyperplane containing the midpoint of that edge. Denote by H_v the hyperplane containing the midpoint of e_v.

For a cube in $\widetilde{X}_\Gamma$, all of the parallel edges are labeled by the same generator v. It follows that all of the edges crossing a hyperplane H have the same label v, and we call this a hyperplane of *type v*. Since A_Γ acts transitively on edges labeled v, a hyperplane is of type v if and only if it is a translate of the standard hyperplane H_v.

For a vertex v of the graph Γ let $\mathrm{lk}(v)$ denote the subgraph of Γ spanned by the vertices adjacent to v and let $\mathrm{st}(v)$ denote the subgraph spanned by v and $\mathrm{lk}(v)$. Obviously, $A_{\mathrm{st}(v)}$ is a join subgroup and $A_{\mathrm{lk}(v)}$ is a subgroup of $A_{\mathrm{st}(v)}$. We call such a subgroup $A_{\mathrm{st}(v)}$ a *star subgroup*.

Lemma 8.9 [6, Lemma 3.1] *Let $H_1 = g_1 H_v$ and $H_2 = g_2 H_w$. Then:*

(1) *H_1 intersects H_2 if and only if v and w commute and $g_1^{-1} g_2 \in A_{\mathrm{lk}(v)} A_{\mathrm{lk}(w)}$.*

(2) *There is a hyperplane H_3 intersecting both H_1 and H_2 if and only if there is a u in $\mathrm{st}(v) \cap \mathrm{st}(w)$ such that $g_1^{-1} g_2 \in A_{\mathrm{lk}(v)} A_{\mathrm{lk}(u)} A_{\mathrm{lk}(w)}$.*

8.2 Lower relative divergence of a right-angled Artin group with respect to a purely loxodromic subgroup

In this section, we compute the lower relative divergence of a right-angled Artin group with respect to a purely loxodromic subgroup (ie a stable subgroup). We first prove the quadratic upper bound for the lower relative divergence of a right-angled Artin group with respect to a loxodromic subgroup and we need the following lemmas.

Lemma 8.10 *Let Γ be a simplicial, finite, connected graph with vertex set S such that Γ does not decompose as a nontrivial join. Let H be a nontrivial, finitely generated purely loxodromic subgroup of A_Γ. There is a positive number K such that for each element g in A_Γ and each pair of commuting generators (s_1, s_2) in S,*

$$d_S(g s_1^i s_2^j, H) \geq \frac{|i| + |j|}{K} - |g|_S - 1.$$

Proof By Theorem 8.6, there is a positive integer N such that for any reduced word w representing $h \in H$, and any join subword w' of w, we have $\ell(w') \leq N$. Let $K = N + 1$ and we will prove that

$$d_S(g s_1^i s_2^j, H) \geq \frac{|i| + |j|}{K} - |g|_S - 1.$$

Let $m = d_S(g s_1^i s_2^j, H)$. Then there is an element g_1 in A_Γ with $|g_1|_S = m$ and an h in H such that $h = g s_1^i s_2^j g_1$. Since $s_1^i s_2^j$ is an element in some join subgroup of A_Γ and $|g_1|_S = m$, the element h can be represented by a reduced word w that is a product of at most $(|g|_S + 1 + m)$ join subwords. Also, the length of each join subword of w is bounded above by N. Therefore, the length of w is bounded above by $N(|g|_S + m + 1)$. Also,

$$\ell(w) \geq |s_1^i s_2^j|_S - |g_1|_S - |g|_S \geq |i| + |j| - m - |g|_S.$$

This implies that

$$|i| + |j| - m - |g|_S \leq N(|g|_S + m + 1).$$

Therefore,

$$d_S(g s_1^i s_2^j, H) = m \geq \frac{|i| + |j|}{N + 1} - |g|_S - \frac{N}{N + 1} \geq \frac{|i| + |j|}{K} - |g|_S - 1. \qquad \square$$

Lemma 8.11 *Let Γ be a simplicial, finite, connected graph with vertex set S such that Γ does not decompose as a nontrivial join. Let H be a nontrivial, finitely generated purely loxodromic subgroup of A_Γ, and let h be an arbitrary element in H. There is a number $L \geq 1$ such that for each positive integer $m \geq L^2$ and generator s in S there is a path α outside the $(m/L-L)$–neighborhood of H connecting s^m and hs^m with length bounded above by Lm.*

Proof Let $M = \operatorname{diam}\Gamma$, let K be the positive integer as in [Lemma 8.10](#), and let $k = |h|_S$. Let $L = 2(k+1)(M+1) + K + k + M + 1$. Choose a reduced word

$$w = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_k^{\epsilon_k}, \quad \text{where } s_i \in S \text{ and } \epsilon_i \in \{-1, 1\},$$

that represents the element h. For each $i \in \{1, 2, \ldots, k\}$ let t_i be a vertex in $\operatorname{st}(s_i)$ and $w_i = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_i^{\epsilon_i}$. Then the length of each word w_i is bounded above by k, $w_{i+1} = w_i s_i^{\epsilon_i}$, and $w_k = w$.

We now construct a path α_0 outside the $(m/L-L)$–neighborhood of H connecting s^m and $w_1 t_1^m$ with length bounded above by $2(M+1)m$. Since $M = \operatorname{diam}\Gamma$, we can choose a positive integer $n \leq M$ and $n+1$ generators $u_0, u_1, \ldots, u_n$ in S such that the following conditions hold:

(1) $u_0 = s$ and $u_n = t_1$.

(2) u_j and u_{j+1} commute for $j \in \{0, 1, 2, \ldots, n-1\}$.

For each $j \in \{0, 1, 2, \ldots, n-1\}$ let β_j be a path connecting u_j^m and u_{j+1}^m of length $2m$ with vertices

$$u_j^m, \ u_j^m u_{j+1}, \ u_j^m u_{j+1}^2, \ \ldots, \ u_j^m u_{j+1}^m, \ u_j^{m-1} u_{j+1}^m, \ u_j^{m-2} u_{j+1}^m, \ \ldots, \ u_{j+1}^m.$$

By [Lemma 8.10](#), the above vertices must lie outside the $(m/K-1)$–neighborhood of H. Therefore, these vertices also lies outside the $(m/L-L)$–neighborhood of H. Therefore, β_j is a path outside the $(m/L-L)$–neighborhood of H connecting u_j^m and u_{j+1}^m. Since $w_1 t_1^m = s_1^{\epsilon_1} t_1^m = t_1^m s_1^{\epsilon_1}$, we can connect t_1^m and $w_1 t_1^m$ by an edge β_n labeled by s_1. Let $\alpha_0 = \beta_0 \cup \beta_1 \cup \cdots \cup \beta_n$. Then it is obvious that the path α_0 outside the $(m/L-L)$–neighborhood of H connecting s^m and $w_1 t_1^m$ has length bounded above by $2(M+1)m$.

By similar constructions as above, for each $i \in \{1, 2, \ldots, k-1\}$ there is a path α_i outside the $(m/L-L)$–neighborhood of H connecting $w_i t_i^m$ and $w_{i+1} t_{i+1}^m$ with length bounded above by $2(M+1)m$. We can also construct a path α_k outside

the $(m/L-L)$–neighborhood of H connecting ht_k^m and hs^m with length bounded above by $2Mm$. Let $\alpha = \alpha_0 \cup \alpha_1 \cup \cdots \cup \alpha_k$. Then it is obvious that the path α outside the $(m/L-L)$–neighborhood of H connecting s^m and hs^m has length bounded above by $2(k+1)(M+1)m$. By the choice of L we observe that the length of α is also bounded above by Lm. $\square$

Proposition 8.12 *Let Γ be a simplicial, finite, connected graph with vertex set S such that Γ does not decompose as a nontrivial join. Let H be a nontrivial, finitely generated purely loxodromic subgroup of A_Γ. Then the lower relative divergence of A_Γ with respect to H is at most quadratic.*

Proof Let h be an arbitrary group element in H and $L \geq 1$ a constant as in Lemma 8.11. Since each cyclic subgroup in a CAT(0) group is undistorted (see Corollary 4.8 and Theorem 4.10 in [8, Chapter III.Γ]), there is a positive integer L_1 such that

$$|h^k|_S \geq \frac{|k|}{L_1} - L_1 \quad \text{for each integer } k.$$

Let $\{\sigma_\rho^n\}$ be the lower relative divergence of $\widetilde{X}_\Gamma^{(1)}$ with respect to H. We will prove that the function $\sigma_\rho^n(r)$ is bounded above by some quadratic function for each $n \geq 2$ and $\rho \in (0, 1]$.

Choose a positive integer $m \in [L(L+r), 2L(L+r)]$ and a generator s in S. Then there is a path α_0 outside the $(m/L-L)$–neighborhood of H connecting s^m and hs^m with length bounded above by Lm. It is obvious that the path α_0 also lies outside the r–neighborhood of H by the choice of m. Choose a positive integer k which lies between $L_1(nr + 8L(L+r) + L_1)$ and $L_1(nr + 8L(L+r) + L_1 + 1)$. Let $\alpha = \alpha_0 \cup h\alpha_0 \cup h^2\alpha_0 \cup \cdots \cup h^{k-1}\alpha_0$. Then α is a path outside the r–neighborhood of H connecting s^m and $h^k s^m$ with length bounded above by kLm. By the choice of k and m, the length of α is bounded above by $2L_1 L^2(L+r)(nr + 8L(L+r) + L_1 + 1)$.

Since $r \leq d_S(s^m, H) \leq m$, there is a path γ_1 outside $N_r(H)$ connecting s^m and some point $x \in \partial N_r(H)$ such that the length of γ_1 is bounded above by m. By the choice of m, the length of γ_1 is also bounded above by $2L(L+r)$. Similarly, there is a path γ_2 outside $N_r(H)$ connecting $h^k s^m$ and some point $y \in \partial N_r(H)$ such that the length of γ_2 is bounded above by $2L(L+r)$. Let $\overline{\alpha} = \gamma_1 \cup \alpha \cup \gamma_2$. Then $\overline{\alpha}$ is a path outside $N_r(H)$ connecting x and y and the length of $\overline{\alpha}$ is bounded above by $2L_1 L^2(L+r)(nr + 8L(L+r) + L_1 + 1) + 4L(L+r)$. Therefore, for each $\rho \in (0, 1]$,

$$d_{\rho r}(x, y) \leq 2L_1 L^2(L+r)(nr + 8L(L+r) + L_1 + 1) + 4L(L+r).$$

Also,

$$d_S(x, y) \geq d_S(s^m, h^k s^m) - d_S(s^m, x) - d_S(h^k s^m, y)$$
$$\geq (|h^k|_S - 2m) - 2L(L+r) - 2L(L+r) \geq \frac{k}{L_1} - L_1 - 8L(L+r)$$
$$\geq (nr + 8L(L+r)) - 8L(L+r) \geq nr.$$

Thus, for each $\rho \in (0, 1]$,

$$\sigma_\rho^n(r) \leq 2L_1 L^2(L+r)(nr + 8L(L+r) + L_1 + 1) + 4L(L+r).$$

This implies that the lower relative divergence of A_Γ with respect to H is at most quadratic. $\square$

We now establish the quadratic lower bound for the lower relative divergence of a right-angled Artin group with respect to a loxodromic subgroup.

Proposition 8.13 *Let Γ be a simplicial, finite, connected graph with vertex set S such that Γ does not decompose as a nontrivial join. Let H be a nontrivial, finitely generated purely loxodromic subgroup of A_Γ. Then the lower relative divergence of A_Γ with respect to H is at least quadratic.*

Proof By Corollary 8.7, there is a positive integer M such that every geodesic in the Cayley graph $\widetilde{X}_\Gamma^{(1)}$ connecting points in H lies in the M–neighborhood of H. By Theorem 8.6, there is another positive integer N such that for any reduced word w representing $h \in H$, and any join subword w' of w, we have $\ell(w') \leq N$. Let $\{\sigma_\rho^n\}$ be the lower relative divergence of $\widetilde{X}_\Gamma^{(1)}$ with respect to H. We will prove that for each $n \geq 9$ and $\rho \in (0, 1]$,

$$\sigma_\rho^n(r) \geq \left(\frac{r-1}{3N+1}\right)(\rho r - 3N) - 2r \quad \text{for each } r > \frac{2M+3N+2}{\rho}.$$

Let u and v be an arbitrary pair of points in $\partial N_r(H)$ such that $d_r(u, v) < \infty$ and $d_S(u, v) \geq nr$. Let γ be an arbitrary path that lies outside the ρr–neighborhood of H connecting u and v. We will prove that the length of γ is bounded below by $((r-1)/(3N+1))(\rho r - 3N) - 2r$.

Let γ_1 be a geodesic of length r in $\widetilde{X}_\Gamma^{(1)}$ connecting u and some point x in H. Let γ_2 be another geodesic of length r in $\widetilde{X}_\Gamma^{(1)}$ connecting v and some point y in H. Let α be a geodesic in $\widetilde{X}_\Gamma^{(1)}$ connecting x and y. Then α lies in the M–neighborhood of H. Choose a positive integer m such that $r < (3N+1)m + 1 < 2r$.

Since $d_S(x, y) \geq d_S(u, v) - 2r \geq (n-2)r \geq 7r$, there is a subpath α_1 with length $(3N+1)m+1$ of α such that $\alpha_1 \cap (B(x, 2r) \cup B(y, 2r)) = \varnothing$. Also, the lengths of γ_1 and γ_2 are both r. This implies that $(\gamma_1 \cup \gamma_2) \cap N_r(\alpha_1) = \varnothing$. Since $\gamma \cap N_{\rho r}(H) = \varnothing$ and $\alpha_1 \subset N_M(H)$, we can conclude that $\gamma \cap N_{\rho r - M}(\alpha_1) = \varnothing$. Also, $\rho r - M > \rho r / 2$. Thus, $\gamma \cap N_{\rho r / 2}(\alpha_1) = \varnothing$. Let $\overline{\gamma} = \gamma_1 \cup \gamma \cup \gamma_2$. Then $\overline{\gamma} \cap N_{\rho r / 2}(\alpha_1) = \varnothing$.

We assume that $\alpha_1 = e_0 w_1 e_1 w_2 \cdots e_m w_m$, where each e_i is an edge labeled by some generator a_i in S, each w_i is a subpath of α_1 of length exactly $3N$. For each $i \in \{0, 1, 2, \ldots, m\}$, let H_i be the hyperplane intersecting e_i, and let v_i be a point in $H_i \cap \overline{\gamma}$. For each $i \in \{1, 2, \ldots, m\}$, let β_i be the subpath of $\overline{\gamma}$ connecting v_{i-1} and v_i.

If there is some hyperplane of type b that intersects two hyperplanes H_{i-1} and H_i for some $i \in \{1, 2, \ldots, m\}$, then $e_{i-1} w_i$ corresponds to a word w that represents an element in $A_{\mathrm{lk}(a_{i-1})} A_{\mathrm{lk}(b)} A_{\mathrm{lk}(a_i)}$ by [Lemma 8.9](). Also, the length of w is $3N+1$. Then w is a product of three join words and so one of the join words has length greater than N. This contradicts the choice of N. Therefore, no hyperplane intersects both H_{i-1} and H_i for each $i \in \{1, 2, \ldots, m\}$. Also, $\overline{\gamma}$ lies outside the $(\rho r / 2)$–neighborhood of α_1. The number of hyperplanes that intersect $H_{i-1} \cup H_i$ is bounded below by $2(\rho r / 2) = \rho r$. Moreover, each of these hyperplanes either intersects w_i or β_i. We note that the number of hyperplanes that intersect w_i is exactly $\ell(w_i) = 3N$ since w_i is a geodesic. This implies that the number of hyperplanes that intersect β_i is bounded below by $\rho r - 3N$. Thus,

$$\ell(\beta_i) \geq \rho r - 3N.$$

Therefore,

$$\ell(\gamma) = \ell(\overline{\gamma}) - 2r \geq \sum_{i=1}^{m} \ell(\beta_i) - 2r \geq m(\rho r - 3N) - 2r \geq \left(\frac{r-1}{3N+1}\right)(\rho r - 3N) - 2r.$$

Thus,

$$\sigma_\rho^n(r) \geq \left(\frac{r-1}{3N+1}\right)(\rho r - 3N) - 2r \quad \text{for each } r > \frac{2M + 3N + 2}{\rho},$$
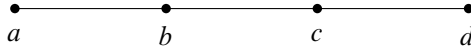
or the lower relative divergence of G with respect to H is at least quadratic. □

The following theorem is obtained from Propositions [8.12]() and [8.13]().

Theorem 8.14 *Let Γ be a simplicial, finite, connected graph with vertex set S such that Γ does not decompose as a nontrivial join. Let H be a nontrivial, finitely generated purely loxodromic subgroup of A_Γ. Then the lower relative divergence of A_Γ with respect to H is exactly quadratic.*

We now construct an example of a right-angled Artin group A_Γ together with a subgroup H that is not virtually cyclic in which the lower relative divergence of the pair (A_Γ, H) is exactly quadratic.

Corollary 8.15 *Let Γ be the following graph:*

$$a \qquad\qquad b \qquad\qquad c \qquad\qquad d$$

Let A_Γ be the associated right-angled Artin group, and let H be the subgroup of A_Γ generated by the two elements ada and dad. Then H is a free subgroup of rank 2, and the lower relative divergence of A_Γ with respect to H is exactly quadratic.

Proof It is obvious that the subgroup K generated by the two elements a and d is a free group of rank 2. Since H is a subgroup of K generated by the two elements ada and dad, it is also a free group of rank 2. It is obvious that any join subword of a reduced word in A_Γ representing a nontrivial group element in H is an element in the set $\{a, a^{-1}, a^2, a^{-2}, d, d^{-1}, d^2, d^{-2}\}$. Therefore, H is a loxodromic subgroup by Theorem 8.6. This implies that the lower relative divergence of A_Γ with respect to H is exactly quadratic by the main theorem. $\qquad\qquad\square$

8.3 Main theorem

Lemma 8.16 *Let K be a simplicial, finite graph (K is not necessarily connected) and Γ the coned-off graph of K with cone point v. If H is a strongly quasiconvex subgroup of the right-angled Artin group A_Γ, then H is trivial or H has finite index in A_Γ.*

Proof We assume that H is nontrivial and we will prove that H has finite index in A_Γ. Since $A_\Gamma = A_K \times \langle v \rangle$, it is sufficient to prove that $H \cap A_K$ has finite index in A_K and v^n lies in H for some $n \neq 0$. We first assume for purposes of contradiction that $H \cap \langle v \rangle$ is a trivial group. Since v commutes with all vertices of Γ, it commutes with all elements in H. Therefore, $v^n H v^{-n} = H$ for all n. Also, $v^i H \neq v^j H$ for $i \neq j$. This is a contradiction by Theorem 4.15. Therefore, v^n lies in H for some $n \neq 0$.

We now assume again for purposes of contradiction that $H \cap A_K$ has infinite index in A_K. Then there is an infinite sequence (g_n) of distinct elements in A_K such that $g_i(H \cap A_K) \neq g_j(H \cap A_K)$ for $i \neq j$. Therefore, $g_i^{-1} g_j$ is not a group element in $H \cap A_K$ for $i \neq j$. This implies that $g_i^{-1} g_j$ is not a group element in H since g_i and g_j are already elements of A_K. Thus, we see that $g_i H \neq g_j H$ for $i \neq j$. Since v^n lies in H and v^n commutes with all g_i, the infinite cyclic group

generated by v^n is a subgroup of each $g_i H g_i^{-1}$. Therefore, $\bigcap g_i H g_i^{-1}$ is an infinite subgroup. This again contradicts Theorem 4.15. Therefore, $H \cap A_K$ has finite index in A_K. This implies that H has finite index in A_Γ. □

Lemma 8.17 *Let Γ be a simplicial, finite, connected graph and K be an induced star subgraph $\mathrm{st}(v)$. For each g_1 and g_2 in A_Γ there is a finite sequence of conjugates of star subgroups $g_1 A_K g_1^{-1} = Q_0, Q_1, \ldots, Q_m = g_2 A_K g_2^{-1}$ such that $Q_{i-1} \cap Q_i$ is infinite for each $i \in \{1, 2, \ldots, m\}$.*

Proof Let S be the vertex set of Γ and $n = |g_1^{-1} g_2|_S$. We will prove the lemma by induction on n. If $n = 0$, then $g_1 = g_2$. Therefore, the conclusion is true obviously. We now assume that $n = 1$. Then there is a vertex u such that $g_2 = g_1 u^\epsilon$, where $\epsilon = 1$ or $\epsilon = -1$. Since Γ is finite and connected, there is a finite sequence of vertices $u = u_0, u_1, u_2, \ldots, u_\ell = v$ such that u_{i-1} is adjacent to u_i for each $i \in \{1, 2, 3, \ldots, \ell\}$. Let $P_i = g_1 A_{\mathrm{st}(u_i)} g_1^{-1}$ and $P_i' = g_2 A_{\mathrm{st}(u_i)} g_2^{-1}$ for each $i \in \{0, 1, 2, 3, \ldots, \ell\}$. It is not hard to check $P_{i-1} \cap P_i$ and $P_{i-1}' \cap P_i'$ are infinite for each $i \in \{1, 2, 3, \ldots, \ell\}$. Moreover, $P_0 = P_0'$ because $g_2 = g_1 u^\epsilon$, where $\epsilon = 1$ or $\epsilon = -1$. Therefore, the conclusion is true for $n = 1$ obviously.

Assume the conclusion is true for all $n \leq k$ for some $k \geq 1$. We will prove that the conclusion is true for $n = k + 1$. In fact, if $|g_1^{-1} g_2|_S = k + 1$, then there is a g_3 in A_Γ such that $|g_1^{-1} g_3|_S = k$ and $|g_3^{-1} g_2|_S = 1$. By the inductive hypothesis, there is a finite sequence of conjugates of star subgroups $g_1 A_K g_1^{-1} = L_0, L_1, \ldots, L_{m_1} = g_3 A_K g_3^{-1}$ such that $L_{i-1} \cap L_i$ is infinite for each $i \in \{1, 2, \ldots, m_1\}$. Similarly, there is a finite sequence of conjugates of star subgroups $g_3 A_K g_3^{-1} = L_0', L_1', \ldots, L_{m_2}' = g_2 A_K g_2^{-1}$ such that $L_{i-1}' \cap L_i'$ is infinite for each $i \in \{1, 2, \ldots, m_2\}$. Therefore, there is a finite sequence of conjugates of star subgroups $g_1 A_K g_1^{-1} = Q_0, Q_1, \ldots, Q_m = g_2 A_K g_2^{-1}$ such that $Q_{i-1} \cap Q_i$ is infinite for each $i \in \{1, 2, \ldots, m\}$. This implies that the conclusion is true for $n = k + 1$. □

Proposition 8.18 *Let Γ be a simplicial, finite, connected graph such that Γ does not decompose as a nontrivial join. Let H be a nontrivial, infinite-index subgroup of the right-angled Artin group A_Γ. If H is a strongly quasiconvex subgroup, then H is free.*

Proof We are going to prove H is *star-free* (ie for each vertex v of Γ and $g \in A_\Gamma$ the subgroup $g H g^{-1} \cap A_{\mathrm{st}(v)}$ is trivial) and then H is a free group by Theorem 1.2 in [26]. We first observe that H is finitely generated and each conjugate of H is also a

strongly quasiconvex subgroup. We now prove that for each vertex v of Γ and $g \in A_\Gamma$ the subgroup $gHg^{-1} \cap A_{\mathrm{st}(v)}$ is trivial. We assume for purposes of contradiction that $g_0 H g_0^{-1} \cap A_{\mathrm{st}(v)}$ is not trivial for some vertex v and some $g_0 \in A_\Gamma$. We claim that $gHg^{-1} \cap A_{\mathrm{st}(v)}$ has finite index in $A_{\mathrm{st}(v)}$ for all $g \in A_\Gamma$.

In fact, since $g_0 H g_0^{-1}$ is a strongly quasiconvex subgroup and $A_{\mathrm{st}(v)}$ is an undistorted subgroup, $g_0 H g_0^{-1} \cap A_{\mathrm{st}(v)}$ is a strongly quasiconvex subgroup of $A_{\mathrm{st}(v)}$ by Proposition 4.11. Therefore, $g_0 H g_0^{-1} \cap A_{\mathrm{st}(v)}$ has finite index in $A_{\mathrm{st}(v)}$ by Lemma 8.16.

We now prove that $gHg^{-1} \cap A_{\mathrm{st}(v)}$ has finite index in $A_{\mathrm{st}(v)}$ for all $g \in A_\Gamma$. By Lemma 8.17, there is a finite sequence of conjugates of star subgroups

$$g_0^{-1} A_{\mathrm{st}(v)} g_0 = Q_0, \quad Q_1, \quad \ldots, \quad Q_m = g^{-1} A_{\mathrm{st}(v)} g$$

such that $Q_{i-1} \cap Q_i$ is infinite for each $i \in \{1, 2, \ldots, m\}$. Since $g_0 H g_0^{-1} \cap A_{\mathrm{st}(v)}$ has finite index in $A_{\mathrm{st}(v)}$, $H \cap g_0^{-1} A_{\mathrm{st}(v)} g_0$ has finite index in $Q_0 = g_0^{-1} A_{\mathrm{st}(v)} g_0$. Also, the subgroup $Q_0 \cap Q_1$ is infinite. Then $H \cap Q_1$ is not trivial. Using a similar argument as above, we obtain that $H \cap Q_1$ has finite index in Q_1. Repeating this process, we have $H \cap g^{-1} A_{\mathrm{st}(v)} g$ has finite index in $g^{-1} A_{\mathrm{st}(v)} g$. In other words, $gHg^{-1} \cap A_{\mathrm{st}(v)}$ has finite index in $A_{\mathrm{st}(v)}$.

By Theorem 4.15, there is a number n such that the intersection of any $(n+1)$ essentially distinct conjugates of H is finite. Since H has infinite index in A_Γ, there are $n+1$ distinct elements $g_1, g_2, \ldots, g_{n+1}$ such that $g_i H \neq g_j H$ for each $i \neq j$. Also, $g_i H g_i^{-1} \cap A_{\mathrm{st}(v)}$ has finite index in $A_{\mathrm{st}(v)}$ for each i. Then $\left(\bigcap g_i H g_i^{-1} \right) \cap A_{\mathrm{st}(v)}$ also has finite index in $A_{\mathrm{st}(v)}$. In particular, $\bigcap g_i H g_i^{-1}$ is infinite, which is a contradiction. Therefore, for each vertex v of Γ and $g \in A_\Gamma$ the subgroup $gHg^{-1} \cap A_{\mathrm{st}(v)}$ is trivial. This implies that H is a free group by Theorem 1.2 in [26]. $\qquad\square$

We are now ready for the main theorem in this section.

Theorem 8.19 *Let Γ be a simplicial, finite, connected graph such that Γ does not decompose as a nontrivial join. Let H be a nontrivial, infinite-index subgroup of the right-angled Artin group A_Γ. Then the following are equivalent:*

(1) *H is strongly quasiconvex.*

(2) *H is stable.*

(3) *The lower relative divergence of A_Γ with respect to H is quadratic.*

(4) *The lower relative divergence of A_Γ with respect to H is completely superlinear.*

Proof The implication (1) $\Rightarrow$ (2) is obtained from Theorem 4.8 and Proposition 8.18. The implication (2) $\Rightarrow$ (3) is deduced from Theorem 3.5. The implication (3) $\Rightarrow$ (4) is trivial and the implication (4) $\Rightarrow$ (1) follows Theorem 4.8. $\qquad\square$

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