Towers of regular self-covers and linear endomorphisms of tori

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Let M be a closed manifold that admits a self-cover $p\colon M \to M$ of degree > 1. We say p is strongly regular if all iterates $p^n\colon M \to M$ are regular covers. In this case, we establish an algebraic structure theorem for the fundamental group of M: We prove that $\pi_1(M)$ surjects onto a nontrivial free abelian group A, and the self-cover is induced by a linear endomorphism of A. Under further hypotheses we show that a finite cover of M admits the structure of a principal torus bundle. We show that this applies when M is Kähler and p is a strongly regular, holomorphic self-cover, and prove that a finite cover of M splits as a product with a torus factor.

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1 Introduction

1.1 Main results

Let M be a closed manifold that admits a smooth covering map $p\colon M \to M$. We say p is *nontrivial* if $\deg(p) > 1$.

Question 1.1 Which closed manifolds M admit a nontrivial self-covering?

Key examples are furnished by tori: if $M = T^n$ and A is an integer $n \times n$ matrix, then the linear map $A\colon \mathbb{R}^n \to \mathbb{R}^n$ descends to a map $\overline{A}\colon T^n \to T^n$ with $\deg(\overline{A}) = \det(A)$. Therefore a torus admits many nontrivial self-coverings.

More generally, suppose N is a simply connected nilpotent Lie group and $\Gamma \subseteq N$ is a lattice, and $\alpha: N \to N$ is an automorphism with $\alpha(\Gamma) \subseteq \Gamma$. Then α descends to a map $N/\Gamma \to N/\alpha(\Gamma)$. Postcomposition with the natural projection $N/\alpha(\Gamma) \to N/\Gamma$ yields a self-cover of N/Γ with degree $[\Gamma : \alpha(\Gamma)]$.

For some choices of N and Γ, there exists α such that this construction yields a nontrivial self-cover (eg choosing N (resp. Γ) to be the real (resp. integer) Heisenberg group). Belegradek [3] has made a detailed investigation of these self-covers, and in particular proved that some nilmanifolds do not nontrivially self-cover.

A modification of the above examples by finite covers gives nontrivial self-covers of flat (resp. infranil) manifolds, eg of the Klein bottle. Finally, we can perform fiber bundle constructions with fibers of the types discussed above, and where the self-cover occurs along the fiber. For example, if T is a torus, and $M \to N$ is a flat principal T–bundle such that the monodromy $\pi_1(N) \to \mathrm{Aut}(T)$ commutes with a self-cover $A: T \to T$, then A induces a self-cover of M.

Are these all the examples (up to finite covers)? Progress on this question has been very limited. The only results are either in dimensions ≤ 3 (by Yu and Wang [39]; also see Wang [36]), or for so-called *expanding maps* (see Section 1.3) and to some extent for algebraic maps on smooth projective varieties (see Section 1.2). All of these results suggest that there are no other examples.

In this paper we start the classification of self-covers without placing geometric constraints on the map or the space. To this end we make the following definition:

Definition 1.2 Let X be a topological space. A self-cover $p: X \to X$ is called *strongly regular* if the covers $p^n: X \to X$ are regular for every $n \geq 1$.

Remark 1.3 It is easy to see that a characteristic cover (ie $p_*(\pi_1(M)) \subseteq \pi_1(M)$ is characteristic) is strongly regular.

Our first result shows that, at least on the level of fundamental groups, any strongly regular self-cover comes from a linear endomorphism on a torus. In fact, this theorem does not just hold for smooth manifolds, but in the more general setting of finite CW–complexes.

Theorem 1.4 Let X be a finite CW–complex, and let $p: X \to X$ be a continuous strongly regular self-cover with degree $1 < d < \infty$. Set $\Gamma := \pi_1(X)$ and let $\varphi: \Gamma \hookrightarrow \Gamma$ be the map induced by p.

Then there exists a surjective morphism $q\colon \Gamma \to \mathbb{Z}^n$ for some $n \geq 1$, and such that φ restricts to an automorphism of $\ker(q)$ and descends to a linear endomorphism of $\mathbb{Z}^n$ with determinant d.

Remark 1.5 The groups appearing in Theorem 1.4 as the sources of strongly regular self-covers are not just nilpotent, but in fact abelian. This is explained by Theorem 1.12, where we characterize strongly regular self-covers of nilmanifolds. We will see that if M is a nilmanifold that admits a strongly regular self-cover, then a finite cover of M splits as $N \times T$ for some nilmanifold N and a torus T. After possibly changing the self-cover by a homotopy, the self-cover respects this splitting and restricts to an automorphism on N and a linear endomorphism on T.

We have the following immediate corollary of Theorem 1.4:

Corollary 1.6 Let X be a finite CW–complex that admits a nontrivial strongly regular self-cover. Then $b_1(X) > 0$.

Recall that for a finite CW–complex X with fundamental group Γ, any homomorphism $\Gamma \to \mathbb{Z}^n$ is induced by some continuous map $X \to T^n$, and two maps $X \to T^n$ are homotopic if and only if they induce the same map on fundamental groups. Therefore, Theorem 1.4 has the following purely topological phrasing:

Corollary 1.7 Let X be a finite CW–complex and $p\colon X \to X$ a nontrivial strongly regular self-cover of degree $1 < d < \infty$. Then there exist

- a map $h\colon X \to T^n$ for some $n \geq 1$, and
- a linear endomorphism $A\colon T^n \to T^n$ with $|\det(A)| = d$

such that the square

$$
\begin{array}{ccc}
X & \xrightarrow{\ h\ } & T^n \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle A} \\
X & \xrightarrow{\ h\ } & T^n
\end{array}
$$

commutes up to homotopy.

Note that in Corollary 1.7, h is only determined up to homotopy. It is then natural to wonder if h can be chosen to be of a particularly nice topological form. We do not know, but let us at least ask:

Question 1.8 Let X be a closed smooth manifold and $p\colon X \to X$ a nontrivial strongly regular smooth self-cover. Can we choose h to be a topological fiber bundle?

Remark 1.9 It is definitely not true that h can be chosen to be a smooth fiber bundle. Namely, Farrell and Jones [10] showed that there are tori with exotic smooth structures that admit expanding maps. Farrell and Gogolev [8] construct expanding maps on exotic tori that are not even PL–homeomorphic to the standard torus. The expanding maps in both of these constructs are (smooth) nontrivial strongly regular self-covers, and $h\colon X \to T$ is homotopic to the identity. In this case one can choose h to be a homeomorphism but not a diffeomorphism.

In the proof of Theorem 1.4, it will be essential to study the actions of the deck groups F_n of the covers $p^n\colon X \to X$. In fact, we will see that as subgroups of $\mathrm{Homeo}(X)$, we have $F_n \subseteq F_{n+1}$ for every n. Hence, we can obtain a locally finite group $F := \bigcup_n F_n$ that acts on X. We call F the *asymptotic deck group* of $p\colon M \to M$ (see Section 3 for more information). Analysis of this group and its action will be the driving force behind the proof of Theorem 1.4.

We will now restrict to the case where $X = M$ is a closed manifold, so that we can introduce a special class of self-covers for which the action of F on M is well-behaved.

Definition 1.10 Let $p\colon M \to M$ be a nontrivial, strongly regular self-cover. We say p is *proper* if $F \subseteq \mathrm{Homeo}(M)$ is precompact. If in addition p is smooth, we say p is *smoothly proper* if F is precompact in $\mathrm{Diff}(M)$ with respect to the C^∞–topology.

For smoothly proper self-covers we have the following structure theorem:

Theorem 1.11 Let M be a closed manifold and suppose $p\colon M \to M$ is a nontrivial, strongly regular, smoothly proper self-cover. Then there exist a finite cover $M' \to M$ and a compact torus T such that

(i) M' is a smooth principal T–bundle $\pi\colon M' \to N$ over some closed manifold N,

(ii) $\pi_1(M') \cong \pi_1(T) \times \pi_1(N)$,

(iii) p lifts to a strongly regular self-cover $p'\colon M' \to M'$,

(iv) p' is a bundle map (with respect to π) that descends to a diffeomorphism of N, and

(v) p' restricts to a covering of degree $\deg(p)$ on each T–orbit in M'.

It seems reasonable to expect that if p preserves some additional structure of a geometric nature, then p will be smoothly proper. Below we will establish this in the case that M is a Kähler manifold and p is holomorphic (see Theorem 1.13).

First we consider the case of strongly regular self-covers of nilmanifolds. It follows from Lie theory of nilpotent groups that any self-cover of a nilmanifold is homotopic to an algebraic self-cover (ie induced by an automorphism of the Lie group). These can be shown to be smoothly proper and in this case we can show the bundle produced by Theorem 1.11 is actually trivial (see Section 4), so that we have:

Theorem 1.12 *Let M be a closed nilmanifold and $p\colon M \to M$ a nontrivial, strongly regular self-cover. Then there exists a finite cover M' of M such that*

(i) *M' splits (as a nilmanifold) as $T \times N$ for some closed nilmanifold N and a torus T,*

(ii) *p lifts to a strongly regular self-cover p' of M', and*

(iii) *p' is homotopic to an algebraic endomorphism f of M' that restricts to an automorphism of N and a linear endomorphism of T.*

1.2 Holomorphic endomorphisms of Kähler manifolds

Recall that if M is a closed complex manifold, then a *Kähler structure* on M is a Hermitian metric whose associated 2–form is closed.

For example, any smooth complex projective variety is a Kähler manifold. If X is a smooth complex projective variety, the natural class of self-covers $f\colon X \to X$ to consider are the *étale* maps. Étale maps have been intensely studied, but a complete classification (up to finite étale cover) is only known in low dimensions; see Fujimoto [14] and Fujimoto and Nakayama [15]. In high dimensions the picture is more fragmented; see eg Beauville [2] and Nakayama and Zhang [26] for more information. For nonalgebraic Kähler manifolds, a complete classification has been obtained in dimensions up to 3 by Höring and Peternell [17].

We prove that strongly regular holomorphic self-covers on Kähler manifolds are smoothly proper. Combined with Theorem 1.11, this allows us to obtain a splitting with a torus factor of a finite cover of M:

Theorem 1.13 *Let M be a closed Kähler manifold and $p\colon M \to M$ a nontrivial, strongly regular, holomorphic self-cover. Then there exist a finite cover $M' \to M$ of M, a complex torus T and a Kähler manifold N such that*

(i) M' is biholomorphic to $T \times N$,

(ii) p lifts to a smoothly proper, strongly regular self-cover $p' : M' \to M'$,

(iii) p' descends to a holomorphic automorphism of $N \cong M'/T$,

(iv) on each slice $T \times \{x\}$ with $x \in N$, the map p' restricts to a holomorphic cover of degree $\deg(p)$.

1.3 Residual covers

Finally we will discuss applications of Theorem 1.4 to so-called expanding maps and strongly scale-invariant groups. First let us make the following definition:

Definition 1.14 Let $p : M \to M$ be a self-cover. Consider the associated tower of covers

$$\cdots \xrightarrow{p} M \xrightarrow{p} M \xrightarrow{p} M.$$

We say p is *residual* if this tower is residual, ie $\bigcap_{n \geq 1} p_*^n(\pi_1(M)) = 1$.

We give the following application of Theorem 1.4 to strongly regular residual self-covers:

Corollary 1.15 *Let X be a finite CW–complex admitting a strongly regular, residual self-cover p. Then $\pi_1(X)$ is abelian. In particular, if X is in addition a closed aspherical manifold, then X is homeomorphic to a torus.*

A group-theoretic version of residual self-covers has been studied by Benjamini and by Nekrashevych and Pete [27]. Let Γ be a finitely generated group. Then we define:

- (Benjamini) Γ is called *scale-invariant* if Γ admits a decreasing residual chain

$$\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots,$$

  where $\bigcap_n \Gamma_n$ is finite, Γ_n is of finite index in Γ and $\Gamma_n \cong \Gamma$ for every $n \geq 0$.

- (Nekrashevych and Pete) Γ is called *strongly scale-invariant* if there exists an embedding $\varphi : \Gamma \hookrightarrow \Gamma$, with image of finite index, such that $\bigcap_n \varphi^n(\Gamma)$ is finite.

Scale-invariant groups were introduced by Benjamini in a blog post, in which he asked if any such group is virtually nilpotent. Counterexamples were given by Nekrashevych and Pete [27], after which they defined the above notion of a strongly scale-invariant group, and asked whether a strongly scale-invariant group is virtually nilpotent.

Hence, Corollary 1.15 can be interpreted as giving a positive answer to a topological version of the question of Nekrashevych and Pete, under the additional assumption that $\varphi^n(\Gamma)$ is normal in Γ for every $n \geq 1$.

As a special case of residual self-covers we consider expanding maps:

Definition 1.16 Let M be a closed manifold and choose an auxiliary Riemannian metric on M. Then a C^1 map $f\colon M \to M$ is *expanding* if there exist $c > 0$ and $\lambda > 1$ such that for every $m \geq 1$ and $v \in TM$, we have

$$\|Df^m(v)\| \geq c\lambda^m \|v\|.$$

Remark 1.17 Whether or not a map $f\colon M \to M$ is expanding does not depend on the choice of Riemannian metric on M.

Any expanding map $f\colon M \to M$ is a local homeomorphism, and hence a covering map. Expanding maps on closed Riemannian manifolds were classified by the combined work of Franks [11], Shub [34] and Gromov [16]: Any expanding map on a closed manifold is topologically conjugate to an algebraic endomorphism of an infranilmanifold.

By an approach that is different from the Franks–Shub–Gromov proof (in particular, without use of Gromov's polynomial growth theorem), we show that strongly regular expanding maps are exactly given by linear endomorphisms of tori (up to topological conjugacy and finite covers).

Corollary 1.18 (characterization of strongly regular expanding maps) *Let M be a closed manifold and $p\colon M \to M$ a C^1 strongly regular expanding map. Then M is diffeomorphic to a torus and p is topologically conjugate to a linear expanding map.*

1.4 Outline of the paper

In Section 2 we review some preliminaries regarding locally finite groups, especially the theory of Artinian locally finite groups. In Section 3 we prove Theorem 1.4, which establishes the connection between strongly regular self-covers factor and linear endomorphisms of tori.

We start by introducing the key object for the proof, the asymptotic deck group F associated to a strongly regular self-cover. The key result is that F is abelian and in fact a finite product of finite cyclic groups and quasicyclic groups. This allows us to construct the desired map $\pi_1(X) \to \mathbb{Z}^n$.

In Section 4 we prove Theorem 1.11, which shows that a smoothly proper self-cover is virtually given by a principal torus bundle. By using the Hilbert–Smith conjecture for smooth maps, we know that the closure of the asymptotic deck group is a compact Lie group whose components are tori. Most of the proof is devoted to showing that there is a finite cover where the torus action is free. The quotient by this free action is the desired principal bundle.

At the end of Section 4, we prove Theorem 1.12, which characterizes strongly regular self-covers of nilmanifolds. Indeed, we show that any such map is homotopic to an algebraic endomorphism, which is smoothly proper, so that Theorem 1.11 applies.

In Section 5 we combine the above characterization of smoothly proper self-covers with Kähler geometry to show that a Kähler manifold with a strongly regular, holomorphic self-cover has a finite cover that splits with a torus factor. Finally, in Section 6 we prove Corollaries 1.15 and 1.18 about residual self-covers and expanding maps.

Acknowledgments

2   Preliminaries on locally finite groups

A key tool in the proofs of our results on self-covers is the emergence of a locally finite group. In this section we review the definition and facts needed for the rest of the paper. For more information we refer to [19].

Definition 2.1   A group G is *locally finite* if any finitely generated subgroup is finite.

Remark 2.2   As any group is the direct limit of its finitely generated subgroups, a locally finite group is the direct limit of a system of finite groups. Conversely, a direct limit of a system of finite groups is easily seen to be locally finite.

Example 2.3   (i)   Any finite group is locally finite.

  (ii)   Any direct sum of finite groups is locally finite.

  (iii) $\mathbb{Q}/\mathbb{Z}$ is locally finite. This is the group of complex roots of unity.

(iv) Let $p \geq 2$ be a prime. The *p–quasicyclic group* (or *Prüfer p–group*) is the group $\mathbb{Z}(p^{\infty}) := \mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}$. This is the group of complex roots of unity whose order is a power of p, and is also isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$.

An especially well-studied class of locally finite groups is formed by those that are Artinian. Recall that a group G is called *Artinian* if any nonincreasing chain

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

of subgroups of G eventually stabilizes. This is also called the *minimality condition*.

Among the above examples, the finite groups and p–quasicyclic groups are Artinian, whereas $\mathbb{Q}/\mathbb{Z}$ and an infinite direct sum of nontrivial finite groups are not Artinian. In 1940, Chernikov posed:

Problem 2.4 (Chernikov's minimality problem) Is any Artinian group a finite extension of an abelian Artinian group?

In general this is false, as shown by Ol'shanskii's construction of Tarski monsters [28]. In view of the above minimality problem, it seems useful to characterize abelian Artinian groups. This was accomplished by Kurosh:

Theorem 2.5 (Kurosh; see eg [33, 4.2.11]) *Let G be an abelian group. Then G is Artinian if and only if G is a finite direct sum of quasicyclic and finite cyclic groups.*

Here we say a group is *quasicyclic* if it is p–quasicyclic for some prime p (see Example 2.3(iv)). For general locally finite groups, Shunkov gave the following criterion for being Artinian:

Theorem 2.6 (Shunkov [35]) *Let G be a locally finite group. Then G is Artinian if and only if any abelian subgroup of G is Artinian.*

Shunkov then used this to solve Problem 2.4 for general locally finite groups. An independent solution (without use of Shunkov's criterion, Theorem 2.6) was obtained by Kegel and Wehrfritz [18].

Theorem 2.7 (Shunkov; Kegel and Wehrfritz) *A locally finite group G is Artinian if and only if G is a finite extension of an abelian Artinian group.*

In view of Shunkov's criterion (Theorem 2.6) it seems useful to have an easy method for determining whether a locally finite abelian group is Artinian. To this end, we recall the notion of rank:

Definition 2.8 Let G be a periodic abelian group. For any prime p, consider the set of elements $G(p)$ of order p. Then $G(p)$ is an elementary abelian p–group, and hence is a vector space over $\mathbb{F}_p$. The p–*rank* of G is $r_p(G) := \dim_{\mathbb{F}_p} G(p)$. The *rank* of G is $r(G) := \sum_p r_p(G)$.

Example 2.9   (i) **Finite groups** A finite abelian group G can be written as $G = \bigoplus_i (\mathbb{Z}/p_i^{k_i}\mathbb{Z})^{n_i}$ where i runs over a finite set, p_i are primes and $k_i, n_i \geq 1$. It is easy to see that for any prime p, we have $r_p(G) = \sum_{i:p_i=p} n_i$, and hence $r(G) = \sum_i n_i$.

(ii) **Quasicyclic groups** If $G = \mathbb{Z}(p^\infty)$ is the p–quasicyclic group (see Example 2.3(iv)), then $G(p) \cong \mathbb{Z}/p\mathbb{Z}$. Indeed, viewing G as the group of complex roots of unity with order given by a power of p, then $G(p)$ is exactly the group of pth roots of unity. Hence, $r_p(G) = 1$ and $r_\ell(G) = 0$ for any prime $\ell \neq p$.

Since finite and quasicyclic groups have bounded rank, and rank is additive under direct sums, we see (by use of Kurosh's theorem) that any periodic, Artinian, abelian group has bounded rank. In fact, the converse is also true. To see this, one first uses the primary decomposition theorem for abelian groups (see eg [33, 4.1.1]) to reduce the problem to p–groups for some prime p, and for p–groups the statement is exactly [33, 4.3.13]. We summarize the result as follows:

Proposition 2.10 *Let A be a periodic abelian group. Then A is Artinian if and only if A has bounded rank.*

3 The asymptotic deck group

The goal of this section is to prove the following:

Theorem 3.1 *Let X be a finite CW–complex, and let $p\colon X \to X$ be a continuous, strongly regular self-cover with degree $1 < d < \infty$. Set $\Gamma := \pi_1(X)$ and let $\varphi\colon \Gamma \hookrightarrow \Gamma$ be the map induced by p.*

Then there exist $n \geq 1$ and a surjective morphism $q\colon \Gamma \to \mathbb{Z}^n$ such that φ restricts to an automorphism of $\ker(q)$ and descends to a linear endomorphism of $\mathbb{Z}^n$ with determinant d.

For the rest of this section we retain the notation of Theorem 3.1. We start by introducing a key object in the proof.

Definition 3.2 Consider the inductive system

$$1 \to \Gamma/\varphi(\Gamma) \xrightarrow{\varphi} \Gamma/\varphi^2(\Gamma) \xrightarrow{\varphi} \Gamma/\varphi^3(\Gamma) \xrightarrow{\varphi} \cdots$$

and write $F_n := \Gamma/\varphi^n(\Gamma)$. The limit $F := \varinjlim F_n$ of the above system is the *asymptotic deck group* of p.

The asymptotic deck group of p is therefore a locally finite group that is the limit of the deck groups F_n of p^n as $n \to \infty$. Since φ is injective, we can write $F = \bigcup_n F_n$, where we identify F_n with a subgroup of F_{n+1} via

(3-1) $$F_n = \Gamma/\varphi^n(\Gamma) \cong \varphi(\Gamma)/\varphi^{n+1}(\Gamma) \subseteq \Gamma/\varphi^{n+1}(\Gamma) = F_{n+1}.$$
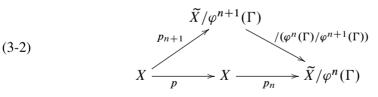
Each F_n acts on X as the group of deck transformations of p^n. A key observation is that these actions are compatible with the identification in (3-1), so that we have:

Lemma 3.3 *There is an action of F on X that extends, for every $n \geq 1$, the action of F_n on X by deck transformations of p^n.*

Proof Let $\widetilde{X}$ be the universal cover of X, so that we can write $X = \widetilde{X}/\Gamma$. For any $n \geq 1$, we have a natural map $X/\varphi^n(\Gamma) \to X/\Gamma$ given by taking the quotient by the action of $F_n = \Gamma/\varphi^n(\Gamma)$. The map $p^n \colon X \to X$ can be lifted to a homeomorphism

$$p_n \colon \widetilde{X}/\Gamma \to \widetilde{X}/\varphi^n(\Gamma).$$

Choose such lifts $(p_n)_{n \geq 1}$ that are compatible, in the sense that p_{n+1} is a lift of $p_n \circ p$, ie such that for every $n \geq 1$ the triangle

(3-2)

commutes. Also choose lifts

$$q_n \colon \widetilde{X}/\varphi^n(\Gamma) \to \widetilde{X}/\varphi^{n+1}(\Gamma)$$

of p such that $p_{n+1} = q_n \circ p_n$.

Let $\varphi_n \colon F_n \to F_{n+1}$ be the map of (3-1). Note that q_n is φ_n–equivariant (because p induces φ on fundamental groups and q_n is a lift of p).

Now define an action of F_n on X by conjugating the action of F_n on $\tilde{X}/\varphi^n(\Gamma)$ by p_n^{-1}, ie for $g \in F_n$ and $x \in X$ we set

$$g \cdot x = p_n^{-1} g p_n(x).$$

It remains to show that for $g \in F_n$, the actions of $g \in F_n$ and $\varphi_n(g) \in F_{n+1}$ coincide. This is immediate, using that $p_{n+1} = q_n \circ p_n$ and that q_n is φ_n–equivariant. □

The natural quotient maps

$$\Gamma/\varphi^{n+1}(\Gamma) \to \Gamma/\varphi^n(\Gamma)$$

induce a surjective map $\alpha \colon F \to F$. We will not need the following result until Section 4, but it will be convenient to prove it here:

Lemma 3.4 *p is α–equivariant.*

Proof We fill in the middle map in the diagram of (3-2) to obtain

(3-3)

$$
\begin{array}{ccc}
 & \tilde{X}/\varphi^{n+1}(\Gamma) & \\
 {\scriptstyle p_{n+1}}\nearrow \quad \big\downarrow \quad \searrow {\scriptstyle /(\varphi^n(\Gamma)/\varphi^{n+1}(\Gamma))} & & \\
X \xrightarrow{\ p\ } X \xrightarrow{\ p_n\ } \tilde{X}/\varphi^n(\Gamma) &
\end{array}
$$

Using the left triangle, we see the vertical map is given by $p \circ p_{n+1}^{-1}$. On the other hand, by using the right triangle, the map is given by $p_n^{-1} \circ \alpha_{n+1}$, where α_{n+1} is the natural quotient map. Thus we obtain that

$$p \circ p_{n+1}^{-1} = p_n^{-1} \circ \alpha_{n+1}.$$

Now the proof of equivariance is straightforward from the definitions and commutativity of diagram (3-3): For $g \in F_{n+1}$, we have

$$
\begin{aligned}
p(gx) &= p(p_{n+1}^{-1} g p_{n+1}(x)) \\
 &= p_n^{-1} \alpha_{n+1} g p_{n+1}(x) \\
 &= p_n^{-1} \alpha(g) \alpha_{n+1} p_{n+1}(x) \\
 &= p_n^{-1} \alpha(g) p_n p(x) \\
 &= \alpha(g) p(x).
\end{aligned}
$$
□

The action given by Lemma 3.3 gives us strong structural constraints on F:

Lemma 3.5 *F is Artinian.*

Proof By Shunkov's criterion (Theorem 2.6), it suffices to prove any abelian subgroup of F is Artinian. Suppose $A \subseteq F$ is an abelian subgroup. Since A is countable and abelian, we can choose an increasing chain

$$A_1 \subsetneqq A_2 \subsetneqq A_3 \subsetneqq \cdots$$

such that A_n is finitely generated for all n and $\bigcup_n A_n = A$. Since A_n is a finitely generated subgroup of the locally finite group F, it follows that A_n is finite. Therefore, there exist $K_n \geq 1$ and primes $p_{n,k}$ and integers $l_{n,k} \geq 1$ (with $1 \leq k \leq K_n$) such that

(3-4) $$A_n \cong \bigoplus_{1 \leq k \leq K_n} \mathbb{Z}/p_{n,k}^{l_{n,k}} \mathbb{Z}.$$

In fact, only finitely many primes are allowed to appear as the primes $p_{n,k}$ in the decomposition (3-4) of A_n. Namely, since A_n is finite and $F = \bigcup_N F_N$, we see that there exists N such that $A_n \subseteq F_N$. Note that F_N is the deck group of the covering map $p^N \colon M \to M$, so that the order of F_N is $d^N = \deg(p)^N$. Since A_n is a subgroup of F_N, its order $|A_n|$ also divides d^N. Therefore, $p_{n,k}$ divides d for every n and k.

Next we will bound the number of factors K_n of the decomposition of A_n given by (3-4). For this, we will need the following result about finite transformation groups, which has been proven for $\ell = 2$ by Carlsson [7] and for $\ell > 2$ by Baumgartner [1]. For simplicity, we will only state a special case of their results for homologically trivial actions.

Theorem 3.6 (Baumgartner, Carlsson) *Let X be a finite CW–complex and ℓ prime. Let $G \cong (\mathbb{Z}/\ell\mathbb{Z})^r$ be an elementary abelian ℓ–group acting effectively and freely on X by cellular maps, and suppose that G acts trivially on $H_*(X;\mathbb{F}_\ell)$.*

Then $H_j(X;\mathbb{F}_\ell) \neq 0$ for at least $r+1$ values of j.

Claim 3.7 *Let K_n be as in (3-4). Then there is a uniform bound on K_n.*

Proof It suffices to prove that for every prime divisor ℓ of $d = \deg(p)$, the ℓ–rank of A_n is uniformly bounded. (For the notion of rank, see Definition 2.8.) Let ℓ be a prime divisor of d and fix $n \geq 1$. Suppose that $r \geq 0$ is such that $G := (\mathbb{Z}/\ell\mathbb{Z})^r \subseteq A_n$. Choose $N \geq 1$ such that $A_n \subseteq F_N$. Since F_N acts by deck transformations on X, we see that G acts freely on X.

Next we show that we can choose a CW–structure on X so that G acts by cellular maps. Indeed, fix any finite CW–structure on X and lift it through the cover $p^N \colon X \to X$. This induces a cell structure on the cover such that deck transformations act by cellular maps. Whitehead proved that the topology induced by this lifted cell structure coincides with the topology of the cover [37, (N), page 231]. Note that the lifted CW–structure is necessarily different from the original one (because it has more cells), but the dimension of X with respect to the two CW–structures is the same.

We want to apply Theorem 3.6 to the action of G on X. We already know that G acts freely, and X admits a cell structure such that G acts cellularly, but of course G may not act trivially on homology. The action of G on homology of X with $\mathbb{F}_\ell$–coefficients is a representation of G over $\mathbb{F}_\ell$,

$$\rho \colon G \to \operatorname{Aut}(H_*(X; \mathbb{F}_\ell)).$$

Since X is a finite CW–complex, $H_*(X; \mathbb{F}_\ell)$ is finite-dimensional, say of dimension n_ℓ. Note that the homology of X is independent of the CW–structure chosen, so n_ℓ does not depend on n. Now we can view ρ as a map with values in $\operatorname{GL}(n_\ell, \mathbb{F}_\ell)$. Write N_ℓ for the order of $\operatorname{GL}(n_\ell, \mathbb{F}_\ell)$, and also write $H := \ker(\rho)$. Let $s \geq 1$ be such that $H \cong (\mathbb{Z}/\ell\mathbb{Z})^s$.

Let D be the dimension of X as a CW–complex with respect to the CW–structure lifted through the cover $p^N \colon X \to X$. As mentioned above, D does not depend on n. Since H acts freely, cellularly and trivially on homology over $\mathbb{F}_\ell$, Theorem 3.6 implies that $D \geq s$. Since H has index at most N_ℓ in G, we find that

$$\ell^r = |G| = [G : H]|H| \leq N_\ell \ell^s \leq N_\ell \ell^D.$$

Since N_ℓ and D are independent of n, we obtain a uniform bound on r.   □

By Claim 3.7, we can choose $C \geq 1$ such that $K_n \leq C$ for every n. For uniformity in the description of A_n, let us introduce the following notation: Let $\ell_1, \ldots, \ell_r$ be the (distinct) prime factors of d. For $1 \leq i \leq r$ and every n, there are at most C values of k such that $p_{n,k} = \ell_i$. Therefore, we can choose integers

$$r_n^{(i,1)} \geq r_n^{(i,2)} \geq \cdots \geq r_n^{(i,C)} \geq 0$$

such that the ℓ_i–torsion of A_n is given by

$$(3\text{-}5) \qquad\qquad A_n \otimes \mathbb{Z}(\ell_i^\infty) \cong \bigoplus_{1 \leq k \leq C} \mathbb{Z}/\ell_i^{r_n^{(i,k)}} \mathbb{Z}.$$

Note that is entirely possible that ℓ_i occurs strictly fewer than C times in the decomposition (3-4) of A_n, in which case $r_n^{(i,k)} = 0$ for some values of k. Combining the description given by (3-5) of the ℓ_i–torsion of A_n for all values of i, we have

$$A_n \cong \bigoplus_{1 \le i \le r} \bigoplus_{1 \le k \le C} \mathbb{Z}/\ell_i^{r_n^{(i,k)}}\mathbb{Z}.$$

It follows that A_n has rank at most rC. It is easy to see that then $A = \bigcup_n A_n$ also has rank at most rC. Since A has bounded rank, A is Artinian (see Proposition 2.10). $\square$

Since F is Artinian, it is a finite extension of an abelian group (by Shunkov's and Kegel and Wehrfritz's solution of the Chernikov problem for locally finite groups; see Theorem 2.7). Our next lemma shows that we can modify the choice of the finite-index abelian subgroup A of F so that A is compatible with the surjection $\alpha\colon F \to F$.

Lemma 3.8 *Let F be a locally finite Artinian group and $\alpha\colon F \to F$ a surjection. Then there exists a finite-index abelian subgroup $A \subseteq F$ such that $\alpha(A) \subseteq A$.*

Proof First choose an abelian finite-index subgroup A_0 of F. For $k \ge 1$, define $A_k := \alpha(A_{k-1}) \cap A_{k-1}$. Then $(A_k)_{k \ge 0}$ is a nonincreasing sequence of subgroups of F. Since F is Artinian, this chain eventually stabilizes, say to the subgroup A. Note that since α is a surjection and A_0 is of finite index in F, the subgroup A_k must also be of finite index in F. For $k \gg 1$ we have $A = A_k$, so A is also of finite index in F.

Finally, note that we have

$$A = \bigcap_{k \ge 0} A_k.$$

It immediately follows that $\alpha(A) \subseteq A$. $\square$

We will actually show that F itself is abelian, not just virtually abelian. The key observation is that α is *locally of finite order*, ie for any $g \in F$, there exists $n \ge 1$ such that $\alpha^n(g) = e$. Indeed, $\alpha\colon F \to F$ is the map induced by the natural quotient maps

(3-6) $$\Gamma/\varphi^{n+1}(\Gamma) \to \Gamma/\varphi^n(\Gamma),$$

so if $g \in F_n$, then applying the quotient maps of (3-6) (at most) n times, we have that the image of g is trivial. After passing to the limit this yields $\alpha^n(g) = e$.

We can now show:

Lemma 3.9 *F is abelian.*

Proof By Lemma 3.8, there exists an abelian finite-index subgroup $A \subseteq F$ with $\alpha(A) \subseteq A$. Choose finitely many coset representatives $g_1, \ldots, g_m$ of F/A. Choose $N \geq 1$ such that $\alpha^N(g_i) = e$ for every i. Then

$$\alpha^N(F) = \alpha^N\left(\bigcup_i g_i A\right) \subseteq A.$$

On the other hand, α is a surjection, so we find that $F \subseteq A$. We conclude that $F = A$ is abelian. $\qquad\square$

We will now finish the proof of Theorem 3.1. At this point we know (by Lemma 3.5 and Theorem 2.7) that the asymptotic deck group of the strongly regular cover p is abelian, and is virtually a finite product of quasicyclic groups. We need to construct a free abelian quotient of Γ such that φ descends to a linear endomorphism of the quotient.

Since $\Gamma/\varphi^n(\Gamma)$ are finite abelian groups whose orders get arbitrarily large as $n \to \infty$, we already see that Γ has infinite abelianization and hence Γ has a free abelian quotient. Our goal now is to construct a free abelian quotient Λ of Γ that is naturally associated to the map φ.

To define Λ, first note that for every n, we have $[\Gamma, \Gamma] \subseteq \varphi^n(\Gamma)$ (because the quotients $\Gamma/\varphi^n(\Gamma)$ are abelian). Define the *stable image*

$$\varphi^\infty(\Gamma) := \bigcap_{n \geq 1} \varphi^n(\Gamma).$$

Then $[\Gamma, \Gamma] \subseteq \varphi^\infty(\Gamma)$, so $\Lambda := \Gamma/\varphi^\infty(\Gamma)$ is a finitely generated abelian group.

Since φ restricts to an automorphism of $\varphi^\infty(\Gamma)$, we know that φ descends to an injective morphism

$$\overline{\varphi} \colon \Lambda \to \Lambda,$$

and

$$[\Lambda : \overline{\varphi}(\Lambda)] = [\Gamma : \varphi(\Gamma)] = d.$$

In particular, Λ is nontrivial. In addition we have:

Lemma 3.10 *The stable image $\overline{\varphi}^\infty(\Lambda) = \bigcap_n \overline{\varphi}^n(\Lambda)$ is trivial.*

Proof Suppose that $v \in \overline{\varphi}^\infty(\Lambda)$, and choose $v_n \in \Lambda$ with $\overline{\varphi}^n(v_n) = v$. Let $\gamma \in \Gamma$ map to $v \in \Lambda = \Gamma/\varphi^\infty(\Gamma)$. Similarly, choose $\gamma_n \in \Gamma$ that map to v_n.

We will show that $\gamma \in \varphi^\infty(\Gamma)$, so that $v = 0$. Let $n \geq 1$ be arbitrary. Since γ and $\varphi^n(\gamma_n)$ both map to $v \in \Lambda$, we know that $\gamma\varphi^n(\gamma_n^{-1}) \in \varphi^\infty(\Gamma)$. Since φ restricts to an automorphism on $\varphi^\infty(\Gamma)$, there exist $\delta_n \in \Gamma$ with

$$\gamma\varphi^n(\gamma_n^{-1}) = \varphi^n(\delta_n).$$

We may rewrite this as

$$\gamma = \varphi^n(\delta_n)\varphi^n(\gamma_n) = \varphi^n(\delta_n\gamma_n),$$

so $\gamma \in \varphi^n(\Gamma)$. Since n was arbitrary, we have $\gamma \in \varphi^\infty(\Gamma)$, as desired. $\qquad\square$

It remains to show that Λ is torsion-free. Let Δ be the torsion subgroup of Λ. Since Λ is finitely generated, Δ is finite. Since $\overline{\varphi}$ is injective and the image of a torsion element under a homomorphism is still torsion, we see that $\overline{\varphi}$ restricts to an automorphism of Δ. Therefore Δ is contained in the stable image of $\overline{\varphi}$, which we have just shown to be trivial. Therefore, Δ is trivial, so that Λ is torsion-free. This completes the proof of Theorem 3.1.

4 Smoothly proper self-covers and principal torus bundles

The goal of this section is to prove Theorem 1.11, a topological classification theorem of smoothly proper, strongly regular self-covers of closed manifolds. Namely, in this case we show that a finite cover of the manifold is a principal torus bundle, and the self-cover lifts to a bundle map that is a cover on each fiber and descends to a diffeomorphism on the base.

At the end of the section, we use this result to prove Theorem 1.12, which characterizes strongly regular self-covers of nilmanifolds.

For the rest of this section, let M be a closed manifold and let $p \colon M \to M$ be a nontrivial, smoothly proper self-cover with asymptotic deck group F. We will first give an outline of the proof of Theorem 1.11. The proof can be divided into roughly three parts.

Part 1 (structure of $\overline{F}$) We claim that $\overline{F}$ is a compact torus. In Lemma 4.1, we establish that (a) $\overline{F}$ is a compact Lie group, and (b) the identity component is a torus.

It is more difficult to prove that $\overline{F}$ is connected (Lemma 4.4). Recall that the self-cover p of M induces a map $\alpha \colon F \to F$. The key tool is an extension of this map to $\overline{F}$ (see Lemmas 4.2 and 4.3). This completes Part 1.

Let us motivate the next step. To obtain a principal torus bundle, we need to show that $\overline{F}$ acts freely on M. In Part 3, we will establish that this holds for a suitable finite cover of M. We already have a strong hint at this, because the dense subgroup F of $\overline{F}$ acts freely. Unfortunately, this does not suffice to prove that $\overline{F}$ acts freely. Therefore, instead we consider the cover $\widehat{M} := \widetilde{M}/\varphi^\infty(\Gamma)$, which has deck group $\Lambda := \Gamma/\varphi^\infty(\Gamma)$, and we consider the group of lifts G of $\overline{F}$. Note that G is an extension

$$1 \to \Lambda \to G \to \overline{F} \to 1.$$

In Part 2, we will show that G has finitely many components, and G^0 is a vector group. In Part 3, we will show that this suffices to prove the desired existence of a finite cover with free torus action.

Part 2 (structure of G)   The control on G is obtained through playing off two pieces of data against each other:

(1) G has an automorphism given by conjugation by a lift of the covering p (see Lemma 4.2), and this automorphism restricts to the linear endomorphism $\overline{\varphi}$ on Λ.

(2)   The linear endomorphism $\overline{\varphi} \colon \Lambda \to \Lambda$ extends to an automorphism of the vector group $V := \Lambda \otimes \mathbb{R}$.

In particular, consider the group $D := \bigcup_n \overline{\varphi}^{-n}(\Lambda)$, which embeds naturally in both G and V, and is dense in G. We show that D is also dense in V (Lemma 4.5). This is the key result needed to prove G has finitely many components (Lemma 4.7). Finally, we prove a technical result on endomorphisms of abelian Lie groups (Lemma 4.8), which, combined with the previous results, yields that $G^0 \cong V$, as desired.

Part 3 (end of the proof)   Using the results of Parts 1 and 2, we prove there exists a finite cover M' of M such that p lifts to a covering map p' of M' with asymptotic deck group F' such that $\overline{F'}$ acts freely on M' (Lemma 4.9).

Indeed, M' is chosen so that the group of lifts of the resulting asymptotic deck group is contained in G^0. The key point is then that any stabilizer of S' in $\overline{F'}$ lifts, and hence embeds in G^0. But, since G^0 is a vector group (by Part 2), we know that G^0 has no nontrivial compact subgroups.

This proves statements (i) and (iii) of Theorem 1.11. We complete the proof by also establishing the remaining claims.

After this outline, let us now start the proof of Theorem 1.11. As discussed, we start by studying the compact group $\overline{F}$:

Lemma 4.1 $\overline{F}$ *is a compact connected Lie group with finitely many components. Its identity component* $\overline{F}^0$ *is isomorphic to a torus.*

Proof The Hilbert–Smith conjecture asserts that a compact group of homeomorphisms of a topological manifold is a Lie group. It has been proven for compact groups of diffeomorphisms by Montgomery [23]. Therefore, $\overline{F}$ is Lie.

It remains to show $\overline{F}^0$ is isomorphic to a torus. Note that $\overline{F}^0$ is abelian, because $F \cap \overline{F}^0$ is a dense abelian subgroup. Therefore, $\overline{F}^0$ is a compact connected abelian Lie group, and hence it must be isomorphic to a torus. □

Let $\widehat{M}$ be the cover of M corresponding to the subgroup $\varphi^\infty(\Gamma) \subseteq \Gamma$. Then $\widehat{M} \to M$ is a regular cover with deck group $\Lambda := \Gamma/\varphi^\infty(\Gamma)$. By Theorem 1.4, Λ is free abelian.

We also know the map $\varphi\colon \Gamma \hookrightarrow \Gamma$ descends to a linear endomorphism $\overline{\varphi}$ of Λ. Since φ restricts to an isomorphism of $\varphi^\infty(\Gamma)$, the self-cover $p\colon M \to M$ lifts to a diffeomorphism $\widehat{p}\colon \widehat{M} \to \widehat{M}$ that is $\overline{\varphi}$–equivariant.

Finally, let G be the group of maps $\widehat{M} \to \widehat{M}$ that are lifts of elements of $\overline{F}$. Then G is an extension

$$1 \to \Lambda \to G \to \overline{F} \to 1.$$

Let D be the preimage of F in G. First, we describe how G and $\widehat{p}$ are related.

Lemma 4.2 *Conjugation by* $\widehat{p}$ *induces an automorphism of* G *that restricts to* $\overline{\varphi}$ *on* Λ.

Proof The action of F_n on M is defined by

$$g \cdot x := p_n^{-1} g p_n(x),$$

where p_n are diffeomorphisms $M \to \widetilde{M}/\varphi^n(\Gamma)$ that lift $p^n\colon M \to M$ (see Lemma 3.3). Since p_n lifts to $\widehat{p}^n$, it is immediate that the group of lifts of F_n is given by $\widehat{p}^{-n} \Lambda \widehat{p}^n$, and

$$D = \bigcup_{n \geq 0} \widehat{p}^{-n} \Lambda \widehat{p}^n.$$

Since $\widehat{p}$ is $\overline{\varphi}$–equivariant, we have, for any $g \in \Lambda$,

$$\widehat{p} g \widehat{p}^{-1} = \overline{\varphi}(g) \widehat{p} \widehat{p}^{-1} = \overline{\varphi}(g).$$

So conjugation by $\widehat{p}$ restricts to $\overline{\varphi}\colon \Lambda \hookrightarrow \Lambda$. Therefore conjugation by $\widehat{p}$ restricts to an automorphism of D. Since D is dense in G, it follows that conjugation by $\widehat{p}$ is an automorphism of G. □

Hence, we can extend $\bar{\varphi}\colon \Lambda \hookrightarrow \Lambda$ to an automorphism of G by setting $\bar{\varphi}(g) := \bar{p} g \bar{p}^{-1}$. Then

$$D = \bigcup_{n \geq 0} \bar{\varphi}^{-n}(\Lambda).$$

Since $\bar{\varphi}(\Lambda) \subseteq \Lambda$, we know that $\bar{\varphi}$ descends to a map of $G/\Lambda \cong \bar{F}$. Write $\bar{\alpha}\colon \bar{F} \to \bar{F}$ for this map.

Recall that there is a surjection $\alpha\colon F \to F$ induced by the natural quotient maps

$$\Gamma/\varphi^{n+1}(\Gamma) \to \Gamma/\varphi^n(\Gamma),$$

and that p is α–equivariant (see Lemma 3.4).

Lemma 4.3 *The map $\bar{\alpha}$ extends $\alpha\colon F \to F$. Further, $p\colon M \to M$ is $\bar{\alpha}$–equivariant.*

Proof Let $g \in F$. By Lemma 3.4, we know that p is α–equivariant, so we have

$$(4\text{-}1) \qquad\qquad\qquad\qquad p \circ g = \alpha(g) \circ p$$

(as diffeomorphisms of M). Choose a lift $\hat{g}$ of g to $\widehat{M}$. By (4-1), there is a lift $\widehat{\alpha(g)}$ of $\alpha(g)$ to $\widehat{M}$ such that

$$\hat{p} \circ \hat{g} = \widehat{\alpha(g)} \circ \hat{p},$$

so that $\widehat{\alpha(g)} = \hat{p}\hat{g}\hat{p}^{-1}$. For the maps on M this means

$$\alpha(g) = \bar{\alpha}(g),$$

as desired.

To see that p is $\bar{\alpha}$–equivariant, note that we already know that p is α–equivariant, and F is dense in $\bar{F}$. Since $\bar{\alpha}$ extends α, it immediately follows that p is $\bar{\alpha}$–equivariant as well. $\square$

Lemma 4.4 *$\bar{F}$ is connected.*

Proof Let G^0 be the connected component of G that contains the identity. Since $\hat{p}(G^0) = G^0$ and G^0 surjects onto $\bar{F}^0$, we know that $\bar{\alpha}(\bar{F}^0) = \bar{F}^0$. Therefore, $\bar{\alpha}$ descends to a map of the group of components $\pi_0(\bar{F}) \cong \bar{F}/\bar{F}^0$ of $\bar{F}$.

The group $\pi_0(\bar{F})$ is finite because $\bar{F}$ is compact. On the other hand, $\bar{\alpha}$ is a surjection (since $\bar{F}$ is compact, α is surjective and F is dense in $\bar{F}$). Therefore, the induced map on components is a surjection and, because $\pi_0(\bar{F})$ is finite, $\bar{\alpha}$ must induce an isomorphism on $\pi_0(\bar{F})$.

We will again use the principle that α being locally of finite order can be used to show that a finite-index subgroup (in this case $\overline{F}^0$) is the entire group (in this case $\overline{F}$): Take coset representatives $g_1, \dots, g_r \in \overline{F}$ for the cosets of $\overline{F}/\overline{F}^0$. Since F is dense, F surjects onto $\overline{F}/\overline{F}^0$. Therefore, we can choose $g_i \in F$. Since α is locally of finite order, we can choose $N \gg 1$ such that $\alpha^N(g_i) = e$. Then we have

$$\overline{\alpha}^N(\overline{F}) = \overline{\alpha}^N \left(\bigcup_i g_i \overline{F}^0 \right) = \overline{F}^0,$$

so $\overline{F}$ is connected. $\qquad\qquad\square$

We will suspend our investigation of G for the moment, because we need some more precise algebraic information about the relationship between Λ and $D = \bigcup_n \overline{\varphi}^{-n}(\Lambda)$.

Consider the vector group $V := \Lambda \otimes \mathbb{R}$. Since $\overline{\varphi}$ is a linear endomorphism of Λ, it naturally extends to some linear transformation

$$\overline{\varphi}\colon V \to V.$$

This allows us to define a natural embedding $j\colon D \hookrightarrow V$ by setting

(4-2) $$j(\overline{\varphi}^{-n}(g)) := \overline{\varphi}^{-n}(g \otimes 1)$$

for $g \in \Lambda$ and $n \geq 0$. It is straightforward to check that j is well-defined.

Lemma 4.5 $j(D)$ *is dense in* V.

Proof The closure $\overline{j(D)}$ is a closed connected subgroup of V. Closed subgroups of connected abelian Lie groups have been classified by Wüstner [38]. In this case, because V has no compact subgroups, this classification yields a discrete subgroup $\Delta \subseteq V$ and a subspace $W \subseteq V$ such that $\overline{j(D)} = \Delta \oplus W$. Therefore, it suffices to show that $\Delta = 0$.

Note that while $W = \overline{j(D)}^0$ is uniquely determined by $j(D)$, the discrete group Δ is not. Indeed, if W is nontrivial then there are many sections of

$$\overline{j(D)} \to \overline{j(D)}/W \cong \Delta$$

and the image of any such section would be an admissible choice of Δ. We will use this freedom to make a particular choice of Δ that is compatible with the lattice $\Lambda \subseteq \overline{j(D)}$.

Write $\Lambda_\Delta := \Lambda \cap \Delta$ and $\Lambda_W := \Lambda \cap W$. Since $\Lambda \subseteq \overline{j(D)}$ is a lattice, we have

(1) Λ_W is a lattice in $W = \overline{j(D)}^0$, and

(2) The composition

$$\Lambda \hookrightarrow \overline{j(D)} = \Delta \oplus W \to \Delta$$

has image that is of finite index in Δ and is a cocompact lattice in V/W.

Now consider the short exact sequence

$$1 \to \Lambda_W \to \Lambda \to \Lambda/\Lambda_W \to 1.$$

Since all of these groups are free abelian, this sequence splits. Choose any section $\sigma\colon \Lambda/\Lambda_W \to \Lambda$. Since Λ/Λ_W is a cocompact lattice in $V/W \cong \Delta \otimes \mathbb{R}$, we can uniquely extend σ to a section $V/W \to V$. Set $\Delta := \sigma(\overline{j(D)}/W)$.

Now we have $\Lambda = \Lambda_\Delta \oplus \Lambda_W$, and we have already seen that

(4-3) $$V = \Lambda \otimes \mathbb{R} = (\Delta \otimes \mathbb{R}) \oplus W.$$

Consider the linear transformation $\overline{\varphi}$. Since $\overline{\varphi}$ restricts to an automorphism of $j(D)$, we know that $\overline{\varphi}$ must preserve W, and hence is triangular with respect to the decomposition of (4-3), say

(4-4) $$\overline{\varphi} = \begin{pmatrix} \overline{\varphi}_\Delta & 0 \\ C & \overline{\varphi}_W \end{pmatrix},$$

where $\overline{\varphi}_\Delta$ (resp. $\overline{\varphi}_W$) is an automorphism of $\Delta \otimes \mathbb{R}$ (resp. W), and

$$C\colon \Delta \otimes \mathbb{R} \to W$$

is a linear map with $C(\Lambda_\Delta) \subseteq \Lambda_W$.

Set $N := [\Delta : \Lambda_\Delta]$. Since Δ is finitely generated, it has only finitely many subgroups of index N. Let Δ_N be the intersection of all subgroups of Δ of index N. Then $\Delta_N \subseteq \Lambda_\Delta$ is a finite-index subgroup and $\overline{\varphi}_\Delta$ restricts to an automorphism on Δ_N.

Our goal is to show that if Δ is nontrivial, we can modify Δ even further so that it is also compatible with $\overline{\varphi}$. Namely, we want to show that we can choose Δ such that $\overline{\varphi}(\Delta) \subseteq \Delta$. For this, we need to block diagonalize the block triangular matrix of (4-4). For this, we need:

Claim 4.6 *Let λ be an eigenvalue of $\overline{\varphi}_\Delta$. Then λ is not an eigenvalue of $\overline{\varphi}_W$.*

Proof Recall that $\Lambda = \Lambda_\Delta \oplus \Lambda_W$. Define $\Lambda_N := \Delta_N \oplus \Lambda_W$. Λ_N is a cocompact lattice in V, so we can choose an isomorphism $V \cong \mathbb{R}^{\dim(V)}$ such that Λ_N corresponds to $\mathbb{Z}^{\dim(V)}$. Since $\overline{\varphi}(\Lambda_N) \subseteq \Lambda_N$, the matrix corresponding to $\overline{\varphi}$ has integer entries. For $\beta \in \mathbb{C}$ an algebraic integer, let K_β be the splitting field of the minimal polynomial of β.

Now let λ be an eigenvalue of $\overline{\varphi}_\Delta$. We argue by contradiction, so suppose that λ is also an eigenvalue of $\overline{\varphi}_W$. First consider $\overline{\varphi}_\Delta$: Since $\overline{\varphi}_\Delta$ is an integer matrix, λ is an algebraic integer and the Galois conjugate of λ by any $\sigma \in \mathrm{Gal}(K_\lambda/\mathbb{Q})$ is also an eigenvalue of $\overline{\varphi}_\Delta$. Therefore, the algebraic norm

$$N_{K_\lambda/\mathbb{Q}}(\lambda) = \prod_{\sigma \in \mathrm{Gal}(K_\lambda/\mathbb{Q})} \sigma(\lambda)$$

divides $\det(\overline{\varphi}_\Delta) = \pm 1$. On the other hand, $N_{K_\lambda/\mathbb{Q}}(\lambda)$ is an integer (because λ is an algebraic integer), and hence we must have $N_{K_\lambda/\mathbb{Q}}(\lambda) = \pm 1$.

Now consider $\overline{\varphi}_W$: First extend $\overline{\varphi}_W$ to a map of the complexification $W_\mathbb{C} := W \otimes_\mathbb{R} \mathbb{C}$. Define the subspace

$$U := \bigoplus_{\sigma \in \mathrm{Gal}(K_\lambda/\mathbb{Q})} \ker(\overline{\varphi}_W - \sigma(\lambda)\,\mathrm{Id})$$

of $W_\mathbb{C}$. Then U is an $\overline{\varphi}_W$–invariant subspace of $W_\mathbb{C}$. Write $\overline{\varphi}_U$ for the restriction of $\overline{\varphi}_W$ to U. Then $\overline{\varphi}_U$ has eigenvalues $\sigma(\lambda)$, where $\sigma \in \mathrm{Gal}(K_\lambda/\mathbb{Q})$, and each eigenvalue occurs with the same algebraic multiplicity $m \geq 1$. Hence,

$$\det(\overline{\varphi}_U) = \prod_{\sigma \in \mathrm{Gal}(K_\lambda/\mathbb{Q})} \sigma(\lambda)^m = (N_{K_\lambda/\mathbb{Q}}(\lambda))^m = \pm 1.$$

Further, because $\ker(\overline{\varphi}_W - \lambda I)$ is defined over $\mathbb{Q}(\lambda)$ and U is spanned by all Galois conjugates of this eigenspace, we know that U is defined over $\mathbb{Q}$.

Set $\Lambda_U := \Lambda_N \cap U$. Then Λ_U is a cocompact lattice in $U(\mathbb{R})$ because U is defined over $\mathbb{Q}$ (hence over $\mathbb{R}$). Since $\overline{\varphi}$ restricts to a linear automorphism of U with determinant ± 1 and $\overline{\varphi}(\Lambda_U) \subseteq \Lambda_U$, we must have that $\overline{\varphi}_U$ restricts to an automorphism of Λ_U. Therefore, Λ_U is contained in the stable image

$$\overline{\varphi}^\infty(\Lambda) := \bigcap_n \overline{\varphi}^n(\Lambda),$$

which is trivial by Lemma 3.10. On the other hand, $\Lambda_U \subseteq U(\mathbb{R})$ is a cocompact lattice, so $U(\mathbb{R})$ must be trivial. This is a contradiction, and completes the proof of Claim 4.6. □

Let $\lambda_1, \ldots, \lambda_r$ be the eigenvalues of $\overline{\varphi}_\Delta$ (counted without multiplicity), and let

$$E_i := \ker(\overline{\varphi} - \lambda_i\,\mathrm{Id})$$

be the corresponding generalized eigenspace in the complexification $V_\mathbb{C}$. As discussed above, $U := \bigoplus_i E_i$ is defined over $\mathbb{Q}$, and hence Λ_N intersects $U(\mathbb{R})$ in a cocompact

lattice. In addition, $U(\mathbb{R})$ is clearly $\bar{\varphi}$–invariant, so

$$\bar{\varphi}(\Lambda_N \cap U(\mathbb{R})) \subseteq \Lambda_N \cap U(\mathbb{R}).$$

We want to show that we must have equality. To do so, it suffices to show that $\det(\bar{\varphi}|_{U(\mathbb{R})}) = \pm 1$: Since none of the eigenvalues λ_i are eigenvalues of $\bar{\varphi}_W$ (by Claim 4.6), we know that $U(\mathbb{R})$ intersects W trivially and hence $U(\mathbb{R})$ projects isomorphically onto V/W. In particular,

$$\det \bar{\varphi}|_{U(\mathbb{R})} = \det \bar{\varphi}_\Delta = \pm 1.$$

We conclude that $\bar{\varphi}(A_N \cap U(\mathbb{R})) = A_N \cap U(\mathbb{R})$. But then $\Lambda_N \cap U(\mathbb{R})$ is contained in the stable image of $\bar{\varphi}$, which is a contradiction. Therefore, we must have that $\Delta = 0$ is trivial, as desired. This completes the proof of Lemma 4.5.   □

We can now continue our investigation of G.

Lemma 4.7 *G has finitely many components.*

Proof The group of components of G is $\pi_0 G \cong G/G^0$. Since Λ is abelian, $D = \bigcup_n \bar{\varphi}^{-n}(\Lambda)$ is the increasing union of abelian groups and hence abelian. Since D is dense in G, we find that G is abelian as well. Therefore, $\pi_0(G)$ is an abelian group. In fact, it must also be finitely generated:

Since Λ is a cocompact lattice in G, the image of Λ under

$$\Lambda \hookrightarrow G \to G/G^0$$

is of finite index in G/G^0. Therefore, G/G^0 contains a finite-index subgroup that is finitely generated and abelian. Since G/G^0 is also abelian itself (because G is abelian), we see that G/G^0 is a finitely generated abelian group. We need to show that G/G^0 is finite, or, equivalently, that there is no nontrivial homomorphism $G/G^0 \to \mathbb{Z}$.

Suppose that there is a homomorphism $G/G^0 \to \mathbb{Z}$. By precomposing with the projection $G \to G/G^0$, we obtain a homomorphism

$$\lambda\colon G \to \mathbb{Z}.$$

Since Λ is free abelian, the restriction of λ to Λ naturally extends to the vector group $V := \Lambda \otimes \mathbb{R}$ and gives a linear functional

$$f\colon V \to \mathbb{R}$$

that is integer-valued on Λ.

Let $j\colon D \hookrightarrow V$ be the embedding of (4-2). Note that we now have two functionals on D, namely λ and $f \circ j$. These two functionals coincide on Λ. By linearity, and since any element of D has a multiple that belongs to Λ, we must have

$$\lambda = f \circ j$$

as maps $D \to \mathbb{R}$. In particular, f is integer-valued on $j(D)$. On the other hand, by Lemma 4.5, $j(D)$ is dense in V. Therefore, f is integer-valued on all of V. But a linear, integer-valued functional on a real vector space is trivial. Hence, the original homomorphism $G/G^0 \to \mathbb{Z}$ is trivial, as desired. □

Before proceeding we need the following technical statement:

Lemma 4.8 *Let H be a connected abelian Lie group and let $\Delta \subseteq H$ be a torsion-free cocompact subgroup. Suppose $\psi\colon H \to H$ is an automorphism with $\psi(\Delta) \subseteq \Delta$ and such that $D := \bigcup_{n \geq 0} \psi^{-n}(\Delta)$ is dense in H. Then H is a vector group.*

Proof We proceed by induction on $\dim H$. The base case $\dim H = 0$ is trivial.

Suppose now that the statement is true for groups of dimension $< N$ for some $N \geq 1$. For $n \geq 0$, set $\Delta_n := \psi^{-n}(\Delta)$. Choose a sequence of open neighborhoods U_k of $e \in H$ such that the exponential map of H is a diffeomorphism onto U_k and $U_{k+1} \subseteq U_k$ and $\psi(U_{k+1}) \subseteq U_k$. For fixed k, consider the sequence

$$V_n^{(k)} := \operatorname{span} \log(\Delta_n \cap U_k)$$

of subspaces of the Lie algebra $\mathfrak{h}$ of H. Since $\Delta_n \subseteq \Delta_{n+1}$, we have $V_n^{(k)} \subseteq V_{n+1}^{(k)}$. Since $\mathfrak{h}$ is finite-dimensional, this sequence stabilizes to some subspace $V^{(k)} \subseteq \mathfrak{h}$. Since $U_{k+1} \subseteq U_k$, we have $V^{(k+1)} \subseteq V^{(k)}$. Therefore, the sequence $V^{(k)}$ stabilizes to some subspace V. Further, since $\psi(U_{k+1}) \subseteq U_k$, we have $\psi_*(V) \subseteq V$. Since ψ is injective and V is finite-dimensional, ψ restricts to an automorphism of $W := \exp(V)$.

We claim that W is a nontrivial closed vector subgroup of H. Indeed, because $D = \bigcup_n \Delta_n$ is dense, we know that for n and k sufficiently large, $\Delta_n \cap U_k \neq \varnothing$. Therefore, $V \neq 0$. This shows that W is nontrivial. Further, again for n and k sufficiently large, $\operatorname{span} \log(\Delta_n \cap U_k)$ is a cocompact lattice in V. Since H is abelian,

$$\exp\colon \mathfrak{h} \to H$$

is a morphism, so $\exp\operatorname{span}\log(\Delta_n \cap U_k)$ is a subgroup of Δ_n, which is discrete. This shows that

$$\exp\colon V \to H$$

is a proper map, and hence $W = \exp(V)$ is closed and, in addition, W is a vector subgroup.

We consider two cases: either Δ is contained in W or not. First suppose Δ is contained in W. Since ψ restricts to an automorphism of W, we have $D \subseteq W$. Since D is dense and W is closed, we find that $W = H$, and hence H is a vector group.

Finally, suppose Δ is not contained in W. Consider the projection map $q_W\colon H \to H/W$. Since W is a nontrivial closed vector subgroup, we see that H/W is a connected abelian Lie group with $\dim(H/W) < \dim H$. We want to apply the inductive hypothesis, so we verify the assumptions in Lemma 4.8:

- Since $\Delta \cap W$ is a cocompact lattice in W, the image of Δ in H/W is also a cocompact lattice.

- ψ restricts to an automorphism of W, and hence descends to some automorphism $\overline{\psi}$ of H/W.

- The union

$$\bigcup_{n \geq 0} \overline{\psi}^{-n}(\Delta/(\Delta \cap W)) = q_W(D)$$

  is dense in H/W because D is dense in H.

All that prevents us from applying the inductive hypothesis is that Δ may not be torsion-free. But Δ is a finitely generated abelian group, so its torsion subgroup $\operatorname{Tor}(\Delta)$ is a finite normal subgroup. It is $\overline{\psi}$–invariant, and since $\overline{\psi}$ is injective and $\operatorname{Tor}(\Delta)$ is finite, $\overline{\psi}$ restricts to an automorphism of $\operatorname{Tor}(\Delta)$. This allows us to consider $(H/W)/\operatorname{Tor}(\Delta)$ instead of H/W. The map

$$H/W \to (H/W)/\operatorname{Tor}(\Delta)$$

is a finite-degree cover, so one easily verifies from the above that Δ is a cocompact lattice, and the image of Δ is now torsion-free. Further, $\overline{\psi}$ descends to an automorphism and the image of D is dense.

By the inductive hypothesis applied to $H/W/\operatorname{Tor}(\Delta)$, we know that $H/W/\operatorname{Tor}(\Delta)$ is a vector group. The map $H/W \to H/W/\operatorname{Tor}(\Delta)$ is a finite connected covering of the

vector group $H/W/\mathrm{Tor}(\Delta)$, and hence an isomorphism. Therefore, H/W is a vector group. Thus, H is an extension of vector groups

$$1 \to W \to H \to H/W \to 1.$$

Since H is abelian, this sequence splits. Therefore, H is also a vector group. $\qquad\square$

Lemma 4.8 readily applies to our situation and shows that G^0 is a vector group. This finishes our investigation of G, and we return to studying M:

Lemma 4.9 *There is a finite cover $M' \to M$ such that:*

(i) *$p\colon M \to M$ lifts to a strongly regular self-cover $p'\colon M' \to M'$.*

(ii) *Let F' be the asymptotic deck group of p'. Then $\overline{F'}$ is a compact connected torus that acts freely on M'.*

Proof Write $\Lambda_0 := \Lambda \cap G^0$. Since G has finitely many components by Lemma 4.7, Λ_0 is of finite index in Λ. Hence, $M' := \widehat{M}/\Lambda_0$ is a finite cover of M. We claim that M' has the required properties. First let us show that $p\colon M \to M$ lifts to a strongly regular cover $p'\colon M' \to M'$. The lift p' exists precisely when $\overline{\varphi}(\Lambda_0) \subseteq \Lambda_0$. This holds because $\overline{\varphi}$ is given by conjugation by $\widehat{p}$, and hence preserves G^0. Further, note that p' is also strongly regular because Λ_0 is abelian, so that $\overline{\varphi}^n(\Lambda_0)$ is automatically normal in Λ_0.

It remains to analyze the asymptotic deck group F' of p'. Write $D_0 := D \cap G^0$. Then we have $F' = D_0/\Lambda_0$, and $\overline{F'} = G^0/\Lambda_0$. In particular, $\overline{F'}$ is a connected torus. We need to show that $\overline{F'}$ acts freely on M'.

Let $x_0 \in M$ and choose $\overline{x}_0 \in \overline{M}$ that projects to x_0. Let S be the stabilizer in $\overline{F'}$ of x_0. Because S fixes a point, the S–action lifts to an S–action on $\widehat{M}$ fixing $\overline{x}_0$ (even if S is not connected!). The group of lifts of $\overline{F'}$ to $\widehat{M}$ is exactly G^0, so we have an embedding $S \hookrightarrow G^0$. But since G^0 is a vector group, it has no nontrivial compact subgroups, so we see that S is trivial, as desired. $\qquad\square$

Armed with Lemma 4.9, we can finish the proof of Theorem 1.11.

Proof of Theorem 1.11 Let $p'\colon M' \to M$ be the cover produced by Lemma 4.9. Let $p'\colon M' \to M'$ be a lift of p and set $T := \overline{F'}$. Since T acts smoothly and freely on M', the map natural quotient map $M' \to M'/T$ is a smooth principal T–bundle.

By Lemma 4.3, applied to p', we have a surjective map $\bar{\alpha}\colon T \to T$ such that p' is $\bar{\alpha}$–equivariant, ie for any $x \in M'$ and $g \in T$, we have

$$p'(gx) = \bar{\alpha}(g)p'(x).$$

So we see that p' is a bundle map, and, identifying the T–orbits of x and $p(x)$ with T, the restriction of p' to a T–orbit is exactly given by $\bar{\alpha}$. The map $\bar{\alpha}$ is linear; indeed, $\bar{\alpha}$ by its very definition lifts to the map $\bar{\varphi}\colon G^0 \to G^0$. As noted above, Lemma 4.8 implies that G^0 is a vector group, and $\bar{\varphi}$ is a linear map with determinant $\pm d$. Therefore, p' restricts to a linear cover of degree d on each T–orbit (which can be identified with $T = G^0/\Lambda_0$).

It only remains to show that we have $\pi_1(M') \cong \pi_1(T) \times \pi_1(M'/T)$. To obtain this splitting, we consider the long exact sequence on homotopy groups for the fiber bundle $M' \to M'/T$. Because $\pi_2(T) = 0$, the end of this long exact sequence reduces to

$$1 \to \pi_1(T) \to \pi_1(M') \to \pi_1(M'/T) \to 1.$$

We claim this sequence splits trivially. To show this, we produce a left splitting $\pi_1(M') \to \pi_1(T)$. Note that since $T = G^0/\Lambda_0$, we have $\pi_1(T) = \Lambda_0$. The left splitting is then just the map $q\colon \pi_1(M') \to \Lambda_0$ produced by Theorem 1.4. This completes the proof of Theorem 1.11.   □

Finally, we establish Theorem 1.12, which characterizes strongly regular self-covers of nilmanifolds.

Proof of Theorem 1.12 Let $M = H/\Gamma$ be a closed nilmanifold, where H is a simply connected nilpotent Lie group and $\Gamma \subseteq H$ is a cocompact lattice, and let $p\colon M \to M$ be a nontrivial strongly regular self-cover. We want to show that there exists a finite cover M' of M that is isomorphic (as a nilmanifold) to $N \times T$ for some nilmanifold N and a torus T. We proceed in three steps: First we show that p is homotopic to an algebraic endomorphism f of M which is necessarily still a strongly regular self-cover. Next, we use that f is algebraic to show that the asymptotic deck group will act by left-translations on $M = H/\Gamma$, which will imply that f is smoothly proper. Finally, we appeal to Theorem 1.11 to obtain that a finite cover M' of M is a torus bundle. We show that in the case of nilmanifolds, this bundle has to be trivial.

Step 1 (p is homotopic to an algebraic endomorphism) Consider the map $\varphi = p_*\colon \Gamma \hookrightarrow \Gamma$. Maltsev proved that lattices in nilpotent groups are superrigid in the

class of nilpotent groups (see eg [32, Theorem II.2.11]), ie if L is a simply connected nilpotent group then any homomorphism $\rho\colon \Gamma \to L$ uniquely extends to a continuous homomorphism $\tilde{\rho}\colon H \to L$.

Hence, the map $\varphi\colon \Gamma \hookrightarrow \Gamma$ uniquely extends to a continuous homomorphism $H \to H$, which we will also denote by φ. Further, since $\varphi(\Gamma)$ is of finite index in Γ, we know that $\varphi(\Gamma)$ is also a lattice in H. Therefore, we can also apply Maltsev's superrigidity theorem to the inverse map

$$\varphi^{-1}|_{\varphi(\Gamma)}\colon \varphi(\Gamma) \to \Gamma.$$

The uniqueness in Maltsev's superrigidity theorem implies that the extension of the inverse is inverse to $\varphi\colon H \to H$. Hence, φ extends to an automorphism of H. Therefore, the map

$$f\colon M = H/\Gamma \xrightarrow{\varphi} H/\varphi(\Gamma) \to H/\Gamma$$

is a covering map. We claim that f is homotopic to p. Indeed, since M is aspherical, it suffices to show that f and p induce the same map on fundamental groups. It is easy to see that $f_* = \varphi\colon \Gamma \hookrightarrow \Gamma$, so f and p are indeed homotopic.

Step 2 (f is smoothly proper) The advantage of working with f instead of p is the ability to control the action of the asymptotic deck group. Namely, it is clear that $\varphi\colon H \to H$ is a lift of f to the universal cover H of M. Hence, for any $n \geq 1$ the deck group of f^n is given by $\varphi^{-n}(\Gamma)/\Gamma$ acting by left-translations on $M = H/\Gamma$. Consider the closed subgroup $G := \overline{\bigcup_n \varphi^{-n}(\Gamma)}$ of H. It is easy to verify that the closure of the asymptotic deck group F in $\mathrm{Diff}(M)$ is given by G/Γ and hence is compact.

Step 3 (end of the proof) Since f is smoothly proper, Theorem 1.11 applies and shows that there exists a finite cover M' of M and a compact torus T acting on M such that

$$(4\text{-}5) \qquad\qquad T \to M' \to M'/T$$

is a fiber bundle. Write $N := M'/T$ for the quotient. Note that T is the closure of the asymptotic deck group in $\mathrm{Diff}(M')$ and hence acts by nilmanifold automorphisms. Therefore, N is also a nilmanifold. Write $\tilde{T}$ and $\tilde{N}$ for the universal covers of T and N, respectively. Note that $\tilde{T}$ is a vector group and $\tilde{N}$ is a simply connected nilpotent Lie group.

We know that $\pi_1(M') = \pi_1(T) \times \pi_1(N)$ by Theorem 1.11(ii). We claim that $H \cong \tilde{T} \times \tilde{N}$. Indeed, lifting the fiber bundle of (4-5) to the universal cover, we have a short exact sequence of Lie groups

$$(4\text{-}6) \qquad\qquad 1 \to \tilde{T} \to H \to \tilde{N} \to 1.$$

It suffices to show this sequence splits by producing a left-splitting $H \to \tilde{T}$. Note that we have such a map on the lattice $\pi_1(M') \subseteq H$ since $\pi_1(M') \cong \pi_1(T) \times \pi_1(N)$ projects to the first factor. Then, by Maltsev's superrigidity for nilpotent Lie groups (see Step 1), this projection extends uniquely to a continuous homomorphism $H \to \tilde{T}$. It is easy to check this produces the required left-splitting of the sequence in (4-6).

It only remains to show that the automorphism $\tilde{f}$ of H that lifts f preserves the subgroup $\tilde{N}$. To see this, first recall the following fact about subgroups of nilpotent Lie groups:

Lemma 4.10 (see eg [32, II.2.5]) *Any subgroup L of a simply connected nilpotent Lie group is contained in a unique minimal connected closed subgroup $\Delta(L)$.*

We use this lemma in the following manner: Observe that $\tilde{f}$ restricts to an automorphism of $\pi_1(N)$, which is a lattice in $\tilde{N}$. It follows that $\tilde{N} = \Delta(\pi_1(N))$. Indeed, $\tilde{N}$ is a connected closed subgroup containing $\pi_1(N)$. Further, for any connected closed subgroup $\Delta' \subseteq \tilde{N}$ containing $\pi_1(N)$, we see that $\Delta'/\pi_1(N)$ is a compact aspherical manifold with fundamental group $\pi_1(N)$, and hence $\dim \Delta'$ is equal to the cohomological dimension of $\pi_1(N)$. But the same holds for $\tilde{N}$, so we must have that $\dim \Delta' = \dim \tilde{N}$. This shows that $\Delta' = \tilde{N}$ and hence $\tilde{N}$ is minimal among all connected closed subgroups containing $\pi_1(N)$.

Since $\tilde{f}$ is an automorphism of $\pi_1(N)$, and by the uniqueness of $\Delta(\pi_1(N))$, we see that $\tilde{f}$ restricts to an automorphism of $\Delta(\pi_1(N)) = \tilde{N}$, as desired. $\qquad\square$

5 Holomorphic endomorphisms of Kähler manifolds

Suppose M is a Kähler manifold and $p\colon M \to M$ is a holomorphic, strongly regular, nontrivial self-cover.

Lemma 5.1 *p is smoothly proper.*

Proof Note that the asymptotic deck group F of p acts by biholomorphisms. Boothby, Kobayashi and Wang [6] proved that the group of biholomorphisms of a closed almost complex manifold is a Lie group (possibly with infinitely many components).

Let $\mathrm{Aut}(M)$ denote the group of biholomorphisms of M and let $\mathrm{Aut}^0(M)$ denote the connected component of $\mathrm{Aut}(M)$ that contains the identity element. Fujiki and Liebermann independently proved that $\mathrm{Aut}^0(M)$ is a subgroup of finite index in the kernel of the action of $\mathrm{Aut}(M)$ on homology [13; 22].

Note that the action of F on homology factors through a finite group. Namely, the action on homology is given by a map $F \to \mathrm{GL}(H_*(M, \mathbb{Z}))$. Since $H_*(M, \mathbb{Z})$ has finite rank, there is a bound on the order of torsion elements of $\mathrm{GL}(H_*(M, \mathbb{Z}))$. It follows that any subgroup of $\mathrm{GL}(H_*(M, \mathbb{Z}))$ every one of whose elements has arbitrarily large roots is finite. Since F is a finite product of finite cyclic and quasicyclic groups, F is generated by elements with arbitrarily large roots. Hence, the image of F in $\mathrm{GL}(H_*(M, \mathbb{Z}))$ is finite.

It follows that $F_0 := F \cap \mathrm{Aut}^0(M)$ is of finite index in F. Let $H := \overline{F_0}$ be the closure of F_0 in $\mathrm{Aut}^0(M)$. Since $\mathrm{Aut}(M)$ is a closed subgroup of $\mathrm{Diff}(M)$, it follows that H is also the closure of F_0 in $\mathrm{Diff}(M)$. Therefore, to show that p is proper, it suffices to show that H is compact. Note that H is an abelian Lie group (not necessarily connected). We will again use the classification by Wüstner of closed abelian subgroups of connected Lie groups [38]. In particular, he showed that the torsion subgroup of $\pi_0 H$ is finite. But $\pi_0 H$ is torsion (because $F_0 \subseteq H$ is dense and torsion), so $\pi_0 H$ is finite. We conclude that H has finitely many components.

Let H^0 be the connected component of the identity of H, and set $F_{00} := F_0 \cap H^0$. Since H^0 is a connected abelian Lie group, there are nonnegative integers $k, l \geq 0$ such that $H^0 \cong T^k \times \mathbb{R}^l$. Since F_{00} is dense in H^0, the image of F_{00} in $H^0/T^k \cong \mathbb{R}^l$ is dense. On the other hand, F_{00} is torsion and $\mathbb{R}^l$ is torsion-free, so the image of F_{00} in $\mathbb{R}^l$ must be trivial. Hence, we must have that $l = 0$, so that H^0 is compact. Since H has finitely many components, it follows that H is compact, as desired. □

We finish the proof of Theorem 1.13. Since p is smoothly proper, Theorem 1.11 shows that there is a finite cover $M' \to M$ such that p lifts to a smoothly proper, strongly regular self-cover $p' \colon M' \to M'$, and M' is a smooth principal T–bundle $M' \to N$. Here $T = \overline{F'}$ is the closure of the asymptotic deck group of p'. Since M is Kähler and p is holomorphic, F' acts on M' by biholomorphisms. Since the biholomorphism group is closed in the group of all diffeomorphisms, T also acts on M' by biholomorphisms, which shows that the bundle $M' \to N$ is holomorphic.

As in the previous section, we know that p' descends to a diffeomorphism of N. Since p' is holomorphic and $M' \to N$ is a holomorphic bundle, the induced diffeomorphism

of N is also holomorphic. In addition, on each T–orbit, p restricts to a holomorphic automorphism of degree d.

It remains to show that N is Kähler and the bundle $M' \to N$ is trivial. For this we need results of Blanchard [5]. He showed that if the total space of a holomorphic fiber bundle is Kähler, then so is the base. In addition he gives criteria for the bundle to admit a flat holomorphic connection. In the case of a holomorphic principal torus bundle with Kähler total space, the results of Blanchard can be used to show the existence of a flat holomorphic connection; a short independent proof of the existence of a flat connection for Kähler principal torus bundles has been given by Biswas [4].

The principal flat T–bundle $M' \to N$ is characterized by its monodromy, which is a representation

$$\rho \colon \pi_1(N) \to T,$$

where we write $T = V/\bar\Lambda$ for a complex vector space V and a cocompact lattice $\bar\Lambda$. Recall that two flat principal T–bundles over N with monodromies ρ_0 and ρ_1 are equivalent if and only if ρ_0 and ρ_1 are isotopic, ie ρ_0 and ρ_1 belong to the same path component of $\mathrm{Hom}(\pi_1(N), T)$. Our goal is to show that, after passing to a further finite cover, ρ is isotopic to the trivial map.

Since ρ takes values in the abelian group T, we have a factorization of ρ through the abelianization $\Delta := \pi_1(N)^{\mathrm{ab}}$ of $\pi_1(N)$. Since Δ is a finitely generated abelian group, we can write

$$\Delta = \mathrm{Tor}(\Delta) \oplus \Delta_\infty,$$

where $\Delta_\infty \subseteq \Delta$ is a free abelian group. If Δ was torsion-free, we could directly isotope ρ to the trivial map. Indeed, for any $k \geq 0$, we have

(5-1) $\mathrm{Hom}(\mathbb{Z}^k, T) \cong T^k.$

Therefore, the following claim will suffice to finish the proof:

Claim 5.2 *There exists a finite cover $M'' \to M'$ of M' such that*

 (i) *p' lifts to a strongly regular self-cover $p'' \colon M'' \to M''$,*

 (ii) *the action of T on M' lifts to an action of T on M'', and*

 (iii) *the monodromy of $M'' \to M''/T$ is given by $\rho|_{\pi_1(M'')}$ and factors through a torsion-free group.*

Proof We need to investigate the splitting $\pi_1(M') = \pi_1(T) \times \pi_1(N)$ more carefully. This splitting is not canonical, but we claim that nevertheless we can choose a splitting that is φ–invariant. Indeed, we identify $\pi_1(T)$ with a subgroup of $\pi_1(M')$ by using an orbit map

$$\theta_{x_0}: T \to M', \quad g \mapsto g x_0,$$

for some fixed $x_0 \in M'$. Any two orbit maps induce the same map on fundamental groups, so the image of θ_{x_0} does not depend on x_0. The image $\theta_{x_0*}\pi_1(T)$ is then φ–invariant because p' maps orbits to orbits.

Further, we can identify $\pi_1(N)$ with the stable image $\varphi^\infty(\pi_1(M'))$ of $\pi_1(M')$ because the composition

$$\varphi^\infty(\pi_1(M')) \hookrightarrow \pi_1(M') \to \pi_1(N)$$

is an isomorphism. The inverse $\pi_1(N) \to \varphi^\infty(\pi_1(M'))$ is therefore a section of $\pi_1(M') \to \pi_1(N)$. So

$$\pi_1(M') \cong \theta_{x_0*}\pi_1(T) \times \varphi^\infty(\pi_1(M'))$$

is the desired φ–invariant splitting. Our next goal is to define the cover $M'' \to M'$.

Identifying $\pi_1(N)$ with $\varphi^\infty(\pi_1(M'))$, we have that φ restricts to an automorphism of $\pi_1(N)$. Therefore, φ descends to some automorphism φ_Δ of the abelianization Δ of $\pi_1(N)$. As above we write

$$\Delta = \mathrm{Tor}(\Delta) \oplus \Delta_\infty.$$

Note that φ_Δ preserves $\mathrm{Tor}(\Delta)$, but does not necessarily preserve Δ_∞. Write $C := [\Delta : \Delta_\infty]$, and let Δ_C be the intersection of all subgroups of Δ of index C. Then Δ_C is torsion-free (because it is contained in Δ_∞). Further, Δ_C is a finite-index characteristic subgroup of Δ, and hence φ_Δ restricts to an automorphism of Δ_C.

Let Γ'' be the preimage in $\pi_1(M')$ of Δ_C under the composition

$$\pi_1(M') \to \pi_1(N) \to \Delta.$$

Then Γ'' is a finite-index subgroup of $\pi_1(M')$ and hence corresponds to a cover $M'' \to M'$. The self-cover $p': M' \to M'$ lifts to a self-cover $p'': M'' \to M''$ because $\varphi(\Gamma'') \subseteq \Gamma''$. This shows that $M'' \to M'$ satisfies property (i) of the claim.

Our next task is to show that $M'' \to M'$ satisfies (ii), ie the action of T on M' lifts to an action of T on M''. Consider the group of lifts H of T to M''. Then H is an

extension given by

$$1 \to \pi_1(M')/\pi_1(M'') \to H \to T \to 1.$$

Let H^0 be the component of H containing the identity. Then the action of T on M' lifts to an action of T on M'' precisely when the map $H \to T$ restricts to an isomorphism $H^0 \to T$. In turn, $H^0 \to T$ is an isomorphism precisely when the induced map $\pi_1(H^0) \to \pi_1(T)$ is an isomorphism.

Note that $\pi_1(H^0) \to \pi(T)$ is automatically injective (because $H^0 \to T$ is a cover). To show it is also surjective, consider an orbit map $\theta \colon H^0 \to M''$. Then the map on fundamental groups induced by θ is a map

$$\pi_1(H^0) \to \Gamma'' \cong \pi_1(T) \times \pi_1(N'')$$

such that the square

$$
\begin{array}{ccc}
\pi_1(H^0) & \longrightarrow & \Gamma'' \cong \pi_1(T) \times \pi_1(N'') \\
\downarrow & & \downarrow \\
\pi_1(T) & \longrightarrow & \Gamma \cong \pi_1(T) \times \pi_1(N)
\end{array}
$$

commutes. This shows that $\pi_1(H^0)$ surjects onto the first factor of $\pi_1(T) \times \pi_1(N)$, so $\pi_1(H^0) \to \pi_1(T)$ is surjective. We already had injectivity, so $\pi_1(H^0) \to \pi_1(T)$ is an isomorphism, as desired.

Finally we show that $M'' \to M''/T$ has trivial monodromy. Indeed, let ρ'' be the monodromy of $M'' \to M''/T$. Then ρ'' is just given by the composition

$$\Gamma'' \hookrightarrow \Gamma' \xrightarrow{\rho} T.$$

Obviously ρ factors through the abelianization $\Gamma' \to \Delta$ and descends to a map

$$\rho_\Delta \colon \Delta \cong \Delta_\infty \oplus \mathrm{Tor}(\Delta) \to T,$$

and we have already seen in (5-1) that ρ_Δ is trivial on Δ_∞ (up to isotopy). On the other hand, we defined M'' so that the composition of the natural inclusion and the abelianization map

$$\pi_1(M''/T) \hookrightarrow \pi_1(M'/T) \xrightarrow{\mathrm{ab}} \Delta$$

has image Δ_C. Since $\Delta_C \subseteq \Delta_\infty$, it follows that ρ'' is trivial (up to isotopy), as desired. This completes the proof of Theorem 1.13. $\qquad\square$

6 Residual self-covers

In this section we prove Corollaries 1.15 (about residual self-covers) and 1.18 (which relates strongly regular expanding maps and linear expanding maps on tori).

Proof of Corollary 1.15 Let X be a finite CW–complex and $p \colon X \to X$ a strongly regular, residual self-cover. As usual we write $\Gamma = \pi_1(X)$ and $\varphi := p_*$ for the map induced by p on fundamental groups. We need to show that Γ is abelian.

By Theorem 1.4, there exists a surjection

$$q \colon \pi_1(X) \to \mathbb{Z}^n$$

(for some $n \geq 1$) such that φ restricts to an automorphism on $\ker(q)$. Therefore, we have $\ker(q) \subseteq \varphi^m(\Gamma)$ for every $m \geq 1$. Since p is residual, we have $\bigcap_m \varphi^m(\Gamma) = 1$, so $\ker(q)$ must be trivial. This shows that q is an isomorphism, as desired.

Now suppose that X is a closed aspherical manifold. Then X is a closed aspherical manifold with abelian fundamental group. The Borel conjecture for manifolds with abelian fundamental groups exactly states that such a manifold is homeomorphic to a torus. It has been proven: In dimensions 1 and 2 it is immediate from the classification of manifolds in these dimensions. In dimensions greater than 4 it is due to Farrell and Hsiang [9]. In dimension 4 the Borel conjecture is closely related to Freedman's disk theorem: a group is called *good in the sense of Freedman* if the disk theorem holds for that group. Freedman proved that free abelian groups are good, and established the Borel conjecture for these groups as a corollary [12, Theorem 8]. In dimension 3, Kreck and Lück [21, Theorem 0.7] have proved that the Borel conjecture is true if Thurston's geometrization conjecture holds. The geometrization conjecture has been proven by Perelman [29; 30; 31] (see eg [24; 25; 20] for expositions of the proof). □

Proof of Corollary 1.18 An argument of Shub shows that any manifold admitting an expanding map is aspherical [34]; namely, p lifts to a homeomorphism $\widetilde{p}$ of $\widetilde{M}$. Now fix any $x \in \widetilde{M}$ and $0 < r < \mathrm{injrad}(M)$. Then $B(x, r) \subseteq \widetilde{M}$ is an embedded ball and

$$\widetilde{M} = \bigcup_{n \geq 0} \widetilde{p}^n(B(x, r))$$

is a union of embedded balls and hence contractible.

Further, it is easy to see that p is a self-cover because it is a local diffeomorphism of the closed manifold M. Since p is expanding, it has to be a residual self-cover:

Indeed, $\widetilde{p}$ is $p_*\colon \pi_1(M) \to \pi_1(M)$ equivariant, and hence

$$p_*^n(\pi_1(M)) = \{\gamma \in \pi_1(M) \mid \gamma x \in \widetilde{p}^n(\pi_1(M) \cdot x)\}.$$

Since $\widetilde{p}$ is globally expanding and the orbit $\pi_1(M) \cdot x$ is a discrete subset of $\widetilde{M}$, it follows that for every $\gamma \in \pi_1(M)$ there exists $n \geq 1$ such that $\gamma x \notin \widetilde{p}^n(\pi_1(M) \cdot x)$.

We characterized nontrivial, strongly regular, residual self-covers in Corollary 1.15. Using the part of this result that applies to aspherical manifolds, we find that M is finitely (topologically) covered by a torus T and p lifts to an expanding map $p'\colon T \to T$ (with respect to a potentially exotic smooth structure). This immediately implies that p' is homotopic to a linear expanding map on T. A well-known result of Shub [34] implies that p' is not just homotopic, but also topologically conjugate to a linear expanding map on T.   □

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