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Lower bounds for Lyapunov exponents of flat bundles on curves

ALEX ESKIN
MAXIM KONTSEVICH
MARTIN MÖLLER
ANTON ZORICH

Consider a flat bundle over a complex curve. We prove a conjecture of Fei Yu that the sum of the top k Lyapunov exponents of the flat bundle is always greater than or equal to the degree of any rank-k holomorphic subbundle. We generalize the original context from Teichmüller curves to any local system over a curve with nonexpanding cusp monodromies. As an application we obtain the large-genus limits of individual Lyapunov exponents in hyperelliptic strata of abelian differentials, which Fei Yu proved conditionally on his conjecture.

Understanding the case of equality with the degrees of subbundle coming from the Hodge filtration seems challenging, eg for Calabi–Yau-type families. We conjecture that equality of the sum of Lyapunov exponents and the degree is related to the monodromy group being a thin subgroup of its Zariski closure.

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To the memory of Jean-Christophe Yoccoz

1 Introduction

Lyapunov exponents are dynamical analogs of characteristic numbers of vector bundles. The Lyapunov exponents for the Teichmüller geodesic flow relate the dynamics on moduli space with the dynamics on flat surfaces. Efficiently computing them is currently still a challenge, both for strata of the moduli space of flat surfaces and for Teichmüller curves, including all the Teichmüller curves generated by square-tiled surfaces. Starting with Kontsevich [20] it was realized that the *sum* of (ie the sum of the positive) Lyapunov exponents equals the normalized degree of the Hodge bundle on Teichmüller curves; see Forni [14], Krikorian [23], Bouw and Möller [3] and Eskin, Kontsevich and Zorich [11] for versions of this formula, including the case of strata. This observation generalizes from the variation of Hodge structures over Teichmüller curves to any

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weight-1 variation of Hodge structures (VHS). Presently, irreducible summands of weight-1 VHS are the only instances where such degree formulas are known. Even the computation of Filip [13] of the top Lyapunov exponent for families of K3 surfaces can be subsumed under this observation, if one refers to his proof using the Kuga–Satake construction.

The main result of this paper is that an *inequality* for the sum of the top k Lyapunov exponents holds in great generality. This was first conjectured by Fei Yu [37], but the scope given here is more general.

Let $C = \mathbb{H}/\Gamma$ be a hyperbolic Riemann surface of finite area (or equivalently, a complex quasiprojective curve) with a representation $\rho\colon \pi_1(C) \to \mathrm{GL}(V)$ such that, if C is noncompact, the monodromies around the cusps $\Delta = \overline{C} \setminus C$ are *nonexpanding*, ie all the eigenvalues lie on the unit circle. This assumption is necessary and also sufficient for Oseledets' theorem; see Sections 2.4 and 2.5. To be more precise, we need to specify a norm on the flat bundle $\mathbb{V}$ determined by ρ. There are two natural choices: the practical choice (for simulations) is a "constant" norm obtained by parallel transport along a Dirichlet fundamental domain for Γ, and the sophisticated choice of an *admissible* norm (see Section 2.3 for the precise definition) that has the right growth at the cusps and compatibility with exterior powers. Oseledets' theorem is very insensitive to such choices: we show (Theorem 2.1 and Proposition 2.2; see also the appendix for the background on measurable cocycles) that both norms satisfy the integrability condition and compute the same Lyapunov exponents.

For VHS of arbitrary weight we show that the Hodge norm is admissible. Along with the proof (Proposition 3.1) we give an upper bound for the Lyapunov exponents that is uniform for all VHS of given weight and rank. However, our estimate is very crude. It is an interesting problem to prove tight upper bounds for Lyapunov exponents for VHS.

In the setting of a local system $\mathbb{V}$ defined by ρ and a norm as above, we can now state our main result, Theorem 4.1:

Theorem *The parabolic degree of any holomorphic rank-k line bundle $\mathcal{E}$ of the Deligne extension of $\mathbb{V}$ provides a lower bound for the sum of the top k Lyapunov exponents, when normalized as follows:*

$$\sum_{i=1}^{k} \lambda_i \geq \frac{2\deg_{\mathrm{par}}(\mathcal{E})}{2g(\overline{C}) - 2 + |\Delta|}. \tag{1}$$

Here $g(\overline{C})$ is the genus of the curve $\overline{C}$ and $|\Delta|$ is the number of cusps.

Here the parabolic degree $\deg_{\mathrm{par}}$ of a vector bundle is equal to the degree in the case of unipotent monodromies and is defined in Section 2.1 in general.

A theorem in a similar spirit in rank two was proven previously by Deroin and Dujardin [8]. The main theorem of their subsequent paper [6] proves a formula similar to (1) using Brownian motion techniques, applicable also to a higher-dimensional base provided that the base is compact.

This theorem has two types of applications. The first is the large-genus limit of Lyapunov exponents for hyperelliptic strata of abelian differentials (Corollary 5.3, proven by Yu conditionally on our main theorem).

As preparation for our second application, we show in the last section that the parabolic degrees of the Hodge bundles of hypergeometric local systems can be easily expressed in terms of the local exponents; see Section 6.4 for the notions and Theorem 6.1 for the precise statement.

This second application concerns families of Calabi–Yau threefolds and conjecturally gives new cases where equality in (1) holds. There is a well-known list of fourteen rank-4 hypergeometric local systems, including the mirror quintic (see Table 1), that could be the middle cohomology of a family of Calabi–Yau threefolds with $h^{2,1} = 1$. In seven out of these fourteen examples the monodromy group is thin in the symplectic group (see Brav and Thomas [4], Singh and Venkataramana [34] and Section 6).

Conjecture *The inequality (1) becomes an equality precisely in the seven out of these fourteen cases where the monodromy group is thin.*[1]

Initially, we stated a more optimistic conjecture on a region in the parameter space for the local exponents where the equality is attained. This initial conjecture can no longer be upheld after more detailed numerical experiments by Fougeron [15]. We discuss this in more detail in Section 6.

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[1] Simion Filip has recently announced a proof of this conjecture.

2 Lyapunov exponents for flat bundles with nonexpanding cusp monodromies

In this section we show that Lyapunov exponents for flat bundles over the geodesic flow on a (base) curve are defined for a very large class of flat bundles. The only restriction that we impose is that the monodromies around the boundary points have eigenvalues of absolute value 1. This (strictly) includes the case of quasi-unipotent monodromies.

We now give the background and the definitions alluded to above. Our base manifold will always be an algebraic curve $C = \Gamma\backslash\mathbb{H}$, not necessarily compact. Let $\overline{C}$ be the smooth compactification and $\Delta = \overline{C} \setminus C$ be the boundary points. The flow will always be the unit-speed geodesic flow g_t on the unit tangent bundle T^1C for the metric of constant curvature -4 (see Remark 4.4 for the history of this convention) and μ will be the corresponding invariant probability measure.

Let $\mathbb{V}$ be a flat bundle over C of rank r. We will denote by $(\mathcal{V}_C, \nabla)$ the associated vector bundle with its flat connection. We say that $\mathbb{V}$ has *nonexpanding cusp monodromies* if, for each element $\gamma \in \pi_1(C, c_0)$ homotopic to a simple loop around a point in Δ, all the eigenvalues of $\rho(\gamma)$ have absolute value 1. Recall that $\mathbb{V}$ has *quasi-unipotent* monodromies if, for each element $\gamma \in \pi_1(C, c_0)$ homotopic to a simple loop around a point in Δ, there exists some n such that $\rho(\gamma)^n - \mathrm{Id}$ is nilpotent. Consequently, having quasi-unipotent monodromy implies nonexpanding cusp monodromies. We show in Sections 2.4 and 2.5 that this condition is necessary and sufficient for integrability of the flat bundle $\mathbb{V}$.

The remaining ingredient we need for the definition of a Lyapunov spectrum is a norm $\|\cdot\|$ on $\mathbb{V}$. We will define in Section 2.3 a notion of *admissible metric* h that we can provide any local system with and that is suitable for metric extensions of line bundles to $\overline{C}$. Such a metric is also the basis for defining Lyapunov exponents for the flat bundle $\mathbb{V}$. These two notions will be our main hypotheses for the existence of Lyapunov exponents for flat bundles. Our aim is to show the following norm bound for the lift G_t of the geodesic flow g_t to $\mathbb{V}$.

Theorem 2.1 *If $\mathbb{V}$ is a flat bundle of $\mathbb{C}$–rank r on C such that the eigenvalues of monodromy around points in Δ all have absolute value 1, then for any admissible metric on $\mathbb{V}$ the induced cocycle is integrable (in the sense of Definition A.1). The corresponding Lyapunov exponents $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ are independent of the choice of admissible metric.*

For practical purposes (eg for numerical simulations) it is useful to be able to compute the Lyapunov exponents with a simpler norm. We define a *constant norm* $\|\cdot\|_{\mathrm{const}}$ on $\mathbb{V}$ to be the parallel transport of any norm at the fiber over some base point c_0 extended to a Dirichlet fundamental domain for Γ on $\mathbb{H}$, or equivalently, on a simply connected complement of some geodesic "boundary" curves in C. Note that the "constant" norm is not continuous across these boundary curves and depends on the choice of the Dirichlet domain.

Proposition 2.2 *Any constant norm $\|\cdot\|_{\mathrm{const}}$ on a flat bundle as in Theorem 2.1 is also integrable and computes the same Lyapunov exponents as any admissible metric.*

2.1 Parabolic bundles and filtered vector bundles

We begin with the definition of a parabolic bundle (see also [24], [31] for the origins of this notion). We first define a $[0,1)$–*filtration* on a complex vector space V to be a collection of (real) weights $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_n < \alpha_{n+1} = 1$ for some $n \ge 1$ together with a filtration of vector subspaces

$$F^\bullet\colon V = V^{\ge \alpha_1} \supsetneq V^{\ge \alpha_2} \supsetneq \cdots \supsetneq V^{\ge \alpha_{n+1}} = V^{\ge 1} = 0.$$

We denote by $\mathrm{gr}_{\alpha_i} V$ the graded piece at weight α_i. The *filtered dimension* of $(V, F^\bullet)$ is defined to be the real number

$$\dim_{F^\bullet}(V) = \sum_{i=1}^{n} \alpha_i \dim \mathrm{gr}_{\alpha_i}(V).$$

The filtration is called *trivial* if $n = 1$ and $\alpha_1 = 0$. This is equivalent to the condition $\dim_{F^\bullet}(V) = 0$.

Let $\mathcal{E}$ be a holomorphic vector bundle on a complex curve $\overline{C}$ and let Δ be a finite set of "boundary" points. A *parabolic structure* $(\mathcal{E}, F^\bullet)$ on $\mathcal{E}$ (with respect to Δ) is a $[0,1)$–filtration $F^\bullet\mathcal{E}_c$ on the fiber $\mathcal{E}_c$ for each $c \in \Delta$. A *parabolic bundle* is simply a holomorphic vector bundle with a parabolic structure.

The *parabolic degree* of $(\mathcal{E}, F^\bullet)$ is defined to be

$$\deg_{\mathrm{par}}(\mathcal{E}, F^\bullet) = \deg(\mathcal{E}) + \sum_{c \in \Delta} \dim_{F^\bullet} \mathcal{E}_c.$$

A *morphism* φ between parabolic bundles $\mathcal{E}$ and $\mathcal{F}$ is a morphism $\varphi\colon \mathcal{E} \to \mathcal{F}$ of holomorphic vector bundles such that for each $c \in \Delta$ and each weight α of $\mathcal{E}_c$, the image $\varphi(\mathcal{E}_c^{\ge \alpha})$ lies in $\mathcal{F}_c^{\ge \beta}$ whenever $\beta \le \alpha$. A *parabolic subbundle* $\mathcal{E}$ of $\mathcal{F}$ is an injective morphism of parabolic bundles with the additional requirements that for each $c \in \Delta$ the weights of $\mathcal{E}$ are a subset of the weights of $\mathcal{F}$ and if β is maximal such that $\varphi(\mathcal{E}_c^{\geq\alpha}) \subseteq \mathcal{F}_c^{\geq\beta}$ then $\beta = \alpha$.

With these notions of degree and subbundles we will recall later that the usual notions of stability and of the Harder–Narasimhan filtration carry over verbatim to the parabolic case.

For taking exterior powers it will be convenient to use the following equivalent notion. A *filtered vector bundle* $\mathcal{E} = \{\mathcal{E}_{\bullet,\bullet}\}$ on $\overline{C}$ is a collection $\mathcal{E}_{c,\alpha}$ of vector bundles in $j_*^c\mathcal{E}_C$ for every $c \in \Delta$ and every $\alpha \in \mathbb{R}$ (where $j^c\colon C \to C \cup \{c\}$ is the inclusion) such that the filtration is descending ($\mathcal{E}_{c,\alpha} \subseteq \mathcal{E}_{c,\beta}$ if $\alpha \geq \beta$), right continuous ($\mathcal{E}_{c,\alpha+\varepsilon} = \mathcal{E}_{c,\alpha}$ for small ε) and such that $\mathcal{E}_{c,\alpha+1} = t\mathcal{E}_{c,\alpha} \subset \mathcal{E}_{c,\alpha}$, where t is a local parameter at c. To retrieve the corresponding bundle with parabolic structure we take the extensions $\mathcal{E}_{c,0}$ at every point $c \in \Delta$ and the filtrations given by the $\alpha \in [0,1)$ where the rank of the fibers of $\mathcal{E}_{c,\alpha}$ at c jumps. In particular, the notions of parabolic degree, etc, defined above apply to filtered vector bundles as well. Obviously a filtered vector bundle is completely determined by the extensions $\mathcal{E}_{c,\alpha}$ for $\alpha \in [0,1)$. Conversely, given a vector bundle with parabolic structure $(\mathcal{E}, F^\bullet)$ we can provide $\mathcal{E}$ with the structure of a filtered bundle $\mathcal{E}_{\bullet,\bullet}$ as follows. For every $\alpha \in \mathbb{R}$ and $c \in \Delta$ we associate to a section s of $\mathcal{E}$ in a neighborhood of c the section $s_\alpha = t^{\lfloor\alpha\rfloor}s$ or $s_\alpha = t^{\lfloor\alpha\rfloor+1}s$ depending on whether the germ of s in the stalk of $\mathcal{E}$ belongs to $V^{\geq\{\alpha\}}$ or not. We define $\mathcal{E}_{c,\alpha}$ to be the subspace generated by all the sections s_α obtained in this way.

2.2 The Deligne extension

Here we recall the construction of Deligne's extension of the bundle $\mathcal{V}_C$ with flat connection to a holomorphic vector bundle $\mathcal{V}$ on $\overline{C}$ with a logarithmic connection. The hypothesis on the nonexpanding cusp monodromies implies that $\mathcal{V}$ has a canonical[2] parabolic structure, as we now explain.

To construct the Deligne extension of $\mathcal{V}_C$ we use a small disc D centered around the point $c \in \Delta$ with coordinate q. We choose a base point $c_0 \in D \setminus \{c\}$; the conjugation resulting from moving the base point will not affect the extension. We let $T = T(\gamma) \in \mathrm{GL}(V_0)$ be the monodromy of the flat bundle $\mathbb{V}$ along a loop γ once around c, where

[2] The choice of the interval $[0,1)$ is an artificial choice of a unit interval in $\mathbb{R}$ and so Deligne [7] calls this extension *quasicanonical*.

$V_0 = (\mathcal{V}_C)_{c_0}$ is the fiber over the base point c_0. For every $\alpha \in [0,1)$ we can define

(2) $\quad W_\alpha = \{v \in V_0 : (T - \zeta_\alpha)^r v = 0\}, \quad$ where $\zeta_\alpha = e^{2\pi i\alpha}$ and $r = \operatorname{rank}(\mathbb{V})$.

These vector spaces are zero for all but finitely many $\alpha_i \in [0,1)$. Finally, we define

$$T_\alpha = \zeta_\alpha^{-1} T|_{W_\alpha} \quad \text{and} \quad N_\alpha = \log T_\alpha,$$

since T_α is unipotent.

Let $q\colon \mathbb{H} \to D^*$, $q(z) = e^{2\pi i z}$ be the covering of $D^* = D \setminus \{c\}$. Choose a basis $v_1, \dots, v_r$ of V_0 adapted to the direct sum decomposition $V_0 = \bigoplus_\alpha W_\alpha$. Since $\mathbb{H}$ is simply connected, we may view the v_i as sections $v_i(z)$ of $q^*(\mathbb{V}_C|_{D^*})$. If $v_i \in W_\alpha$, then we define

$$\widetilde{v}_i(z) = \exp(2\pi i\alpha z + zN_\alpha)v_i. \tag{3}$$

These sections are constructed to be equivariant under $z \mapsto z+1$, hence they give global sections of $\mathcal{V}_C(D^*)$. The Deligne extension $\mathcal{V}$ of $\mathcal{V}_C$ is the vector bundle whose space of sections over D is the $\mathcal{O}_D$–module spanned by $\widetilde{v}_1, \dots, \widetilde{v}_r$.

This construction naturally gives a parabolic structure on the special fiber $V_x = (\mathcal{V}_C)_x$. We let V_α be the subspace generated by the $\widetilde{v}_i$ with $v_i \in W_\alpha$ and we let $V^{\geq\alpha} = \bigoplus_{\beta \geq \alpha} V_\beta$ to obtain a filtration $F_\gamma^\bullet$ on V_x.

2.3 Metric extension, acceptable and admissible metrics

The notion of an admissible metric serves two technical purposes. On one hand it should specify the correct metric extension by imposing appropriate growth near the cusp, while on the other hand it should give an integrable flat bundle. This section follows the treatment of metric extensions of vector bundles and local systems given in [32, Section 10] and [33].

As preparation for the definition, we first recall the notion of *metric extension* $\Xi(\mathcal{E}_C)$ of a vector bundle $\mathcal{E}_C$ on C. Let $j\colon C \to \overline{C}$ be the inclusion. Given a metric h on $\mathcal{E}$ we define $\Xi(\mathcal{E}_C)$ to be the family of subsheaves of $j_*\mathcal{E}_C$ indexed by $\alpha \in \mathbb{R}$ such that sections $s(q)$ of $\Xi(\mathcal{E}_C)^{\geq\alpha}$ are those holomorphic sections that satisfy the following "growth" condition: for all $\varepsilon \geq 0$ there exists C_ε such that[3]

$$|s(q)|_h \leq C_\varepsilon |q|^{\alpha-\varepsilon}. \tag{4}$$

[3] [32, Section 10] has a typo: the exponent there is erroneously $\alpha + \varepsilon$.

In general, the metric extension of a vector bundle is a coherent sheaf, not a vector bundle. We will, however, use metric extensions only when they are vector bundles, in fact Deligne extensions of local systems; see Lemma 2.4 below.

Following [33] we say that a smooth metric $h = \langle \cdot , \cdot \rangle$ on the bundle $\mathcal{E}_C$ on the curve C (provided with the Poincaré metric) is *acceptable* if the curvature of the metric h admits locally near every $x \in \Delta$ a bound

$$(5) \qquad |R_h| \le f + \frac{C}{|q|^2 |\log(q)|^2} \quad \text{with } f \in L^p \text{ for some } p > 1.$$

We also say that h is an acceptable metric on a filtered vector bundle $\mathcal{E} = \{\mathcal{E}_{\bullet,\bullet}\}$ if the metric h is acceptable on $\mathcal{E}|_C$ and $\mathcal{E} = \Xi(\mathcal{E}|_C)$.

For integrability purposes we require for admissibility growth rates that are slightly more restrictive than (4), but obviously imply this bound.

Definition 2.3 A smooth metric $h = \langle \cdot , \cdot \rangle$ on the bundle $\mathcal{V}_C$ with underlying local system $\mathbb{V}$ is called *admissible* if for every cusp $c \in \Delta$ with local coordinate q

(i) the metric extension $\Xi(\mathcal{V}_C)$ with respect to h is isomorphic as a filtered vector bundle to the Deligne extension $\mathcal{V}$ of $\mathcal{V}_C$,

(ii) for any $e \in \Xi(\mathcal{V}_C)^{\ge\alpha}$ and any $e' \in \Xi(\mathcal{V}_C)^{\ge\alpha'}$ there is some $n \in \mathbb{N}$ and $C_1 = C_1(e, e') > 0$ independent of q such that

$$\langle e, e' \rangle \le C_1 |q|^{\alpha+\alpha'} (\log|q|)^{2n},$$

(iii) there is some $n \in \mathbb{N}$ and $C_2 > 0$ such that a generating section e of $\det(\mathcal{V})$ has the lower bound

$$\|e\|^2 \ge C_2 |q|^{2 \dim_{F^\bullet} \mathcal{V}_c} (\log|q|)^{-2n},$$

(iv) and, moreover, if the metric is acceptable.

In our situation, the relevant existence statement is the following lemma, which follows from Theorem 4 in [33].

Lemma 2.4 *A local system $\mathbb{V}$ with nonexpanding cusp monodromies has a metric which is admissible for its Deligne extension $\mathcal{V}$.*

Proof It suffices to construct such metrics locally and patch them with the help of a partition of unity. On the complement of cusp neighborhoods we can take any metric. On the cusp neighborhoods it suffices to treat each eigenspace for the monodromy separately and declare the different eigenspaces to be pairwise orthogonal. The basis elements $\widetilde{v}_i$ of the α–eigenspace of the Deligne extension are given the norm $|q|^\alpha$ in the local coordinate q around the cusp and defined to be pairwise orthogonal. This implies that the Deligne extension is the metric extension and that the norm bounds (ii) and (iii) hold. The fact that such a metric satisfies the curvature bound for being acceptable can be shown directly; see also [33, Section 5]. □

In the proof of the main theorem it will be convenient to pass to exterior powers. We now provide the necessary background in the case of parabolic bundles. First note that if the metric h is acceptable on a bundle $\mathcal{E}$, then the induced metric on any exterior power of $\mathcal{E}$ is again acceptable. There are two natural ways to define its exterior powers as filtered vector bundles. One is to declare $v_1 \wedge \cdots \wedge v_k$ to lie in $\left(\bigwedge^k \mathcal{E}\right)_{c,\alpha}$ if and only if $\alpha \le \sum \alpha_i$, where α_i is maximal with $v_i \in \mathcal{E}_{c,\alpha_i}$. The second possibility is to take $\Xi\left(\bigwedge^k (\mathcal{E}|_C)\right)$. It is obvious from the definition that $\bigwedge^k (\mathcal{E})_\alpha \subseteq \Xi\left(\bigwedge^k (\mathcal{E}|_C)\right)_\alpha$. It was shown by Simpson (Proposition 3.1 of [33], using the calculations leading to Corollary 10.4 of [32] and in particular the remark on page 911 of [32]) that accessibility of h implies that the converse inequality also holds, ie

$$\bigwedge^k (\Xi(\mathcal{E}|_C)) = \Xi\left(\bigwedge^k (\mathcal{E}|_C)\right),$$

and so both definitions of the exterior power agree.

Proposition 2.5 *If $\mathcal{E}$ is a vector bundle of rank k then $\deg_{\mathrm{par}} \mathcal{E} = \deg_{\mathrm{par}}\left(\bigwedge^k \mathcal{E}\right)$. Moreover, any acceptable metric h computes the parabolic degree of $\mathcal{E}$, ie*

$$\deg_{\mathrm{par}}(\mathcal{E}, F^\bullet) = \frac{1}{2\pi i} \int_C \partial\overline{\partial} \log(\det h_{ij}),$$

where $h_{ij} = \langle e_i, e_j \rangle$ are the coefficients of the acceptable metric.

Proof The first statement is a direct consequent of the first definition of the exterior power.

By the first statement and since $\det h_{ij}$ is the coefficient of the induced acceptable metric on the k^{th} power (obvious from the second definition), we may suppose that $\mathcal{E}$ is a line bundle. For any choice of a generating local section $e = e(q)$ near a point

$c \in \Delta$ and a smooth metric h_ε that agrees with h outside ε–neighborhoods of the cusps, we have (see eg [18, pages 60–61] for details)

$$\begin{aligned} \deg(\mathcal{E}) &= \frac{1}{2\pi i}\int_C \partial\overline{\partial}\log h_\varepsilon \\ &= \frac{1}{2\pi i}\biggl(\int_C \partial\overline{\partial}\log h + \sum_{c\in\Delta}\lim_{\varepsilon\to 0}\int_{|q|=\varepsilon}\overline{\partial}\log\langle e(q), e(q)\rangle\biggr) \\ &= \frac{1}{2\pi i}\int_C \partial\overline{\partial}\log h - \sum_{c\in\Delta}\dim_{F_c^\bullet}\mathcal{E}_c, \end{aligned} \tag{6}$$

and this proves the claim. □

2.4 Proof of the integrability statements

It is obvious that in order to prove Theorem 2.1 and Proposition 2.2 it suffices to prove the following two lemmas. We use the cocycle language for the flat bundle, as introduced in the appendix.

Lemma 2.6 *The cocycle A induced by the geodesic flow on a hyperbolic surface with cusps on a normed flat bundle with nonexpanding cusp monodromies is integrable for a constant norm.*

Proof We have to estimate the growth of the norm over a geodesic segment of length 1. Consider a complement C_ε to a neighborhood of cusps. If the starting point is located in C_ε, then the geodesic segment of unit length starting at this point can cross the boundary of the Dirichlet domain only finitely many times, and the bound is uniform for all starting points. Thus, the growth of the constant norm is uniformly bounded for such segments (where the bound depends on the flat bundle, on the Dirichlet domain and on the choice of ε).

It remains to estimate the growth of the norm for a geodesic segment of unit length starting in a small neighborhood of a cusp. Since the boundary of the Dirichlet domain near a cusp is represented by a geodesic ray going straight to the cusp, we have to count how many times such a geodesic segment could turn around the cusp. Consider standard coordinates in the neighborhood of the cusp; namely, take a half-strip $-\frac{1}{2} \le x \le \frac{1}{2}$, $y \ge y_0 \gg 1$ in the upper half-plane with coordinates $z = x + iy$ and with hyperbolic metric g of constant negative curvature -4,

$$g = \frac{|dz|^2}{4(\operatorname{Im} z)^2} = \frac{dx^2 + dy^2}{4y^2}. \tag{7}$$

The upper bound of the number of turns around the cusp of a geodesic segment of unit length starting at a point $x+iy$ with $y \geq y_0 \gg 1$ is given by the path which first goes straight to the cusp for time 1 and then follows the closed horocycle around the cusp for time 1.

The first segment starts at a point $x+iy$ and goes vertically up to the point $x+ie^2y$. The hyperbolic length of the closed horocycle around the cusp located at the height $y=e^2y$ is $1/(2e^2y)$, so the path following the closed horocycle for time 1 makes at most $2e^2y+1$ turns around the cusp. The condition on nonexpanding cusp monodromy implies that the norm of a constant vector transported N times around the cusp grows linearly in N. Hence for $x+iy \in C_\varepsilon$ the growth of the constant norm is bounded by

$$\max_{t\in[-1,1]} \log^+ \|A(x+iy,t)\| < c_1 \log y + c_2$$

for some constants $c_1, c_2 \in \mathbb{R}$ depending on the flat bundle. Clearly,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{y_0}^{\infty} (c_1 \log y + c_2) \frac{dy}{4y^2} < \infty,$$

and the integrability of the cocycle for the constant norm follows. □

The notion of *equivalent norms* for integrable cocycles is definitely known; see, for example, the corresponding remark in [29]. However, since this notion is important in the context of this paper, for the sake of completeness we collect all necessary details in the appendix.

Lemma 2.7 *A constant norm and an admissible norm h are $L^1(\mu)$–equivalent.*

Proof Consider standard coordinates in the neighborhood of the cusp; namely, take a half-strip $-\frac{1}{2} \leq x \leq \frac{1}{2}$, $y \geq y_0$ in the upper half-plane with coordinates $z = x+iy$ and hyperbolic metric g as in (7). Consider a geodesic ray $\{x_0+iy \mid y \geq y_0\}$ going straight to the cusp. Consider a section $\vec{v}_{x_0+iy}$ of the flat bundle over the geodesic ray constant with respect to the flat connection. The coordinate q in a punctured disk around the cusp is related to our coordinate z as above as

$$2\pi i z = \log q, \quad \text{that is,} \quad \log|q| = -2\pi y.$$

By condition (i) of admissibility, the flat section $\vec{v}_{x_0+iy}$ can be expressed on the half-strip as a linear combination of either the basis elements $\tilde{v}_i$ or the basis elements v_i introduced along with the definition of the Deligne extension. By condition (ii) of admissibility, the sections $\exp(-2\pi i\alpha z)\tilde{v}_i$ are bounded above by $C\log|q|^{2n}$ for some C and n. By the conversion (3), the flat sections v_i and hence also $\|\vec{v}_{x_0+iy}\|_{\mathrm{adm}}$ are bounded above by $C\log|q|^{2n'}$ (for an appropriate choice of the constant, depending on the monodromies N_α). The lower bound for the determinant given by condition (iii) of admissibility and Cramer's rule imply that the norm of such a nonzero flat section is bounded below by $C'\log|q|^{-2n''}$. Thus the ratio of a constant norm and an admissible norm is uniformly bounded in the complement of neighborhoods of the cusps and has the form

$$\max_{\vec{v}\in\mathcal{V}_{x_0+iy}\setminus\vec{0}}\left|\log\frac{\|\vec{v}\|_{\mathrm{adm}}}{\|\vec{v}\|_{\mathrm{const}}}\right| = \max_{\vec{v}\in\mathcal{V}_{x_0+iy}\setminus\vec{0}}\left|\log\frac{\|\vec{v}\|_{\mathrm{const}}}{\|\vec{v}\|_{\mathrm{adm}}}\right| \le K\log y$$

in the local coordinates in the neighborhood of a cusp. The integral

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{y_0}^{+\infty} \log y\frac{dy}{4y^2}$$

converges, so the constant norm and the admissible norm are L^1–equivalent, and Theorem A.5 implies that the cocycle corresponding to the admissible norm is integrable and defines the same Lyapunov exponents as the one corresponding to the constant norm. □

2.5 Necessity of the nonexpanding condition

We remark that if there exists a cusp c_0 of C such that at least one of the eigenvalues of the monodromy around c_0 has absolute value different from 1, then the flat bundle $\mathbb{V}$ is not integrable with respect to the constant norm.

Proof Suppose that the starting point $p=x+iy$ of a geodesic segment is located sufficiently high in the cusp, that is, $y\ge y_0\gg 1$ in coordinates (7). We consider the geodesic launched from p in direction ξ from the subset $\left[\frac{\pi}{6},\frac{\pi}{3}\right]\cup\left[\frac{2\pi}{3},\frac{5\pi}{6}\right]\subset[0,2\pi]$. The direction is chosen to make the geodesic spiral toward the cusp so that its y–coordinate still grows at least for some uniform starting time $\varepsilon(y_0)>0$ depending only on the parameter y_0. We have chosen our geodesic to go not too steeply to the cusp. The angle between the geodesic $\gamma_t(p,\xi)$ as above and the vertical direction only grows for $t\in[0,\varepsilon]$, so the horizontal projection of the geodesic has speed at least $\frac{1}{2}$ for the entire interval of time $[0,\varepsilon]$. Since the cusp at height y has width $1/(2y)$ and for the time ε the geodesic does not get below the initial height y, we conclude that in the interval of time $[0,\varepsilon]$ it makes at least $y\varepsilon-1$ turns around the cusp.

Suppose that there is an eigenvalue of the monodromy around the cusp whose absolute value is different from 1. Let $a \neq 0$ be the logarithm of this absolute value. The calculation above shows that for any geodesic as above we have

$$\sup_{t\in[-\varepsilon,\varepsilon]} \log^+ \|A(\gamma_t(p,\xi))\|_{\mathrm{const}} \geq (y\varepsilon - 1)\cdot a.$$

The subset of starting directions allowed above has $\frac{1}{6}$ of the measure of the full unit circle. Since the integral of the function y diverges with respect to the measure (7) near the cusp, this implies that the integral

$$\int_{T^1C} \sup_{t\in[-1,1]} \log^+ \|A(\gamma_t(p,\xi))\|_{\mathrm{const}}\, d\mu(x)$$

diverges and, hence, that the flat bundle $\mathbb{V}$ is not integrable. □

3 Existence of Lyapunov exponents for variations of Hodge structures

In this section we show that the Hodge metric for families of varieties or more generally for a real variation of Hodge structures satisfies the admissibility assumption of Section 2.3. For a variation of Hodge structures we sketch, moreover, that there are *uniform* bounds for the Lyapunov exponents depending only on the rank and weight of the VHS. An interesting open problem is to prove sharp estimates and interpret the families that reach the upper bounds geometrically.

We recall the definition of real and complex variations of Hodge structures, and also introduce the Hodge metric. A $\mathbb{C}$–*VHS* on the curve C consists of a complex local system $\mathbb{V}_{\mathbb{C}}$ with connection ∇ and a decomposition of the Deligne extension $\mathcal{V} = \bigoplus_{p\in\mathbb{Z}} \mathcal{E}^p$ into C^∞–bundles, such that

(i) $\mathcal{F}^p := \bigoplus_{i\geq p} \mathcal{E}^i$ are holomorphic subbundles and $\overline{\mathcal{F}^p} := \bigoplus_{i\leq p} \mathcal{E}^p$ are anti-holomorphic subbundles for every $p \in \mathbb{Z}$, and

(ii) the connection shifts the grading by at most 1, ie $\nabla(\mathcal{F}^p) \subset \Omega^1_C \otimes \mathcal{F}^{p-1}$ and $\nabla(\overline{\mathcal{F}^p}) \subset \Omega^1_C \otimes \overline{\mathcal{F}^{p+1}}$.

To define the notion of $\mathbb{R}$–VHS we first recall that for a *real Hodge structure of weight* ℓ on W, we require a decomposition $W \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p=0}^{\ell} W^{p,\ell-p}$ such that $\overline{W^{p,q}} = W^{q,p}$. An $\mathbb{R}$–*VHS of weight* ℓ over the base C consists of an $\mathbb{R}$–local system $\mathbb{V}$ and a filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_{\ell-1} \subset \mathcal{F}_\ell \subset \mathcal{V}$$

on the Deligne extension of $\mathbb{V}$ with the property that the bundles $\mathcal{H}^{p,q} = \mathcal{F}^p \cap \overline{\mathcal{F}^q}$ fiberwise define an $\mathbb{R}$–Hodge structure.

An $\mathbb{R}$–VHS $\mathbb{W}$ is *polarized* if there exists a nondegenerate, locally constant bilinear form $Q(\cdot,\cdot)$ on $\mathbb{W}$, skew for ℓ odd and symmetric for ℓ even, such that $Q(\mathcal{H}^{p,q},\mathcal{H}^{r,s}) = 0$ unless $p = s$ and $q = r$, and such that $i^{p-q} Q(v,\bar{v}) > 0$ for every nonzero $v \in \mathcal{H}^{p,q}$. Consequently, if we define an endomorphism S of $\mathbb{V} \otimes_{\mathbb{R}} \mathcal{O}_C$ by $S(v) = i^{p-q} v$ for $v \in \mathcal{H}^{p,q}$, then the *Hodge scalar product* $h(v,w) = Q(Sv,\overline{w})$ is positive definite. We let $\|\cdot\|_h$ be the associated *Hodge norm* of $\mathbb{V}_{\mathbb{C}}$. It is obtained by interpreting $\mathbb{V}$ as the direct sum of the smooth subbundles $\mathcal{H}^{p,q}$ and by using the positive definite metric on each of them.

For any family of projective varieties $f\colon X \to C$ the ℓ^{th} cohomology gives a polarized $\mathbb{R}$–VHS of weight ℓ in this sense.

Note that by a theorem of Borel (see eg [30, Lemma 4.5]) the nonexpanding cusp monodromy hypothesis holds. If the local system underlying the VHS has a $\mathbb{Z}$–structure (or arises as a direct summand of the cohomology of a family of varieties) then the monodromies around the cusps are moreover *quasi-unipotent*.

Proposition 3.1 *The Hodge metric on $\mathbb{V}$ is admissible.*

Proof The corresponding estimates were first derived by Schmid [30]. They are restated in [27]; see Proposition 2.2.1 for the growth rates and Example 3.2 for how to derive the curvature estimate for acceptability. □

The following result gives a second proof of integrability in this case. Recall that G_t denotes the lift of the geodesic flow g_t.

Proposition 3.2 *For a VHS, the function $x \mapsto \sup_{t\in[0,1]} \log^+ \|G_t\|_x$ is bounded by a constant depending on the rank and the weight only. Consequently, the Lyapunov exponents of a VHS are bounded by a constant depending on the rank and the weight only.*

We make no attempt here to make the estimate precise, since the bound from the estimate below is very rough.

Proof Let D be the period domain for polarized weight-ℓ Hodge structures with dimensions of the filtration pieces as given by $\mathbb{V}$. In general, D is not a symmetric domain but just a homogeneous space. The tangent bundle to D has the so-called horizontal subbundle T_h with two properties. First, by Griffiths transversality the tangent vectors to the period map $p\colon \mathbb{H} \to D$ for $\mathbb{V}$ lie in $T_h \subset T_D$. Second, the holomorphic sectional curvature of directions in T_h is negative and bounded away from zero [16, Theorem 9.1; 5, Chapter 13], say by K. This contractivity along the horizontal distribution implies the integrability, as we now elaborate.

To provide a universal bound, it suffices to bound for $(\partial/\partial t)\log h(v(t),v(t))|_{t=0}$, where $v(t)$ is the parallel transport of a unit-norm vector v along g_t. We decompose $v(t)=\sum v^{p,q}(t)$ into its Hodge components and write $\sigma=\sum \sigma_p$ for the graded pieces $\sigma_p\colon \mathcal{H}_p \to \mathcal{H}_{p+1}$ of the Gauss–Manin connection contracted against a unit tangent vector at $t=0$ in the direction of g_t. Expanding into components, we obtain

$$\text{(8)}\quad \frac{\partial}{\partial t}\log h(v(t),v(t))|_{t=0} = \frac{\frac{\partial}{\partial t}h(v(t),v(t))}{h(v,v)}$$
$$\leq 2\frac{\sum_{p=0}^{\ell-1} h(\sigma_p(v^{p,\ell-p}), v^{p+1,\ell-p-1}) + h(\sigma_{p+1}^{\dagger}(v^{p+1,\ell-p-1}), v^{p,\ell-p})}{\sum_{p=0}^{\ell} h(v^{p,\ell-p}, v^{p,\ell-p})},$$

where $\sigma^{\dagger}\colon \mathcal{H}_{p+1} \to \mathcal{H}_p$ is the adjoint of σ. From this expression it is obvious that it suffices to bound from above the operator norms of all the maps σ_p, hence of σ. Since D is homogeneous and finite-dimensional, any two norms are comparable, so we may as well bound the euclidean norm σ. But since σ is just the derivative of the period map p, we can now invoke the Ahlfors lemma in the version of [28, Theorem 2] to obtain the bound $\|\sigma\|_2 \leq \|dp\|_2 \leq \sqrt{|k|}/\sqrt{|K|}$, where $k=-4$ is the curvature of $\mathbb{H}$ in the convention we use. □

4 The bad locus and the main estimate

Suppose that we are given a $\mathbb{C}$–local system $\mathbb{V}$ of rank r over a curve C with nonexpanding cusp monodromies. Let Δ be the set of boundary points of C and recall that by assumption $\chi(C) = -\deg \Omega_C^1(\Delta) < 0$. Denote by $\lambda_1 \geq \cdots \geq \lambda_r$ the Lyapunov exponents of $\mathbb{V}$ with respect to the norm $\|\cdot\| = \|\cdot\|_h$ stemming from an admissible metric h as given by Theorem 2.1. Note that the metric on $\mathbb{V}$ naturally equips the dual bundle $\mathbb{V}^{\vee}$ with an admissible metric $\|\cdot\|_{\vee}$ defined by $\|u\|_{\vee} = \sup_{v\neq 0} |u(v)|/\|v\|_h$, which is admissible as well [33, Theorem 4], and can be used to compute the Lyapunov exponents of $\mathbb{V}^{\vee}$.

In this section we prove a conjecture of Yu [37], or, more precisely, a generalization from the context of VHS to the case of local systems with nonexpanding cusp monodromies.

Theorem 4.1 *If $\mathcal{E} \subset \mathcal{V}$ is a holomorphic parabolic subbundle of rank k of the Deligne extension $\mathcal{V}$ of $\mathbb{V} \otimes_{\mathbb{C}} \mathcal{O}_C$, then*

$$\sum_{i=1}^{k} \lambda_i \geq \frac{2\deg_{\mathrm{par}}(\mathcal{E})}{\deg \Omega^1_{\overline{C}}(\Delta)} = \frac{2\deg_{\mathrm{par}}(\mathcal{E})}{2g(\overline{C}) - 2 + |\Delta|}. \tag{9}$$

We do not assume that the flat bundle $\mathbb{V}$ is irreducible. Clearly, the theorem is applicable to every irreducible summands of $\mathbb{V}$, so if $\mathbb{V}$ is reducible, we can decompose $\mathbb{V}$ into a direct sum of irreducible summands and obtain finer estimates by applying the theorem individually to each irreducible summand.

The condition "parabolic subbundle" refers to the parabolic structure on $\mathcal{V}$ introduced in Section 2.2. This condition is void for unipotent monodromies. For Teichmüller curves one can always restrict to this case. In fact, we can in this case (tacitly) replace C by a finite unramified covering such that the local monodromies around the cusps in Δ are unipotent. This is always possible since, in general, local monodromies are quasi-unipotent and since $\pi_1(C)$ is finitely generated and free if C is not compact. This base change does not modify Lyapunov exponents, and it multiplies numerator and denominator of the right-hand side of (9) by the degree of the covering.

We prepare for the proof with three reduction steps. First, note that the Lyapunov spectrum of $\mathbb{V}$ is symmetric with respect to zero, ie $-\lambda_{r+1-\ell} = \lambda_\ell$ for any ℓ. This follows since the geodesic flow in negative time has on the one hand the negative of the Lyapunov spectrum for every flow and on the other hand (due to the $\mathrm{SL}_2(\mathbb{R})$–action on $\mathbb{H}$) the flows in positive and negative time are conjugate and consequently have the same Lyapunov spectrum.

Second, we remark that $\lambda_i(\mathbb{V}) = -\lambda_{r+1-i}(\mathbb{V}^\vee)$. Combining these two observations, it suffices to prove that

$$\sum_{i=1}^{k} \lambda_i(\mathbb{V}^\vee) \geq \frac{2\deg_{\mathrm{par}}(\mathcal{E})}{\deg \Omega^1_{\overline{C}}(\Delta)}. \tag{10}$$

Moreover, we remark that it suffices to prove the theorem for the case when $\mathcal{E}$ is a *line* bundle, ie to treat the case $k = 1$. In fact, given a parabolic subbundle $\mathcal{E} \subseteq \mathcal{V}$ as in the statement of the theorem, the exterior power $\mathcal{L} = \bigwedge^k \mathcal{E}$ is a parabolic subbundle of $\bigwedge^k \mathcal{V}$ (obvious from the second definition of exterior powers as defined in Section 2.3) and $\bigwedge^k \mathcal{V}$ is the Deligne extension of $\bigwedge^k \mathbb{V} \otimes \mathcal{O}_C$. (This also follows from the first defining property of an admissible metric and the compatibility of the metric with taking exterior powers.) Since $\deg_{\mathrm{par}}(\mathcal{E}) = \deg_{\mathrm{par}}(\mathcal{L})$ by Proposition 2.5 and since the top Lyapunov exponent of $\bigwedge^k \mathbb{V}$ is just $\sum_{i=1}^k \lambda_i$, the claimed reduction to $k=1$ follows.

Mimicking the idea of [20], we define an auxiliary norm on the dual bundle $\mathcal{V}^\vee$ by defining for any point (c,u) in the total space of $\mathcal{V}^\vee$

(11) $$\|u\|_{\mathcal{E}} := \frac{|\omega_c(u)|}{\sqrt{|h(\omega_c,\omega_c)|}} = \frac{|\omega_c(u)|}{\|\omega_c\|_h},$$

where ω_c is a nonzero element of the fiber $\mathcal{E}_c$ over the point c in C. This seminorm is well defined, ie it does not depend on the choice of the nonzero vector ω_c in $\mathcal{E}_c$, since numerator and denominator are homogeneous of the same degree in ω_c.

The difference with the standard case of weight 1 (see [20; 14; 3; 10]) is that the numerator can indeed become zero. We call the locus where the numerator in (11) vanishes the *bad locus* with respect to $\mathcal{E}$, that is, we define

$$T^{\mathrm{bad}} = \{(c,u) : \omega_c(u) = 0\}$$

as a subset of the total space of the bundle $\mathcal{V}_C$ over C.

Since Lyapunov exponents are defined by parallel transport, we really need a definition of the bad locus that records all translates of a given vector. Let $p\colon \mathbb{H} \to C$ denote the universal cover. The flat structure on $\mathcal{V}_C^\vee$ provides a trivialization of $\varphi^*\mathcal{V}_C^\vee$. Using this trivialization, we define for $u \in \mathcal{V}_c^\vee$ the *bad locus of* u as

(12) $$T^{\mathrm{bad}}(u) = \{z \in \mathbb{H} : \omega_z(u) = 0\}.$$

Here ω_z is a generator of the fiber $(p^*\mathcal{E})_z$ of the induced bundle $p^*\mathcal{E}$ over $\mathbb{H}$. In other words, $T^{\mathrm{bad}}(u)$ is the set of points z in $\mathbb{H}$ for which the fiber $(p^*\mathcal{E})_z$ of the line bundle $p^*\mathcal{E}$ is contained in the hyperplane $\operatorname{Ann} u$.

Lemma 4.2 *For every $c \in C$ there is a countable union H of hyperplanes in $\mathcal{V}_c^\vee$ such that for $u \in \mathcal{V}_c^\vee \setminus H$ the bad locus $T^{\mathrm{bad}}(u)$ is a discrete subset of $\mathbb{H}$.*

Proof Since $\mathcal{E}$ is a holomorphic subbundle, locally $T^{\mathrm{bad}}(u)$ is given as the vanishing locus of a holomorphic function. Thus, for any given u the locus $T^{\mathrm{bad}}(u)$ is either discrete in $\mathbb{H}$ or equal to $\mathbb{H}$. The second possibility implies that $u \in \operatorname{Ann}\omega_c$. (The countable union results from the choice of a p–preimage.) □

In fact, one can prove that if the flat bundle $\mathbb{V}$ is irreducible over C, the subbundle of $\mathcal{V}^\vee$ given by those u for which $T^{\mathrm{bad}}(u) = \mathbb{H}$ is actually the zero bundle. Next, we compare the admissible metric and the $\|\cdot\|_{\mathcal{E}}$–seminorm.

Lemma 4.3 *For any point c of the curve C and for any u in the fiber $\mathcal{V}_c^\vee$ over c,*

$$\frac{\|u\|_\vee}{\|u\|_{\mathcal{E}}} \geq 1. \tag{13}$$

Proof The definition of the norm on the dual bundle implies $|\omega_c(u)| \leq \|\omega_c\|_h \cdot \|u\|_\vee$, implying the claim. □

Proof of Theorem 4.1 Pull back the flat bundle and the holomorphic linear subbundle $\mathcal{E}$ to the universal cover $\mathbb{H}$ over C. For any $z \in \mathbb{H}$ and for almost any u in the fiber $\mathbb{V}_z^\vee$ over z one can express the Lyapunov exponent $\lambda_1(\mathbb{V}^\vee)$ (see [11, Section 3.2]) as

$$\lambda_1(\mathbb{V}^\vee) = \lim_{T\to\infty} \frac{1}{T}\frac{1}{2\pi}\int_0^{2\pi} \log\|g_T r_\theta u\|_\vee \, d\theta.$$

Now we replace the admissible norm $\|\cdot\|_\vee$ by the seminorm $\|\cdot\|_{\mathcal{E}}$. Lemma 4.3 implies the inequality

$$\lim_{T\to\infty} \frac{1}{T}\frac{1}{2\pi}\int_0^{2\pi} \log\|g_T r_\theta u\|_\vee \, d\theta \geq \lim_{T\to\infty} \frac{1}{T}\frac{1}{2\pi}\int_0^{2\pi} \log\|g_T r_\theta u\|_{\mathcal{E}} \, d\theta. \tag{14}$$

A priori, the limit in the right-hand side might be equal to $-\infty$. As an outline for the remaining proof, we want to run the standard argument (compare eg [14], or [11, Sections 3.2–3.3], see [17, proof of Theorem 3.3] with $n = 1$ and $\kappa = \frac{1}{2}$, for details allowing to trace the origin of the normalizing factor 2 in the numerator given our curvature conventions) for computing Lyapunov exponents in terms of degree of holomorphic subbundle. We use that from the definition of the seminorm $\|g_t r_\theta L\|_{\mathcal{E}}$ in (11) we get

$$\log\|g_t r_\theta u\|_{\mathcal{E}} = \log|\omega_{g_t r_\theta z}(g_t r_\theta u)| - \log\|\omega_{g_t r_\theta z}\|_h.$$

In contrast to the classical case, we need to consider the Laplacian of the first summand on the right-hand side. Away from $T^{\mathrm{bad}}(g_t r_\theta u)$, the argument of the logarithm is a nonzero holomorphic function, and since Δ_{hyp} is proportional to $\partial\bar\partial$ this contribution vanishes. Near a bad point, the local contribution is the integral of $\Delta_{\mathrm{hyp}} \log(|z|^n)$ for some positive n, hence positive. Altogether, we argued (by integrating over the hyperbolic disc $D(u)$ around the basepoint of u swept out by $g_t r_\theta$) that, for almost every u,

$$\int_{D(u)} \Delta_{\mathrm{hyp}} \log \|u_z\|_{\mathcal{E}}\, dg_{\mathrm{hyp}}(z) \geq -\int_{D(u)} \Delta_{\mathrm{hyp}} \log \|\omega_z\|_h\, dg_{\mathrm{hyp}}(z). \tag{15}$$

Here u_z is the parallel transport of u to the point $z \in D(u)$. This inequality will imply that $\lambda_1(\mathbb{V}^\vee)$ is greater than or equal to the parabolic degree of $\mathcal{E}$, suitably normalized.

To be self-contained, we reproduce this computation in detail. Let D_t be the hyperbolic disc of radius t and Δ_{hyp} be the Laplacian for the hyperbolic metric g_{hyp} on D_t. In the following chain of (in)equalities, we first apply an extra averaging over the unit tangent bundle T^1C. Next, we apply a version of Green's formula ([14, Lemma 3.1] or [11, Lemma 3.6]) for the disc $D_t(u)$ centered around the basepoint of u of hyperbolic radius t. The subsequent inequality follows from (15). Then we exchange the T–limit and the C–integration, justified by dominated convergence since the metric h is accessible. The resulting double integration over C and $D_t(u)$ both just shift the basepoint and can be subsumed into a single integration. To pass to the next line, we use that the integrand no longer depends on T and interchange the order of integration again. Finally we pass from Δ_{hyp} to $\partial\overline{\partial}$:

$$\begin{aligned}
&\mathrm{vol}(C)\lambda_1(\mathbb{V}^\vee)\\
&\quad\geq \int_{T^1C} \lim_{T\to\infty} \frac{1}{T}\frac{1}{2\pi}\int_0^{2\pi} \log \|g_T r_\theta u\|_{\mathcal{E}} d\theta\, \mathrm{d}\mu_{T^1C}(u)\\
&\quad= \int_{T^1C} \lim_{T\to\infty} \frac{1}{T}\frac{1}{2\pi}\int_0^{T} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi} \log \|g_t r_\theta u\|_{\mathcal{E}}\, \mathrm{d}\theta\, \mathrm{d}t\, \mathrm{d}\mu_{T^1C}(u)\\
&\quad= \int_{T^1C} \lim_{T\to\infty} \frac{1}{T}\int_0^{T} \frac{\tanh(t)}{2\,\mathrm{vol}(D_t)}\int_{D_t(u)} \Delta_{\mathrm{hyp}} \log \|u_z\|_{\mathcal{E}}\, \mathrm{d}g_{\mathrm{hyp}}(z)\, \mathrm{d}t\, \mathrm{d}\mu_{T^1C}(u)\\
&\quad\geq \int_{C} \lim_{T\to\infty} \frac{1}{T}\int_0^{T} \frac{\tanh(t)}{2\,\mathrm{vol}(D_t)}\int_{D_t(u)} -\Delta_{\mathrm{hyp}} \log \|\omega_z\|_h\, \mathrm{d}g_{\mathrm{hyp}}(z)\, \mathrm{d}t\, \mathrm{d}g_{\mathrm{hyp}}\\
&\quad= \lim_{T\to\infty} \frac{1}{T}\int_0^{T} \tanh(t)\, \mathrm{d}t \int_C -\tfrac{1}{2}\Delta_{\mathrm{hyp}} \log \|\omega_z\|_h\, \mathrm{d}g_{\mathrm{hyp}}(z)\\
&\quad= -\frac{1}{2}\int_C \Delta_{\mathrm{hyp}} \log \|\omega_z\|_h\, \mathrm{d}g_{\mathrm{hyp}}(z) = -\frac{1}{4}\int_C \Delta_{\mathrm{hyp}} \log |\det h_{ij}|\, \mathrm{d}g_{\mathrm{hyp}}(z)\\
&\quad= -\frac{1}{4}\int_C 4\frac{\partial^2}{\partial z\partial\bar{z}} \log |\det h_{ij}| \frac{i}{2}\, dz\wedge d\bar{z} = \frac{1}{2i}\int_C \partial\overline{\partial} \log |\det h_{ij}|\\
&\quad= \pi \deg_{\mathrm{par}}(\Xi_h(\mathcal{E}|_C)) \geq \pi \deg_{\mathrm{par}}(\mathcal{E}),
\end{aligned}$$

where the last inequality is justified as follows. By the hypothesis $\mathcal{E} \subset \mathcal{V}$, the hypothesis $\mathcal{V} = \Xi_h(\mathcal{V}|_C)$ on the metric h, and the definition of a parabolic subbundle, the metric h is acceptable for $\Xi_h(\mathcal{E}|_C)$, and hence $\Xi_h(\mathcal{E}|_C)$ contains $\mathcal{E}$ as parabolic subbundle. The degree decreases upon passing to subbundles. (In fact, the last inequality would even be an equality by Proposition 2.5 if the metric h restricted from $\mathcal{V}$ to $\mathcal{E}$ was acceptable for $\mathcal{E}$.)

Taking into consideration that the hyperbolic area $\mathrm{vol}(C)$ in the hyperbolic metric of constant negative curvature -4 has the form $\mathrm{vol}(C) = \frac{\pi}{2}(2g(\overline{C}) - 2 + |\Delta|)$, we obtain the desired inequality. □

Remark 4.4 The normalization of the constant negative curvature on the Riemann surface C to -4 is a matter of pure convention coming, partly, from the tradition to associate Teichmüller geodesic flow to the action of the 1-parameter group $\left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right)$ and to have $\lambda_1 = 1$ for the top Lyapunov exponent of the Hodge bundle over the Teichmüller geodesic flow. The choice of the constant negative curvature -1 would impose time normalization which is half as fast, so the 1-parameter subgroup corresponding to geodesic time for curvature -1 would be $\left(\begin{smallmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{smallmatrix}\right)$. In other words, the Lyapunov exponents for the geodesic flow in constant negative curvature $-k^2$ are k times the Lyapunov exponents for the geodesic flow in constant negative curvature -1. The hyperbolic area of the Riemann surface in the metric of constant negative curvature $-k^2$ is k^{-2} times the hyperbolic area of the same Riemann surface in the metric of constant negative curvature -1. The latter is equal to $2\pi(2g(\overline{C}) - 2 + |\Delta|)$.

5 Application: Lyapunov exponents for the Hodge bundle over the Teichmüller geodesic flow

Here we give applications of the main theorem to the Teichmüller geodesic flow. The first is a comparison of slope polygons and the second is a contribution towards the large-genus asymptotics of individual Lyapunov exponents. Both results were observed in [37], and proved there conditionally on our main theorem. We assume in this section that the reader is familiar with the stratification of the moduli space of abelian differentials and with the notion of Teichmüller curves; see eg [22; 39; 25].

5.1 Two polygons

The *slope* of a vector bundle $\mathcal{F}$ on a curve is defined as $\mu(\mathcal{F}) = \deg(\mathcal{F}) / \operatorname{rank}(\mathcal{F})$. A bundle is called *semistable* if it contains no subbundle of strictly larger slope. A filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_g = \mathcal{F}$$

is called a *Harder–Narasimhan filtration* if the successive quotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ are semistable and the slopes are strictly decreasing, ie

$$\mu_i := \mu(\mathcal{F}_i/\mathcal{F}_{i-1}) > \mu_{i+1} := \mu(\mathcal{F}_{i+1}/\mathcal{F}_i).$$

The Harder–Narasimhan filtration is the unique filtration with these properties. Given such a filtration, one can record the numerical data in a "Harder–Narasimhan polygon" with vertices $(\mathrm{rank}(\mathcal{F}_i), 2\deg(\mathcal{F}_i)/|\chi|)$, where $|\chi| = 2g-2+|\Delta|$.

Here, we apply these considerations to a Teichmüller curve C and to $\mathcal{F} = f_*\omega_{\overline{X}/\overline{C}}$, the direct image of the relative dualizing sheaf of the family of stable curves $f\colon \overline{X} \to \overline{C}$. This agrees with the Deligne extension of the first filtration piece of the weight-1 VHS associated with f.

Similarly, one can record the numerical data of the Lyapunov exponents in a "Lyapunov polygon" with vertices $\bigl(k, \sum_{i=1}^k \lambda_i\bigr)$.

The Harder–Narasimhan polygon and the Lyapunov polygon share the endpoints $(0,0)$ and $(g, 2\deg f_*\omega_{\overline{X}/\overline{C}}/|\chi|)$. Applying the main theorem to the subbundles in the Harder–Narasimhan filtration immediately gives the following result, originally conjectured by Yu [37].

Corollary 5.1 *The Lyapunov polygon of a Teichmüller curve always lies above the Harder–Narasimhan polygon (with equality permitted).*

5.2 Lyapunov exponents for strata

So far, we only have been working over curves. From this we can deduce properties of Lyapunov exponents for strata thanks to a convergence result in [2] for individual Lyapunov exponents.

Theorem 5.2 [2] *If $\sum_{i=1}^k \lambda_i \geq M$ for a dense set of Teichmüller curves in some connected component stratum $\mathcal{H}^*(\kappa)$ of the moduli space of abelian differentials $\mathcal{H}(\kappa)$, then the Lyapunov exponents $\lambda_i(\kappa)$ for the Teichmüller geodesic flow on the entire component $\mathcal{H}^*(\kappa)$ also satisfy $\sum_{i=1}^k \lambda_i(\kappa) \geq M$.*

This theorem applies also to any $\mathrm{GL}_2^+(\mathbb{R})$–invariant suborbifold that contains a dense set of Teichmüller curves.

In [21] the two authors conjectured that the large-genus limit of the Lyapunov is

$$\lim_{g\to\infty} \lambda_2 = 1$$

for the hyperelliptic components of the strata $\mathcal{H}(2g-2)$ and $\mathcal{H}(g-1, g-1)$ and that for all other strata and their components

$$\lim_{g\to\infty} \lambda_2 = \tfrac{1}{2}.$$

This first part of this conjecture now follows. The proof of this corollary was given by Yu, assuming the validity of Theorem 4.1 and Theorem 5.2.

Corollary 5.3 [37, Conjecture 5.13] *For the hyperelliptic components of the series of strata $\mathcal{H}(2g-2)$ and $\mathcal{H}(g-1, g-1)$, the large-genus limits of Lyapunov exponents are*

$$\lim_{g\to\infty} \lambda_k = 1$$

for any fixed $k \geq 1$.

(The Lyapunov exponents in the preceding statement are defined for $g \geq k$.)

Proof For those hyperelliptic strata the Harder–Narasimhan filtration over any Teichmüller curve is computed in [38] to be given by the subbundles

$$\mathcal{E}_k = f_*\omega_{\overline{X}/\overline{C}}(-(2g-2k)S) \quad \text{and} \quad \mathcal{E}_k = f_*\omega_{\overline{X}/\overline{C}}(-(g-k)(S_1+S_2))$$

respectively, for $k = 1, \dots, g$, and the degrees of the successive quotient line bundles $\mathcal{E}_k/\mathcal{E}_{k-1}$ are equal to

$$\deg(\mathcal{E}_k/\mathcal{E}_{k-1}) = \frac{|\chi|}{2}\Big(1 - \frac{2(k-1)}{2g-1}\Big) \quad \text{and} \quad \deg(\mathcal{E}_k/\mathcal{E}_{k-1}) = \frac{|\chi|}{2}\Big(1 - \frac{(k-1)}{g}\Big).$$

The implies that $2\deg(\mathcal{E}_k)/|\chi|$ tends to k in both cases as g tends to infinity. Together with Theorem 5.2, our main theorem implies the result. □

A similar statement holds for any family of hyperelliptic loci in a sequence of strata where the order of at least one singularity tends to infinity.

6 Application: Lyapunov exponents for some hypergeometric groups and Calabi–Yau threefolds

Now we apply our main theorem to a class of VHS of rank greater than 1. Our example is the well-studied class of hypergeometric local systems that arise from Calabi–Yau threefolds with $h^{2,1} = 1$. The irreducible local systems that meet the additional requirements imposed by physics (existence of a MUM-point and a conifold point; see Section 6.3 for details) depend on two parameters μ_1, μ_2 called local exponents (see Section 6.4 for the definition). For any pair $0 < \mu_1 \le \mu_2 \le \frac{1}{2}$ with $\mu_i \in \mathbb{R}$, the corresponding local system admits an $\mathbb{R}$–VHS. We compute the degrees of the Hodge bundles and, consequently, lower bounds for the Lyapunov exponents. The terminology will be explained in the sequel.

Theorem 6.1 *Suppose that the local exponents $0 < \mu_1 \le \mu_2 \le \frac{1}{2}$ at the point $z = \infty$ of a Calabi–Yau-type hypergeometric group with $h^{2,1} = 1$ are $(\mu_1, \mu_2, 1-\mu_2, 1-\mu_1)$. Then the degrees of the Hodge bundles are*

$$\deg_{\mathrm{par}} \mathcal{E}^{3,0} = \mu_1 \quad \textit{and} \quad \deg_{\mathrm{par}} \mathcal{E}^{2,1} = \mu_2.$$

For families of Calabi–Yau threefolds the local monodromies are quasi-unipotent, hence $\mu_i \in \mathbb{Q}$. In Table 1 we reproduce from [9] the well-known list of possible parameters (μ_1, μ_2) that meet the physically relevant conditions. We present in the same table approximations for the Lyapunov exponents. Explanations for the first three columns are given in Section 6.3.

The most remarkable conclusion from the numerical approximation of Lyapunov exponents is the following. In the first seven cases the sum of Lyapunov exponents matches the lower bound predicted by Theorem 4.1. The table lists the corresponding sum as exact fractions, but note that only three digits seem to be reliable in the experiments. In the remaining cases, the sum $\lambda_1 + \lambda_2$ of Lyapunov exponents is strictly larger than predicted by the lower bound in Theorem 4.1. Note that in precisely the seven cases of (numerical) equality the monodromy groups of the hypergeometric local systems are of infinite index ("thin") in $\mathrm{Sp}(4, \mathbb{Z})$ while in the other seven cases the monodromy group is of finite index in $\mathrm{Sp}(4, \mathbb{Z})$. This follows from combining the results in [4] and [34]. It would be interesting to decide if in these seven cases actually equality holds and to explain the relation to the arithmeticity of the monodromy groups. We provide further conjectures in this direction in Section 6.5 below.

There is another commonly used normalization of the degrees and Lyapunov exponents. Instead of working with parabolic degrees and over $\mathbb{P}^1$ with three singular points, we can view the above local systems as representations of the Fuchsian triangle group $\Delta(n, \infty, \infty)$, where n is the least common multiple of the denominators of μ_1 and μ_2 if $0 < \mu_1 < \mu_2 < \frac{1}{2}$ and where $n = \infty$ if at least one of the inequalities is

#	model	C	d	μ_1, μ_2	λ_1	$\lambda_1+\lambda_2$	$-\chi$
1		46	1	$1/12$, $5/12$	0.97	1	$11/12$
2		44	2	$1/8$, $3/8$	0.95	1	$7/8$
3		52	4	$1/6$, $1/2$	1.27	$4/3$	1
4	$\mathbb{P}^4[5]$	50	5	$1/5$, $2/5$	1.12	$6/5$	$4/5$
5		56	8	$1/4$, $1/2$	1.40	$3/2$	1
6	$\mathbb{P}^6[2^2,3]$	60	12	$1/3$, $1/2$	1.53	$5/3$	1
7	$\mathbb{P}^7[2^4]$	64	16	$1/2$, $1/2$	1.75	2	1
8		22	1	$1/6$, $1/6$	0.75	0.92	1
9		34	1	$1/10$, $3/10$	0.77	0.83	$9/10$
10		32	2	$1/6$, $1/4$	0.84	0.97	$11/12$
11		42	3	$1/6$, $1/3$	0.96	1.06	$5/6$
12		40	4	$1/4$, $1/4$	1.07	1.30	1
13		48	6	$1/4$, $1/3$	1.15	1.31	$11/12$
14		54	9	$1/3$, $1/3$	1.34	1.60	1

Table 1: Table of CY-VHS and approximate values of their Lyapunov exponents

replaced by an equality. Geometrically, this corresponds to viewing the local systems over the orbifold $C = \mathbb{H}/\Delta(n,\infty,\infty)$. The orbifold Euler characteristic $-\chi$ of C is given in each case in the last column. Note that $0 < -\chi \le 1$ in all the cases. One can also define and compute Lyapunov exponents $\lambda_1^{\mathrm{orb}}, \lambda_2^{\mathrm{orb}}$ of the corresponding local systems over the orbifold $C = \mathbb{H}/\Delta(n,\infty,\infty)$. They are related to the Lyapunov exponents λ_i over the thrice-punctured sphere by

$$\lambda_i = \lambda_i^{\mathrm{orb}} \cdot |\chi|.$$

The corresponding orbifold degrees of Hodge bundles can be computed as ordinary degrees of line bundles on a cyclic cover where all the monodromies are unipotent, as indicated in Section 6.3. The orbifold normalization $\lambda_1^{\mathrm{orb}}, \lambda_2^{\mathrm{orb}}$ was used in previous computations for Teichmüller curves (eg in [3] and [10]).

6.1 Hypergeometric differential equations

We fix two sequences of real numbers, $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_n)$ and $\boldsymbol{\beta} = (\beta_1,\dots,\beta_n)$, with

$$0 \le \alpha_1 \le \dots \le \alpha_n < 1, \quad 0 \le \beta_1 \le \dots \le \beta_n < 1 \tag{16}$$

and with the property that $\alpha_i \ne 1-\beta_j$ for any i and j. The regular hypergeometric differential operator is the operator

$$(17)\qquad P = P(\boldsymbol{\alpha},\boldsymbol{\beta}) = \prod_{i=1}^{n}(D-\alpha_i) - t\prod_{i=1}^{n}(D-\beta_i), \quad \text{where } D = t\frac{d}{dt}.$$

It gives rise to a flat connection ∇ on the trivial vector bundle $\mathcal{V}_0$ on $\mathbb{P}^1$ with regular singularities precisely at the points $\{0,1,\infty\}$. We refer to this local system as the hypergeometric local system $\mathbb{V} = \mathbb{V}(\boldsymbol{\alpha},\boldsymbol{\beta})$.

A *hypergeometric group* with parameters $\boldsymbol{a} = (a_1,\dots,a_n)$ and $\boldsymbol{b} = (b_1,\dots,b_n)$ subject to the conditions $|a_i| = 1 = |b_j|$ and $a_i \neq 1/b_j$ for all (i,j) is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ generated by three elements

$$(18)\qquad h_0, h_1, h_\infty \in \mathrm{GL}_n(\mathbb{C}) \quad \text{with } h_\infty h_1 h_0 = \mathrm{Id}$$

such that

$$(19)\qquad \det(X\,\mathrm{Id} - h_\infty) = \prod_{i=1}^{n}(X-a_i), \quad \det(X\,\mathrm{Id} - h_0^{-1}) = \prod_{i=1}^{n}(X-b_i)$$

and such that h_1 is a pseudoreflection. Here, a *pseudoreflection* is an element $g \in \mathrm{GL}_n(\mathbb{C})$ such that $g - \mathrm{Id}$ has rank one.

Up to conjugation there is a unique hypergeometric group for a given set of parameters. The proof (due to Levelt) and monodromy matrices can be found, for example, in [1, Theorem 3.5]. The hypothesis $a_i \neq 1/b_j$ guarantees that the flat bundle $\mathbb{V}$ is irreducible [1, Proposition 3.3].

The monodromy group of $\mathbb{V}(\boldsymbol{\alpha},\boldsymbol{\beta})$ is the hypergeometric group with parameters $\boldsymbol{a}$ and $\boldsymbol{b}$ where $e^{2\pi i\alpha_j} = a_j$ and $e^{2\pi i\beta_j} = b_j$ for $j = 1,\dots,n$.

6.2 Simpson's correspondence in the parabolic case

In order to state Simpson's correspondence, we need to extend the definition of parabolic structure and stability from vector bundles to the cases of parabolic vector bundles, local systems and Higgs bundles.

A *regular parabolic Higgs bundle* is a parabolic vector bundle $(\mathcal{E}, F^\bullet)$ together with a Higgs field, ie a map of sheaves of $\mathcal{O}_C$–modules

$$(20)\qquad \theta\colon \mathcal{E} \to \mathcal{E} \otimes \Omega^1_C$$

that respects the parabolic structure in the sense that for every $c \in \Delta$ the map θ extends for every $\alpha \in [0,1)$ to

$$\theta_{c,\alpha}\colon \mathcal{E}_c^{\geq\alpha} \to \mathcal{E}_c^{\geq\alpha} \otimes \Omega^1_{\overline{C}}(\Delta). \tag{21}$$

A *regular parabolic system of Hodge bundles* is a regular parabolic Higgs bundle whose underlying vector bundle admits a decomposition $\mathcal{E} = \bigoplus_{p\in\mathbb{Z}} \mathcal{E}^p$ such that θ has degree -1 with respect to the grading given by this decomposition.

Recall that a vector bundle $\mathcal{V}$ is called stable if for every subbundle $\mathcal{M} \subset \mathcal{V}$ the condition

$$\frac{\deg(\mathcal{M})}{\operatorname{rank}(\mathcal{M})} < \frac{\deg(\mathcal{V})}{\operatorname{rank}(\mathcal{V})} \tag{22}$$

holds. Similarly, a *parabolic vector bundle* (resp. *a local system*, resp. a *Higgs bundle*) is called *stable* if the condition (22) holds for every parabolic subbundle (resp. every subbundle preserved by the connection, resp. every subbundle preserved by the Higgs field).

Simpson's correspondence [33] for the noncompact case states that there is a natural one-to-one correspondence between stable regular parabolic Higgs bundles and stable parabolic local systems of degree zero.

There is an action of $\mathbb{C}^*$ on the set of regular parabolic Higgs bundles of degree zero, where $s \in \mathbb{C}^*$ sends (E,θ) to $(E, s\theta)$ while preserving the filtration. Fixed points of this action are precisely the regular parabolic systems of Hodge bundles of degree zero.

Since hypergeometric local systems are rigid (see eg [1], Proposition 3.5), Simpson's correspondence implies the following (see [33, Corollary 8.1]).

Corollary 6.2 *A hypergeometric local system $\mathbb{V} = \mathbb{V}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ carries a complex variation of Hodge structures.*

The Hodge numbers, ie the ranks h^p of the summands $\mathcal{E}^p$, are known by a theorem of Fedorov. If we set $\rho(k) = \#\{j : \alpha_j < \beta_k\} - k$, then the main theorem of [12] (Theorem 1) states that

$$h^p = \#\rho^{-1}(p) \tag{23}$$

after an appropriate shifting of the weight (or the grading).

6.3 Families of Calabi–Yau threefolds with $h^{2,1}=1$ and generalizations

Families of Calabi–Yau threefolds with $h^{2,1}=1$ carry a weight-3 variation of Hodge structures, and by definition of Calabi–Yau threefolds, the Hodge numbers of these families are $(1,1,1,1)$, ie $\dim \mathcal{E}^{p,q}=1$ for $p=0,1,2,3$. In a VHS arising from geometry, the VHS has an $\mathbb{R}$–structure and quasi-unipotent monodromies. Motivated by physics requirements, the most intensely investigated families satisfy the following properties: they are over $\mathbb{P}^1$, smooth outside three points, have one point of *maximal unipotent monodromy* (MUM, ie there is only one Jordan block of maximal size) and one rank-1 unipotent point. There are fourteen possible cases, as derived in [9]. They are given in Table 1. In some cases, these families have been realized geometrically (eg as complete intersections in weighted projective spaces) and the first column of the table lists this realization (if available; eg $\mathbb{P}^4[5]$ refers to the (mirror) quintic).

The local exponents of such a hypergeometric system, with real structure, with a MUM-point and with a point where the monodromy is unipotent of rank one, are

$$\begin{aligned} \boldsymbol{\beta} &= (0,0,0,0) && \text{at } t=0,\\ &\ (0,1,1,2) && \text{at } t=1,\\ \boldsymbol{\alpha} &= (\mu_1,\mu_2,1-\mu_2,1-\mu_1) && \text{at } t=\infty; \end{aligned} \tag{24}$$

see eg [35; 36; 1; 12] for general background.

A realization of monodromy groups of the hypergeometric local systems listed in Table 1 is given by

$$T_0=\begin{pmatrix} 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 1/2 & 1 & 1 & 0\\ 1/6 & 1/2 & 1 & 1 \end{pmatrix},\quad T_1=\begin{pmatrix} 1 & -C/12 & 0 & -d\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with the parameters (C,d) as in the table. Here, the symplectic form defining the polarization of the Hodge structure on $\mathbb{V}(\boldsymbol{\alpha},\boldsymbol{\beta})$ is given by

$$\Omega=\begin{pmatrix} 0 & C/12 & 0 & d\\ -C/12 & 0 & -d & 0\\ 0 & d & 0 & 0\\ -d & 0 & 0 & 0 \end{pmatrix}$$

and this symplectic form can be conjugated into $\mathrm{Sp}(4,\mathbb{Z})$. The proof of Theorem 6.1 does not use properties of these realizations. In fact, the representation is real by

[12, Theorem 2] if

$$\alpha_m + \alpha_{4+1-m} \in \mathbb{Z} \quad \text{and} \quad \beta_m + \beta_{4+1-m} \in \mathbb{Z}.$$

The basic principle for the proof of Theorem 6.1 is the following. We consider the Kodaira–Spencer maps (graded pieces of the Higgs fields)

$$\tau_{p-1}\colon \mathcal{E}^{p,q} \to \mathcal{E}^{p-1,q+1} \otimes \Omega^1_{\overline{C}}(\Delta). \tag{25}$$

In our situation, these are maps between line bundles. The maps τ_0, τ_1 and τ_2 are nonzero by Lemma 6.3 below, hence are inclusions. To compute the (parabolic) degrees it suffices to compute the lengths of the cokernels of these maps and to determine the parabolic structures. We prove the following lemma (which applies not only to hypergeometric systems, but to any self-dual flat bundle) and explain the notions about differential equations in the next subsection.

Lemma 6.3 *If $x \in C$ is a regular point of the local system $\mathbb{V}$ on $\overline{C}$, then all the Kodaira–Spencer maps τ_i are isomorphisms at x.*

More generally, if $x \in C$ and the local exponents $\mu_1 < \mu_2 < \mu_3 < \mu_4$ are distinct and integral, then τ_0 has a cokernel of length $\mu_2 - \mu_1 - 1$, and so does τ_2 by duality. The map τ_1 has a cokernel of length $\mu_3 - \mu_2 - 1$.

If $c \in \Delta \subset \overline{C}$ and the local exponents satisfy $\mu_1 \le \mu_2 \le \mu_3 \le \mu_4$, then τ_0 has a cokernel of length $\lfloor \mu_2 \rfloor - \lfloor \mu_1 \rfloor$, and so does τ_2 by duality. The map τ_1 has a cokernel of length $\lfloor \mu_3 \rfloor - \lfloor \mu_2 \rfloor$.

Proof of Theorem 6.1 We can apply the first observation in Lemma 6.3 to every point different from $0, 1, \infty$ and we can apply the observation for boundary points in Lemma 6.3 at the MUM-point $t = 0$ and at the point $t = \infty$ to conclude that at all these points all the τ_i are isomorphisms. Finally, the last statement in Lemma 6.3 tells us that at the unipotent rank-1 point at $t = 1$, the map τ_1 is still an isomorphism, while τ_0 and τ_2 have cokernels of length 1.

Next, we consider the $[0,1)$–filtrations, which are nontrivial only at the point $t = \infty$. There, since all the $\mathcal{E}^{p,q}$ are line bundles and since θ must shift the degree by -1 for a system of Hodge bundles, the only possibility for a filtration respecting the regularity hypothesis (21) is

$$V_\infty^{\ge \mu_i} = \bigoplus_{p=4-i}^{3} \mathcal{E}_\infty^{p,3-p}.$$

We deduce from properties of an $\mathbb{R}$–VHS that $\deg_{\mathrm{par}} \mathcal{E}^{p,q} = -\deg_{\mathrm{par}} \mathcal{E}^{q,p}$. Hence the fiber at $t = \infty$ of $\mathcal{E}^{p,3-p}$ is the graded piece of weight μ_{4-p} of the filtration. This implies that

$$-\deg(\mathcal{E}^{2,1}) = \deg(\mathcal{E}^{1,2}) + 1, \quad -\deg(\mathcal{E}^{3,0}) = \deg(\mathcal{E}^{0,3}) + 1.$$

Since τ_1 is an isomorphism, we conclude from (25) that $\deg(\mathcal{E}^{2,1}) = \deg(\mathcal{E}^{1,2}) + 1$ and hence $\deg(\mathcal{E}^{2,1}) = 0$ Since τ_0 has a cokernel of length 1 we conclude from (25) again that $\deg(\mathcal{E}^{3,0}) = 0$. This gives the parabolic degrees as claimed in the theorem. □

6.4 Local exponents, weight filtration and the cokernel lemmas

We need two general concepts about a flat bundle $\mathbb{V}$: local exponents and the monodromy weight filtration. We let $n = \mathrm{rank}(\mathbb{V})$, and later we specialize to the case $n = 4$ of primary interest.

To recall the definition and the properties of local exponents, fix a point $c \in \overline{C}$, let t be a coordinate of C such that c is the point $t = 0$, and fix a section $\omega(t)$ of $\mathbb{V}$ whose first n derivatives (with respect to $\nabla_{d/dt}$) generate $\mathbb{V}$ in a neighborhood of c. Then there are meromorphic functions $P_i(t)$ such that

$$L(\omega) = \left(\nabla_{d/dt}^n + \sum_{i=0}^{n-1} P_i(t) \nabla_{d/dt}^i\right)(\omega) = 0.$$

Since the local system $\mathbb{V}$ is supposed to have regular singularities, $t^{n-1-i} P_i(t)$ is holomorphic at 0. It will be convenient to rewrite the differential equation in terms of the differential operator $D = t \cdot d/dt$ as

$$t^n \left(\nabla_{d/dt}^n + \sum_{i=0}^{n-1} P_i(t) \nabla_{d/dt}^i\right) = \nabla_D^n + \sum_{i=0}^{n-1} Q_i(t) \nabla_D^i,$$

for some $Q_i(t)$ that are holomorphic at $t = 0$. Now consider in general a linear differential operator

$$L(y) = \frac{d^n y}{d\, t^n} + \sum_{i=0}^{n-1} Q_i(t) \frac{d^i y}{d\, t^i}. \tag{26}$$

The *local exponents* $\{\mu_1(c), \dots, \mu_n(c)\}$ of L at $c \in C$ are the solutions of the equation

$$y^n + \sum_{i=0}^{n-1} Q_i(0) y^i = 0.$$

The local exponents at a point c are well-defined up to a simultaneous shift by some integer. This ambiguity is due to the possibility of replacing the section $\omega(t)$ by $t^k\omega(t)$; see Frobenius' theorem (eg in [35]) and below.

A point $c \in C$ is called *regular* if the functions $P_i(t)$ are regular at c. The regular points are precisely those points where the local exponents are of the form $\{k, k+1, \dots, k+n-1\}$ for some k.

The local exponents $\{\mu_1(c), \dots, \mu_n(c)\}$ determine the exponents needed to write local solutions of the differential equation as a power of a uniformizer times a power series expansion. More precisely, if the difference of any two local exponents is nonintegral, then the theorem of Frobenius states that the solutions of the differential equation (26) are

$$s_i = t^{\mu_i} P_i \quad \text{with } P_i \in 1 + \mathbb{C}[\![t]\!].$$

We refer to this basis of solutions as the *Frobenius basis*. If some difference of local exponents is integral, then one has to add logarithmic terms, according to the monodromy at c. We give an example for $n = 4$.

Suppose that the monodromy is maximal unipotent (hence all the μ_i are the same). Then the solutions are of the form

$$\begin{aligned}
s_1 &= t^{\mu_1} P_1, \\
s_2 &= \log(t) s_1 + t^{\mu_2} P_2, \\
s_3 &= \tfrac{1}{2}\log(t)^2 s_2 + \log(t) s_1 + t^{\mu_3} P_3, \\
s_4 &= \tfrac{1}{6}\log(t)^3 s_3 + \tfrac{1}{2}\log(t)^2 s_2 + \log(t) s_1 + t^{\mu_4} P_4.
\end{aligned}$$

We deduce that, by definition, $t^{\mu_1} P_1$, $t^{\mu_2} P_2$, $t^{\mu_3} P_3$, $t^{\mu_4} P_4$ forms a basis of local sections of the Deligne extension. In fact, this last conclusion holds for any local monodromy matrix. For this reason the proof of Lemma 6.3 does not depend on the form of the monodromy matrix.

We have expressed above the local exponents in terms of a (polynomial associated to a) differential operator L, which in turns depends on the choice of a local section ω of $\mathbb{V}$. We recall how to retrieve $(\mathbb{V}, \omega)$ up to isomorphism from L. In fact, let $\mathrm{Sol} \subset \mathcal{O}_C$ be the rank-n local system of solutions of L. Then $\mathrm{Sol} \cong \mathbb{V}^\vee$, since in fact the multiplication map

$$m\colon \mathrm{Sol} \otimes \mathcal{O}_C \to \mathcal{O}_C$$

defines a section of $\mathbb{V}$ and the pair (Sol, m) is isomorphic to the pair $(\mathbb{V}, \omega)$ we started with.

In terms of a basis of Sol and its dual basis we can compute the effect of the covariant derivative. To simplify notation, we restrict to the case $n = 4$ of primary interest here. Let $\{s_1, s_2, s_3, s_4\}$ be a basis of Sol around c, and denote the dual basis by $\{s_1^\vee, s_2 1^\vee, s_3^\vee, s_4^\vee\}$, so that

$$s_j^\vee\Bigl(\sum_{i=1}^4 s_i \otimes g_i\Bigr) = g_j \in \mathrm{Sol}^\vee \cong \mathbb{V} \quad (\text{where } g_i \in \mathcal{O}_C(U) \text{ for some } U).$$

In this basis, $m = \sum_{i=1}^4 s_i s_i^\vee$ as a section of $\mathrm{Sol}^\vee \cong \mathbb{V}$. Moreover,

$$\nabla_{d/dt}(m)\Bigl(\sum_{i=1}^4 s_i \otimes g_i\Bigr) = d\Bigl(\sum_{i=1}^4 s_i \otimes g_i\Bigr) - \sum_{i=1}^4 s_i g_i = \sum_{i=1}^4 s_i' \otimes g_i,$$

that is,

$$\nabla_{d/dt}(m) = \sum_{i=1}^4 s_i' s_i^\vee.$$

This completes the preparation for the main lemma.

Proof of Lemma 6.3 We start with the case of a regular point. Without changing the length of the cokernels, we may choose the section ω to be nonvanishing at c, hence the local exponents are $\{\mu_1 = 0, 1, 2, 3\}$. The length of the cokernel of τ_0 at the point c, ie at $t = 0$, is the vanishing order of

$$\nabla_{d/dt}(m) = \sum_{i=1}^4 s_i' s_i^\vee \in V/\langle m\rangle, \tag{27}$$

where $V = \langle s_1^\vee, s_2^\vee, s_3^\vee, s_4^\vee\rangle$ is the fiber of $\mathcal{V}$ over c. We use the Frobenius basis $\{s_1, s_2, s_3, s_4\}$ from now on. Consider the matrix $M = M(t)$ with entries $M_{ij}(t) = s_i^{(j-1)}(t)$. Since the 2×2–minor M_{12}^{12} of M has a determinant with nonzero constant term (considered as element of $\mathbb{C}[t]$), the vanishing order of (27) is zero, ie the map τ_0 is an isomorphism at c. Similarly, the minor M_{123}^{123} and also the determinant M itself have nonzero constant terms by our hypothesis on the local exponents. Since

$$\nabla_{d/dt}^{(j)}(m) = \sum_{i=1}^4 s_i^{(j)} s_i^\vee,$$

this is precisely what we need to deduce that also the Kodaira–Spencer maps τ_1 and τ_2 are isomorphisms at x.

The case of general (but still integral) local exponents follows similarly. In fact, the minimal order of vanishing of a 2×2–minor of the first two rows of M is given by M_{12}^{12}, which starts with t^{μ_2-1}. Hence the length of the cokernel of τ_0 is as claimed. The minor M_{123}^{123} starts with t^{μ_3-2}. This is the length of the cokernel of the composition of Kodaira–Spencer maps $\tau_1\circ\tau_0\colon \mathcal{E}^{3,0}\to\mathcal{E}^{1,2}\to\Omega^1_{\overline{C}}(\Delta)^{\otimes 2}$ and it implies the claim about τ_1. The same argument with the determinant M implies the claim about τ_2.

The discussion so far was concerned with points $c\in C$. If $c\in\Delta\subset\overline{C}$, then the calculations above are the same with M replaced by the matrix with entries $M_{ij}(t)=(t\,\partial/\partial t)^{j-1}s_i(t)$. This increases the length of each of the cokernels by 1 with respect to the previous calculations.

Finally, in the case of nonintegral local exponents, recall that the sections of the Deligne extension are given by $t^{\{\mu_i\}}s_i$ in terms of the Frobenius basis, where $\{\mu\}=\mu-\lfloor\mu\rfloor$ denotes the fractional part of μ. Consequently, the preceding calculation applies again, now with μ_i replaced by $\lfloor\mu_i\rfloor$. □

6.5 Conjectural region of equality

It seems likely that the seven cases of Calabi–Yau type families with equality are not isolated examples. Initially we conjectured that the equality $\lambda_1+\lambda_2=2(\mu_1+\mu_2)$ is attained in the entire region in the (μ_1,μ_2)–plane defined by the linear inequality $3\mu_2\ge\mu_1+1$. After more detailed numerical experiments by Fougeron, this conjecture cannot be upheld in this form any more (see [15]). According to these experiments, it appears, rather, that equality is attained at an infinite number of rational points in the (μ_1,μ_2)–plane. It would be very interesting to relate, in general, thinness of the monodromy group and the equality $\lambda_1+\lambda_2=2(\mu_1+\mu_2)$; see also [6] for some results in this direction.

Special cases of the conjecture can be equivalently formulated as a number-theoretic problem. This new hypothetical nonvanishing property is similar to the nonvanishing of the classical modular form $\Delta(q):=q\prod_{n\ge 1}(1-q^n)^{24}$ for $0<|q|<1$. Consider the mirror quintic (row 4 in Table 1), normalized so that the MUM-point is zero, the conifold point is at $t=\infty$, and the remaining singular point is $t=\left(\frac{1}{5}\right)^5$ instead of $t=1$. Since strict inequality in Theorem 4.1 is caused by the presence of bad points (see (12)), it seems natural to look for a flat section of $p^*\left(\bigwedge^2\mathbb{V}\right)$ (where $p\colon\mathbb{H}\to C$

is the universal cover) that avoids the bad locus. As a first attempt we take L to be the Lagrangian 2–plane that is invariant under the monodromy around $t=0$ and the flat section it defines by parallel transport along the upper half-plane. In fact, since the condition of having empty bad locus is open, it suffices to find a single flat section of $p^*(\bigwedge^2 \mathbb{V}^\vee)$ whose pairing with L is everywhere nonzero on $\mathbb{H}$. Here, again, we try the 2–plane invariant under the monodromy around $t=0$ and its parallel transport. Near $t=0$, the 2–plane L is generated by the differential 3–forms $\Omega^{3,0}$ and $\Omega^{2,1}$ generating $\mathcal{E}^{3,0}$ and $\mathcal{E}^{2,1}$, respectively. The homology 2–plane is generated by the two "shortest" 3–cycles γ_0, γ_1. It is well-known (see eg [19]) that

$$\psi_0(t) := \int_{\gamma_0} \Omega^{3,0}(t) = \sum_{n\geq 0} \frac{(5n)!}{n!^5} t^n$$

and

$$\psi_1(t) := \int_{\gamma_1} \Omega^{3,0}(t) = \log(t)\psi_0 + \sum_{n\geq 0} \frac{(5n)!}{n!^5} \Bigg(\sum_{k=n+1}^{5n} \frac{1}{k} \Bigg) t^n.$$

Since the Kodaira–Spencer map is nonvanishing (on $\mathbb{P}^1 \setminus \{0, (\frac{1}{5})^5, \infty\}$), the integral against $\Omega^{2,1}(t)$ is given by the t–derivatives of ψ_0 and ψ_1, respectively. Consequently, the contraction of L against $\langle \gamma_0, \gamma_1 \rangle$ is given by the Wronskian

$$W(t) = \psi_0(t)\psi_1'(t) - \psi_0'(t)\psi_1(t).$$

We consider the composition $F(q) = W \circ \lambda(q)$ with the λ–function

$$\lambda\colon \Delta^* \to \mathbb{C}, \quad \lambda(q) = \frac{q}{5^5} \cdot \left(\frac{\sum_{n\in\mathbb{Z}} q^{n^2+n}}{\sum_{n\in\mathbb{Z}} q^{n^2}} \right)^4,$$

where $\Delta^* = \{q \in \mathbb{C} : 0 < |q| < 1\}$ denotes the punctured unit disc. By the choice of L and $\{\gamma_0, \gamma_1\}$, the function F extends meromorphically with a simple pole across $q=0$.

Altogether, the nonvanishing of L contracted against $\langle \gamma_0, \gamma_1 \rangle$ on the whole upper half-plane and the mirror quintic case $\mu_1 = \frac{1}{5}$, $\mu_2 = \frac{2}{5}$ of the conjecture stated in the introduction follows from the following statement.

Conjecture 6.4 *The pullback F of the Wronskian $W(t)$ via λ vanishes nowhere on the punctured unit disc Δ^*.*

Strong numerical evidence for this conjecture is given by considering the growth rate of the coefficients of $1/F$. They appear to grow like $\exp(C\sqrt{n})$ for some C, whereas, for the reciprocal of a function with a zero in the disc (for example $1/\psi_0(\lambda(q))$), the radius of convergence is strictly smaller than 1 and the coefficients grow like $\exp(n)$.

Appendix: The multiplicative ergodic theorem and equivalent norms for measurable cocycles

Suppose that we have a smooth or continuous (or just measurable) finite-dimensional complex vector bundle $\mathcal{V}$ of rank r over the base B, where the smooth (or topological) manifold B is endowed with a probability measure μ. Suppose that a map $T\colon B \to B$, ergodic with respect to the measure μ, extends to a smooth (continuous, measurable) automorphism A of the vector bundle $\mathcal{V}$. In other words, we suppose that the map T of the base to itself lifts to a map A of the total space of the vector bundle to itself, preserving the bundle structure, such that A is fiberwise $\mathbb{C}$–linear, and such that the induced linear transformations $A_x\colon \mathcal{V}_{(x)} \to \mathcal{V}_{T(x)}$ of the fibers is invertible for any $x \in B$. Suppose finally that each fiber $\mathcal{V}_{(x)}$ of the vector bundle $\mathcal{V}$ is endowed with a norm $\|\cdot\|_{(x)}$ which depends smoothly (continuously, measurably) on the base point $x \in B$.

Consider the usual operator norm

$$\|A_x\| := \max_{\vec{v}\in\mathcal{V}_x\setminus\vec{0}} \frac{\|A_x\vec{v}\|_{(T(x))}}{\|\vec{v}\|_{(x)}}.$$

Define $\log^+(y) = \max(0, \log(y))$.

Definition A.1 The above data $(B, T, \mu, \mathcal{V}, \|\cdot\|, A)$ defines a *measurable cocycle* if $\log^+\|A_x\|$ is integrable over B with respect to the measure μ, ie

$$\int_B \log^+\|A_x\|\, d\mu(x) < \infty.$$

We state the multiplicative ergodic theorem in a form close to the original formulation in [26].

Theorem A.2 (Oseledets' theorem) *Suppose that $(B, T, \mu, \mathcal{V}, \|\cdot\|, A)$ is an integrable cocycle. Then there exist real numbers $\lambda_{(1)} > \lambda_{(2)} > \cdots > \lambda_{(k)}$ and T–equivariant complex subbundles of $\mathcal{V}$ defined for almost every $x \in B$, denoted by*

$$0 \subsetneq \mathcal{V}^{\le\lambda_{(k)}} \subsetneq \cdots \subsetneq \mathcal{V}^{\le\lambda_{(1)}} = \mathcal{V},$$

such that for vectors $v \in \mathcal{V}^{\leq\lambda_{(i)}} \setminus \mathcal{V}^{\leq\lambda_{(i+1)}}$ *we have*

$$\lim_{N\to\infty} \frac{1}{N} \log \|T^N(v)\| \to \lambda_{(i)}.$$

We also write $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ for the Lyapunov spectrum consisting of the numbers $\lambda_{(i)}$ from Oseledets' theorem repeated with multiplicities $\operatorname{rank}(\mathcal{V}^{\leq\lambda_{(i)}}/\mathcal{V}^{\leq\lambda_{(i+1)}})$.

Instead of a discrete ergodic transformation of the vector bundle one can consider an ergodic flow g_t on the base B and a smooth (continuous, measurable) connection ∇ on the vector bundle, where ∇ is not necessarily assumed to be flat. Denote by $A(x,t)\colon \mathcal{V}_{(x)} \to \mathcal{V}_{g_t(x)}$ the linear transformation of the fibers induced by the holonomy along the trajectory of the flow.

Definition A.3 The cocycle $(B, g_t, \mu, \mathcal{V}, \nabla, \|\cdot\|)$ is called *integrable* if the function $\sup_{t\in[-1,1]} \log^+ \|A(x,t)\|$ is integrable over B with respect to the measure μ, ie

$$\int_B \sup_{t\in[-1,1]} \log^+ \|A(x,t)\|\, d\mu(x) < \infty.$$

In this situation we also say that $(\mathbb{V}, g_t, \|\cdot\|)$ is an *integrable flat bundle*.

The multiplicative ergodic theorem stated above generalizes naturally to multiplicative cocycles over flows.

It is clear from the definition that integrability of the cocycle and the Lyapunov spectrum do not depend on the choice of the norm in the vector bundle for a large class of norms. To provide a convenient sufficient condition for two norms to be equivalent, we start with the following definition.

Definition A.4 Let $\mathcal{V}$ be a vector bundle over the base B; let μ be a probability measure on B. We say that two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on the vector bundle $\mathcal{V}$ are $L^1(\mu)$–equivalent if the quantity

$$\max_{\vec{v}\in\mathcal{V}_x\setminus\vec{0}} \left|\log \frac{\|\vec{v}\|_{2\,(x)}}{\|\vec{v}\|_{1\,(x)}}\right| = \max_{\vec{v}\in\mathcal{V}_x\setminus\vec{0}} \left|\log \frac{\|\vec{v}\|_{1\,(x)}}{\|\vec{v}\|_{2\,(x)}}\right| \tag{28}$$

is integrable over B with respect to the measure μ, ie

$$\int_B \max_{\vec{v}\in\mathcal{V}_x\setminus\vec{0}} \left|\log \frac{\|\vec{v}\|_{2\,(x)}}{\|\vec{v}\|_{1\,(x)}}\right| d\mu(x) < \infty. \tag{29}$$

The relation of $L^1(\mu)$–equivalence is, clearly, reflexive, symmetric, and transitive.

Theorem A.5 *Suppose that data $(B,T,\mu,\mathcal{V},\|\cdot\|_1,A)$ define a measurable cocycle. For any norm $\|\cdot\|_2$ which is $L^1(\mu)$–equivalent to $\|\cdot\|_1$, the data $(B,T,\mu,\mathcal{V},\|\cdot\|_2,A)$ also define a measurable cocycle, and it has the same Lyapunov filtration and the same Lyapunov exponents as the original one.*

Suppose that data $(B,g_t,\mu,\mathcal{V},\nabla,\|\cdot\|_1)$ define a measurable cocycle. For any norm $\|\cdot\|_2$ which is $L^1(\mu)$–equivalent to $\|\cdot\|_1$ the data $(B,g_t,\mu,\mathcal{V},\nabla,\|\cdot\|_2)$ also define a measurable cocycle, and it has the same Lyapunov filtration and the same Lyapunov exponents as the original one.

Proof We prove the theorem for the cocycle with the discrete time; the proof for the cocycles with continuous time is completely analogous. We have

$$
\begin{aligned}
&\log^+ \max_{\vec v\in\mathcal{V}_x\setminus\vec 0} \frac{\|A_x\vec v\|_{2\,(T(x))}}{\|\vec v\|_{2\,(x)}}\\
&\quad= \log^+ \max_{\vec v\in\mathcal{V}_x\setminus\vec 0} \frac{\|A_x\vec v\|_{2\,(T(x))}}{\|A_x\vec v\|_{1\,(T(x))}}\cdot\frac{\|A_x\vec v\|_{1\,(T(x))}}{\|\vec v\|_{1\,(x)}}\cdot\frac{\|\vec v\|_{1\,(x)}}{\|\vec v\|_{2\,(x)}}\\
&\quad\le \log^+ \max_{\vec w\in\mathcal{V}_{T(x)}\setminus\vec 0} \frac{\|\vec w\|_{2\,(T(x))}}{\|\vec w\|_{1\,(T(x))}} + \log^+ \max_{\vec v\in\mathcal{V}_x\setminus\vec 0}\frac{\|A_x\vec v\|_{1\,(T(x))}}{\|\vec v\|_{1\,(x)}} + \log^+ \max_{\vec v\in\mathcal{V}_x\setminus\vec 0}\frac{\|\vec v\|_{1\,(x)}}{\|\vec v\|_{2\,(x)}}\\
&\quad\le \max_{\vec w\in\mathcal{V}_{T(x)}\setminus\vec 0}\left|\log\frac{\|\vec w\|_{2\,(T(x))}}{\|\vec w\|_{1\,(T(x))}}\right| + \log^+ \max_{\vec v\in\mathcal{V}_x\setminus\vec 0}\frac{\|A_x\vec v\|_{1\,(T(x))}}{\|\vec v\|_{1\,(x)}} + \max_{\vec v\in\mathcal{V}_x\setminus\vec 0}\left|\log\frac{\|\vec v\|_{1\,(x)}}{\|\vec v\|_{2\,(x)}}\right|.
\end{aligned}
$$

It remains to note that, since $T\colon B\to B$ is measure-preserving, we have

$$
\int_B \max_{\vec w\in\mathcal{V}_x\setminus\vec 0}\left|\log\frac{\|\vec w\|_{2\,(T(x))}}{\|\vec v\|_{1\,(T(x))}}\right| d\mu(x) = \int_B \max_{\vec v\in\mathcal{V}_x\setminus\vec 0}\left|\log\frac{\|\vec v\|_{2\,(x)}}{\|\vec v\|_{1\,(x)}}\right| d\mu(x).
$$

Thus, the first and the third terms in the latter sum are $L^1(\mu)$–integrable by definition of $L^1(\mu)$–equivalent norms, and the second term is $L^1(\mu)$–integrable since the cocycle represented by the data $(B,T,\mu,\mathcal{V},\|\cdot\|_1,A)$ is integrable by the assumption of the theorem. We have proved that $L^1(\mu)$–equivalence of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ implies that if the cocycle represented by the data $(B,T,\mu,\mathcal{V},\|\cdot\|_1,A)$ is integrable, then the cocycle represented by the data $(B,T,\mu,\mathcal{V},\|\cdot\|_2,A)$ is also integrable. It remains to prove that the Lyapunov filtrations and the Lyapunov spectra of the two cocycles coincide.

For almost all points $x\in B$, the Lyapunov filtrations and Lyapunov exponents are well defined for both cocycles, and the ergodic sum of the quantity (28) along the trajectory $x, T(x), T(T(x)),\dots$ converges to the integral (29). Namely, let

$$a_N(x) := \frac{1}{N}\Bigg(\max_{\vec{v}\in\mathcal{V}_x\setminus\vec{0}}\left|\log\frac{\|\vec{v}\|_{2\,(x)}}{\|\vec{v}\|_{1\,(x)}}\right| + \max_{\vec{v}\in\mathcal{V}_{T(x)}\setminus\vec{0}}\left|\log\frac{\|\vec{v}\|_{2\,(T(x))}}{\|\vec{v}\|_{1\,(T(x))}}\right| + \cdots + \max_{\vec{v}\in\mathcal{V}_{T^{N-1}(x)}\setminus\vec{0}}\left|\log\frac{\|\vec{v}\|_{2\,(T^{N-1}(x))}}{\|\vec{v}\|_{1\,(T^{N-1}(x))}}\right|\Bigg).$$

The ergodic theorem implies that, for almost all $x \in B$,

$$\lim_{N\to+\infty} a_N(x) = \int_B \max_{\vec{v}\in\mathcal{V}_x\setminus\vec{0}}\left|\log\frac{\|\vec{v}\|_{2\,(x)}}{\|\vec{v}\|_{1\,(x)}}\right| d\mu(x) < \infty,$$

which implies for almost all $x \in B$ the vanishing of the limits

$$\lim_{N\to+\infty}(a_N - a_{N-1}) = 0 \quad \text{and} \quad \lim_{N\to+\infty}\frac{1}{N}a_{N-1} = 0,$$

and hence

$$\lim_{N\to+\infty}\frac{1}{N}\max_{\vec{v}\in\mathcal{V}_{T^{N-1}(x)}\setminus\vec{0}}\left|\log\frac{\|\vec{v}\|_{2\,(T^N(x))}}{\|\vec{v}\|_{1\,(T^N(x))}}\right| = \lim_{N\to+\infty} a_N - \frac{N-1}{N}a_{N-1} = 0. \tag{30}$$

Thus, for almost any $x \in B$ and for any $\vec{v} \in \mathcal{V}_x \setminus \vec{0}$ we have

$$\begin{aligned}\lambda_{(1)}(\vec{v}) &= \lim_{N\to+\infty}\frac{1}{N}\log\|T^N\vec{v}(x)\|_1 = \lim_{N\to+\infty}\frac{1}{N}\log\left(\frac{\|T^N\vec{v}(x)\|_1}{\|T^N\vec{v}(x)\|_2}\cdot\|T^N\vec{v}(x)\|_2\right)\\ &= \lim_{N\to+\infty}\frac{1}{N}\log\frac{\|T^N\vec{v}(x)\|_1}{\|T^N\vec{v}(x)\|_2} + \frac{1}{N}\log\|T^N\vec{v}(x)\|_2\\ &= 0 + \lambda_{(2)}(\vec{v}),\end{aligned}$$

where $\lambda_{(1)}(\vec{v})$ (resp. $\lambda_{(2)}(\vec{v})$) is the Lyapunov exponent associated to the vector $\vec{v}$ defined by the first (resp. second) cocycle, and where the equality

$$\lim_{N\to+\infty}\frac{1}{N}\log\frac{\|T^N\vec{v}(x)\|_1}{\|T^N\vec{v}(x)\|_2} = 0$$

is the corollary of (30). □

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AE: *Department of Mathematics, University of Chicago*
Chicago, IL, United States

MK: *Institut des Hautes Études Scientifiques*
le Bois Marie, Bures-sur-Yvette, France

MM: *Institut für Mathematik, Goethe-Universität Frankfurt*
Frankfurt am Main, Germany

AZ: *Center for Advanced Studies, Skolkovo Institute of Science and Technology*
Moscow, Russia

AZ: *Institut de Mathématiques de Jussieu, Université Paris 6*
Paris, France

eskin@math.uchicago.edu, maxim@ihes.fr, moeller@math.uni-frankfurt.de, anton.zorich@gmail.com

Proposed: Benson Farb
Seconded: Ian Agol, Anna Wienhard

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