msp

Geometry & Topology 22 (2018) 2145–2186

Rotation intervals and entropy on attracting annular continua

ALEJANDRO PASSEGGI
RAFAEL POTRIE
MARTÍN SAMBARINO

We show that if f is an annular homeomorphism admitting an attractor which is an irreducible annular continua with two different rotation numbers, then the entropy of f is positive. Further, the entropy is shown to be associated to a C^0–robust *rotational horseshoe*. On the other hand, we construct examples of annular homeomorphisms with such attractors for which the rotation interval is uniformly large but the entropy approaches zero as much as desired.

The developed techniques allow us to obtain similar results in the context of *Birkhoff attractors*.

37E30; 37B40, 37B45, 37E45, 54H20

1 Introduction

The study of annular dynamics goes back at least to Poincaré, who used suitable (Poincaré) sections in the restricted three-body problem to reduce the initial dynamics to an annulus. This study turned out to be crucial in understanding the problem of stability (see Chenciner [12]) and gave rise to what nowadays is known as KAM theory; see Broer [11].

In this theory the considered dynamics are volume-preserving, reflecting the conservation laws of the particular mechanical system. On the other hand, when physical problems involving nonconservative forces are analyzed, sometimes one is led to study dissipative versions of the former class of systems (see for instance Abraham and Stewart [1], Akhmet and Fen [2] and Thompson and Stewart [41]). In this setting *strange attractors* emerge as natural objects related to the underlying dynamics (for the definitions and basic examples see Milnor [34]). These were proved to exist by Birkhoff [7], who actually showed that they appear associated to the wide class of differentiable annular maps given by *dissipative twist maps* (see P Le Calvez [28]

Published: 5 April 2018

DOI: 10.2140/gt.2018.22.2145

for a comprehensive exposition). They were also found numerically by R Shaw [39], associated to the dynamics induced by differential equations such as the *forced Van der Pol systems*; see also Thompson and Stewart [41]. Since then, annular attractors have been studied both from the mathematical and physical point of view (see [28; 41]).

In order to study this kind of attractors, there are two important dynamical invariants: the *rotation set* and the *topological entropy*. The former is given by averages of displacements of points in the attractor, information that is expressed by an interval of real numbers (see below). The latter is a quantity which measures how *chaotic* the attractor is.[1] It is then natural to try to understand whether these two invariants are related and this motivates our article: we prove that a nontrivial rotation set implies positive topological entropy, and, in contrast, provide examples of systems which have uniformly large rotation intervals and arbitrary small topological entropy.

From the pure mathematical point of view, this problem can be thought of as a version of the well-known Shub's entropy conjecture [40] for maps in the homotopy class of the identity: some geometric property of the dynamical system detectable from "large scale" imposes some lower bound on its complexity (eg topological entropy). In this case we focus on the rotation set of a dynamical system (see Franks [15]), motivated by previous results providing a relationship between the *shape and size* of this set and the topological entropy in some particular settings (degree one circle maps, torus homeomorphisms isotopic to the identity). Searching for similar relationships in the setting of dissipative annular homeomorphisms, we came into a rather surprising outcome: it is possible to show positive entropy assuming that the rotation set is nontrivial, yet it is not possible to obtain lower bounds depending on the shape and size of the rotation set.

The following subsection presents an account of the results in this paper to prepare for the precise statements.

1.1 Presentation of the results

The *rotation set* is an invariant for dynamical systems which has been shown to contain essential information of the dynamics when the underlying space has low dimension, in particular in dimensions one and two.

Poincaré's theory for orientation-preserving homeomorphisms on the circle is the paradigmatic case: the rotation number turns out to be a number which provides a

[1] A weaker version is the study of existence of positive *Lyapunov exponents* — when the dynamics is smooth — which is implied by positivity of topological entropy.

complete description of the underlying dynamics (see for example Katok and Hasselblatt [22, Chapter 11]). Still in dimension one, there is a natural generalization of the rotation number for degree one endomorphisms of the circle, given by an interval (possibly trivial) called the *rotation set*. From this set, crucial information of the dynamics can be deduced, providing, for instance, criteria for the existence of periodic orbits with certain relative displacements among other interesting properties (see Alsedà, Llibre, Mañosas and Misiurewicz [3]).

In dimension two, the dynamics of certain surface homeomorphisms homotopic to the identity is usually described by means of this topological invariant. In particular, for the annulus $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$ and the two-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ it can be said that a theory has been built supported on the rotation set. In these contexts, for a dynamics f given by a homeomorphisms in the homotopy class of the identity and any compact, forward-invariant set K, the rotation set associated to a lift $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined as

$$\rho_K(F) = \left\{ \lim_k \frac{\pi_1(F^{n_k}(x_k) - x_k)}{n_k} \;\middle|\; x_k \in \pi^{-1}(K),\, n_k \nearrow +\infty \right\} \subset \mathbb{R},$$
$$\rho_K(F) = \left\{ \lim_k \frac{F^{n_k}(x_k) - x_k}{n_k} \;\middle|\; x_k \in \pi^{-1}(K),\, n_k \nearrow +\infty \right\} \subset \mathbb{R}^2,$$

respectively, where π denotes for both cases the quotient map and π_1 is the projection over the first coordinate in $\mathbb{R}^2$. In the case $K = \mathbb{T}^2$ one writes $\rho(F)$ instead $\rho_K(F)$.

When $K \subset \mathbb{A}$ is also connected, the shape of this set is given by an (possibly degenerated) interval in the annular case. For the toral case the foundational result by Misiurewicz and Ziemian [35] shows that $\rho(F)$ is a (possibly degenerated) compact and convex set. From these facts, there exists a vast list of interesting results, where, assuming possible geometries for the rotation set, descriptions of the underlying dynamics are obtained. We refer the interested reader to Beguín [6] and Passeggi [37] for a more complete[2] account of this theory.

The *topological entropy* measures how chaotic a prescribed dynamical system is. It measures the rate of exponential growth of different orbits in a dynamical system when observed at a given (arbitrarily small) scale. We shall not provide a formal definition of topological entropy here (see eg Katok and Hasselblatt [22, Chapter 3]). The precise formulation of this notion is rather technical, but it is unimportant to our paper as our proof of positivity of topological entropy relies on obtaining certain dynamical

[2]These surveys are not completely updated as there has been some fast progress in recent years.

configurations which are interesting by themselves (and which are known to imply positive topological entropy).

When the dimension of the rotation set equals the dimension of the space where the dynamics acts, there exists a relation between the *geometry and arithmetic* of the rotation set and the topological entropy of the system. For instance, for degree one maps on the circle, the topological entropy is bounded from below by an explicit (and optimal) function of the extremal points of the rotation set as shown in Alsedà, Llibre, Mañosas and Misiurewicz [3]. In the toral case, the quantity considered for such a lower bound is less explicit and, as far as the authors are aware, not optimal. See Kwapisz [27], Le Calvez and Tal [30] and Llibre and MacKay [31].

In the annulus $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$, a large rotation set is not necessarily associated with large entropy. Integrable twist maps, eg maps of the form $(x, y) \to (x + r(y) \pmod 1, y)$, preserving a foliation by essential circles, have zero entropy but may have rotation sets of arbitrarily large size. One can look at the rotation set restricted to certain invariant regions of the annulus and hope to draw better conclusions.

For this purpose, the class of invariant sets which turns out to be interesting to observe are the essential annular continua: a continuum $K \subset \mathbb{A}$ is called an *essential annular continuum* if $\mathbb{A} \setminus K$ has exactly two connected components and both of them are unbounded (and hence K must disconnect both ends of $\mathbb{A}$). These sets are natural objects in surface dynamics, which model, for instance, the mentioned attractors, and have been the focus of several works in the field. The topology of essential annular continua can be very simple, as for the circle or the closed annulus itself, and very complex, as in the case of indecomposable annular continua, for instance the pseudocircle.

We mentioned above that for the case where $K \subset \mathbb{A}$ is a closed essential annulus, there is no relation between the length of the rotation interval and the topological entropy. As a next step, one can look at those annular continua containing no essential annulus. For this class of continua, there exists an interesting example by Walker [42], in which an invariant annular continua having empty interior K is constructed having zero entropy and arbitrary large rotation set. Nevertheless, this continuum contains an essential circle inside, that is, K is not irreducible. Irreducible annular continua (see Section 2.3), often called *circloids*, with nontrivial rotation sets are known as interesting examples, and it is possible to construct them so that they are robust in the C^0 topology (see Boroński and Oprocha [10] and Le Calvez [28]). Further, as we mentioned before, this kind of dynamics occur as global attractors of dissipative twist maps given by the so-called *Birkhoff attractors* [28], and are the canonical model for the *strange attractors* of annular diffeomorphisms.

In this article we show the following complementary facts. For an orientation-preserving homeomorphism f and an attracting invariant circloid $\mathcal{C}$:

- We show in Theorem A that if $\mathcal{C}$ has a nontrivial rotation set, then some power of f has a topological horseshoe with a nontrivial rotation set (see Section 1.2 for the definition of *rotational horseshoe*). Moreover, this situation is C^0–robust, that is, any homeomorphism C^0–close to f has a *rotational horseshoe*.
- In Theorem B we show that there is no relation between the entropy and the length of the rotation set, so the power of f needed in order to find the horseshoe in Theorem A can be arbitrary large for a prescribed rotation set.

The first result answers positively (assuming the circloid is a global attractor) a folklore problem about the relation between entropy and rotation intervals on circloids (see for instance Koropecki [24] and Question 3 in Boroński and Oprocha [10]). Moreover, the result shows that these kind of attractors are associated to C^0–robust topological horseshoes with rotational information (see the definition below). The second result is quite surprising: one might expect that the size of the rotation set could impose a lower bound on the topological entropy, as is the case for degree one maps of the circle.

The techniques in the proofs allow us to deal with the related class of Birkhoff attractors (see the definition below).

Next, we give precise statement of the results.

1.2 Precise statements

In what follows we list the obtained results. Recall that $\mathbb{A}$ stands for the infinite annulus $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$. We denote by $\mathrm{Homeo}_+(\mathbb{A})$ the set of homeomorphisms of the annulus which preserve orientation.

Given a homeomorphism $f\colon \mathcal{X} \to \mathcal{X}$ and a partition by $m > 1$ elements $R_0, \dots, R_{m-1}$ of $\mathcal{X}$, the *itinerary* function $\xi\colon \mathcal{X} \to \{0, \dots, m-1\}^{\mathbb{Z}} := \Sigma_m$ is defined by $\xi(x)(j) = k$ if and only if $f^j(x) \in R_k$ for every $j \in \mathbb{Z}$.

We say that a compact invariant set $\Lambda \subset \mathbb{A}$ of $f \in \mathrm{Homeo}_+(\mathbb{A})$ is a *rotational horseshoe* if it admits a finite partition $\mathcal{P} = \{R_0, \dots, R_{m-1}\}$ with R_i open sets of Λ such that:

(1) The itinerary ξ defines a semiconjugacy between $f|_\Lambda$ and the full-shift $\sigma\colon \Sigma_m \to \Sigma_m$, that is, $\xi \circ f = \sigma \circ \xi$ with ξ continuous and onto.

(2) For any lift F of f, there exist a positive constant κ and vectors $v_0, \dots, v_{m-1} \in \mathbb{Z} \times \{0\}$ such that

$$\left\| (F^n(x) - x) - \sum_{i=0}^{n} v_{\xi(x)} \right\| < \kappa \quad \text{for every } x \in \pi^{-1}(\Lambda),\ n \in \mathbb{N}.$$

Clearly, the existence of a rotational horseshoe for a map implies positive topological entropy larger than $\log(m) \geq \log(2)$. Other interesting implications can be obtained, for instance the realization[3] of every rational rotation vector in $\rho_\Lambda(F)$.

Theorem A *Assume that $f \in \mathrm{Homeo}_+(\mathbb{A})$ has a global attractor $\mathcal{C}$ given by a circloid for which $\rho_{\mathcal{C}}(F)$ is a nontrivial interval, where F is a lift of f. Then there exists n_0 such that f^{n_0} has a rotational horseshoe Λ contained in $\mathcal{C}$. Moreover, there exists a C^0–neighborhood $\mathcal{N}$ in $\mathrm{Homeo}_+(\mathbb{A})$ of f such that for every $g \in \mathcal{N}$ we have a rotational horseshoe Λ_g for g^{n_0}. In particular, $h_{\mathrm{top}}(g) > \varepsilon_0$ for all $g \in \mathcal{N}$ and some positive constant ε_0.*

This result and Theorem C below can be derived from a more general statement, given by Theorem 3.14 in Section 3.5.

The complementary result is given by the following:

Theorem B *Given $\varepsilon > 0$ there exists a smooth diffeomorphism $f \in \mathrm{Homeo}_+(\mathbb{A})$ admitting a global attractor $\mathcal{C}$, which is a circloid, such that $\rho_{\mathcal{C}}(F) \supset [0,1]$ for some lift F of f, while $h_{\mathrm{top}}(f) < \varepsilon$.*

As we mentioned above, this implies that for a prescribed positive length of the rotation interval, the minimum positive integer n_0 as in Theorem A (for which f^{n_0} has a rotational horseshoe) could be arbitrary large.

Recall that given a riemannian manifold M a diffeomorphism $f\colon M \to M$ is said to be *dissipative* whenever there exists $\varepsilon > 0$ such that $|\det(Df_x)| < 1 - \varepsilon$ for every $x \in M$. Further, recall that a diffeomorphism $f\colon \mathbb{A} \to \mathbb{A}$ is said to be a *twist map* if for some lift F of f there is $\varepsilon > 0$ such that $DF_x((0,1)) = (a(x), b(x))$ with $\varepsilon < a(x) < \frac{1}{\varepsilon}$.

[3] A periodic point x *realizes* a rational rotation vector $\frac{p}{q}$ (with $p \in \mathbb{Z}^2$ and $q \in \mathbb{Z}_{>0}$) if there is a lift $\tilde{x}$ of x such that $F^q(\tilde{x}) = \tilde{x} + p$.

Given a dissipative twist map of the annulus which maps an essential closed annulus into its interior one can associate a global attractor Λ, given by the intersection of the iterates of the annulus. This is an annular continua with empty interior and contains a unique circloid $\mathcal{C}$, which is the so-called Birkhoff attractor (see [28]). Notice, however, that this situation differs from the situation in Theorem A, as the Birkhoff attractor $\mathcal{C}$ might not be an attractor in the usual sense. In other words, it could be the case that $\Lambda \neq \mathcal{C}$. In this setting, we show the following result:

Theorem C *Assume that $f\colon \mathbb{A} \to \mathbb{A}$ is an orientation-preserving diffeomorphism which is dissipative, verifies the twist condition and $f(\mathcal{A}) \subset \mathcal{A}$ for some compact essential annulus $\mathcal{A} \subset \mathbb{A}$. Further, assume that $\rho_{\mathcal{C}}(F)$ is a nontrivial interval, where $\mathcal{C}$ is the Birkhoff attractor of f. Then there exists n_0 such that f^{n_0} has a rotational horseshoe Λ. Moreover, there exists a C^0–neighborhood $\mathcal{N}$ of f in* $\mathrm{Homeo}_+(\mathbb{A})$ *such that for every $g \in \mathcal{N}$ we have a rotational horseshoe Λ_g for g^{n_0}. In particular, $h_{\mathrm{top}}(g) > \varepsilon_0$ for all $g \in \mathcal{N}$ and some positive constant ε_0.*

We finish adapting the proof of Theorem B to show that the topological entropy and the lengths of rotation intervals are again not related for Birkhoff attractors. The difference with Theorem B is that although in this case we have dissipation, we cannot ensure that the global attracting set coincides with the unique invariant circloid it contains.

Theorem D *For every $\varepsilon > 0$ there exists a dissipative twist smooth diffeomorphisms $f\colon \mathbb{A} \to \mathbb{A}$ having a Birkhoff attractor $\mathcal{C}$ with $\rho_{\mathcal{C}}(F) \supset [0,1]$ and $h_{\mathrm{top}}(f|_{\mathcal{C}}) < \varepsilon$.*

Remark 1.1 There is a certain analogy between Birkhoff attractors and *regions of instability* of conservative annulus homeomorphisms (see for example Franks and Le Calvez [17]). Recall that an *instability region* R for an area-preserving annular homeomorphism is an invariant compact connected set whose boundary is given by two disjoint essential annular continua C_- and C_+, having a point with α–limit in C_- and ω–limit in C_+, and a point with ω–limit in C_- and α–limit in C_+. In a recent article, Le Calvez and F Tal [30] (see also Franks and Handel [16]) have shown that whenever an instability region has a nontrivial interval as rotation set, then the map has positive entropy. In the process of proving Theorems B and D we must construct an instability region (of a smooth twist map) with rotation set containing $[0,1]$ and arbitrarily small entropy, showing that in this context again, there is no relation between the size of the rotation interval and the topological entropy of the map.

1.3 The techniques

We present here some key points in the proofs of Theorems A and B, avoiding technicalities.

The main idea behind the proof of Theorem A is to work in the universal cover and use the fact that there are periodic points turning at different speeds in order to construct a topological rectangle R which has an iterate intersecting itself and a translate of itself as well in a *Markovian* way. Using this configuration and the results of Kennedy and Yorke [23], we obtain a rotational horseshoe as defined above.

We are not able to control the number of iterates we need to obtain this intersection (and it would be impossible in view of Theorem B) but we give some geometric criteria that forces a lower bound. The construction of this rectangle requires entering into properties of the topology of noncompactly generated continua (a generalization of indecomposable continua). The two key points are the construction of "stable" sets for periodic points, obtained by approaching the dynamics by hyperbolic dynamics in the C^0 topology (this step works in quite large generality; see Theorem 3.3), and then show that for periodic points having different rotation vectors, these "stable sets" intersect both boundaries of a given annulus containing the circloid (Proposition 3.6).

In order to construct the examples of Theorem B, the idea is to work with C^1 perturbations of a twist-map, which are based on the C^1–connecting lemma for pseudoorbits in the conservative setting, due to M C Arnaud, C Bonatti and S Crovisier [4]. The use of this theorem in this case is not completely straightforward, as it is a result of generic nature, and we need to take care of some nongeneric properties of our examples. However, by an inspection of the proof in Crovisier [13], one can state a suitable version in order to obtain our desired perturbations. We remark that similar kind of perturbative techniques were already considered in Girard [18] for different purposes. Using these perturbations one can construct a smooth diffeomorphism of the closed annulus which is conservative and for which points in each of the boundary components are homoclinically related (and have different rotation numbers). A further perturbation allows us to destroy the annulus and an attracting circloid emerges, which still has the same rotation set. As the derivative of the original map had small growth, the same holds for the perturbations which ensures small entropy.

Theorem B shows that the usual arguments dealing with Nielsen–Thurston theory as used for instance in Llibre and MacKay [31] and Kwapisz [27] do not work for proving Theorem A. On the other hand, recently Le Calvez and Tal [30] developed a *forcing* technique based in Le Calvez's foliation by Brouwer lines (see Le Calvez [29]), which could provide an alternative proof of the positive entropy in Theorem A.

Let us end this introduction by mentioning that Crovisier, Kocsard, Koropecki and Pujals have announced progress in the study of a particular family of diffeomorphisms of the annulus which they call *strongly dissipative*. In this class, they are able, among other things, to prove positive entropy if there are two rotation vectors and the maximal invariant set is transitive. We notice that even if our proof does not give lower bounds on the entropy in all generality (and it cannot give one because of Theorem B), it is possible that for some families such a lower bound exists. In particular, we reemphasize that our method does give a lower bound after some configuration is attained (see Lemma 3.1).

1.4 Organization of the paper

The structure of the article is the following. We start with some preliminaries in Section 2. From those, Sections 2.1 and 2.2 are used in the proof of Theorem B while Sections 2.3 and 2.4 are used for the proof of Theorem A.

Theorems A and B have independent proofs and can be read in any order. Theorems A and C are proved in Section 3, whereas Theorems B and D are proved in Section 4. In Section 3.5 a generalization of Theorem A is obtained, from which Theorem C can be derived.

Acknowledgments

We would like to thank M C Arnaud, S Crovisier, T Jäger, A Koropecki, P Le Calvez and F Tal and the referees for their input to this paper. The authors were partially supported by CSIC Grupo 618 *Sistemas dinámicos*. Potrie and Sambarino were partially supported by the Palis-Balzan project.

2 General preliminaries

We introduce in this section some preliminary well-known results which will be used later. Some results hold in higher dimensions too but we will always restrict to the surface case. The reader can safely skip this section and come back when results are referred to.

2.1 A remark on continuity of entropy in the C^1 topology

For a C^1–surface map $f\colon M \to M$ there is a bound on the topological entropy given by

$$h_{\mathrm{top}}(f) \le 2\log \sup_{x\in M} \|Df_x\| = 2\log\|Df\|.$$

See for example [22, Corollary 3.2.10]. Since $h_{\mathrm{top}}(f) = \frac{1}{n} h_{\mathrm{top}}(f^n)$, we have

$$h_{\mathrm{top}}(f) \le \frac{2}{n}\log\|Df^n\| \quad \text{for all } n\in\mathbb{N}.$$

We deduce the following:

Proposition 2.1 *Let $f\colon M\to M$ be a C^1–surface map such that*

$$\lim_{n\to\infty}\frac{2}{n}\log\|Df^n\| = 0.$$

Then, for every $\varepsilon>0$ there exists a C^1–neighborhood $\mathcal{N}$ of f such that if $g\in\mathcal{N}$, one has that $h_{\mathrm{top}}(g)<\varepsilon$.

Proof Fix $\varepsilon>0$ and choose $n>0$ such that $\frac{2}{n}\log\|Df^n\|<\varepsilon$. Choose a C^1–neighborhood $\mathcal{N}$ of f so that for every $g\in\mathcal{N}$ one has $\frac{2}{n}\log\|Dg^n\|<\varepsilon$. By the estimate above, it follows that for every $g\in\mathcal{N}$ one has that $h_{\mathrm{top}}(g)<\varepsilon$. □

2.2 Connecting lemma for pseudoorbits

In this section we state a C^1 perturbation lemma for pseudoorbits in the conservative setting in the spirit of the well-known pseudoorbit connecting lemma [8; 4].

Let M be a surface, ν an area form in M and let $\mathrm{Diff}^1_\nu(M)$ be the space of C^1 area-preserving diffeomorphisms, with the C^1 topology. We recall that given ε, a finite sequence $(z_k)_{k=0}^n$ is an *ε–pseudoorbit* (or *ε–chain*) from $p\in M$ to $q\in M$ when $z_0=p$, $z_n=q$ and

$$d(f(z_k), z_{k+1}) < \varepsilon \quad \text{for all } k=0,\dots,n-1.$$

Consider a compact set $K\subset M$. For $x,y\in M$ we write $x \dashv_K y$ if for every $\varepsilon>0$ there exists a ε–pseudoorbit $(z_k)_{k=0}^n$ with $z_0=x$, $z_n=y$ and

$$f(z_k),\, z_{k+1}\in K \quad \text{whenever } f(z_k)\ne z_{k+1}.$$

Denote by $\mathrm{Diff}^1_{\nu,\mathrm{per}}(M)$ the set of those $f \in \mathrm{Diff}^1_\nu(M)$ for which the set of periodic points of period k is finite for all $k \in \mathbb{N}$. Recall that the *support* of a perturbation g of f is the set of points $x \in M$ where $g(x) \neq f(x)$.

Theorem 2.2 (a version of the C^1–connecting lemma for pseudoorbits [13]) *Let M be a compact surface possibly with boundary and $f \in \mathrm{Diff}^1_{\nu,\mathrm{per}}(M)$. Given a neighborhood $\mathcal{N} \subset \mathrm{Diff}^1_\nu$ of f, there exists $N = N(f, \mathcal{N})$ such that if*

- *K is a compact set disjoint from the boundary,*
- *U is an arbitrary small neighborhood of $K \cup \cdots \cup f^{N-1}(K)$, and*
- *$p, q \in M$ with $p \dashv_K q$,*

then there exist a perturbation $g \in \mathcal{N}$ of f supported in U and $n > 0$ such that $g^n(p) = q$.

This result follows with the same proof as that of Theorem III.1 presented in [13] via [13, Theorem III.4], where the choice of N appears. The difference is that in [13] the statement requires the complete pseudoorbit to be contained in K while here we demand only the *jumps* to be contained there. By an inspection of the proofs in [13] one can see that the perturbations are only performed when the pseudoorbit has jumps, so our statement holds with only minor modifications.

Remark 2.3 The diffeomorphism g can be considered to be as smooth as f since it is obtained by composing a finite number of elementary perturbations with small support, all of which are smooth (though their C^r–size with $r > 1$ might be large).

2.3 Some properties of separating continua

We first recall some basic facts about continua and separation properties in surfaces. We refer the reader to [5] for more information. After this, we will show a property of irreducible annular continua that will be useful in the proof of Theorem A.

Throughout this article we consider the annulus $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$ and $\pi\colon \mathbb{R}^2 \to \mathbb{A}$ the usual covering map. Further, we will fix a two-point compactification of $\mathbb{A}$ given by the sphere S^2 and two different points $+\infty, -\infty \in S^2$.

Recall that a continuum is a compact nonempty connected metric space. We say a continuum $E \subset \mathbb{A}$ is *essential* whenever there are two unbounded connected components in $\mathbb{A} \setminus E$. These connected components are denoted in general by $\mathcal{U}^+$ and $\mathcal{U}^-$, where the first one accumulates in $+\infty$ and the second one in $-\infty$, when considered in S^2. Notice that there could be also several bounded connected components in E^c. Continua that are not essential in $\mathbb{A}$ are called *inessential*, and can be characterized as those continua contained in some topological disk in $\mathbb{A}$.

An annular continuum $K \subset \mathbb{A}$ is an essential continuum such that K^c contains no bounded connected components. Finally, an *irreducible annular continuum* or *circloid* $\mathcal{C}$ is an annular continuum which does not contain properly any other annular continua. As is well known, the topology of these continua can be very simple, as the one of the circle, or extremely complicated as the case of the pseudocircle. It can be the case that the circloid has nonempty interior; an example (and figure) can be found, for instance, in [38].

When a circloid has empty interior it is called a *cofrontier* as it coincides with the boundaries of $\mathcal{U}^+$ and $\mathcal{U}^-$. A partial converse result holds: whenever an annular continuum $\mathcal{C}$ verifies that $\partial\mathcal{C} = \partial\mathcal{U}^+ \cap \partial\mathcal{U}^-$, we have that $\mathcal{C}$ is a circloid (with possible nonempty interior). See [20, Corollary 3.3].

There is an important class of continua, which is associated to a complicated topology, defined as follows. An *indecomposable* continuum $\mathcal{C}$ is a continuum such that whenever C_1 and C_2 are a pair of continua included in $\mathcal{C}$ with $\mathcal{C} = C_1 \cup C_2$, we have that $C_1 = \mathcal{C}$ or $C_2 = \mathcal{C}$. In particular, one can define the *indecomposable cofrontier*. This definition is not suitable for circloids having nonempty interior, as one can observe that in this case the continua can be always decomposed. Nevertheless, a suitable generalization of indecomposability for this situation can be considered, given by the following (see [21]).

Let $\mathcal{C} \subset \mathbb{A}$ be an essential annular continua. We say that $\mathcal{C}$ is *compactly generated* if there exists a compact connected set $\widehat{\mathcal{C}}$ in $\mathbb{R}^2$ such that $\pi(\widehat{\mathcal{C}}) = \mathcal{C}$ (such a continuum $\widehat{\mathcal{C}}$ is called a *compact generator*). In particular this definition can be applied to essential circloids. The annular continua which are *not compactly generated* and *indecomposable* annular continua have strong relationships even if their properties are slightly different (see eg [21, Remarks 1.1 and 5.5]). For this paper, the notion of being noncompactly generated is the most suitable.

In this article we deal only with *noncompactly generated circloids*, as compactly generated ones do not support two rotation vectors for a given dynamics. This result was originally proved for Birkhoff attractors in [28] and then generalized for cofrontiers in [5]. Finally, it was extended in [21] to deal with circloids. Although in this last reference the proof is not explicitly given for the nonempty interior case, as is remarked by the authors, the proof they give works exactly as it is written for circloids with nonempty interior (see [21, Remark 5.5]).

Theorem 2.4 [5; 21] *Let $f \in \mathrm{Homeo}_+(\mathbb{A})$ having an invariant circloid $\mathcal{C}$ such that $\rho_{\mathcal{C}}(F)$ contains two different rotation vectors for some lift F of f. Then $\mathcal{C}$ is noncompactly generated.*

We establish next a proposition concerning the topology of noncompactly generated circloids. Given a circloid $\mathcal{C} \subset \mathbb{A}$, $x \in \mathcal{C}$, $\widetilde{\mathcal{C}} = \pi^{-1}(\mathcal{C})$ and a lift $\hat{x}$ of x we define[4]

$$\widehat{C}_{\hat{x}} = \bigcup_{k \in \mathbb{N}} \mathrm{c.c.}_{\hat{x}}\big[\widetilde{\mathcal{C}} \cap \pi_1^{-1}([-k,k])\big].$$

For indecomposable cofrontiers, these are connected sets which lift the *composants* (see [19]).

Proposition 2.5 *Let $\mathcal{C}$ be a noncompactly generated circloid. Then $\widehat{C}_{\hat{x}}$ is an unbounded connected set which does not contain any point $\hat{x}+j$ with $j \in \mathbb{Z} \setminus \{0\}$.*

Proof By definition, $\widehat{C}_{\hat{x}}$ is an increasing union of compact connected sets $C_k = \mathrm{c.c.}_{\hat{x}}\big[\widetilde{\mathcal{C}} \cap \pi_1^{-1}([-k,k])\big]$ containing $\hat{x}$. Moreover, as $\widetilde{\mathcal{C}}$ is connected and unbounded, one can observe that every connected component of $\widetilde{\mathcal{C}} \cap \pi_1^{-1}([-k,k])$ must intersect $\partial\pi_1^{-1}([-k,k])$, so C_k meets $\partial\pi_1^{-1}([-k,k])$ for every $k \in \mathbb{N}$. This implies that $\widehat{C}_{\hat{x}}$ is unbounded and connected.

Assume for a contradiction we have $\hat{x}+j \in \widehat{C}_{\hat{x}}$ with $j \in \mathbb{Z} \setminus \{0\}$ Then we have that both $\hat{x}$ and $\hat{x}+j$ belong to C_k for some $k \in \mathbb{N}$. Thus, $\pi(C_k)$ is an annular continuum, so it must coincide with $\mathcal{C}$ as it is a circloid. But this implies that $\mathcal{C}$ has a compact generator. □

We are interested in studying inessential continua intersecting a noncompactly generated circloid $\mathcal{C}$ which do not meet one of the unbounded components in the complement of the circloid. Fix a noncompactly generated circloid $\mathcal{C}$ and let $K \not\subset \mathcal{C}$ be an inessential continuum in $\mathbb{A}$, so that $K \cap \mathcal{U}^- = \varnothing$. Everything we show for this situation also holds for the complementary case where $K \cap \mathcal{U}^+ = \varnothing$.

In general for a continuum $C \subset \mathbb{A}$ we say that an injective curve $\gamma\colon [0,+\infty) \to \mathbb{A}$ lands at $z \in C$ from $+\infty$ if $\gamma(t) \in C^c$ for all $t \neq 0$, $\gamma(0) = z$, and $\lim_{t\to+\infty}\gamma(t) = +\infty$

[4] We denote for any point x in a topological space $\mathcal{X}$ its connected component by $\mathrm{c.c.}_x(\mathcal{X})$.

when viewed in S^2. When C is an essential continuum, the points z which admit a curve landing on them are called *accessible points*, and it is easy to prove that they form a dense set in $C \cap \partial \mathcal{U}^+$. Thus, in our situation we can consider a curve γ as before such that:

- $\gamma \cap K = \varnothing$.[5]
- γ lands at $z \in \mathcal{C}$.
- $\pi_1(\hat{\gamma})$ is a bounded set for any lift $\hat{\gamma}$ of γ.

Let $\hat{A}$ be a connected component of $\pi^{-1}(\mathcal{U}^+ \setminus \gamma)$. Our main goal is to show the following property, which is important to prove Theorem A.

Proposition 2.6 *It holds that $\pi^{-1}(K) \cap \hat{A}$ is bounded.*

Consider $\widetilde{\mathcal{C}} = \pi^{-1}(\mathcal{C})$, $\widetilde{\mathcal{U}}^+ = \pi^{-1}(\mathcal{U}^+)$ and $\widetilde{\mathcal{U}}^- = \pi^{-1}(\mathcal{U}^-)$. Fix a lift $\hat{K}$ of K which intersects $\hat{A}$. In order to prove Proposition 2.6, it is enough to show that only finitely many horizontal integer translations of $\hat{A}$ meets $\hat{K}$.

We prove the following lemma. Recall that $z \in \mathcal{C}$ is the landing point of the curve γ.

Lemma 2.7 *Fix $\hat{z} \in \pi^{-1}(z)$. If $\hat{K}$ intersects $\hat{C}_{\hat{z}} + k$ and $\hat{C}_{\hat{z}} + k'$ then $|k - k'| \leq 1$.*

Proof Assume otherwise. Without loss of generality we can assume that $k' > k$.

By the definition of $\hat{C}_{\hat{z}}$ we can consider continua $\Lambda_k \subset \hat{C}_{\hat{z}} + k$ containing $\hat{z} + k$ and intersecting $\hat{K}$, and $\Lambda_{k'} \subset \hat{C}_{\hat{z}} + k'$ containing $\hat{z} + k'$ and intersecting $\hat{K}$. Furthermore, as $\mathcal{C}$ is not compactly generated, Proposition 2.5 implies that none of them contain $\hat{z} + k + 1$.

Let $\hat{\gamma}$ be the lift of γ containing $\hat{z}$. We have that $\Lambda_k \cap (\hat{\gamma} + k) = \{\hat{z} + k\}$ and $\Lambda_k \cap (\hat{\gamma} + j) = \varnothing$ for every $j \in \mathbb{Z} \setminus \{k\}$, and the symmetric conditions hold for $\Lambda_{k'}$. See Figure 1.

Let $\Gamma = (\hat{\gamma} + k) \cup \Lambda_k \cup (\hat{\gamma} + k') \cup \Lambda_{k'} \cup \hat{K}$, which is a closed and connected set. Further, consider a horizontal segment $H \subset \widetilde{\mathcal{U}}^+$ whose endpoints are contained one in $\hat{\gamma} + k$, the other one in $\hat{\gamma} + k'$, and there are no other intersection between H and Γ. Notice that this can be easily constructed since the vertical coordinate of points in $\widetilde{\mathcal{C}}$ are uniformly bounded.

[5] We here abuse notation by identifying the curve with its image using the same name.

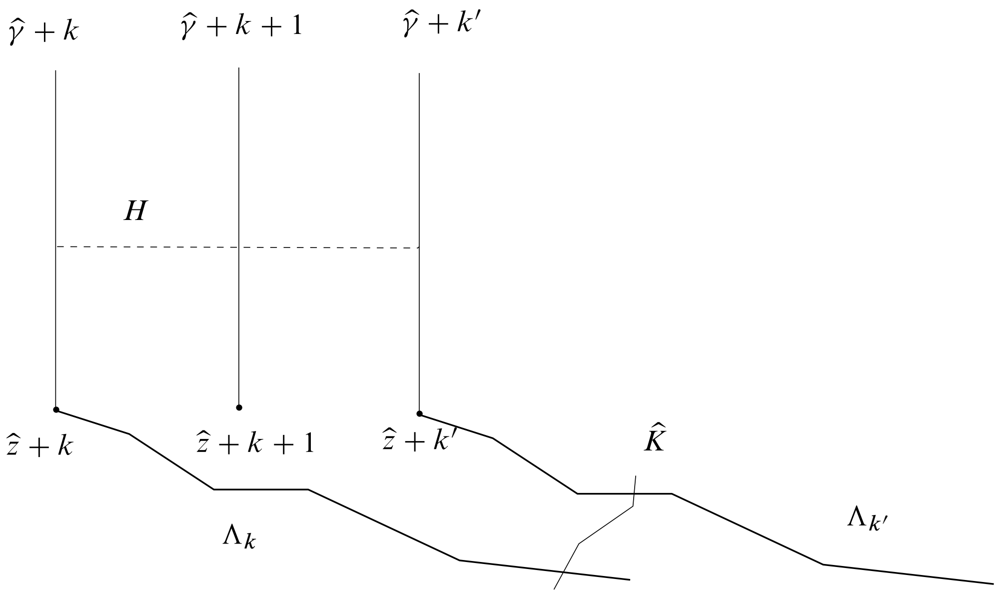

Figure 1: Proof of Lemma 2.7

As $\Gamma\cap\widetilde{\mathcal{U}}^-=\varnothing$, we have that $\widetilde{\mathcal{U}}^-$ is contained in one connected component of Γ^c, which we call U^-. Moreover, H must be contained in a different connected component of Γ^c, as any curve from H to $-\infty$ which does not intersect Γ would allow us to separate Γ into two connected components, one containing $\widehat{\gamma}+k$ and another one containing $\widehat{\gamma}+k'$. We denote the connected component of Γ^c containing H in its closure by U^+.

Due to our assumption, we have that $\widehat{\gamma}+k+1$ intersects H. Therefore, $\widehat{z}+k+1$ is in the interior of U^+ and therefore is not accumulated by $\widetilde{\mathcal{U}}^-$, which contradicts that $\widetilde{\mathcal{C}}$ is the lift of a circloid. □

Now we are ready to prove Proposition 2.6.

Proof of Proposition 2.6 Working with $\widehat{\gamma}$ as before, we can assume without loss of generality that the closure of $\widehat{A}$ contains both $\widehat{\gamma}$ and $\widehat{\gamma}+1$.

We will show that if a connected component $\widehat{K}$ of $\pi^{-1}(K)$ intersects $\widehat{A}+k$ and $\widehat{A}+k'$ for some $k\neq k'$, then it must intersect either $\widehat{C}_{\widehat{z}}+k$ or $\widehat{C}_{\widehat{z}}+k+1$. Thus, Lemma 2.7 implies that $\widehat{K}$ meets only finitely many lifts of $\widehat{A}$ (in fact, at most three consecutive lifts), which implies that $\pi^{-1}(K)\cap\widehat{A}$ is bounded.

Without loss of generality, we assume that $\widehat{K}$ intersects $\widehat{A}$ and $\widehat{A}+k$ for some $k \neq 0$, and assume by contradiction that $\widehat{K}$ does not intersect $\widehat{C}_{\hat{z}}$ nor $\widehat{C}_{\hat{z}}+1$. Choose a curve η contained in $\widehat{A}+k$ landing at a point $y \in \widehat{K}$ (recall that $\widehat{K} \cap \gamma = \varnothing$). Choose also a point $x \in \widehat{K} \cap \widehat{A}$.

Recall that we denote by $\pi_1\colon \mathbb{R}^2 \to \mathbb{R}$ the projection onto the first coordinate. By Proposition 2.5 one has that $\pi_1(\widehat{C}_{\hat{z}})$ is unbounded, and we assume without loss of generality that it has no upper bound. Notice that π_1 is bounded both on $\hat{\gamma}$ and η.

Choose a very large $r > 0$ and consider a vertical line $v_r = \pi_1^{-1}(r)$ which intersects $\widehat{C}_{\hat{z}}$ and $\widehat{C}_{\hat{z}}+1$ in points w_0^r and w_1^r, respectively. Choose a nonseparating continuum Λ_0^r in $\widehat{C}_{\hat{z}}$ containing $\hat{z}$ and w_0^r and similarly consider $\Lambda_1^r \subset \widehat{C}_{\hat{z}}+1$ containing $\hat{z}+1$ and w_1^r, which can be done due to the arguments we did before. Define $I_r \subset v_r$ as the segment joining w_0^r with w_1^r.

Let $\Gamma_r = \hat{\gamma} \cup \Lambda_0^r \cup (\hat{\gamma}+1) \cup \Lambda_1^r \cup I_r$. Then, by the same argument we did before, one can consider $U^+(r)$ as the connected component of Γ_r^c containing an horizontal segment H joining $\hat{\gamma}$ and $\hat{\gamma}+1$. One can observe that, by construction, $\bigcup_{r>0} U^+(r) \supset \widehat{A}$, as every point u in $\widehat{A}$ can be joined to a point in H, with a compact arc J, so that for r_u large enough we have that $J \cap \mathrm{pr}_1^{-1}([r_u, +\infty)) = \varnothing$, so $J \subset U^+(r)$ for $r \geq r_u$. Thus, for every r big enough, we find a point of $\widehat{K} \cap \widehat{A}$ in $U^+(r)$. Therefore, as $\widehat{K}$ is compact and connected, there exists $r_0 \in \mathbb{R}$ such that $\widehat{K} \subset U^+(r)$ for every $r > r_0$. Notice that we do not claim that $U^+(r) \subset \widehat{A}$, which is false in general.

On the other hand, as η cannot intersect H, one can see that η meets $U^+(r)^c$ for all $r > r_0$. Thus, if one considers $r' > r_0$ such that $\eta \cap v_{r'} = \varnothing$ (which can be done as $\pi_1(\eta)$ is bounded), we have that $U^+(r') \cap \eta = \varnothing$, otherwise η intersects $\Gamma_{r'} \setminus v_{r'}$, which is impossible by construction. This is a contradiction as $\eta \cap \widehat{K} \neq \varnothing$ and $\widehat{K} \subset U^+(r')$. □

2.4 Prime end compactification

Consider a homeomorphism $f\colon \mathbb{A} \to \mathbb{A}$ which we can compactify to a homeomorphism $\hat{f}\colon S^2 \to S^2$ by adding two fixed points at infinity. In our context, there is a global attractor $\mathcal{C}$ in $\mathbb{A}$ which is a circloid; this implies that the points at infinity are sources for $\hat{f}$ and the boundary of their basins coincide with $\partial\mathcal{C}$.

Let $\mathcal{U}^+$ and $\mathcal{U}^-$ be the connected components of $\mathbb{A} \setminus \mathcal{C}$ (which are unbounded). Denote by $\widetilde{\mathcal{U}}^{\pm}$ their lifts to the universal cover $\mathbb{R}^2$ which are connected sets. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$

be a lift of f to $\mathbb{R}^2$; it follows that $F \circ T = T \circ F$, where T is any integer translation in the first coordinate.

We denote by $\widehat{\mathcal{U}}^{\pm} = \mathcal{U}^{\pm} \cup \{\pm\infty\}$ the corresponding components in S^2. These are f– and $\hat{f}$–invariant, respectively, simply connected open sets and the dynamics coincides with that of the basin of a source in each $\widehat{\mathcal{U}}^{\pm}$. We introduce here some very basic facts from prime end theory used in this paper and refer to the reader to [32; 25; 33] or [24, Section 2.2] for more details and references.

The prime end compactification of $\widehat{\mathcal{U}}^{\pm}$ is a closed topological disk $U^{\pm} \cong \mathbb{D}^2$ obtained as a disjoint union of $\widehat{\mathcal{U}}^{+}$ and a circle with an appropriate topology (see [32]).

If one lifts the inclusion $\mathcal{U}^{\pm} \hookrightarrow \widehat{\mathcal{U}}^{\pm} \setminus \{\pm\infty\}$ one obtains a homeomorphism

$$p^{\pm} \colon \widetilde{\mathcal{U}}^{\pm} \to \mathbb{H}^2,$$

and by considering $\widehat{F}^{\pm}$, the homeomorphism of $\mathbb{H}^2$ induced by F on $\widetilde{\mathcal{U}}^{\pm}$ (ie such that $p^{\pm} \circ F = \widehat{F}^{\pm} \circ p^{\pm}$), one sees that $\widehat{F}^{\pm}$ extends to a homeomorphism of the closure $\mathrm{cl}[\mathbb{H}^2]$ in $\mathbb{R}^2$ and still commutes with horizontal integer translations. This allows one to compute the *upper and lower prime end rotation numbers* of $\mathcal{C}$ (see [24] for more details). However, we shall not use this, but just use the following fact about $\widehat{F}^{\pm}$ and its relation with F:

- The map $\widehat{F}^{\pm}$ restricted to $\partial\mathbb{H}^2 \cong \mathbb{R}$ is the lift of a circle homeomorphism where the horizontal integer translations act as deck transformations.

We finish with a last topological property for the prime end compactification. Let $\mathcal{U}$ be a topological disk bounded by a continuum $\mathcal{C}$ contained in some surface. For any curve $\gamma \colon [0,1] \to \mathcal{U} \cup \mathcal{C}$, with $\gamma(t) \in \mathcal{C}$ if and only if $t = 0$, we have that the corresponding curve $\eta \colon (0,1] \to \mathbb{D}$ of γ restricted to $(0,1]$ admits a unique continuous extension to a curve $\overline{\eta} \colon [0,1] \to \mathbb{D}$, with $\overline{\eta}(0) \in \partial\mathbb{D}$.

3 Attracting circloids and entropy

In this section we give a proof of Theorem A, stating that an attracting circloid with two different rotation numbers for a map f has a rotational horseshoe associated to some power f^{n_0}. We first present a proof of Theorem A. Then, in Section 3.5 we show how the hypothesis in Theorem A can be relaxed to obtain a more general statement; see Theorem 3.14, from which we can obtain Theorem C.

To fix the context, we introduce the following hypothesis:

(GA) $f\colon \mathbb{A} \to \mathbb{A}$ is an orientation-preserving homeomorphism of the infinite annulus $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$ such that it has a global attractor $\mathcal{C}$ which is a circloid and the rotation set of f restricted to $\mathcal{C}$ is a nontrivial interval.

Theorem A states that if f verifies 3 then there is a rotational horseshoe for some power f^{n_0}. Notice that by Theorem 2.4, property 3 implies that the circloid $\mathcal{C}$ must be noncompactly generated.

3.1 Some previous definitions

Choose $\mathcal{A} \subset \mathbb{A}$ any annular neighborhood of $\mathcal{C}$ (ie homeomorphic to $\mathbb{S}^1 \times [-1, 1]$, containing $\mathcal{C}$ in its interior) such that $f(\mathcal{A}) \subset \mathcal{A}$. Since $\mathcal{C}$ is a global attractor, we have $\mathcal{C} = \bigcap_{n\in\mathbb{N}} f^n(\mathcal{A})$.

Denote by $\mathcal{U}^+$ and $\mathcal{U}^-$ the connected components of $\mathbb{A} \setminus \mathcal{C}$ whose projections into the second coordinate is not bounded from above and below, respectively, and by $\partial^+\mathcal{A}$ and $\partial^-\mathcal{A}$ the connected components of $\partial\mathcal{A}$, contained in $\mathcal{U}^+$ and $\mathcal{U}^-$, respectively.

Given any essential annulus $\mathcal{A}$ in $\mathbb{A}$, with boundary components $\partial^-\mathcal{A}$ and $\partial^+\mathcal{A}$, we say that a continuum D *joins the boundaries of* $\mathcal{A}$ if it verifies the following conditions:

(1) $D \subset \mathcal{A}$ and it intersects both boundaries, ie $D \cap \partial^+\mathcal{A} \neq \varnothing$ and $D \cap \partial^-\mathcal{A} \neq \varnothing$.

(2) D is inessential (ie it is contained in a topological disk).

Let D_0 and D_1 be two disjoint continua in $\mathcal{A}$ joining the boundaries. It follows that $\mathcal{A} \setminus (D_0 \cup D_1)$ has at least one connected component R which contains a curve joining the boundaries of $\mathcal{A}$. Such a component must verify that its closure intersects both D_0 and D_1 and it will be called a *rectangle adapted* to D_0 and D_1. It is easy to show that it is an open connected subset of $\mathcal{A}$ whose boundary (relative to $\mathcal{A}$) is contained in $D_0 \cup D_1$.

Recall that we have considered $\pi\colon \mathbb{R} \times \mathbb{R} \to \mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$, the canonical projection, where $\mathbb{S}^1$ is identified with $\mathbb{R}/\mathbb{Z}$. Given an inessential continuum $D \subset \mathcal{A}$ which joins the boundaries of $\mathcal{A}$, one considers $\widehat{D}$ to be a connected component[6] of $\pi^{-1}(D)$ in $\widehat{\mathcal{A}} = \pi^{-1}(\mathcal{A})$. One defines *the right* of $\widehat{D}$ to be the (unique) unbounded component of $\widehat{\mathcal{A}} \setminus \widehat{D}$ accumulating in $+\infty$ in the first coordinate. One defines *the left* of $\widehat{D}$ symmetrically.

[6]Notice that since $\mathcal{A}$ is essential, one has that $\pi^{-1}(\mathcal{A})$ is connected.

Notice that if D_0 and D_1 are two disjoint continua joining the boundaries of $\mathcal{A}$, and R is a rectangle adapted to D_0 and D_1, then, if $\widehat{R}$ is a connected component of the lift of R, there is a unique connected component of the lift of D_0 (resp. D_1) such that it intersects the closure of $\widehat{R}$. Call these components by $\widehat{D}_0$ and $\widehat{D}_1$.

3.2 A criteria for producing rotational horseshoes

We start with a lemma which guarantees the existence of a rotational horseshoes. Then we prove that under the hypothesis of Theorem A, we can apply this result. The proof of the lemma is given by the well-known construction of the *Smale's horseshoe*, which is generalized in [23].

Lemma 3.1 *Let $\mathcal{A} \subset \mathbb{A}$ be an essential annulus as before, and $h: \mathcal{A} \to \mathcal{A}$ be a continuous map with $h(\mathcal{A}) \subset \operatorname{int}(\mathcal{A})$. Denote by $\widetilde{h}$ the lift of h to the universal cover. Assume we have two disjoint continua D_0 and D_1 joining the boundaries of $\mathcal{A}$ such that for some rectangle R adapted to D_0 and D_1 and some connected component $\widehat{R}$ of the lift of R, there is a positive integer j with the following properties:*

1. *If $\widehat{D}_0$ and $\widehat{D}_1$ denote the connected components of the lift of D_0 and D_1 intersecting the closure of $\widehat{R}$ we have that $\widetilde{h}(\widehat{D}_0)$ is at the left of the closure of $\widehat{R}$.*
2. *$\widetilde{h}(\widehat{D}_1)$ is at the right of the closure of $\widehat{R}+j$.*

Then there exists a C^0–neighborhood $\mathcal{N}$ of h in $\mathrm{Homeo}_+(\mathbb{A})$ such that every $g \in \mathcal{N}$ has a rotational horseshoe such that the associated partition has at least $j+1$ symbols.

Proof The proof is given by a simple inspection of [23]. The hypothesis we have for D_0 and D_1 implies that an adapted rectangle R is under the *horseshoe hypothesis* (together with $\mathbb{A}$ and the map h), so applying Theorem 1 in the quoted article, we already have the existence of a compact h–invariant set $\Lambda \subset \mathbb{A}$ for which the first condition of the definition of rotational horseshoe is verified (see Figure 2). As noticed by the authors, the semiconjugacy is constructed by using a partition $S_0, \dots, S_{m-1}$ with $m-1 \geq j$ by a finite pairwise disjoint compact sets (which relative to Λ are open). Moreover, the construction in our particular case implies that this partition can be considered such that

1. for every $i = 0, \dots, m-1$ we have a lift $\widehat{S}_i$ of S_i in $\widehat{R}$,
2. $\widetilde{h}(\widehat{S}_i) \subset \widehat{R} + v_i$ for some integer vector $v_i = (l_i, 0)$, where $l_0 \leq 0$ and $l_{m-1} > j$.

As the semiconjugacy is constructed by the itinerary function ξ associated to the sets $S_0, \dots, S_{m-1}$, one can deduce by a simple induction that given any lift $\hat{x} \in \pi^{-1}(\Lambda) \cap \hat{R}$ of x, we have

$$\left\| (\tilde{h}^n(\hat{x}) - \hat{x}) - \sum_{i=0}^{n-1} v_{\xi(x)(i)} \right\| < \kappa,$$

where κ is the diameter of $\hat{R}$. This implies that Λ is a rotational horseshoe. □

In order to prove Theorem A, the crucial idea is the following: using the fact that the dynamics is given on a circloid, and that it is an attractor, we will construct a sort of stable manifolds for some periodic points p_0 and p_1, given by two continua C_0 and C_1, such that they have to intersects both components $\mathcal{U}^+$ and $\mathcal{U}^-$. These continua will play the role of D_0 and D_1 in the hypothesis of the last lemma, and this will provide the rotational horseshoe.

We see in the next lemma how the existence of the continua as above allows us to use the previous lemma.

Lemma 3.2 *Let $f\colon \mathbb{A} \to \mathbb{A}$ verify 3 and assume that there exist two periodic points p_0 and p_1 with different rotation numbers and two contractible continua C_0 and C_1 containing p_0 and p_1, respectively, such that, for $i = 0, 1$:*

(1) *$f^{n_i}(C_i) \subset C_i$, where n_i is the period of p_i.*

(2) *C_i is inessential and intersects both boundaries of $\mathcal{A}$.*

Then f^{n_0} has a rotational horseshoe for some $n_0 \in \mathbb{N}$. Moreover, there exists a C^0–neighborhood $\mathcal{N}$ of f in $\mathrm{Homeo}_+(\mathbb{A})$ *such that for any $h \in \mathcal{N}$ we have that h^{n_0} has a topological horseshoe.*

We remark that we are not assuming that the sets C_i are contained in $\mathcal{A}$, so we cannot consider them as joining boundary components of $\mathcal{A}$.

Proof Consider an iterate g of f and a lift G to the universal cover $\tilde{\mathcal{A}}$ such that both p_0 and p_1 are fixed and their lifts $\tilde{p}_0$ and $\tilde{p}_1$ verify $G(\tilde{p}_0) = \tilde{p}_0 - j$ and $G(\tilde{p}_1) = \tilde{p}_1 + l$ for some positive integers j and l (ie p_0 rotates negatively and p_1 rotates positively).

As C_i and $\mathcal{A}$ are forward-invariant by f^{n_i} (and therefore also for g) we have for $i = 0, 1$ that $g(C_i \cap \mathcal{A}) \subset C_i \cap \mathcal{A}$. Further, as C_i intersects both boundaries of $\mathcal{A}$

and p_i are contained in the interior of $\mathcal{A}$, there exist some continua $D_i \subset C_i$ in $\mathcal{A}$ for $i = 0, 1$, joining the boundary components of $\mathcal{A}$; see Figure 2 (for a proof of this folklore topological fact, see for instance Theorem 14.3 in [36]).

We pick now some rectangle R adapted to D_0 and D_1, and $\widehat{R}$ a connected component of the lift of R. Let $\widehat{C}_0$ and $\widehat{C}_1$ be the lifts of C_0 and C_1, containing $\widehat{D}_0$ and $\widehat{D}_1$ as defined above. It is easy to see that the sets C_0 and C_1 must be disjoint, as they are both forward-invariant for g and have different rotation vectors.

As both $\widehat{C}_0$ and $\widehat{C}_1$ have bounded diameter, and rotate negatively and positively, we must have for some sufficiently large $n \in \mathbb{N}$ that

- $G^n(\widehat{D}_0)$ is at the left $\widehat{R}$,
- $G^n(\widehat{D}_1)$ is at the right of $\widehat{R}+1$.

Lemma 3.1 now implies that g^n has a rotational horseshoe, hence so does a power of f. Furthermore, as the configuration above remains for small perturbations of g, we obtain the same result in a C^0–neighborhood of f. □

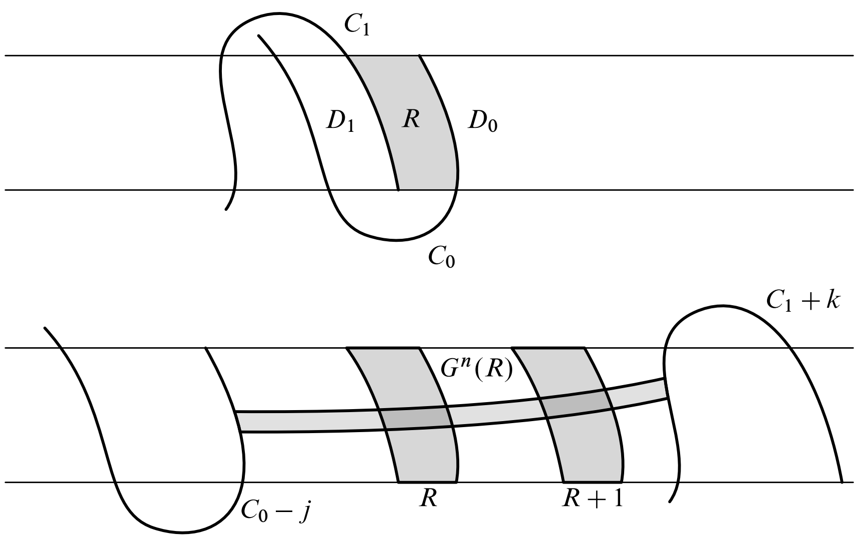

Figure 2: The rotational horseshoe

3.3 A first reduction

The next result, whose importance we believe transcends the context, will be proved in the next subsection. We will use it here in order to complete the proof of Theorem A.

Theorem 3.3 *Let $f\colon \mathbb{A} \to \mathbb{A}$ verifies 3 and let $p \in \partial\mathcal{C}$ be a periodic point. Then there exist an inessential continuum C_p containing p such that $f^{n_p}(C_p) \subset C_p$, where n_p is the period of p and $C_p \cap \partial\mathcal{A} \neq \varnothing$.*

Notice that the continuum C_p might not intersect a priori both boundary components of $\mathcal{A}$. Moreover, although C_p meets $\mathcal{C}^c$, it may happens that C_p intersects only one of the unbounded connected components $\mathcal{U}^+$ and $\mathcal{U}^-$, that is, $C_p \subset (\mathcal{U}^-)^c$ or $C_p \subset (\mathcal{U}^+)^c$.

We now proceed with the proof of Theorem A assuming Theorem 3.3. By Lemma 3.2 it is enough to find two periodic points p_0 and p_1 with different rotation vectors for which C_{p_0} and C_{p_1} intersect both boundary components of $\mathcal{A}$, which is equivalent to the following condition since $\mathcal{C}$ is a global attractor:

(1) $C_{p_0} \cap \mathcal{U}^+ \neq \varnothing$ and $C_{p_0} \cap \mathcal{U}^- \neq \varnothing$, $\qquad C_{p_1} \cap \mathcal{U}^+ \neq \varnothing$ and $C_{p_1} \cap \mathcal{U}^- \neq \varnothing$.

We conclude the section by proving the existence of periodic points p_0 and p_1 such that (1) holds.

Let us state the following realization theorem of [26], which improves previous results [24; 5]. Here is one of the essential points where we use that $\mathcal{C}$ is irreducible (see [42]). Notice that if one wishes to use [5] instead of [26], similar results hold but one needs to add the assumption that the circloid $\mathcal{C}$ in Theorem A has empty interior.

Theorem 3.4 [26, Theorem G] *Let $h\colon \mathbb{A} \to \mathbb{A}$ be a homeomorphism of the annulus preserving a circloid $\mathcal{C}$ such that $\rho_{\mathcal{C}}(H)$ is nonsingular for any lift H of h. Then every rational point in the rotation set $\rho_{\mathcal{C}}(H)$ is realized by a periodic orbit in $\partial\mathcal{C}$.*

The idea is to use points which are in $\partial\mathcal{C}$ but are not accessible from $\mathcal{U}^+$ and $\mathcal{U}^-$ such that a connected set which intersects the boundary of $\mathcal{A}$ will necessarily intersect both boundaries. Recall that a point $x \in \partial\mathcal{C}$ is accessible if there exists a continuous arc $\gamma\colon [0,1] \to \mathcal{A}$ such that $\gamma([0,1)) \subset \mathcal{A} \setminus \mathcal{C}$ and $\gamma(1) = x$.

Here we shall use a weaker form of accessibility which, moreover, involves the dynamics of f in the annulus. We will say that a periodic point $p \in \mathcal{C}$ is *dynamically continuum accessible from above* (resp. *dynamically continuum accessible from below*) if there exist a continuum C_p such that:

- $p \in C_p$.
- $C_p \setminus \mathcal{C}$ is nonempty and contained in $\mathcal{U}^+$ (resp. $\mathcal{U}^-$).

- C_p is inessential in $\mathbb{A}$.
- $f^{n_p}(C_p) \subset C_p$ for n_p the period of p.

Using the prime end theory and the result stated in Section 2.3, one can show the following result:

Proposition 3.5 *Let p and q in $\partial\mathcal{C}$ be periodic points of f which are both dynamically continuum accessible from above (resp. from below). Then, for any lift of f to $\mathbb{R}^2$, both p and q have the same rotation number.*

Proof Assume by contradiction that p and q have different rotation numbers for some lift. Considering an iterate f^j and a suitable lift G of f^j to $\mathbb{R}^2$, we can assume that $G(\tilde{p}) = \tilde{p}$ and $G(\tilde{q}) = \tilde{q} + k$ with $k \neq 0$.

Let C_p and C_q be given by the fact that p and q are continuum accessible from above. By definition, we have that they are disjoint and inessential. Thus, we can consider a proper arc $\gamma\colon [0, +\infty) \to \mathcal{U}^+ \cup \mathcal{C}$ such that $\gamma(0) = z \in \mathcal{C}$ and $\gamma(t) \in (\mathcal{C} \cup C_p \cup C_q)^c$ for every $t \in (0, +\infty)$ and $\lim_{t\to+\infty} \gamma(t) = +\infty$ (see Section 2.3).

Consider $\widetilde{\mathcal{U}}^+ = \pi^{-1}(\mathcal{U}^+)$, $\hat{\gamma}$ a lift of γ and $\hat{A}$ the lift of $\pi^{-1}(\mathcal{U}^+ \setminus \gamma)$ containing $\hat{\gamma}$ and $\hat{\gamma} + 1$ in its boundary. Further, consider the connected components $\hat{C}_p$ and $\hat{C}_q$ of $\pi^{-1}(C_p)$ and $\pi^{-1}(C_q)$ intersecting $\hat{A}$, respectively, with $\hat{K}_p = \hat{C}_p \cap \widetilde{\mathcal{U}}^+$ and $\hat{K}_q = \hat{C}_q \cap \widetilde{\mathcal{U}}^+$. We have that $G(\hat{K}_p) \subset \hat{K}_p$ and $G(\hat{K}_q) \subset \hat{K}_q + k$.

As $\hat{C}_p$ and $\hat{A}$ are in the situation of Proposition 2.6, we have that $\hat{C}_p$ intersects only finitely many of the sets $\hat{A} + j$ for $j \in \mathbb{Z}$, and the same holds for $\hat{C}_q$.

Consider the map $H\colon \mathbb{H}^2 \to \mathbb{H}^2$ induced by G and the prime end compactification of $\mathcal{U}^+$ as stated in Section 2.4, and let $\theta\colon \widetilde{\mathcal{U}}^+ \to \mathbb{H}^2$ be the induced conjugacy between $G|_{\widetilde{\mathcal{U}}^+}$ and $\hat{H}|_{\mathbb{H}^2}$. As γ lands at an accessible point z, we have that $\eta = \theta(\hat{\gamma} \setminus \hat{\gamma}(0))$ can be extended continuously in $t = 0$, so that $\eta(0) \in \partial\mathbb{H}^2$ with respect to the usual topology of $\mathbb{R}^2$ (see Section 2.4).

Then we have that the sets $\overline{K}_p = \mathrm{cl}[\theta(\hat{K}_p)]$ and $\overline{K}_q = \mathrm{cl}[\theta(\hat{K}_q)]$ are contained in a region of $\mathrm{cl}[\mathbb{H}^2]$ between the extended curves $\eta - j_0$ and $\eta + j_1$ for some $j_0, j_1 \in \mathbb{N}$. Furthermore, if $\hat{H}$ is the continuous extension of H to $\mathrm{cl}[\mathbb{H}^2]$, we can assume without loss of generality that $\hat{H}^n(\overline{K}_p) \subset \overline{K}_p$ and $\hat{H}^n(\overline{K}_q) \subset \overline{K}_q + nk$ for all $n \in \mathbb{N}$. Let h be the restriction of $\hat{H}$ to $\partial\mathbb{H}^2$, which is known to lift an orientation-preserving circle homeomorphism as stated in Section 2.4.

Thus we obtain two compact sets $L_p = \overline{K}_p \cap \partial\mathbb{H}^2$ and $L_q = \overline{K}_q \cap \partial\mathbb{H}^2$ such that $h^n(L_p) \subset L_p$ and $g^n(L_q) \subset L_q + kn$ for all $n \in \mathbb{N}$, which is impossible, as h lifts an orientation-preserving circle homeomorphism. □

We are now ready to complete the proof of Theorem A by showing the following proposition:

Proposition 3.6 *There exist two periodic points p_0 and p_1 in $\partial\mathcal{C}$ with different rotation numbers such that C_{p_0} and C_{p_1} satisfy (1).*

Proof Pick four rational points $r_0, r_1, r_2, r_3 \in \rho_{\mathcal{C}}(F)$ with different denominators in their irreducible form (in particular, different from each other). Using Theorem 3.4 we know that all four are realized by periodic points p_i in $\partial\mathcal{C}$, and using Proposition 3.5 we know that at least two of them, say p_0 and p_1, are not dynamically continuum accessible.

Consider the compact connected sets C_{p_0} and C_{p_1} given by Theorem 3.3; since p_0 and p_1 are not continuum accessible, it follows directly that (1) is verified, as desired. □

3.4 Proof of Theorem 3.3

Let $\partial^+\mathcal{A}$ and $\partial^-\mathcal{A}$ be the two boundaries of $\mathcal{A}$. Let $\mathcal{F}_0^+$ be a foliation by essential simple closed curves in the upper connected component of $\overline{\mathcal{A} \setminus f(\mathcal{A})}$ such that they coincide in the boundary with $\partial^+\mathcal{A}$ and $f(\partial^+\mathcal{A})$ and let

$$\mathcal{F}^+ = \bigcup_{n \geq 0} f^n(\mathcal{F}_0^+).$$

In a symmetric way we define $\mathcal{F}^-$. Notice that any annulus $\mathcal{A}_1$ whose boundary is given by a curve of $\mathcal{F}^+$ and curve of $\mathcal{F}^-$ satisfies $f(\mathcal{A}_1) \subset \operatorname{int}(\mathcal{A}_1)$.

From now on we fix a periodic point $p \in \partial\mathcal{C}$ as in Theorem 3.3. Replacing f by an iterate and choosing an appropriate lift F, we may assume that p is fixed and has zero rotation vector. Let $q \in \partial\mathcal{C}$ be another periodic point with different rotational speed. We may assume without loss of generality that q is fixed and rotates one.

Lemma 3.7 *There exist $\eta > 0$, an annulus $\mathcal{A}_1$ bounded by leaf of $\mathcal{F}^+$ and a leaf of $\mathcal{F}^-$ and an arc $I_q \subset \mathcal{A}_1$ containing q and joining both boundaries of $\mathcal{A}_1$ such that, if g is η–C^0–close to f and G is the lift η–close to F, we have that $G(\widehat{I}_q)$ is to the right of $\widehat{I}_q$ and $G^2(\widehat{I}_q)$ is to the right of $\widehat{I}_q + 1$ in $\pi^{-1}(\mathcal{A}_1)$, where $\widehat{I}_q$ denotes a connected component of the lift of I_q.*

Proof Let $\epsilon > 0$ be such that $B(p,\epsilon) \cap B(q,\epsilon) = \varnothing$. Let δ be small enough that $f(B(q,\delta))$ and $f^2(B(q,\delta))$ are contained in $B\big(q, \frac{1}{2}\epsilon\big)$ (recall $f(q) = q$). Let $B = B(q,\delta)$.

One can choose unique leaves $\mathcal{F}^+_\delta$ and $\mathcal{F}^-_\delta$ of $\mathcal{F}^+$ and $\mathcal{F}^-$ which intersect ∂B, and do not intersect B.

We may assume (reducing δ if necessary) that both leaves also intersect $B(p,\epsilon)$ and consider the annulus $\mathcal{A}_1$ determined by $\mathcal{F}^+_\delta$ and $\mathcal{F}^-_\delta$. Denote by K the connected component of $B(q,\epsilon) \cap \mathcal{A}_1$ that contains B. Notice that K is inessential in $\mathcal{A}_1$ since it is disjoint from $B(p,\epsilon)$ (and there is an arc in $B(p,\epsilon)$ joining the two boundaries of $\mathcal{A}_1$).

Let $\eta > 0$ be small enough that if g is η–C^0–close to f in $\mathcal{A}$ then:

- $g(\mathcal{A}_1) \subset \operatorname{int}(\mathcal{A}_1)$.
- $g(B(q,\delta))$ and $g^2(B(q,\delta))$ are contained in $B(q,\epsilon)$.
- $g(B(q,\delta)) \cap B(q,\delta) \neq \varnothing$ and $g^2(B(q,\delta)) \cap B(q,\delta) \neq \varnothing$.

Let I_q be an arc inside $B(q,\delta)$ joining the two boundaries of $\mathcal{A}_1$. Notice that $g(I_q)$ and $g^2(I_q)$ are both contained in K. Now, fix a lift $\widehat{q}$ of q and a lift $\widehat{I}_q$ of I_q containing $\widehat{q}$ and let $\widehat{K}$ be the connected component of $\pi^{-1}(K)$ that contains $\widehat{I}_q$. Let G be the lift of g which is η–close to the lift F of f. Since $F(\widehat{q}) = \widehat{q} + 1$, we have that $F(\widehat{I}_q) \subset \widehat{K} + 1$ and $F^2(\widehat{I}_q) \subset \widehat{K} + 2$, and the same holds for G, which completes the proof. □

Remark 3.8 By continuity, one can assume without loss of generality that there exists a neighborhood $N(I_q)$ of I_q in the annulus $\mathcal{A}_1$ such that if $N(\widehat{I}_q)$ denotes a connected component of the lift, then $G(N(\widehat{I}_q))$ is to the right of $\widehat{I}_q$ and $G^2(N(\widehat{I}_q))$ is to the right of $\widehat{I}_q + 1$.

From now on we fix the annulus $\mathcal{A}_1$ given by the previous lemma. The idea will be to approach f by homeomorphisms presenting a stable manifold of p escaping $\mathcal{A}_1$ and not intersecting I_q, so that we will control its convergence in the limit.

Lemma 3.9 *There exists a sequence of homeomorphisms f_n converging to f in the C^0 topology such that:*

(1) *p is a hyperbolic fixed point of f_n.*

(2) *$W^s(p, f_n)$ intersects the boundary of $\mathcal{A}_1$.*

Proof Let ϵ_n be a positive sequence converging to zero. We may assume that $B(p, 2\epsilon_n) \subset \mathcal{A}_1$ for every n. Let $\mathcal{F}^+_{\epsilon_n}$ and $\mathcal{F}^-_{\epsilon_n}$ be the unique leaves of the foliations $\mathcal{F}^+$ and $\mathcal{F}^-$ which intersect $\partial B(p, \epsilon_n)$, and do not intersect $B(p, \epsilon_n)$. Let $\mathcal{A}_{\epsilon_n}$ be the annulus determined by those leaves. Now consider g_n such that $g_n = f$ outside $B\big(p, \frac{1}{2}\epsilon_n\big)$ and p is a hyperbolic fixed point of g_n. The C^0 distance between g_n and f is bounded by ϵ_n.

Fix a fundamental domain D^s of $W^s(p, g_n)$ inside $B\big(p, \frac{1}{2}\epsilon_n\big)$ and join an interior point z of D^s with a point y in $\mathcal{F}^+_{\epsilon_n} \cap \partial B(p, \epsilon_n)$ by a polygonal arc inside $B(p, \epsilon_n)$; see Figure 3. Let U be a neighborhood of this arc which does not intersect the forward iterates $g_n^m(D^s)$ for $m \geq 1$ and such that $\overline{U}$ is contained in the interior of $g_n^{-1}(\mathcal{A}_{\epsilon_n})$, which is equal to $f^{-1}(\mathcal{A}_{\epsilon_n})$. We may assume that $U \subset B(p, 2\epsilon_n)$ as well. See Figure 3.

Consider $\varphi\colon \mathcal{A} \to \mathcal{A}$ such that $\varphi \equiv \mathrm{id}$ outside U and $\varphi(y) = z$. The C^0 distance between φ and the identity is bounded by $2\epsilon_n$. Let $f_n = g_n \circ \varphi$. We have that $y \in W^s(p, f_n)$ and $f_n^{-1}(y)$ belongs to the boundary of $f^{-1}(\mathcal{A}_{\epsilon_n})$. Since $g_n = f_n$ outside U and $g_n = f$ outside $B\big(p, \frac{1}{2}\epsilon_n\big)$, iterating backwards we eventually have that $W^s(p, f_n)$ intersects the boundary of $\mathcal{A}_1$.

Finally, it is clear that the C^0 distance from f_n to f goes to zero with ϵ_n, as desired. □

Denote by $W^s_1(p, f_n)$ the connected component of $W^s(p, f_n) \cap \mathcal{A}_1$ that contains p.

Remark 3.10 The set $W^s_1(p, f_n)$ verifies that $f_n(W^s_1(p, f_n)) \subset W^s_1(p, f_n)$. Indeed, $f_n(\mathcal{A}_1) \subset \mathcal{A}_1$ and $W^s(p, f_n)$ is also f_n–invariant.

We now use Lemma 3.7 to control the diameter of $W^s_1(p, f_n)$ in order to be able to consider a limit continuum through p which will be forward-invariant by f.

Lemma 3.11 *Let $\mathcal{A}_1$ and I_q be as in Lemma 3.7. Then there is a neighborhood $N(I_q)$ of I_q such that $W^s_1(p, f_n) \cap N(I_q) = \varnothing$ for every large enough n.*

Proof In the lift $\widetilde{\mathcal{A}}_1$ of $\mathcal{A}_1$, we choose $\hat{p}$ in the fundamental domain D determined by a connected component $\hat{I}_q$ of the lift of I_q and $\hat{I}_q - 1$.

Consider a lift $\widehat{W}^s_1(\hat{p}, f_n)$ of $W^s_1(p, f_n)$ through $\hat{p}$. Let W be the connected component of $\widehat{W}^s_1(\hat{p}, f_n) \cap D$ that contains $\hat{p}$ and let F_n be a lift of f_n close to the lift F of f. Notice that $F_n(W) \subset W$. We may assume that f_n is η–close to f, where η is as in Lemma 3.7.

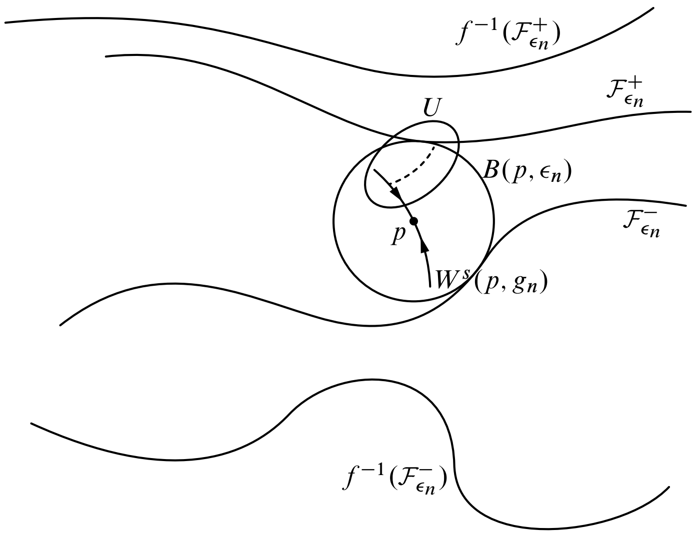

Figure 3: Construction of small perturbations having fixed hyperbolic saddles with stable manifolds accumulating at $-\infty$ or $+\infty$

Choose $N(I_q)$ as in Remark 3.8. Assume that $\widehat{W}^s_1(\hat{p}, f_n) \cap N(\widehat{I}_q) \neq \varnothing$. Then $W \cap N(\widehat{I}_q) \neq \varnothing$. But then $F_n(W) \subset W \subset D$. Since $F_n(N(\widehat{I}_q))$ is to the right of $\widehat{I}_q$, Lemma 3.7 implies that $F_n(W)$ is not contained in D, a contradiction. If $\widehat{W}^s_1(\hat{p}, f_n) \cap (\widehat{I}_q - 1) \neq \varnothing$, we arrive at a contradiction as well, since then $F_n^2(W)$ is contained in W and contains a point in $F_n^2(\widehat{I}_q - 1)$ which is to the right of $\widehat{I}_q$ and so it must intersect $\widehat{I}_q$. □

End of the proof of Theorem 3.3 We say that a set $S \subset \mathbb{R}^2$ has *bounded horizontal diameter* if its projection to the first coordinate is bounded. In this case, let us write $\operatorname{diam}_H(S) = \operatorname{diam}(\pi_1(S))$. We consider the lift $\widehat{\mathcal{A}}$ of $\mathcal{A}$. Let $\mathcal{A}_1$ be as in Lemma 3.7 and let $\widehat{\mathcal{A}}_1$ be its lift inside $\widehat{\mathcal{A}}$.

In this context, we have that the fundamental domain in $\widehat{\mathcal{A}}_1$ determined by $\widehat{I}_q - 1$ and $\widehat{I}_q$ has bounded horizontal diameter, say by $a > 0$. This implies, by Lemma 3.11, that $\operatorname{diam}_H(\widehat{W}^s_1(p_n, f_n))$ is also bounded by a.

Let m be the first positive integer such that $f^m(\mathcal{A}) \subset \mathcal{A}_1$. Notice that $f_n^m(\mathcal{A}) \subset \mathcal{A}_1$ by construction. Let F be the lift of f and F_n the lift of f_n. Then $F_n^{-m}(\widehat{W}^s_1(p, f_n))$ has bounded diameter in $\mathbb{R}^2$. Let $\widehat{C}_n = F_n^{-m}(\widehat{W}^s_1(p, f_n))$. We have that:

(1) $\widehat{C}_n$ is a continuum containing $\hat{p}$.

(2) $\widehat{C}_n$ is forward-invariant by F_n (see Remark 3.10).

(3) $\widehat{C}_n$ intersects the boundary of $\widehat{\mathcal{A}}$.

(4) $\widehat{C}_n$ has uniformly bounded diameter.

(5) $F_n \rightrightarrows F$.

Then, by taking the Hausdorff limit $\widehat{C}_p$ of $(\widehat{C}_n)_{n\in\mathbb{N}}$, we have a continuum which is forward-invariant under F and contains $\hat{p}$. Moreover, it intersects $\partial\widehat{\mathcal{A}}$, and its projection into $\mathbb{A}$ must be inessential, since otherwise it would intersect I_q, which is not possible. Taking $C_p = \pi(\widehat{C}_p)$, we are done. □

3.5 General statement for Theorem A

In this section we comment on the proof of Theorem A to see that weaker hypotheses are enough to obtain the existence of rotational horseshoes. We state a general version of the result, from which Theorem C can be obtained.

Consider an $f \in \mathrm{Homeo}_+(\mathbb{A})$ such that $f(\mathcal{A}) \subset \mathrm{int}(\mathcal{A})$ for some compact and essential annulus $\mathcal{A}$. In this situation an attractor $K_{\mathcal{A}} = \bigcap_{n\in\mathbb{N}} f^n(\mathcal{A})$ exists and is an essential annular continuum.

The proof of Theorem 3.3 extends to the following with the same proof.

Theorem 3.12 *Assume we are in the situation above, and $p, q \in K_{\mathcal{A}}$ are two periodic points of f such that*

- *p and q have different rotation vectors for any lift F of f,*
- *p and q are both an accumulation point of $\bigcup_{n\in\mathbb{N}} f^n(\partial^+\mathcal{A})$ and $\bigcup_{n\in\mathbb{N}} f^n(\partial^-\mathcal{A})$.*

Then there exists an inessential continuum C_p containing p such that $f^k(C_p) \subset C_p$, where k is the period of p and $C_p \cap \partial A \neq \varnothing$. A similar statement holds for q.

The following is an easy application of Zorn's lemma and Theorem 2.4.

Lemma 3.13 *Let $f \in \mathrm{Homeo}_+(\mathbb{A})$ and a closed essential annulus $\mathcal{A}$ such that $f(\mathcal{A}) \subset \mathrm{int}(\mathcal{A})$. Further, assume that there are at least four periodic points p_1, p_2, p_3 and p_4 in $\mathcal{A}$ having pairwise different rotation vectors for any lift F of f, and that $\bigcup_{n\in\mathbb{N}} f^n(\partial^i\mathcal{A})$ accumulates in p_1, p_2, p_3 and p_4 for $i = +, -$. Then there exists an invariant noncompactly generated circloid $\mathcal{C} \subset K_{\mathcal{A}}$ such that $p_i \in \mathcal{C}$ for $i = 1, 2, 3, 4$.*

With these two results, following exactly the proof of Theorem A, we obtain a more general result:

Theorem 3.14 *Let $f \in \mathrm{Homeo}_+(\mathbb{A})$ and a closed essential annulus $\mathcal{A}$ such that $f(\mathcal{A}) \subset \mathrm{int}(\mathcal{A})$. Further, assume that there are at least four periodic points p_1, p_2, p_3 and p_4 in $\mathcal{A}$ having pairwise different rotation vectors for any lift F of f, and that $\bigcup_{n\in\mathbb{N}} f^n(\partial^i \mathcal{A})$ accumulates in p_1, p_2, p_3 and p_4 for $i = +, -$. Then there exists $n_0 \in \mathbb{N}$ and a C^0–neighborhood $\mathcal{N}$ of f in $\mathrm{Homeo}_+(\mathbb{A})$ such that for every element $g \in \mathcal{N}$ the power g^{n_0} has a rotational horseshoe $\Lambda_g \subset \mathcal{A}$. In particular, every element in $\mathcal{N}$ has topological entropy larger than $\log(2)/n_0$.*

We finish this section by showing that Theorem C can be derived from this last theorem as well. In the hypothesis of Theorem C we have that, for the closed annulus $\mathcal{A}$, the attractor $K_{\mathcal{A}}$ must have empty interior as the map is dissipative. Furthermore, as the Birkhoff attractor $\mathcal{C}$ is by definition the unique circloid contained in $K_{\mathcal{A}}$ and has nonempty interior, it must be that $\mathcal{C} = \overline{U^+} \cap \overline{U^-}$ where U^+ and U^- are the connected components of $\mathbb{A} \setminus K_{\mathcal{A}}$. This implies that both sets $\bigcup_{n\in\mathbb{N}} f^n(\partial^+ \mathcal{A})$ and $\bigcup_{n\in\mathbb{N}} f^n(\partial^+ \mathcal{A})$ accumulate on every point of $\mathcal{C}$. As the rotation set on $\mathcal{C}$ is not trivial, the realization results [24; 5] imply that we have infinitely many periodic points in $\mathcal{C}$ realizing every rational number in $\rho_{\mathcal{C}}(F)$ for any lift F of f. Hence, the last theorem can be applied, so we obtain Theorem C.

4 Entropy versus rotation set for circloids

Let us recall the basic definitions. We considered $\mathbb{A} = \mathbb{S}^1 \times \mathbb{R}$, where $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, and the usual covering $\pi\colon \mathbb{R}^2 \to \mathbb{A}$ given by $\pi(x, y) = (x \pmod{\mathbb{Z}}, y)$.

Consider the *integrable twist map* $\tau\colon \mathbb{A} \to \mathbb{A}$ given by the lift

$$T(x, y) = (x + y, y).$$

If we denote by $\mathcal{F} = \{C_y\}_{y\in\mathbb{R}}$ the foliation of $\mathbb{A}$ by essential circles given by $C_y = \pi(\mathbb{R} \times \{y\})$, we have that $\tau|_{C_y}$ is a rotation of angle $2\pi y$.

Remark 4.1 A simple computation gives that $\lim_n \frac{1}{n} \log \|D\tau^n\| = 0$. This can be combined with Proposition 2.1 to get that, given $\varepsilon > 0$, there is a C^1–neighborhood $\mathcal{N}_\varepsilon$ of τ such that $h_{\mathrm{top}}(f) < \varepsilon$ for all $f \in \mathcal{N}_\varepsilon$.

We will prove the following theorem:

Theorem 4.2 *For every C^1–neighborhood $\mathcal{N}$ of τ there exists $f \in \mathcal{N}$ such that f has a global attractor given by an essential circloid $\mathcal{C}$ with $\rho_{\mathcal{C}}(F) \supset [0, 1]$ for some lift F of f.*

Combining this with Remark 4.1, we show that there are circloids with rotation sets containing [0, 1] whose entropy approaches zero as much as desired, therefore proving Theorem B. Notice that the twist condition is C^1–open, so we can assume also that the obtained diffeomorphism verifies the twist condition. To obtain a dissipation hypothesis (as required in Theorem D) one has to perform a slightly different perturbation, which is explained at the end of this section.

We fix $\mathcal{N}$ and construct $f \in \mathcal{N}$ by means of a sequence of C^1 perturbations of τ. We remark that all the perturbations are just C^1 small, but the map itself can be considered to be smooth (see Remark 2.3).

4.1 First perturbation

We first fix some notation. For $y < y'$ we denote by $[C_y, C_{y'}]$ the compact region between these two circles, and by $(C_y, C_{y'})$ its interior.

As usual, given a map $f\colon M \to M$ and a point $x \in M$, we define for $\varepsilon > 0$ the local stable set of x to be $W^s_\varepsilon(x, f) = \{y \in M \mid d(f^n(y), f^n(x)) < \varepsilon \text{ for all } n \in \mathbb{N}\}$, and define the stable set of x as $W^s(x, f) = \{y \in M \mid \lim_n d(f^n(x), f^n(y)) = 0\}$. The local unstable and unstable sets are defined by considering f^{-1} instead of f, ie $W^u(x, f) = W^s(x, f^{-1})$. When x is a hyperbolic periodic point, the local stable set is a submanifold tangent to the stable subspace at x, and $W^s(x, f) = \bigcup_{n\in N} f^{-kn}(W^s_\varepsilon(x, f))$, where k is the period of x (see [22, Section 6]). A similar result holds for unstable manifolds.

The first perturbation will be $f_1 \in \mathcal{N}$ such that (see Figure 4):

(1) f_1 is conservative restricted to the annulus $[C_0, C_1]$.

(2) $f_1(C_r) = C_r$ for $r \in \{0, 1\}$.

(3) f_1 has a saddle $x_0 \in C_0$ and a saddle-node $p_0 \in C_0$, so that $W^u(x_0, f_1) = C_0 \setminus p_0$, which implies that $W^s(p_0, f_1) \supseteq C_0 \setminus \{x_0\}$.

(4) f_1 has a saddle $x_1 \in C_1$ and a saddle-node $p_1 \in C_1$, so that $W^u(x_1, f_1) = C_1 \setminus p_1$, which implies as before that $W^s(p_1, f_1) \supseteq C_1 \setminus \{x_1\}$.

(5) There is a forward-invariant arc $I^s_0 \subset W^s(p_0, f_1) \cap (-\infty, C_0]$ with one endpoint at p_0, and a small backward-invariant compact arc $I^u_0 \subset W^u(p_0, f_1) \cap [C_0, C_1]$ with one endpoint in p_0.

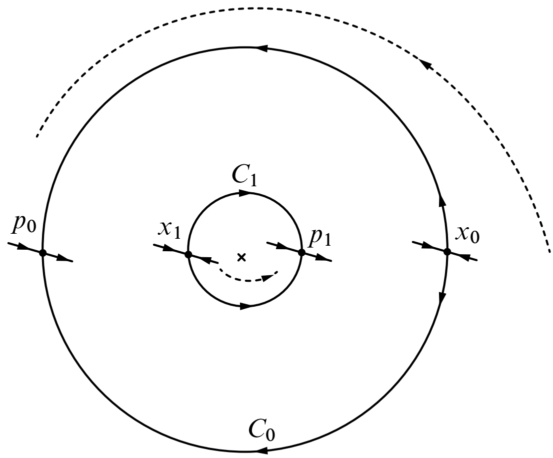

Figure 4: The map f_1

(6) There is a forward-invariant arc $I_1^s \subset W^s(p_1, f_1) \cap [C_1, +\infty)$ with one endpoint at p_1, and a small backward-invariant compact arc $I_1^u \subset W^u(p_1, f_1) \cap [C_0, C_1]$ with one endpoint in p_1.

(7) $[C_0, C_1]$ is a global attractor for f_1.

(8) For every $n \in \mathbb{N}$, f_1 has finitely many points of period n.

This can be done by C^1 small smooth perturbations around the circles C_0 and C_1 and Franks' lemma [14] — see [9, Proposition 7.4] for the conservative version — for suitable perturbations of the derivative in the conservative setting. To obtain (7), one can just take a dissipative perturbation supported in $(C_0, C_1)^c$. Item (8) can be achieved by means of standard arguments in generic dynamics: a simple Baire argument allows one to find a smooth diffeomorphism nearby for which all periodic points in the interior of the annulus have no eigenvalues equal to ± 1, and this implies that the set of those having period n is finite for all $n \in \mathbb{N}$. This first perturbation is depicted in Figure 4.

4.2 Second perturbation

For the second perturbation, we make use of Theorem 2.2. We construct $f_2 \in \mathcal{N}$ such that

(1) f_2 is conservative in $[C_0, C_1]$,

(2) $f_2(x) = f_1(x)$ outside (C_{r_1}, C_{r_2}) for some values $0 < r_1 < r_2 < 1$,

(3) there is a transverse intersection between the connected component of the intersection $I_0^u \cap [C_0, C_{r_1}]$ containing p_0 and $W^s(x_1, f_2)$ and a transverse intersection between the connected component of $I_1^u \cap [C_{r_2}, C_1]$ containing p_1 and $W^s(x_0, f_2)$.

Remark 4.3 The diffeomorphism f_2 restricted to $[C_0, C_1]$ is a conservative annulus diffeomorphism which deviates the vertical and the whole annulus is an instability region. In particular, the rotation set in this instability region is [0, 1] and the entropy can be chosen to be as small as desired.

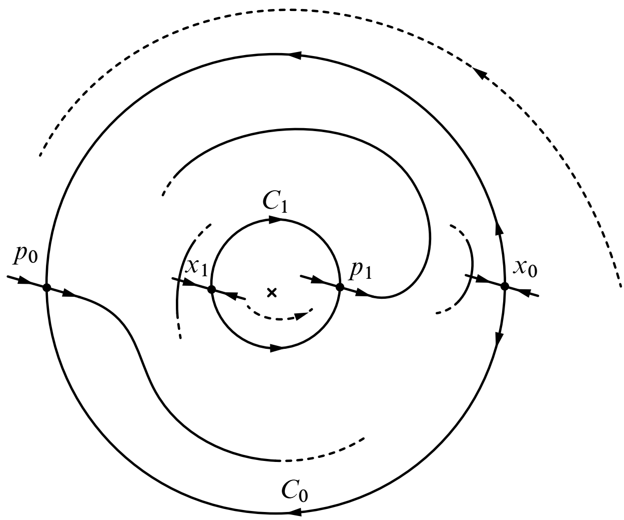

Figure 5: The map f_2

In order to produce f_2 we just have to choose a perturbation of f_1 in $\mathcal{N}$ which is conservative in $[C_0, C_1]$, supported outside a neighborhood of C_0 and C_1 in $[C_0, C_1]$, and connects the forward orbit of a small arc in I_0^u (inside the neighborhood where the perturbation is made) with the stable manifold of x_1 and symmetrically connects the forward orbit of I_1^u with the stable manifold of x_0. See Figure 5.

This will be achieved by means of Theorem 2.2. But first we need to show an abstract lemma to put ourselves in the hypothesis of the theorem.

Lemma 4.4 *Assume h: $[C_0, C_1] \to [C_0, C_1]$ is an area-preserving diffeomorphism and D is a connected open subset whose closure is contained in (C_0, C_1). Let z and w be points in (C_0, C_1) such that there are integers $n_z > 0$ and $n_w > 0$ such that $h^{n_z}(z)$ and $h^{-n_w}(w)$ are contained in D. Then $z \dashv_{\mathrm{cl}[D]} w$.*

Proof Notice that it is enough to show that for every pair of points p and q in D and $\varepsilon > 0$ one can construct a pseudoorbit with jumps in D going from p to q, since one can go without jumps from the interior of D to the points z and w.

We fix p in D, and consider for every $\varepsilon > 0$ the set P_ε of those points $q \in D$ such that there exists an ε–pseudoorbit $(z_k)_{k=0}^n$ with $z_0 = p$, $z_n = q$ and $h(z_k), z_{k+1} \in D$ whenever $h(z_k) \neq z_{k+1}$. It is enough to prove that P_ε is a nonempty open and closed set in D.

For $q \in P_\varepsilon$ we can consider an ε–pseudoorbit $(z_k)_{k=0}^n$ as before. Then there exists ε' such that $d(h(z_{n-1}), q) < \varepsilon' < \varepsilon$. Pick a neighborhood V of q in D such that $V \subset B(q, \varepsilon - \varepsilon')$ and take $z \in V$.

- If $h(z_{n-1}) = q$, we have that $z_0, \dots, z_{n-1}, z$ is an ε–pseudoorbit whose jumps are in D. Thus $z \in P_\varepsilon$, and $V \subset P_\varepsilon$, so P_ε.
- If $h(z_{n-1}) \neq q$, then both $h(z_{n-1})$ and z are contained in D. Thus the pseudoorbit $z_0, \dots, z_{n-1}, z$ is an ε–pseudoorbit who has its jumps in D. Thus, we have again $V \subset P_\varepsilon$.

Therefore, we can conclude that P_ε is open. In order to check that it is also closed in D, we consider a sequence of points $q_n \in P_\varepsilon$ converging to a point q in D. Fix q_n such that $d(q_n, q) < \varepsilon$ and let V be a neighborhood of q_n in D, contained in $B(q, \varepsilon)$.

Consider an ε–pseudoorbit $p = z_0, \dots, z_m = q_n$ with jumps inside D. Hence, $d(h(z_{m-1}), q_n) < \varepsilon$. Poincaré's recurrence theorem (see [22, Section 4.1]) implies that we can consider a recurrent point $r \in V$ such that $d(h(z_{m-1}), r) < \varepsilon$. Let $h^l(r) \in V$ and define the pseudoorbit

$$p = z_0, \dots, z_{m-1}, r, h(r), \dots, h^{l-1}(r), q.$$

Then we have an ε–pseudoorbit from p to q whose jumps are all contained in D.

To show that P_ε is nonempty, notice that again that, by Poincaré's recurrence theorem, one has that $p \in P_\varepsilon$. □

Now let us construct the desired perturbation of f_1.

For $f_1 \in \mathrm{Diff}^1_{\nu,\mathrm{per}}(\mathbb{A})$ and the prescribed neighborhood $\mathcal{N}$, let $N = N(f_1, \mathcal{N})$ be the positive integer given by Theorem 2.2. We consider first the set $D \subset (C_0, C_1)$, given by $D_0 = (C_{a_0}, C_{b_0})$, such that the arc I_0^u and the invariant manifold $W^s(x_1, f_1)$

intersects D_0. Choose $0 < a_1 < a_0$ and $b_0 < b_1 < 1$ so that $D_1 = (C_{a_1}, C_{b_1})$ contains $\mathrm{cl}[D_0 \cup \cdots \cup f_1^{N-1}(D_0)]$.

Choose a point $z \in I_0^u \setminus D_1$ and $w \in W^s(x_1, f_1) \setminus D_1$. It follows from Lemma 4.4 that one has $z \dashv_{D_0} w$. Theorem 2.2 implies that there exists $g \in \mathcal{U}$ such that $g^n(z) = w$ and such that $g = f_1$ outside D_1. Due to the way w is chosen and since $g = f_1$ outside D_1, it follows that w still belongs to $W^s(x_1, g)$ after perturbation[7] and the same holds for I_0^u, so we deduce that I_0^u intersects $W^s(x_1, g)$. A further small perturbation makes this intersection transversal. Being transversal, the intersection will persist for sufficiently small C^1 perturbations even if the involved points are moved,

Now we do the same argument again but reducing a_1 and b_1 further so that we can connect I_1^u with the stable manifold of x_0 and again make the intersection transversal. We can choose the perturbation small enough that the intersection we had already created persists thanks to transversality. This concludes the proof that $f_2 \in \mathcal{N}$ can be constructed.

4.3 Final perturbation

For our last move, we fix z_0 in one of the connected components of $C_0 \setminus \{x_0, p_0\}$ and z_1 in one of the connected components of $C_1 \setminus \{x_1, p_1\}$. Consider for $k = 0, 1$ an open ball $B(z_k, \delta)$ such that $B(z_k, \delta) \cap C_k = I_k$ is a wandering interval, ie $I_k \cap \bigcup_{n \in \mathbb{Z} \setminus \{0\}} f_2^n(I_k) = \varnothing$.

We now take two C^∞–diffeomorphisms b_0 and b_1 which are arbitrarily C^∞–close to the identity and supported in $B(z_0, \delta)$ and $B(z_1, \delta)$, defined as follows.

If we set for every $p \in \mathbb{R}^2$ the coordinates $\tilde{x} = \pi_1(p - z_0)$ and $\tilde{y} = \pi_2(p - z_0)$,[8] the first map is given by

$$b_0(p) = (\tilde{x}, \tilde{y} + \mu(\tilde{x}, \tilde{y})),$$

where $\mu\colon \mathbb{R}^2 \to [0, 1]$ is some C^∞ *bump* function which is zero in $B(0, \delta)^c$ and positive in $B(0, \delta)$. Note that $I_0 \cup b_0(I_0)$ is the boundary of an open disk contained in (C_0, C_1).

For b_1, if we now set for every $p \in \mathbb{R}^2$ the coordinates $\tilde{x} = \pi_1(p - z_1)$ and $\tilde{y} = \pi_2(p - z_1)$, we define

$$b_1(p) = (\tilde{x}, \tilde{y} - \mu(\tilde{x}, -\tilde{y})).$$

[7] Technically one has to choose $z \neq p_0$ in the connected component of $I_0^u \setminus D_1$ containing p_0 and $w \neq x_1$ in the connected component of $W^s(x_1, f_1) \setminus D_1$ containing x_1.

[8] Here π_1 and π_2 stand for the projections over the first and second coordinate in $\mathbb{R}^2$.

Note that $I_1 \cup b_1(I_1)$ is the boundary of an open disk contained in (C_0, C_1). Let us call by L_0 the open disk between I_0 and $b_0(I_0)$ and L_1 the open disk in-between I_1 and $b_1(I_1)$.

We are ready now to perform our final perturbation. We consider $f \in \mathcal{N}$ such that

$$f = b_1 \circ b_0 \circ f_2,$$

where the following holds:

(1) Property (3) of the second perturbation f_2 still holds.

(2) $\lim_n f^{-n}(l) = -\infty$ for all $l \in L_0$.

(3) $\lim_n f^{-n}(l) = +\infty$ for all $l \in L_1$.

Indeed, the choice of b_0 and b_1 imply immediately the last two properties and if b_0 and b_1 are small enough, then the transverse intersections required in (3) of f_2 still holds. Notice that $f = f_2$ in a neighborhood of x_0, x_1, p_0 and p_1. See Figure 6 for a schematic drawing.

4.4 The perturbation verifies the announced properties

We must now show that f verifies our Theorem 4.2. Consider the set

$$\mathcal{B} = \mathrm{cl}[W^u(x_0, f)].$$

Observe that it is a closed connected set. The next lemma shows that it coincides with $\mathrm{cl}[W^u(x_1, f)]$, and by construction $\mathcal{B} \subset [C_0, C_1]$. Thus, we actually have that $\mathcal{B}$ is an essential continuum with $\rho_{\mathcal{B}}(F) \supseteq [0, 1]$ for some suitable lift F of f. Let us call $\mathcal{U}^-$ and $\mathcal{U}^+$ the two unbounded connected components of $\mathbb{A} \setminus \mathcal{B}$.

Lemma 4.5 *The points x_0 and x_1 are homoclinically related.*

Proof This follows by applying a small variation of the λ–lemma [22] in a neighborhood of p_0 and p_1, respectively. Notice that the usual λ–lemma does not apply since p_0 is not hyperbolic but by looking at the local dynamics of p_0 and the way we have performed the perturbation b_0 (far from p_0) one has that the new unstable manifold of x_0 will approach for forward iterates the unstable manifold of p_0 which is connected to the stable manifold of x_1. The symmetric argument gives that the unstable manifold of x_1 must intersect transversally the stable manifold of x_0. □

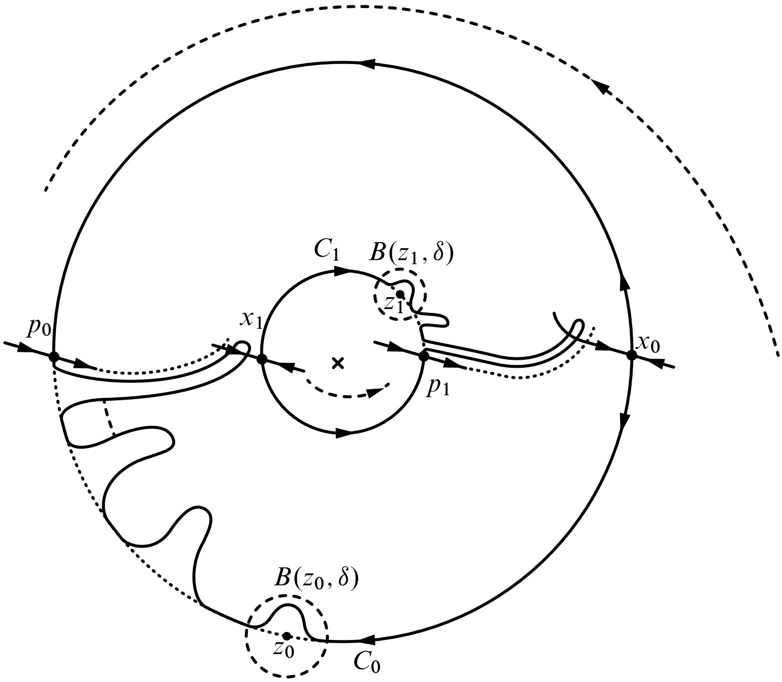

Figure 6: The final map f. We perform a small perturbation near z_0 and z_1 so that x_0, p_0, x_1 and p_1 belong to the same homoclinic class. The closure of $W^u(x_0, f)$ will give our desired circloid.

Furthermore, as we have one branch of $W^s(x_0, f)$ contained in $\mathcal{U}^-$, Lemma 4.5 shows that $\partial\mathcal{U}^- \supseteq \mathcal{B}$. In the same way, as the saddle x_1 is homoclinically related to x_0 and one branch of $W^s(x_1, f)$ is contained in $\mathcal{U}^+$, we have that $W^s(x_0, f)$ must intersect $\mathcal{U}^+$. Therefore, arguing with the λ–lemma, we find that $\partial\mathcal{U}^+ \supseteq \mathcal{B}$. So

$$\mathcal{B} \subset \partial\mathcal{U}^- \cap \partial\mathcal{U}^+.$$

On the other hand, since $\mathcal{U}^\pm$ is a connected component of $\mathbb{A} \setminus \mathcal{B}$, the set $\mathcal{U}^\pm \cup \mathcal{B}$ is closed, and in particular, $\partial\mathcal{U}^\pm \subset \mathcal{B}$, therefore, $\mathcal{B} = \partial\mathcal{U}^- = \partial\mathcal{U}^+$.

This implies, that $\mathcal{B}$ is the boundary of a circloid $\mathcal{C}$ with $\mathbb{A} \setminus \mathcal{C} = \mathcal{U}^- \cup \mathcal{U}^+$, as is proved for instance in [20, Corollary 3.3]. In order to obtain Theorem 4.2, we need to prove that $\mathcal{C}$ is the global attractor of f.

For this, it is enough to show that every point $u \in \mathcal{U}^-$ has its α–limit in $-\infty$ and that every point $v \in \mathcal{U}^+$ has it α–limit in $+\infty$. We work with $\mathcal{U}^-$; the other case is similar. Recall the definition of the open disk L_0 associated to the wandering interval I_0. We have by construction that L_0 is bounded by the concatenation of curves I_0 and $b_0(I_0)$. Denote by $\widetilde{I}_0$ the maximal open interval in I_0.

In order to show that $-\infty = \lim_n f^{-n}(u)$ for all $u \in \mathcal{U}^-$, it is enough to show the following lemma:

Lemma 4.6 *We have that* $\mathcal{U}^- = (-\infty, C_0) \cup \bigcup_{n\in\mathbb{N}} f^n(L_0 \cup \widetilde{I}_0)$.

Proof It is easy to see that $(-\infty, C_0) \subset \mathcal{U}^-$. Further, as $L_0 \cup \widetilde{I}_0 \subset \mathcal{U}^-$, we have that $\mathcal{U}^- \supseteq (-\infty, C_0) \cup \bigcup_{n\in\mathbb{N}} f^n(L_0)$. We must look now for the symmetric inclusion.

Observe that $f^n(I_0) \subset C_0$ for all $n \in \mathbb{N}$ and that $f^n(b_0(I_0)) \subset [C_0, +\infty) \cap \mathcal{C}$ for all $n \in \mathbb{N}$.

Let W be the interior of the arc in C_0 joining x_0 and p_0 and containing I_0. Observe that the closure of the complementary connected component is contained in $\mathcal{C}$. Then

$$C_0 \cap \mathcal{C} = C_0 \setminus \biggl(\bigcup_{n\in\mathbb{N}} f^n(\widetilde{I}_0) \biggr).$$

Assume $x \in \mathcal{U}^- \cap [C_0, +\infty)$; hence, we can connect x to $-\infty$ through a simple curve $\Gamma' \subset \mathcal{U}^-$, which must contain a compact arc $\Gamma \subset [C_0, +\infty)$ from x to a certain point in $f^{n_0}(I_0)$. Thus, Γ must be contained in a disk bounded by the concatenation of $f^{n_0}(I_0)$ and $f^{n_0}(b_0(I_0))$, otherwise Γ meets $f^n(b_0(I_0)) \subset \mathcal{C}$.

Therefore we get that $x \in f^{n_0}(L_0)$, and we have

$$\mathcal{U}^- = (-\infty, C_0) \cup \bigcup_{n\in\mathbb{N}} f^n(L_0 \cup \widetilde{I}_0). \qquad \square$$

We conclude that the nonwandering set of f is contained in $\mathcal{C}$, so $\mathcal{C}$ must be a global attractor for f, and we are done with the proof of Theorem 4.2 (and consequently of Theorem B).

4.5 Proof of Theorem D

We here perform some modifications to the construction developed above to obtain a proof of Theorem D. In the construction of f_2, it is not hard to construct another pair of saddle periodic points inside (C_0, C_1) so that they are homoclinically related and have different rotation numbers which are as close as desired to 0 and 1, respectively. This can be achieved using Theorem 2.2. See Figure 7.

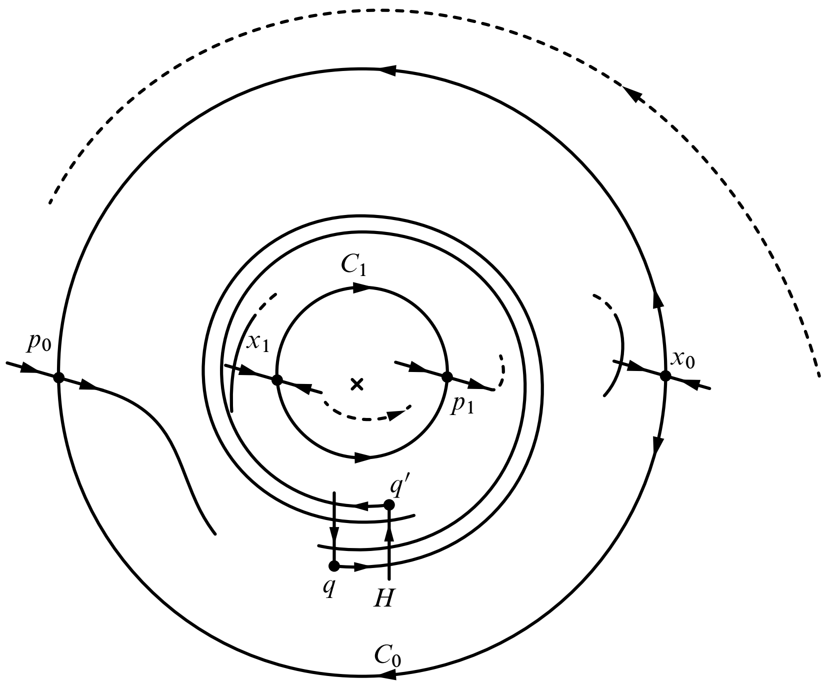

Figure 7: The map f_2 for the examples in Theorem D. We consider a homoclinic class $H = H(q, q')$ contained in (C_0, C_1) with a rotation set arbitrarily close to $[0, 1]$.

Then, for an arbitrary small $\delta > 0$ we can choose f_2 so that there is a homoclinic class[9] $H = H(q, q') \subset (C_0, C_1)$ with $(\delta, 1-\delta) \subset \rho_H(F_2) \subset [0, 1]$ for a lift F_2 of f_2. Notice that f_2 verifies $f_2(\mathbb{S}^1 \times [-1, 2]) \subset \mathbb{S}^1 \times (-1, 2)$ and we can assume that the determinant of the derivative of f_2 is everywhere smaller than $1-\delta$ outside $\mathbb{S}^1 \times [-1, 2]$.

Now, instead of pushing the unstable manifolds of x_0 and x_1, we will consider smooth diffeomorphisms h_n which coincide with the identity outside the region (C_{-n}, C_{n+1}), and having the form $h_n(x, y) = (x, \widehat{h}_n(y))$ and $\widehat{h}_n \colon \mathbb{R} \to \mathbb{R}$ is a function such that

- $\widehat{h}'_n(y) \in (1-1/n, 1+1/n)$ for every $y \in \mathbb{R}$ and $\widehat{h}'_n(y) < 1-1/2n$ if $y \in [-1, 2]$,
- the C^1 distance between h_n and the identity tends to 0 as $n \to \infty$.

We will consider the perturbations $g_n = h_n \circ f_2$.

[9] In our context the homoclinic class is the minimal f–invariant set containing the closure of the transversal heteroclinic intersections associated to the periodic points q and q'.

Since f_2 has the homoclinic class H, it follows that for large enough n, this class has a continuation H_n which contains in its rotation set the interval $[\delta, 1-\delta]$. Moreover, for large enough n there will still be a global attractor as one has $g_n([C_{-1}, C_2]) \subset (C_{-1}, C_2)$, and the dynamics is dissipative since the jacobian of g_n in $[C_{-1}, C_2]$ is everywhere less than $1 - 1/2n < 1$. Since f_2 satisfies the twist condition, which is open, the same holds for g_n when n is large. Thus g_n presents Birkhoff attractors $\mathcal{C}_n$ for large $n \in \mathbb{N}$.

By the same arguments we did before, the closure of the unstable manifold $W^u(q, g_n)$ must be a circloid $\mathcal{C}'_n$ which is invariant for some power of g_n. As any power of g_n is also a dissipative twist map, it has a unique invariant circloid, so it must be that $\mathcal{C}'_n = \mathcal{C}_n$ (the same holds for $W^u(q', g_n)$). Therefore, the homoclinic class H_n is contained in the Birkhoff attractor $\mathcal{C}_n$, so $\rho_{\mathcal{C}_n}(G_n) \supset [\delta, 1-\delta]$ for some lift G_n of g_n.

On the other hand, as g_n can be considered in an arbitrary small C^1 neighborhood of τ, the entropy of g_n can is arbitrary small (Remark 4.1), say smaller than $\frac{1}{3}\varepsilon$, and then[10] choosing g_n^3 we obtain the proof of Theorem D.

Remark 4.7 It might be possible that the global attractor Λ in this case is equal to the Birkhoff attractor $\mathcal{C}$. However, we did not find a simple argument to prove this fact.

References

[1] **R H Abraham**, **H B Stewart**, *A chaotic blue sky catastrophe in forced relaxation oscillations*, Phys. D 21 (1986) 394–400 MR

[2] **M U Akhmet**, **M O Fen**, *Entrainment by chaos*, J. Nonlinear Sci. 24 (2014) 411–439 MR

[3] **L Alsedà**, **J Llibre**, **F Mañosas**, **M Misiurewicz**, *Lower bounds of the topological entropy for continuous maps of the circle of degree one*, Nonlinearity 1 (1988) 463–479 MR

[4] **M-C Arnaud**, **C Bonatti**, **S Crovisier**, *Dynamiques symplectiques génériques*, Ergodic Theory Dynam. Systems 25 (2005) 1401–1436 MR

[5] **M Barge**, **R M Gillette**, *Rotation and periodicity in plane separating continua*, Ergodic Theory Dynam. Systems 11 (1991) 619–631 MR

[6] **F Beguín**, *Ensembles de rotations des homéomorphismes du tore* $\mathbb{T}^2$, lecture notes (2007) Available at `https://www.math.univ-paris13.fr/~beguin/Publications_files/cours-2.pdf`

[10] The iterate is just to ensure that the rotation set of a well-chosen lift contains $[0, 1]$. Notice that the entropy of g_n^3 will be smaller than ε.

[7] **G D Birkhoff**, *Sur quelques courbes fermées remarquables*, Bull. Soc. Math. France 60 (1932) 1–26 MR Reprinted in "Collected mathematical papers, II: Dynamics (continued), physical theories", Amer. Math. Soc., New York (1950) 444-461

[8] **C Bonatti**, **S Crovisier**, *Récurrence et généricité*, Invent. Math. 158 (2004) 33–104 MR

[9] **C Bonatti**, **L J Díaz**, **E R Pujals**, *A C^1–generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources*, Ann. of Math. 158 (2003) 355–418 MR

[10] **J P Boroński**, **P Oprocha**, *Rotational chaos and strange attractors on the 2–torus*, Math. Z. 279 (2015) 689–702 MR

[11] **H W Broer**, *KAM theory: the legacy of A N Kolmogorov's 1954 paper*, Bull. Amer. Math. Soc. 41 (2004) 507–521 MR

[12] **A Chenciner**, *Poincaré and the three-body problem*, from "Henri Poincaré, 1912–2012" (B Duplantier, V Rivasseau, editors), Prog. Math. Phys. 67, Springer (2015) 51–149 MR

[13] **S Crovisier**, *Perturbation of C^1–diffeomorphisms and generic conservative dynamics on surfaces*, from "Dynamique des difféomorphismes conservatifs des surfaces: un point de vue topologique", Panor. Synthèses 21, Soc. Math. France, Paris (2006) 1–33 MR

[14] **J Franks**, *Necessary conditions for stability of diffeomorphisms*, Trans. Amer. Math. Soc. 158 (1971) 301–308 MR

[15] **J Franks**, *Rotation vectors for surface diffeomorphisms*, from "Proceedings of the International Congress of Mathematicians" (S D Chatterji, editor), volume II, Birkhäuser, Basel (1995) 1179–1186 MR

[16] **J Franks**, **M Handel**, *Entropy zero area preserving diffeomorphisms of S^2*, Geom. Topol. 16 (2012) 2187–2284 MR

[17] **J Franks**, **P Le Calvez**, *Regions of instability for non-twist maps*, Ergodic Theory Dynam. Systems 23 (2003) 111–141 MR

[18] **M Girard**, *Sur les courbes invariantes par un difféomorphisme C^1–générique symplectique d'une surface*, PhD thesis, Université d'Avignon (2009) Available at `https://tel.archives-ouvertes.fr/tel-00461234/`

[19] **J G Hocking**, **G S Young**, *Topology*, Addison-Wesley, Reading, MA (1961) MR

[20] **T Jäger**, *Linearization of conservative toral homeomorphisms*, Invent. Math. 176 (2009) 601–616 MR

[21] **T Jäger**, **A Passeggi**, *On torus homeomorphisms semiconjugate to irrational rotations*, Ergodic Theory Dynam. Systems 35 (2015) 2114–2137 MR

[22] **A Katok**, **B Hasselblatt**, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications 54, Cambridge Univ. Press (1995) MR

[23] **J Kennedy**, **J A Yorke**, *Topological horseshoes*, Trans. Amer. Math. Soc. 353 (2001) 2513–2530 MR

[24] **A Koropecki**, *Realizing rotation numbers on annular continua*, Math. Z. 285 (2017) 549–564 MR

[25] **A Koropecki**, **P Le Calvez**, **M Nassiri**, *Prime ends rotation numbers and periodic points*, Duke Math. J. 164 (2015) 403–472 MR

[26] **A Koropecki**, **A Passeggi**, *A Poincaré–Bendixson theorem for translation lines and applications to prime ends*, preprint (2017) arXiv

[27] **J M Kwapisz**, *Rotation sets and entropy*, PhD thesis, State University of New York at Stony Brook (1995) MR Available at `https://search.proquest.com/docview/304251332`

[28] **P Le Calvez**, *Propriétés des attracteurs de Birkhoff*, Ergodic Theory Dynam. Systems 8 (1988) 241–310 MR

[29] **P Le Calvez**, *Une version feuilletée équivariante du théorème de translation de Brouwer*, Publ. Math. Inst. Hautes Études Sci. 102 (2005) 1–98 MR

[30] **P Le Calvez**, **F A Tal**, *Forcing theory for transverse trajectories of surface homeomorphisms*, preprint (2015) arXiv

[31] **J Llibre**, **R S MacKay**, *Rotation vectors and entropy for homeomorphisms of the torus isotopic to the identity*, Ergodic Theory Dynam. Systems 11 (1991) 115–128 MR

[32] **J N Mather**, *Topological proofs of some purely topological consequences of Carathéodory's theory of prime ends*, from "Selected studies: physics-astrophysics, mathematics, history of science" (T M Rassias, G M Rassias, editors), North-Holland, Amsterdam (1982) 225–255 MR

[33] **S Matsumoto**, *Prime end rotation numbers of invariant separating continua of annular homeomorphisms*, Proc. Amer. Math. Soc. 140 (2012) 839–845 MR

[34] **J W Milnor**, *Attractor*, Scholarpedia 1 (2006) art. id. 1815

[35] **M Misiurewicz**, **K Ziemian**, *Rotation sets for maps of tori*, J. London Math. Soc. 40 (1989) 490–506 MR

[36] **M H A Newman**, *Elements of the topology of plane sets of points*, 2nd edition, Cambridge Univ. Press (1951) MR

[37] **A Passeggi**, *Contributions in surface dynamics: a classification of minimal sets of homeomorphisms and aspects of the rotation theory on the torus*, PhD thesis, Friedrich-Alexander-Universität, Erlangen (2013) Available at `https://d-nb.info/1064996493/34`

[38] **A Passeggi**, **J Xavier**, *A classification of minimal sets for surface homeomorphisms*, Math. Z. 278 (2014) 1153–1177 MR

[39] **R Shaw**, *Strange attractors, chaotic behavior, and information flow*, Z. Naturforsch. A 36 (1981) 80–112 MR

[40] **M Shub**, *All, most, some differentiable dynamical systems*, from "Proceedings of the International Congress of Mathematicians" (M Sanz-Solé, J Soria, J L Varona, J Verdera, editors), volume III, Eur. Math. Soc., Zürich (2006) 99–120 MR

[41] **J M T Thompson**, **H B Stewart**, *Nonlinear dynamics and chaos*, 2nd edition, Wiley, Chichester (2002) MR

[42] **R B Walker**, *Periodicity and decomposability of basin boundaries with irrational maps on prime ends*, Trans. Amer. Math. Soc. 324 (1991) 303–317 MR

Facultad de Ciencias, Centro de Matemática, Universidad de la República
Montevideo, Uruguay

alepasseggi@gmail.com, rpotrie@cmat.edu.uy, samba@cmat.edu.uy

Proposed: Danny Calegari — Received: 18 July 2016
Seconded: Leonid Polterovich, Jean-Pierre Otal — Revised: 22 May 2017