

Product formulae for Ozsváth–Szabó 4–manifold invariants

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We give formulae for the Ozsváth–Szabó invariants of 4–manifolds X obtained by fiber sum of two manifolds M_1, M_2 along surfaces Σ_1, Σ_2 having trivial normal bundle and genus $g \geq 1$. The formulae follow from a general theorem on the Ozsváth–Szabó invariants of the result of gluing two 4–manifolds along a common boundary, which is phrased in terms of relative invariants of the pieces. These relative invariants take values in a version of Heegaard Floer homology with coefficients in modules over certain Novikov rings; the fiber sum formula follows from the theorem that this “perturbed” version of Heegaard Floer theory recovers the usual Ozsváth–Szabó invariants, when the 4–manifold in question has $b^+ \geq 2$. The construction allows an extension of the definition of Ozsváth–Szabó invariants to 4–manifolds having $b^+ = 1$ depending on certain choices, in close analogy with Seiberg–Witten theory. The product formulae lead quickly to calculations of the Ozsváth–Szabó invariants of various 4–manifolds; in all cases the results are in accord with the conjectured equivalence between Ozsváth–Szabó and Seiberg–Witten invariants.

[57R58](#); [57M99](#)

1 Introduction

At the time of writing, there is no example of a smoothable topological 4–manifold whose smooth structures have been classified. Indeed, no smooth 4–manifold is known to support only finitely many smooth structures, and in virtually every case a 4–manifold that admits more than one smooth structure is known to admit infinitely many such structures. A substantial amount of ingenuity by a large number of authors—see Stern [19] for a brief survey—has been required to produce these exotic 4–manifolds, though ultimately the list of topological tools used in the constructions is perhaps surprisingly short. The standard approach to distinguishing smooth structures on 4–manifolds has been to make use of gauge-theoretic invariants, which requires an understanding of how these invariants behave under the cut-and-paste operations used in constructing examples. In the case of the Seiberg–Witten invariants, this understanding was provided by Morgan–Mrowka–Szabó [7], Morgan–Szabó–Taubes [8], D Park [17], Li–Liu [5] and many others, and the Seiberg–Witten invariants have become the tool

of choice for studying smooth manifolds. Beginning in 2000, Ozsváth and Szabó [12; 11; 15] introduced invariants of 3- and 4-dimensional manifolds meant to mimic the Seiberg–Witten invariants but also avoid the technical issues that for many years prevented the expected Seiberg–Witten–Floer theory from taking shape. Their theory has been remarkably successful and has had a number of important consequences in the study of 3-manifolds and knot theory. The 4-dimensional side of the story has been developed to a somewhat lesser extent, however, and the existing gauge-theoretic technology means it is still the case that Seiberg–Witten invariants are often the easiest to use in the study of smooth 4-manifolds. Our aim here is to develop reasonably general cut-and-paste principles for Ozsváth–Szabó invariants, that will be useful in a variety of situations. A central tool in many constructions of exotic 4-manifolds is the normal connected sum or “fiber sum,” in which neighborhoods of diffeomorphic surfaces are excised from closed 4-manifolds and the resulting complements glued together along their boundary. As an application of the formalism we introduce here, and as a motivating test case, we give formulae that essentially determine the behavior of the Ozsváth–Szabó 4-manifold invariants under fiber sum along surfaces of trivial normal bundle.

To realize this goal we are obliged to introduce a substantial amount of machinery, including the development of Heegaard Floer homology with coefficients in certain power series (Novikov) rings. This can be viewed in analogy with Seiberg–Witten Floer homology perturbed by a 2-dimensional cohomology class, and in many ways exhibits parallel behavior. It is our hope that this “perturbed” Heegaard Floer theory will be of interest in other applications as well.

For the sake of exposition, we state our results in this introduction in order of increasing technicality. In particular, Ozsváth and Szabó defined their invariants initially for 4-manifolds M with $b^+(M) \geq 2$, and since the theory is simplest in that case we begin there.

1.1 Constructions and statements of results when $b^+ \geq 2$

The Ozsváth–Szabó invariants [15; 9] are defined using a “TQFT” construction, meaning that they are built from invariants of 3-dimensional manifolds (the Heegaard Floer homology groups) and cobordisms between such manifolds. To a closed oriented 4-manifold M with $b^+(M) \geq 2$, with a spin^c structure \mathfrak{s} , Ozsváth and Szabó associate a linear function

$$\Phi_{M,\mathfrak{s}}: \mathbb{A}(M) \rightarrow \mathbb{Z},$$

where $\mathbb{A}(M)$ is the algebra $\Lambda^*(H_1(M; \mathbb{Z})/\text{torsion}) \otimes \mathbb{Z}[U]$, graded such that elements of $H_1(M)$ have degree 1 and U has degree 2. This invariant has the property that

$\Phi_{M,\mathfrak{s}}$ is nonzero for at most finitely many spin^c structures \mathfrak{s} , and furthermore, with \mathfrak{s} fixed, vanishes on all elements of $\mathbb{A}(M)$ not having degree

$$d(\mathfrak{s}) = \frac{1}{4}(c_1^2(\mathfrak{s}) - 3\sigma(M) - 2e(M)),$$

where σ denotes the signature of the intersection form on M and e is the Euler characteristic. Ozsváth and Szabó conjecture [15] that $\Phi_{M,\mathfrak{s}}$ is identical with the Seiberg–Witten invariant.

We remark that there is a sign ambiguity in the definition of $\Phi_{M,\mathfrak{s}}$, so that the results to follow are true up to an overall sign.

The fiber sum of two smooth 4–manifolds is defined as follows. Let M_1 and M_2 be closed oriented 4–manifolds, and suppose $\Sigma_i \hookrightarrow M_i$, $i = 1, 2$, are smoothly embedded closed oriented surfaces of the same genus g . We assume throughout this paper that g is at least 1 and that the Σ_i have trivial normal bundles. In this case, Σ_i has a neighborhood $N(\Sigma_i)$ diffeomorphic to $\Sigma_i \times D^2$. Choose an orientation-preserving diffeomorphism $f: \Sigma_1 \rightarrow \Sigma_2$, and lift it to an orientation-reversing diffeomorphism $\phi: \partial N(\Sigma_1) \rightarrow \partial N(\Sigma_2)$ via conjugation in the normal fiber. We define the fiber sum $X = M_1 \#_{\Sigma} M_2$ by

$$X = (M_1 \setminus N(\Sigma_1)) \cup_{\phi} (M_2 \setminus N(\Sigma_2)).$$

In general, the manifold X can depend on the choice of ϕ . We assume henceforth that the homology classes $[\Sigma_1]$ and $[\Sigma_2]$ are nontorsion elements of $H_2(M_i; \mathbb{Z})$ (though the results of this paper can in principle be adapted to other situations).

To state the results, it is convenient to express the Ozsváth–Szabó invariant in terms of the group ring $\mathbb{Z}[H^2(M; \mathbb{Z})]$. That is to say, we write

$$\text{OS}_M = \sum_{\mathfrak{s} \in \text{Spin}^c(M)} \Phi_{M,\mathfrak{s}} e^{c_1(\mathfrak{s})},$$

where $e^{c_1(\mathfrak{s})}$ is the formal variable in the group ring corresponding to the first Chern class of the spin^c structure \mathfrak{s} (note that $c_1(\mathfrak{s}) = c_1(\mathfrak{s}')$ for distinct spin^c structures \mathfrak{s} and \mathfrak{s}' if and only if $\mathfrak{s} - \mathfrak{s}'$ is of order 2 in $H^2(M; \mathbb{Z})$, so the above formulation may lose some information if 2–torsion is present). The coefficients of the above expression are functions on $\mathbb{A}(M)$, so that OS_M is an element of $\mathbb{Z}[H^2(M; \mathbb{Z})] \otimes \mathbb{A}(M)^*$. The value of the invariant on $\alpha \in \mathbb{A}(M)$ is denoted $\text{OS}_M(\alpha) \in \mathbb{Z}[H^2(M; \mathbb{Z})]$.

The behavior of $\Phi_{M,\mathfrak{s}}$ under fiber sum depends on the value of $\langle c_1(\mathfrak{s}), [\Sigma] \rangle$ (since $c_1(\mathfrak{s})$ is a characteristic class, this value is always even when $[\Sigma]^2 = 0$). Thus, we

partition OS_M accordingly: for an embedded surface $\Sigma \hookrightarrow M$ with trivial normal bundle, let

$$\text{OS}_M^k = \sum_{\langle c_1(s), [\Sigma] \rangle = 2k} \Phi_{M,s} e^{c_1(s)}.$$

The adjunction inequality for Ozsváth–Szabó invariants implies that $\text{OS}_M^k \equiv 0$ if $|k| > g - 1$.

The topology of fiber sums is complicated in general by the presence of *rim tori*. A rim torus is a submanifold of the form $\gamma \times S^1 \subset \Sigma \times S^1$, where γ is an embedded circle on Σ . Such tori are homologically trivial in the fiber summands M_i , but typically essential in $X = M_1 \#_{\Sigma} M_2$. Let \mathcal{R} denote the subspace of $H^2(X; \mathbb{Z})$ spanned by the Poincaré duals of rim tori, and let $\rho: H^2(X; \mathbb{Z}) \rightarrow H^2(X; \mathbb{Z})/\mathcal{R}$ denote the natural projection. If $b_i \in H^2(M_i; \mathbb{Z})$, $i = 1, 2$, are cohomology classes with the property that $b_1|_{\partial N(\Sigma_1)}$ agrees with $b_2|_{\partial N(\Sigma_2)}$ under ϕ , then Mayer–Vietoris arguments show that there exists a class $b \in H^2(X; \mathbb{Z})$ whose restrictions to $M_i \setminus N(\Sigma_i)$ agrees with the corresponding restrictions of b_i , and furthermore that b is determined uniquely up to elements of \mathcal{R} and multiples of the Poincaré dual of Σ . If b , b_1 and b_2 satisfy these conditions on their respective restrictions, we say that the three classes are *compatible* with the fiber sum. We can eliminate part of the ambiguity in b given (b_1, b_2) by requiring that

$$b^2 = b_1^2 + b_2^2 + 4|m|,$$

where $m = \langle b_1, [\Sigma_1] \rangle = \langle b_2, [\Sigma_2] \rangle$. With this convention, the pair (b_1, b_2) gives rise to a well-defined element of $H^2(X; \mathbb{Z})/\mathcal{R}$ (see [Section 10.3](#) for details).

Theorem 1.1 *Let $X = M_1 \#_{\Sigma} M_2$ be obtained by fiber sum along a surface Σ of genus $g > 1$ from manifolds M_1, M_2 satisfying $b^+(M_i) \geq 2$, $i = 1, 2$. If $|k| > g - 1$ then $\text{OS}_X^k = \text{OS}_{M_1}^k = \text{OS}_{M_2}^k = 0$. In general, we have*

$$(1) \quad \rho \left(\text{OS}_X^k(\alpha) \right) = \sum_{\beta \in \mathcal{B}_k} \text{OS}_{M_1}^k(\alpha_1 \otimes \beta) \cdot \text{OS}_{M_2}^k(f_*(\beta^\circ) \otimes \alpha_2) \cdot u_{\beta,k}$$

where $\alpha_i \in \mathbb{A}(M_i \setminus N(\Sigma_i))$ are any elements such that $\alpha_1 \otimes \alpha_2$ maps to α under the inclusion-induced homomorphism.

The notation of the theorem requires some explanation. First, the product of group ring elements appearing on the right makes use of the construction outlined above, producing elements of $H^2(X; \mathbb{Z})/\mathcal{R}$ from compatible pairs (b_1, b_2) . The set \mathcal{B}_k denotes a basis over \mathbb{Z} for the group $H_*(\text{Sym}^d(\Sigma); \mathbb{Z}) \cong \bigoplus_{i=0}^d \Lambda^i H_1(\Sigma) \otimes \mathbb{Z}[U]/U^{d-i+1}$, thought

of as a subgroup of $\mathbb{A}(\Sigma)$, where $d = g - 1 - |k|$. Likewise, $\{\beta^\circ\}$ denotes the dual basis to \mathcal{B}_k under a certain nondegenerate pairing (see Section 10.3). The terms $\alpha_1 \otimes \beta$ and $f_*(\beta^\circ) \otimes \alpha_2$ are understood to mean the images of those elements in $\mathbb{A}(M_1)$ and $\mathbb{A}(M_2)$, using the inclusion-induced maps. Finally, $u_{\beta,k}$ is a polynomial in the variable $e^{\text{PD}[\Sigma]}$ whose constant coefficient is 1, and which is equal to 1 except possibly in the case $k = 0$.

The left hand side of (1) lies in the group ring of $H^2(X; \mathbb{Z})/\mathcal{R}$, and its coefficients are “rim torus averaged” Ozsváth–Szabó invariants. That is to say, each coefficient of $\rho(\text{OS}_X^k)$ is a term of the form

$$\Phi_{X,\mathfrak{s}}^{\text{Rim}} = \sum_{\substack{\mathfrak{s}' \in \text{Spin}^c(X) \\ \mathfrak{s}' - \mathfrak{s} \in \mathcal{R}}} \Phi_{X,\mathfrak{s}'}$$

A 4–manifold X is said to have (Ozsváth–Szabó) *simple type* if any spin^c structure \mathfrak{s} for which $\Phi_{X,\mathfrak{s}} \neq 0$ has $d(\mathfrak{s}) = 0$. We have:

Corollary 1.2 *If M_1 and M_2 have simple type, then the fiber sum $X = M_1 \#_{\Sigma} M_2$ has the property that if $\Phi_{X,\mathfrak{s}}^{\text{Rim}} = 0$ whenever $d(\mathfrak{s}) \neq 0$. Furthermore,*

$$(2) \quad \rho(\text{OS}_X^k) = 0 \quad \text{if } |k| < g - 1,$$

while

$$\rho(\text{OS}_X^{\pm(g-1)}(\alpha)) = \begin{cases} \text{OS}_{M_1}^{\pm(g-1)}(1) \cdot \text{OS}_{M_2}^{\pm(g-1)}(1) & \text{if } \alpha = 1 \\ 0 & \text{if } \deg(\alpha) > 1. \end{cases}$$

In other words, the fiber sum of manifolds of simple type has simple type after sum over rim tori. We note that Equation (2) holds if M_1 and M_2 are assumed only to have $\mathbb{A}(\Sigma)$ –simple type, that is, if $\Phi_{M,\mathfrak{s}}(\alpha) = 0$ whenever α lies in the ideal of $\mathbb{A}(M)$ generated by U and the image of $H_1(\Sigma)$.

We should remark that Taubes [20] has shown that symplectic 4–manifolds with $b^+ \geq 2$ have Seiberg–Witten simple type. It seems safe, therefore, to conjecture the following:

Conjecture 1.3 *If X is a symplectic 4–manifold with $b^+(X) \geq 2$ then X has Ozsváth–Szabó simple type.*

Leaving this issue for now, we turn to the case of a fiber sum along a torus, where the product formula is slightly different.

Theorem 1.4 *Let $X = M_1 \#_{\Sigma} M_2$ be obtained by fiber sum along a surface Σ of genus $g = 1$, such that M_1 , M_2 , and X each have $b^+ \geq 2$. Let \tilde{T} denote the Poincaré dual of the class in $H_2(X; \mathbb{Z})$ induced by $[\Sigma_i]$, and write T for the image of \tilde{T} in $H^2(X; \mathbb{Z})/\mathcal{R}$. Then for any $\alpha \in \mathbb{A}(X)$ we have*

$$\rho(\text{OS}_X(\alpha)) = (T - T^{-1})^2 \text{OS}_{M_1}(\alpha_1) \cdot \text{OS}_{M_2}(\alpha_2)$$

where $\alpha_1 \otimes \alpha_2 \in \mathbb{A}(M_1) \otimes \mathbb{A}(M_2)$ maps to α as before.

Here the product between OS_{M_1} and OS_{M_2} uses the previous construction, while multiplication with T takes place in the group ring of $H^2(X; \mathbb{Z})/\mathcal{R}$.

We will show (Proposition 11.1) that any 4-manifold M containing an essential torus T of self-intersection 0 has $\mathbb{A}(T)$ -simple type, in analogy with a result of Morgan, Mrowka, and Szabó in Seiberg–Witten theory [7].

It is interesting to compare these results with those in Seiberg–Witten theory. Taubes proved an analogue of Theorem 1.4 in [21], generalizing work of Morgan–Mrowka–Szabó [7], and D Park [17] gave an independent proof of that result. The higher-genus case was considered by Morgan, Szabó and Taubes [8], but only under the condition that $|k| = g - 1$. In this case the sum appearing in Theorem 1.1 is trivial since $\mathcal{B}_{g-1} = \{1\}$, and the result here gives a product formula directly analogous to that of [8]. To our knowledge, no product formulae at the level of generality of Theorem 1.1 have yet appeared in the literature on Seiberg–Witten theory.

1.2 Relative invariants and a general gluing result

The theorems above are proved as particular cases of a general result on the Ozsváth–Szabó invariants of 4-manifolds obtained by gluing two manifolds along their boundary. In its most general form, the form that is useful in the context of fiber sums (Theorem 1.6 below), the statement involves perturbed Heegaard Floer invariants. If one is interested in gluing two manifolds-with-boundary that both have $b^+ \geq 1$, however, the perturbed theory is unnecessary and there is a slightly simpler “intermediate” result. To state it, recall that the construction of the 4-manifold invariant $\Phi_{M, \mathfrak{s}}$ is based on the Heegaard Floer homology groups associated to closed spin^c 3-manifolds (Y, \mathfrak{s}) . These groups have various incarnations; the relevant one for our immediate purpose is denoted $HF_{\text{red}}^-(Y, \mathfrak{s})$. Below, we recall the construction of Heegaard Floer homology with “twisted” coefficients, whereby homology groups are obtained whose coefficients are modules M over the group ring $R_Y = \mathbb{Z}[H^1(Y)]$ (here and below, ordinary (co)homology is considered with integer coefficients). If $Y = \partial Z$ is the boundary of an oriented 4-manifold Z , then such a module is provided by

$$M_Z = \mathbb{Z}[\ker(H^2(Z, \partial Z) \rightarrow H^2(Z))],$$

where $H^1(Y)$ acts by the coboundary homomorphism $H^1(Y) \rightarrow H^2(Z, \partial Z)$. The intermediate product formula alluded to above can be formulated as follows.

Theorem 1.5 *If (Z, \mathfrak{s}) is a spin^c 4–manifold with connected spin^c boundary (Y, \mathfrak{s}_Y) and if $b^+(Z) \geq 1$, then there exists a relative Ozsváth–Szabó invariant $\Psi_{Z, \mathfrak{s}}$ which is a function*

$$\Psi_{Z, \mathfrak{s}}: \mathbb{A}(Z) \rightarrow HF_{\text{red}}^-(Y, \mathfrak{s}_Y; M_Z),$$

a diffeomorphism invariant of (Z, \mathfrak{s}) well-defined up to multiplication by a unit in $\mathbb{Z}[H^1(Y)]$.

Furthermore, if (Z_1, \mathfrak{s}_1) and (Z_2, \mathfrak{s}_2) are two such spin^c 4–manifolds with spin^c boundary $\partial Z_1 = (Y, \mathfrak{s}) = -\partial Z_2$, write $X = Z_1 \cup_Y Z_2$. Then there exists an R_Y –sesquilinear pairing

$$(\cdot, \cdot): HF_{\text{red}}^-(Y, \mathfrak{s}; M_{Z_1}) \otimes_{R_Y} HF_{\text{red}}^-(-Y, \mathfrak{s}; M_{Z_2}) \rightarrow M_{X, Y},$$

where $M_{X, Y} = \mathbb{Z}[K(X, Y)]$ and $K(X, Y) = \ker(H^2(X) \rightarrow H^2(Z_1) \oplus H^2(Z_2))$. The pairing has the property that for any spin^c structure \mathfrak{s} on X restricting to \mathfrak{s}_i on Z_i , we have an equality of group ring elements:

$$\sum_{h \in K(X, Y)} \Phi_{X, \mathfrak{s}+h}(\alpha) e^h = (\Psi_{Z_1, \mathfrak{s}_1}(\alpha_1), \Psi_{Z_2, \mathfrak{s}_2}(\alpha_2)),$$

up to multiplication by a unit in $\mathbb{Z}[K(X, Y)]$. Here $\alpha \in \mathbb{A}(X)$, $\alpha_1 \in \mathbb{A}(Z_1)$ and $\alpha_2 \in \mathbb{A}(Z_2)$ are related by inclusion-induced multiplication as before.

To understand the term “ R_Y –sesquilinear,” observe that $R_Y = \mathbb{Z}[H^1(Y)]$ is equipped with an involution $r \mapsto \bar{r}$ induced by $h \mapsto -h$ in $H^1(Y)$. To say that the pairing in the theorem is sesquilinear means that

$$(g\xi, \eta) = g(\xi, \eta) = (\xi, \bar{g}\eta)$$

for $g \in R_Y$, $\xi \in HF_{\text{red}}^-(Y, \mathfrak{s}; M_{Z_1})$ and $\eta \in HF_{\text{red}}^-(-Y, \mathfrak{s}; M_{Z_2})$.

We note that the reason for the assumption $b^+(Z) \geq 1$ in the theorem above is that this condition guarantees that the homomorphism in HF^- induced by $Z \setminus B^4$ (which gives rise to the relative invariant $\Psi_{Z, \mathfrak{s}}$ above) takes values in the reduced Floer homology $HF_{\text{red}}^-(Y, \mathfrak{s}; M_Z) \subset HF^-(Y, \mathfrak{s}; M_Z)$. That fact in turn is necessary to make sense of the pairing (\cdot, \cdot) . In the notation of later sections, $(\cdot, \cdot) = \langle \tau^{-1}(\cdot), \cdot \rangle$ where $\tau: HF^+ \rightarrow HF^-$ is the natural map; τ is invertible only on the reduced groups.

The utility of [Theorem 1.5](#) is limited somewhat by the difficulty of determining the relative invariants $\Psi_{Z, \mathfrak{s}}$ in general. Furthermore, in the case of a fiber sum it is natural

to hope to relate the relative invariants of the complement of the neighborhood $\Sigma \times D^2$ of the summing surface in M to the absolute invariants of M ; however the manifold $\Sigma \times D^2$ has $b^+ = 0$ and it is not clear that the relative invariant is well-defined. This issue is addressed by the introduction of a “perturbation.”

1.3 Perturbed Heegaard Floer theory and results when $b^+ \geq 1$

Let Y be a closed oriented 3–manifold and $\eta \in H^2(Y; \mathbb{R})$ a given cohomology class. The Novikov ring associated to η is the set of formal series

$$\mathcal{R}_{Y,\eta} = \left\{ \sum_{g \in H^1(Y; \mathbb{Z})} a_g \cdot g \mid a_g \in \mathbb{Z} \right\} \subset \mathbb{Z}[[H^1(Y; \mathbb{Z})]]$$

subject to the condition that for each $N \in \mathbb{Z}$, the set of $g \in H^1(Y; \mathbb{Z})$ with a_g nonzero and $\langle g \cup \eta, [Y] \rangle < N$ is finite. This means $\mathcal{R}_{Y,\eta}$ consists of “semi-infinite” series with variables in $H^1(Y; \mathbb{Z})$, with the usual convolution product.

In Section 8 below, we develop the theory of Heegaard Floer homology for 3–manifolds Y and 4–dimensional cobordisms W equipped with 2–dimensional cohomology classes η , having coefficients in a module \mathcal{M}_η over $\mathcal{R}_{Y,\eta}$. We refer to this theory as Heegaard Floer homology perturbed by η . Many features of the unperturbed theory carry over to this setting with minimal modification, but one key simplification is that if η is chosen “generically” in a suitable sense (in particular $\eta \neq 0$), then $HF^\infty(Y, \mathfrak{s}; \mathcal{M}_\eta) = 0$ for any $\mathcal{R}_{Y,\eta}$ –module \mathcal{M}_η . In fact, one can arrange this latter fact to hold for any nonzero perturbation η by a further extension of coefficients: Heegaard Floer homology is naturally a module over a polynomial ring $\mathbb{Z}[U]$, and we form a “ U –completion” by extension to the power series ring $\mathbb{Z}[[U]]$. The U –completed Floer homology is written $HF_\bullet(Y, \mathfrak{s}; \mathcal{M}_\eta)$ by notational analogy with a similar construction in monopole Floer homology (see Kronheimer and Mrowka [4]). The vanishing of $HF_\bullet^\infty(Y, \mathfrak{s}; \mathcal{M}_\eta)$ means that $HF_\bullet^-(Y, \mathfrak{s}; \mathcal{M}_\eta) = HF_{\text{red}}^-(Y, \mathfrak{s}; \mathcal{M}_\eta)$ for all such \mathcal{M}_η , and allows us to define a relative invariant

$$\Psi_{Z,\mathfrak{s},\eta} \in HF_{\text{red}}^-(Y, \mathfrak{s}; \mathcal{M}_{Z,\eta})$$

that has the desired properties so long as $\eta|_Y \neq 0$. Note, however, that $\Psi_{Z,\mathfrak{s},\eta}$ is defined only up to sign and multiplication by an element of $H^1(Y)$. We remark that if $\eta|_Y = 0$ then $\mathcal{R}_{Y,\eta} = R_Y$, and we recover the unperturbed theory.

Now suppose that X is a closed 4–manifold, $Y \subset X$ a separating submanifold, and $\eta \in H^2(X; \mathbb{R})$ a cohomology class such that either $\eta|_Y \neq 0$, or in the decomposition $X = Z_1 \cup_Y Z_2$ we have $b^+(Z_i) \geq 1$. (Such a submanifold Y is said to be an *allowable*

cut for η .) Then we can define the *perturbed Ozsváth–Szabó invariant* associated to X , Y , η , and a spin^c structure \mathfrak{s} to be

$$(3) \quad \Phi_{X,Y,\eta,\mathfrak{s}}(\alpha) = \langle \tau^{-1}(\Psi_{Z_1,\eta,\mathfrak{s}}(\alpha_1)), \Psi_{Z_2,\eta,\mathfrak{s}}(\alpha_2) \rangle.$$

This invariant takes values in a module $\mathcal{M}_{X,Y,\eta}$, which is a suitable Novikov completion of $M_{X,Y}$ introduced previously. In [Section 8](#) we show ([Theorem 8.17](#)) that if $b^+(X) \geq 2$ then $\Phi_{X,Y,\eta,\mathfrak{s}}$ is in fact a polynomial lying in $M_{X,Y}$, whose coefficients are the Ozsváth–Szabó invariants of X in the various spin^c structures having restrictions to Z_1 and Z_2 that agree with the restrictions of \mathfrak{s} . The precise statement is the following:

Theorem 1.6 *Let X be a closed oriented 4–manifold with $b^+(X) \geq 2$, and $Y \subset X$ a connected submanifold determining a decomposition $X = Z_1 \cup_Y Z_2$, where Z_i are 4–manifolds with boundary. Fix a class $\eta \in H^2(X; \mathbb{R})$, and assume that Y is an allowable cut for η . If $b^+(Z_1)$ and $b^+(Z_2)$ are not both 0, then for any spin^c structure \mathfrak{s} on X and element $\alpha \in \mathbb{A}(X)$,*

$$(4) \quad \sum_{t \in K(X,Y)} \Phi_{X,\mathfrak{s}+t}(\alpha) e^t = \Phi_{X,Y,\eta,\mathfrak{s}}(\alpha) = \langle \tau^{-1} \Psi_{Z_1,\eta,\mathfrak{s}}(\alpha_1), \Psi_{Z_2,\eta,\mathfrak{s}}(\alpha_2) \rangle$$

up to sign and multiplication by an element of $K(X, Y)$, where $\alpha_1 \otimes \alpha_2 \mapsto \alpha$ as before. If $b^+(Z_1) = b^+(Z_2) = 0$ then the same is true after possibly replacing η by another class $\tilde{\eta}$, where $\tilde{\eta}|_{Z_i} = \eta|_{Z_i}$ for $i = 1, 2$.

The above definition (3) of $\Phi_{X,Y,\eta,\mathfrak{s}}$ makes sense for any allowable pair (Y, η) and spin^c structure \mathfrak{s} , but its dependence on the choice of (Y, η) is not clear. When $b^+(X) \geq 2$ it follows from [Theorem 1.6](#) that since $\Phi_{X,\mathfrak{s}}$ is independent of Y and η , so is $\Phi_{X,Y,\eta,\mathfrak{s}}$. However when $b^+(X) = 1$ the situation is not so simple; indeed, different choices of (Y, η) for a given (X, \mathfrak{s}) can lead to different results. This situation is analogous to the chamber structure of Seiberg–Witten invariants for 4–manifolds with $b^+ = 1$; partial results in this direction are given in [Section 8](#).

Note that the existence of a separating 3–manifold $Y \subset X$ and a class $\eta \in H^2(X, \mathbb{R})$ restricting nontrivially to Y implies that X is indefinite, in particular $b^+(X) \geq 1$.

We also point out a minor difference between [Theorem 1.1](#) and [Theorem 1.4](#) from the first section, and [Theorem 1.6](#) above and [Theorem 1.8](#) and [Theorem 1.9](#) below. In the former results, the various spin^c structures are labeled by their Chern classes, while in the latter they are identified in an affine way with two-dimensional cohomology classes. Thus the results in the present situation do not lose information corresponding to classes whose difference is of order 2, and to translate from results in this subsection to those in the first one we must square the variables.

An immediate consequence of [Theorem 1.6](#) is the following result on the Ozsváth–Szabó invariants of a manifold obtained by gluing two 4–manifolds along a boundary 3–torus. To state it, note first that if Z is a 4–manifold with boundary diffeomorphic to T^3 and $\eta \in H^2(Z; \mathbb{R})$ is a class whose restriction to T^3 is nontrivial then the relative invariant $\Psi_{Z, \mathfrak{s}, \eta}$ is well-defined, and takes values in the ring $\mathcal{K}(Z, \eta) \subset \mathbb{Z}[[K(Z)]]$, where $K(Z) = \ker(H^2(Z, \partial Z) \rightarrow H^2(Z))$ and $\mathcal{K}(Z, \eta)$ is a Novikov completion of the group ring $\mathbb{Z}[K(Z)]$. (If $b^+(Z) \geq 1$ then $\Psi_{Z, \mathfrak{s}, \eta}$ lies in $\mathbb{Z}[K(Z)]$.) Indeed, $\mathcal{K}(Z, \eta)$ is precisely the perturbed Floer homology of T^3 in the appropriate coefficient system. Note that $\mathcal{K}(Z, \eta)$ can be identified with a multivariable Laurent series ring, which is polynomial in variables that pair trivially with η (and some variables may have finite order, if there is torsion in the cokernel of $H^1(Z) \rightarrow H^1(\partial Z)$).

If $X = Z_1 \cup Z_2$ is obtained by gluing two 4–manifolds Z_1 and Z_2 with boundary T^3 , and $\eta \in H^2(X; \mathbb{R})$ restricts nontrivially to the splitting 3–torus, then the pairing appearing in [\(4\)](#) is naturally identified with a multiplication map

$$\mathcal{K}(Z_1, \eta) \otimes \mathcal{K}(Z_2, \eta) \xrightarrow{\sim} \mathcal{M}_{X, T^3, \eta} \subset \mathbb{Z}[[H^2(X; \mathbb{Z})]]$$

induced by the maps $j_i^*: H^2(Z_i, \partial Z_i) \rightarrow H^2(X)$ Poincaré dual to the inclusion homomorphisms. Thus [Theorem 1.6](#) gives:

Corollary 1.7 *Let $X = Z_1 \cup_{\partial} Z_2$ be a 4–manifold obtained as the union of two manifolds Z_1 and Z_2 whose boundary is diffeomorphic to the 3–torus T^3 , $\eta \in H^2(X; \mathbb{R})$ a class restricting nontrivially to T^3 , and \mathfrak{s} a spin^c structure on X . Then*

$$\Phi_{X, T^3, \mathfrak{s}, \eta} = j_1^*(\Psi_{Z_1, \eta, \mathfrak{s}}) j_2^*(\Psi_{Z_2, \eta, \mathfrak{s}}).$$

In particular if $b^+(X) \geq 2$ then

$$\sum_{k \in \delta H^1(T^3)} \Phi_{X, \mathfrak{s}+k} e^k = j_1^*(\Psi_{Z_1, \eta, \mathfrak{s}}) j_2^*(\Psi_{Z_2, \eta, \mathfrak{s}})$$

up to sign and translation by an element of $\delta H^1(T^3)$, where $\delta: H^1(T^3) \rightarrow H^2(X)$ is the Mayer–Vietoris coboundary.

We deduce the fiber sum formulae in [Theorem 1.1](#) and [Theorem 1.4](#) from the following somewhat more general results, which apply in particular to the situation in which M_1 , M_2 , and/or X have $b^+ = 1$. In each case, the perturbed invariants $\Phi_{M_i, \Sigma \times S^1, \eta}$ take values in $\mathcal{M}_{M_i, \Sigma \times S^1, \eta}$, which is isomorphic to the ring $\mathcal{L}(t)$ of Laurent series in the variable t corresponding to the Poincaré dual of the surface Σ . Each of the following is obtained by an application of [\(3\)](#), combined with knowledge of the relative invariants

of manifolds of the form $\Sigma \times D^2$. In particular, [Theorem 1.8](#) follows quickly from the fact that up to multiplication by $\pm t^n$,

$$\Psi_{T^2 \times D^2, \eta, \mathfrak{s}} = \frac{1}{t-1}$$

where \mathfrak{s} is the spin^c structure with trivial first Chern class and $\eta \in H^2(T^2 \times D^2; \mathbb{R})$ has $\int_{T^2} \eta > 0$ ([Proposition 10.3](#)). Note that this implies that the complement Z of a torus of square 0 in a closed 4–manifold M has relative invariant satisfying

$$\rho(\Psi_{Z, \eta, \mathfrak{s}}) = (t-1) \mathbb{Q}_{M, T^3, \eta, \mathfrak{s}},$$

where η is a class as above. To understand the homomorphism $\rho: \mathcal{K}(Z, \eta) \rightarrow \mathcal{L}(t)$, recall that $\mathcal{K}(Z, \eta)$ can be identified with a Laurent series ring in variables generating $K(Z) \cong H^1(\partial Z)/H^1(Z)$. Since η is defined on M (in particular η extends from $\partial Z = T^3$ to $T^2 \times D^2$), its restriction to T^3 is a multiple of the Poincaré dual of the meridian circle $\text{pt} \times D^2$. Thus we can choose generators for $K(Z)$ in such a way that $\eta \cup t$ is nonzero, while η pairs trivially with the other generators. Then there is a well-defined homomorphism $\rho: \mathcal{K}(Z, \eta) \rightarrow \mathcal{L}(t)$ that corresponds to setting the variables other than t equal to 1 (for more detail on homomorphisms between Novikov rings see [Section 8.1](#); this particular case is considered in [Section 10](#), specifically [Equation \(32\)](#)).

Theorem 1.8 *Let $X = M_1 \#_{T_1=T_2} M_2$ be the fiber sum of two 4–manifolds M_1, M_2 along tori T_1, T_2 of square 0. Assume that there exist classes $\eta_i \in H^2(M_i; \mathbb{R})$, $i = 1, 2$, such that the restrictions of η_i to $T_i \times S^1 \subset M_i$ correspond under the gluing diffeomorphism $f: T_1 \times S^1 \rightarrow T_2 \times S^1$, and assume that $\int_{T_i} \eta_i > 0$. Let $\eta \in H^2(X; \mathbb{R})$ be a class whose restrictions to $Z_i = M_i \setminus (T_i \times D^2)$ agree with those of η_i , and choose spin^c structures $\mathfrak{s}_i \in \text{Spin}^c(M_i)$, $\mathfrak{s} \in \text{Spin}^c(X)$ whose restrictions correspond similarly. Then for any $\alpha \in \mathbb{A}(X)$, the image of $\alpha_1 \otimes \alpha_2$ under the map $\mathbb{A}(Z_1) \otimes \mathbb{A}(Z_2) \rightarrow \mathbb{A}(X)$, we have*

$$\rho(\mathbb{Q}_{X, T \times S^1, \eta, \mathfrak{s}}(\alpha)) = (t^{1/2} - t^{-1/2})^2 \mathbb{Q}_{M_1, T_1 \times S^1, \eta_1, \mathfrak{s}_1}(\alpha_1) \cdot \mathbb{Q}_{M_2, T_2 \times S^1, \eta_2, \mathfrak{s}_2}(\alpha_2)$$

up to multiplication by $\pm t^n$.

In the higher-genus case we have the following.

Theorem 1.9 *Let $X = M_1 \#_{\Sigma_1=\Sigma_2} M_2$ be the fiber sum of two 4–manifolds M_1, M_2 along surfaces Σ_1, Σ_2 of genus $g > 1$ and square 0. Let η_1, η_2, η be 2–dimensional cohomology classes satisfying conditions analogous to those in the previous theorem, and choose spin^c structures $\mathfrak{s}_1, \mathfrak{s}_2$, and \mathfrak{s} restricting compatibly as before. If the*

Chern classes of each spin^c structure restrict to $\Sigma \times S^1$ as a class other than $2k \text{PD}[S^1]$ with $|k| \leq g - 1$ then the Ozsváth–Szabó invariants of all manifolds involved vanish. Otherwise, writing f for the gluing map $\Sigma_1 \times S^1 \rightarrow \Sigma_2 \times S^1$, we have

$$\begin{aligned} \rho(\Phi_{X, \Sigma \times S^1, \eta, \mathfrak{s}}(\alpha)) \\ = \sum_{\beta} \Phi_{M_1, \Sigma_1 \times S^1, \eta_1, \mathfrak{s}_1}(\alpha_1 \otimes \beta) \cdot \Phi_{M_2, \Sigma_2 \times S^1, \eta_2, \mathfrak{s}_2}(\alpha_2 \otimes f_*(\beta^\circ)) \cdot u_{\beta, k} \end{aligned}$$

up to multiplication by $\pm t^n$.

In this theorem, $\{\beta\}$ is a basis for $H_*(\text{Sym}^d \Sigma)$, $d = g - 1 - |k|$, as before, and $u_{\beta, k}$ is a polynomial in t with constant coefficient 1, which is equal to 1 except possibly if $k = 0$.

The maps $\rho: \mathcal{M}_{X, \Sigma \times S^1, \eta} \rightarrow \mathcal{L}(t)$ appearing in [Theorem 1.8](#) and [Theorem 1.9](#) are “change of coefficient” homomorphisms as before, defined in [\(32\)](#) below.

1.4 Examples

1.4.1 Elliptic surfaces For $n \geq 1$, let $E(n)$ denote the smooth 4–manifold underlying a simply connected minimal elliptic surface with no multiple fibers and holomorphic Euler characteristic n . In [\[13\]](#), Ozsváth and Szabó calculated that $\text{OS}_{E(2)} = 1$, meaning that $\Phi_{E(2), \mathfrak{s}}$ is trivial on all spin^c structures \mathfrak{s} with $c_1(\mathfrak{s}) \neq 0$, while if $c_1(\mathfrak{s}) = 0$ then $\Phi_{E(2), \mathfrak{s}} = 1$. We infer *a posteriori* that $E(2)$ has simple type.

In general, we have that $E(n)$ is diffeomorphic to the fiber sum of n copies of the rational elliptic surface $E(1) = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$, summed along copies of the torus fiber F of the elliptic fibration, using the fibration structure to identify neighborhoods of the fibers. From [Theorem 1.8](#) we infer that the perturbed Ozsváth–Szabó invariant of $E(1)$, calculated with respect to the splitting along the boundary of a neighborhood of F and using a spin^c structure whose Chern class restricts trivially to the complement of F , is given by the Laurent series $(t - 1)^{-1}$, up to multiplication by $\pm t^n$. For other spin^c structures the perturbed invariant vanishes.

It is straightforward to deduce from this and [Theorem 1.4](#) that for $n \geq 2$,

$$\text{OS}_{E(n)} = (T - T^{-1})^{n-2},$$

where T is the class Poincaré dual to a regular fiber. In fact, [Theorem 1.4](#) gives this after summing over rim tori using the homomorphism ρ on the left hand side. Arguments based on the adjunction inequality [\[15, Theorem 1.5\]](#), familiar from Seiberg–Witten theory [\[2\]](#), show that only multiples of T can contribute to $\text{OS}_{E(n)}$ and therefore

application of ρ is unnecessary. Likewise, the only ambiguity remaining in the formula above is an overall sign; the conjugation-invariance of $\Phi_{X,\mathfrak{s}}$ when $b^+(X) \geq 2$ due to Ozsváth and Szabó [15, Theorem 3.6] shows that $\text{OS}_{E(n)}$ must be a symmetric polynomial.

1.4.2 Higher-genus sums The elliptic surface $E(n)$ can be realized as the double branched cover of $S^2 \times S^2$, branched along a surface obtained by smoothing the union of 4 parallel copies of $S^2 \times \{\text{pt}\}$ and $2n$ copies of $\{\text{pt}\} \times S^2$. The projection $\pi_1: S^2 \times S^2 \rightarrow S^2$ to the first factor lifts to an elliptic fibration on $E(n)$, while projection π_2 on the second factor realizes $E(n)$ as a fibration with typical fiber a surface Σ of genus $n-1$, which can be perturbed to be a Lefschetz fibration if desired. Note that Σ intersects the fiber F of the elliptic fibration in two (positive) points. Let $X_n = E(n) \#_{\Sigma} E(n)$ denote the fiber sum of two copies of $E(n)$ along Σ , and suppose $n \geq 3$. We wish to use Theorem 1.1 to calculate the Ozsváth–Szabó invariants of X_n .

A useful observation is that $E(n)$ has simple type by the example above. Corollary 1.2 then shows that we can have a nontrivial contribution to $\rho(\text{OS}_{X_n})$ only when $|k| = g-1$, ie, from spin^c structures \mathfrak{s} with $|\langle c_1(\mathfrak{s}), [\Sigma] \rangle| = 2g-2 = 2n-4$. From the preceding example and the fact that $[\Sigma].[F] = 2$, the right-hand side of (1) in the case $|k| = g-1$ is equal to ± 1 , being the product of the invariants arising from $T^{\pm(n-2)}$. Since $T^{\pm(n-2)}$ is equal (up to sign) to the first Chern class $c_1(E(n))$, a convenient way to express these conclusions is that $\text{OS}_{X_n} = \pm K \pm K^{-1}$, where K is the canonical class on X_n . This formula is true after summing over rim tori.

Note that X_n is diffeomorphic to a minimal complex surface of general type, and therefore this calculation agrees with the corresponding one in Seiberg–Witten theory; see Witten [22].

1.5 Organization

The first goal of the paper is to set up enough machinery for the proof of Theorem 1.5. To this end, the next section recalls the definition of Heegaard Floer homology with twisted coefficients from [12] and the corresponding constructions associated to 4–dimensional cobordisms in [15]. Section 3 discusses a refinement of the relative grading on Heegaard Floer homology, available with twisted coefficients. Sections 4, 5 and 6 extend other algebraic features of Heegaard Floer homology to the twisted-coefficient setting, including the pairing mentioned in Theorem 1.5 and the action on Floer homology by $H_1(Y; \mathbb{Z})/\text{tors}$ which is useful in later calculations. With this machinery in place, Section 7 proves Theorem 1.5. Section 8 defines perturbed Heegaard Floer theory, and deals with the extension of many of the results in preceding

sections to that case; in particular [Theorem 1.6](#). After making the necessary Floer homology calculations in [Section 9](#), [Section 10](#) gives the proofs of [Theorem 1.8](#) and [Theorem 1.9](#), and thence [Theorem 1.1](#) and [Theorem 1.4](#). We conclude with some remarks on manifolds of simple type in [Section 11](#).

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2 Preliminaries on twisted coefficients

2.1 Definitions

We briefly recall the construction of the Heegaard Floer homology groups with “twisted” coefficients. For more details, the reader is referred to Ozsváth–Szabó [[12](#); [11](#)]. To a closed oriented 3–manifold Y we can associate a pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ where Σ is a surface of genus $g \geq 1$ and $\boldsymbol{\alpha} = \alpha_1, \dots, \alpha_g$ and $\boldsymbol{\beta} = \beta_1, \dots, \beta_g$ are sets of attaching circles for the two handlebodies in the Heegaard decomposition. We consider intersection points between the g –dimensional tori $T_{\boldsymbol{\alpha}} = \alpha_1 \times \dots \times \alpha_g$ and $T_{\boldsymbol{\beta}} = \beta_1 \times \dots \times \beta_g$ in the symmetric power $\text{Sym}^g(\Sigma)$, which we assume intersect transversely. Recall that the basepoint z , chosen away from the α_i and β_i , gives rise to a map $s_z: T_{\boldsymbol{\alpha}} \cap T_{\boldsymbol{\beta}} \rightarrow \text{Spin}^c(Y)$. Given a spin^c structure \mathfrak{s} on Y , and under suitable admissibility hypotheses on the Heegaard diagram, the generators for the Heegaard Floer chain complex $CF^\infty(Y, \mathfrak{s})$ are pairs $[\mathbf{x}, i]$ where $i \in \mathbb{Z}$ and $\mathbf{x} \in T_{\boldsymbol{\alpha}} \cap T_{\boldsymbol{\beta}}$ satisfies $s_z(\mathbf{x}) = \mathfrak{s}$.

The differential in CF^∞ counts certain maps $u: D^2 \rightarrow \text{Sym}^g(\Sigma)$ of the unit disk in \mathbb{C} that connect pairs of intersection points \mathbf{x} and \mathbf{y} . That is to say, we consider maps u satisfying the boundary conditions:

$$\begin{aligned} u(e^{i\theta}) \in T_{\boldsymbol{\alpha}} \text{ for } \cos \theta \geq 0 & & u(i) = \mathbf{y} \\ u(e^{i\theta}) \in T_{\boldsymbol{\beta}} \text{ for } \cos \theta \leq 0 & & u(-i) = \mathbf{x}. \end{aligned}$$

For $g > 2$ we let $\pi_2(\mathbf{x}, \mathbf{y})$ denote the set of homotopy classes of such maps; for $g = 2$ we let $\pi_2(\mathbf{x}, \mathbf{y})$ be the quotient of the set of such homotopy classes by a further equivalence, the details of which need not concern us (see Ozsváth–Szabó [[12](#)]).

There is a topological obstruction to the existence of any such disk connecting \mathbf{x} and \mathbf{y} , denoted $\epsilon(\mathbf{x}, \mathbf{y}) \in H_1(Y; \mathbb{Z})$. To any homotopy class $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ we can associate the quantity $n_z(\phi)$, being the algebraic intersection number between ϕ and the subvariety $\{z\} \times \text{Sym}^{g-1}(\Sigma)$. The following is a basic fact in Heegaard Floer theory:

Proposition 2.1 [12, Proposition 2.15] Suppose $g > 1$ and let $\mathbf{x}, \mathbf{y} \in T_\alpha \cap T_\beta$. If $\epsilon(\mathbf{x}, \mathbf{y}) \neq 0$ then $\pi_2(\mathbf{x}, \mathbf{y})$ is empty, while if $\epsilon(\mathbf{x}, \mathbf{y}) = 0$ then there is an affine isomorphism

$$\pi_2(\mathbf{x}, \mathbf{y}) = \mathbb{Z} \oplus H^1(Y; \mathbb{Z}),$$

such that the projection $\pi_2(\mathbf{x}, \mathbf{y}) \rightarrow \mathbb{Z}$ is given by the map n_z .

We remark that if $\mathbf{x} = \mathbf{y}$, then the isomorphism in the above proposition is natural (not merely affine).

There is a natural “splicing” of homotopy classes

$$\pi_2(\mathbf{x}, \mathbf{y}) \times \pi_2(\mathbf{y}, \mathbf{z}) \rightarrow \pi_2(\mathbf{x}, \mathbf{z}),$$

as well as an action

$$\pi_2'(\text{Sym}^g(\Sigma_g)) \times \pi_2(\mathbf{x}, \mathbf{y}) \rightarrow \pi_2(\mathbf{x}, \mathbf{y}),$$

where π_2' denotes the second homotopy group divided by the action of the fundamental group. (For $g > 1$, $\pi_2'(\text{Sym}^g(\Sigma_g)) \cong \mathbb{Z}$, generated by a class S with $n_z(S) = 1$. When $g > 2$, $\pi_2'(\text{Sym}^g(\Sigma_g)) = \pi_2(\text{Sym}^g(\Sigma_g))$.) The isomorphism in the above proposition is affine in the sense that it respects the splicing action by $\pi_2(\mathbf{x}, \mathbf{x})$, under the natural identification $\pi_2(\mathbf{x}, \mathbf{x}) = \mathbb{Z} \oplus H^1(Y)$.

The ordinary “untwisted” version of Heegaard Floer homology takes CF^∞ to be generated (over \mathbb{Z}) by pairs $[\mathbf{x}, i]$ as above, equipped with a boundary map such that the coefficient of $[\mathbf{y}, j]$ in the boundary of $[\mathbf{x}, i]$ is the number of pseudo-holomorphic maps in all homotopy classes $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ having moduli spaces of formal dimension 1 and $n_z(\phi) = i - j$. The twisted version is similar, but where one keeps track of all possible homotopy data associated to ϕ . In light of the above proposition, this means that we should form a chain complex freely generated by intersection points \mathbf{x} as a module over the group ring of $\mathbb{Z} \oplus H^1(Y)$, or equivalently by pairs $[\mathbf{x}, i]$ over the group ring of $H^1(Y)$. Following [11, Section 8.1], we define:

Definition 2.2 An *additive assignment* for the diagram $(\Sigma, \alpha, \beta, z)$ is a collection of functions

$$A_{\mathbf{x}, \mathbf{y}}: \pi_2(\mathbf{x}, \mathbf{y}) \rightarrow H^1(Y; \mathbb{Z})$$

that satisfies the following:

- (1) $A_{\mathbf{x}, \mathbf{z}}(\phi * \psi) = A_{\mathbf{x}, \mathbf{y}}(\phi) + A_{\mathbf{y}, \mathbf{z}}(\psi)$ whenever $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ and $\psi \in \pi_2(\mathbf{y}, \mathbf{z})$.
- (2) $A_{\mathbf{x}, \mathbf{y}}(S * \phi) = A_{\mathbf{x}, \mathbf{y}}(\phi)$ for $S \in \pi_2'(\text{Sym}^g(\Sigma_g))$.

We will drop the subscripts from $A_{\mathbf{x},\mathbf{y}}$ whenever possible. It is shown in [11] how a certain finite set of choices (a “complete set of paths”) gives rise to an additive assignment in the above sense. We can also assume that $A_{\mathbf{x},\mathbf{x}}: \pi_2(\mathbf{x}, \mathbf{x}) \cong \mathbb{Z} \oplus H^1(Y) \rightarrow H^1(Y)$ is the natural projection on the second factor.

Definition 2.3 Let $(\Sigma, \alpha, \beta, z)$ be a pointed Heegaard diagram for Y and $\mathfrak{s} \in \text{Spin}^c(Y)$. Fix an additive assignment A for the diagram. The twisted Heegaard Floer chain complex $CF^\infty(Y, \mathfrak{s}; \mathbb{Z}[H^1(Y)])$ is the module freely generated over $\mathbb{Z}[H^1(Y)]$ by pairs $[\mathbf{x}, i]$, with differential ∂^∞ given by

$$\partial^\infty[\mathbf{x}, i] = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, y) \\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}(\phi) \cdot e^{A(\phi)}[\mathbf{y}, i - n_z(\phi)],$$

where the symbol $e^{A(\phi)}$ indicates the variable in $\mathbb{Z}[H^1(Y)]$ corresponding to $A(\phi)$.

Here $\mathcal{M}(\phi)$ denotes the space of holomorphic disks in the homotopy class ϕ , where “holomorphic” is defined relative to an appropriately generic path of almost-complex structure on $\text{Sym}^g(\Sigma_g)$. For such a path, $\mathcal{M}(\phi)$ is a smooth manifold of dimension given by a Maslov index $\mu(\phi)$. There is an action of \mathbb{R} on $\mathcal{M}(\phi)$ by reparametrization of the disk, and $\widehat{\mathcal{M}}(\phi)$ denotes the quotient of $\mathcal{M}(\phi)$ by this action. When $\mu(\phi) = 1$, $\widehat{\mathcal{M}}(\phi)$ is a compact, zero-dimensional manifold. An appropriate choice of “coherent orientation system” serves to orient the points of $\widehat{\mathcal{M}}(\phi)$ in this case, and $\#\widehat{\mathcal{M}}(\phi)$ denotes the signed count of these points. It is shown in [12; 11] that under appropriate admissibility hypotheses on the diagram $(\Sigma, \alpha, \beta, z)$ the chain homotopy type of $CF^\infty(Y, \mathfrak{s}; \mathbb{Z}[H^1(Y)])$ is an invariant of (Y, \mathfrak{s}) .

As in the introduction, in much of what follows we will write R_Y for the ring $\mathbb{Z}[H^1(Y)]$, or simply R when the underlying 3-manifold is apparent from context. Note that by choosing a basis for $H^1(Y)$ we can identify R with the ring of Laurent polynomials in $b_1(Y)$ variables.

By following the usual constructions of Heegaard Floer homology, we obtain other variants of the above with coefficients in R_Y : namely by considering only generators $[\mathbf{x}, i]$ with $i < 0$ we obtain a subcomplex $CF^-(Y, \mathfrak{s}; R)$ whose quotient complex is $CF^+(Y, \mathfrak{s}; R)$, with associated homology groups HF^- and HF^+ respectively. There is an action $U: [\mathbf{x}, i] \mapsto [\mathbf{x}, i - 1]$ on CF^∞ as usual; the kernel of the induced action on CF^+ is written \widehat{CF} with homology $\widehat{HF}(Y, \mathfrak{s}; R)$. There is a relative grading on the Floer complex with respect to which U decreases degree by 2; we will discuss gradings further in Section 3.

Given any module M for R_Y we can form Heegaard Floer homology with coefficients in M by taking the homology of the complex $CF \otimes_R M$. In particular if $M = \mathbb{Z}$, equipped with the action of R_Y by which every element of $H^1(Y)$ acts as the identity, we recover the ordinary untwisted theory.

For use in later sections, we introduce the following notion of *conjugation* of R_Y –modules. First, observe that the automorphism $x \mapsto -x$ of $H^1(Y)$ induces an automorphism $R_Y \rightarrow R_Y$ that we refer to as conjugation, and write as $r \mapsto \bar{r}$ for $r \in R_Y$. Now if M is a module for R_Y , we let \bar{M} denote the additive group M equipped with the “conjugate” module structure in which module multiplication is given by

$$r \otimes m \mapsto \bar{r} \cdot m$$

for $r \in R_Y$ and $m \in \bar{M}$.

2.2 Twisted cobordism invariants

We now sketch the construction and main properties of twisted-coefficient Heegaard Floer invariants associated to cobordisms, which can be found in greater detail in [15]. Recall that if $W: Y_1 \rightarrow Y_2$ is an oriented 4–dimensional cobordism and M is a module for $R_1 := R_{Y_1} = \mathbb{Z}[H^1(Y_1)]$, then there is an induced module $M(W)$ for $R_2 = R_{Y_2}$ defined as follows. Let

$$K(W) = \ker(H^2(W, \partial W) \rightarrow H^2(W))$$

be the kernel of the map in the long exact sequence for the pair $(W, \partial W)$: then $\mathbb{Z}[K(W)]$ is a module for R_1 and R_2 via the coboundary maps $H^1(Y_i) \rightarrow K(W) \subset H^2(W, \partial W)$. Define

$$M(W) = \bar{M} \otimes_{R_1} \mathbb{Z}[K(W)].$$

Then $M(W)$ is a module for R_2 in the obvious way. The reason for the appearance of the conjugate module \bar{M} above has to do with the fact that the orientation of W induces the opposite orientation on Y_1 from the given one, and will be explained more fully in the next section.

Ozsváth and Szabó show in [15] how to associate to a cobordism W as above with spin^c structure \mathfrak{s} a homomorphism

$$F_{W, \mathfrak{s}}^\circ: HF^\circ(Y_1, \mathfrak{s}_1; M) \rightarrow HF^\circ(Y_2, \mathfrak{s}_2; M(W))$$

(where \mathfrak{s}_i denotes the restriction of \mathfrak{s} to Y_i , and \circ indicates a map between each of the varieties of Heegaard Floer homology, respecting the long exact sequences relating

them). This is defined as a composition

$$F_W^\circ = E^\circ \circ H^\circ \circ G^\circ,$$

where G° is associated to the 1–handles in W , H° to the 2–handles, and E° to the 3–handles. Note that the coefficient module remains unchanged by cobordisms consisting of 1– or 3–handle additions. Indeed, such cobordisms induce homomorphisms in an essentially formal way, so we simply refer the reader to [15, Section 4.3] for the definition of E° and G° .

Suppose that W is a cobordism consisting of 2–handle additions, so that we can think of W as associated to surgery on a framed link $L \subset Y_1$. In this situation, Ozsváth and Szabó construct a “Heegaard triple” $(\Sigma, \alpha, \beta, \gamma, z)$ associated to W . This diagram describes three 3–manifolds $Y_{\alpha\beta}$, $Y_{\beta\gamma}$ and $Y_{\alpha\gamma}$ obtained by using the indicated circles on Σ as attaching circles, such that

$$Y_{\alpha\beta} = Y_1, \quad Y_{\beta\gamma} = \#^k S^1 \times S^2, \quad Y_{\alpha\gamma} = Y_2,$$

where k is the genus of Σ minus the number of components of L . In fact the diagram $(\Sigma, \alpha, \beta, \gamma, z)$ describes a 4–manifold $X_{\alpha\beta\gamma}$ in a natural way, whose boundaries are the three manifolds above. Furthermore, in the current situation, $X_{\alpha\beta\gamma}$ is obtained from W by removing the regular neighborhood of a 1–complex (see [15, Proposition 4.3]).

We can arrange that the top-dimensional generator of

$$HF^{\leq 0}(Y_{\beta\gamma}, \mathfrak{s}_0; \mathbb{Z}) \cong \Lambda^* H^1(Y_{\beta\gamma}; \mathbb{Z}) \otimes \mathbb{Z}[U]$$

is represented by an intersection point $\Theta \in T_\beta \cap T_\gamma$ (here \mathfrak{s}_0 denotes the spin^c structure on $\#^k S^1 \times S^2$ having $c_1(\mathfrak{s}_0) = 0$). The map F° is defined by counting holomorphic triangles, with the aid of another additive assignment. To describe this, suppose $\mathbf{x} \in T_\alpha \cap T_\beta$, $\mathbf{y} \in T_\beta \cap T_\gamma$, and $\mathbf{w} \in T_\alpha \cap T_\gamma$ are intersection points arising from a Heegaard triple $(\Sigma, \alpha, \beta, \gamma, z)$. Let Δ denote a standard 2–simplex, and write $\pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$ for the set of homotopy classes of maps $u: \Delta \rightarrow \text{Sym}^g(\Sigma)$ that send the boundary arcs of Δ into T_α , T_β , and T_γ respectively, under a clockwise ordering of the boundary arcs e_α , e_β , and e_γ of Δ , and such that

$$u(e_\alpha \cap e_\beta) = \mathbf{x}, \quad u(e_\beta \cap e_\gamma) = \mathbf{y}, \quad u(e_\alpha \cap e_\gamma) = \mathbf{w}.$$

Again there is a topological obstruction $\epsilon(\mathbf{x}, \mathbf{y}, \mathbf{w}) \in H_1(X_{\alpha\beta\gamma}; \mathbb{Z})$ that vanishes if and only if $\pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$ is nonempty. The analogue of Proposition 2.1 in this context is the following.

Proposition 2.4 [12, Proposition 8.3] *Let $(\Sigma, \alpha, \beta, \gamma, z)$ be a pointed Heegaard triple as above, and $X_{\alpha\beta\gamma}$ the associated 4–manifold. Then whenever $\epsilon(\mathbf{x}, \mathbf{y}, \mathbf{w}) = 0$ we have an (affine) isomorphism*

$$\pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w}) \cong \mathbb{Z} \oplus H_2(X_{\alpha\beta\gamma}; \mathbb{Z})$$

where the projection to \mathbb{Z} is given by $\psi \mapsto n_z(\psi)$.

There is an obvious “splicing” action on homotopy classes of triangles by disks on each corner; the above identification respects this action.

Recall from [12, Section 8.1] that the basepoint z gives rise to a map

$$s_z: \coprod_{\mathbf{x}, \mathbf{y}, \mathbf{w}} \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w}) \rightarrow \text{Spin}^c(X_{\alpha\beta\gamma}),$$

such that triangles $\psi \in \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$ and $\psi' \in \pi_2(\mathbf{x}', \mathbf{y}', \mathbf{w}')$ have $s_z(\psi) = s_z(\psi')$ if and only if there exist disks $\phi_x \in \pi_2(\mathbf{x}, \mathbf{x}')$, $\phi_y \in \pi_2(\mathbf{y}, \mathbf{y}')$ and $\phi_w \in \pi_2(\mathbf{w}, \mathbf{w}')$ with $\psi' = \psi + \phi_x + \phi_y + \phi_w$. In this case ψ and ψ' are said to be *spin^c equivalent*. Note that in case $(\Sigma, \alpha, \beta, \gamma, z)$ describes a 2–handle cobordism W as previously, we can think of s_z as a function

$$s_z: \coprod_{\mathbf{x}, \mathbf{w}} \pi_2(\mathbf{x}, \Theta, \mathbf{w}) \rightarrow \text{Spin}^c(W).$$

Definition 2.5 *An additive assignment for a Heegaard triple $(\Sigma, \alpha, \beta, \gamma, z)$ describing a 2–handle cobordism $W: Y_1 \rightarrow Y_2$ as above is a function*

$$A_W: \coprod_{\mathfrak{s} \in \text{Spin}^c(W)} s_z^{-1}(\mathfrak{s}) \rightarrow K(W)$$

obtained in the following manner. For a fixed $\psi_0 \in s_z^{-1}(\mathfrak{s})$, let $\psi = \psi_0 + \phi_{\alpha\beta} + \phi_{\beta\gamma} + \phi_{\alpha\gamma}$ be an arbitrary element of $s_z^{-1}(\mathfrak{s})$. Then set

$$A_W(\psi) = \delta(-A_1(\phi_{\alpha\beta}) + A_2(\phi_{\alpha\gamma}))$$

where A_i are additive assignments for Y_i and $\delta: H^1(\partial W) \rightarrow H^2(W, \partial W)$ is the coboundary from the long exact sequence of $(W, \partial W)$.

We are now in a position to define the map on Floer homology induced by W (given additive assignments on Y_1 , Y_2 , and W). We again refer to [15, Section 4] for the details required to make full sense of the following.

Definition 2.6 For a triple $(\Sigma, \alpha, \beta, \gamma, z)$ describing a 2–handle cobordism W with spin^c structure \mathfrak{s} , we define

$$F_{W, \mathfrak{s}}^\circ: HF^\circ(Y_1, \mathfrak{s}_1; M) \rightarrow HF^\circ(Y_2, \mathfrak{s}_2; M(W)),$$

where $\mathfrak{s}_i = \mathfrak{s}|_{Y_i}$, to be the map induced on homology by the chain map

$$[\mathbf{x}, i] \mapsto \sum_{\mathbf{w} \in T_\alpha \cap T_\gamma} \sum_{\substack{\psi \in \pi_2(\mathbf{x}, \Theta, \mathbf{w}) \\ \mu(\psi) = 0}} \# \mathcal{M}(\psi) \cdot [\mathbf{w}, i - n_z(\psi)] \otimes e^{A_W(\psi)}.$$

Here $\mu(\psi)$ denotes the expected dimension of the moduli space $\mathcal{M}(\psi)$ of pseudo-holomorphic triangles in the homotopy class ψ , and $\# \mathcal{M}(\psi)$ indicates the signed count of points in a compact oriented 0–dimensional manifold.

We should note that while the Floer homology $HF^\circ(Y, \mathfrak{s}; M)$ does not depend on the additive assignment A_Y , the map $F_{W, \mathfrak{s}}$ does depend on the choice of A_W as in [Definition 2.5](#) through the reference triangle ψ_0 . Changing this choice has the effect of precomposing (postcomposing) F_W with the action of an element of $H^1(Y_1)$ (resp $H^1(Y_2)$), which in turn act in $M(W)$ via the coboundary. Likewise the generator Θ is determined only up to sign, so that F_W has a sign indeterminacy as well. Following [\[15\]](#), we let $[F_{W, \mathfrak{s}}^\circ]$ denote the orbit of $F_{W, \mathfrak{s}}^\circ$ under the action of $H^1(Y_1) \oplus H^1(Y_2)$.

With the conventions employed here $F_{W, \mathfrak{s}}^\circ$ is “antilinear” with respect to the action of R_{Y_1} , meaning that $F_{W, \mathfrak{s}}^\circ(r \xi) = \bar{r} F_{W, \mathfrak{s}}^\circ(\xi)$ for $r \in R_{Y_1}$.

2.3 Composition law

An advantage to using twisted coefficient modules for Heegaard Floer homology is the availability of a refined composition law in this situation. To describe this, we must first understand the behavior of the coefficient modules themselves under composition of cobordisms. The following lemma will be useful in formulating results to come; as usual, ordinary (co)homology is taken with integer coefficients.

Lemma 2.7 *Let $W = W_1 \cup_{Y_1} W_2$ be the composition of two cobordisms $W_1: Y_0 \rightarrow Y_1$ and $W_2: Y_1 \rightarrow Y_2$. Define*

$$K(W, Y_1) = \ker[\rho_1 \oplus \rho_2: H^2(W, \partial W) \rightarrow H^2(W_1) \oplus H^2(W_2)],$$

where ρ_i denotes the restriction map $H^2(W, \partial W) \rightarrow H^2(W_i)$. Then

$$\mathbb{Z}[K(W_1)] \otimes_{\mathbb{Z}[H^1(Y_1)]} \mathbb{Z}[K(W_2)] \cong \mathbb{Z}[K(W, Y_1)]$$

as modules over $\mathbb{Z}[H^1(Y_0)]$ and $\mathbb{Z}[H^1(Y_2)]$.

Proof We have

$$\mathbb{Z}[K(W_1)] \otimes_{\mathbb{Z}[H^1(Y_1)]} \mathbb{Z}[K(W_2)] \cong \mathbb{Z} \left[\frac{K(W_1) \oplus K(W_2)}{H^1(Y_1)} \right],$$

so the claim amounts to exhibiting an isomorphism

$$\frac{K(W_1) \oplus K(W_2)}{H^1(Y_1)} \cong K(W, Y_1).$$

To see this, consider the diagram

$$\begin{array}{ccc} H^1(Y_1) & \longrightarrow & H^2(W_1, \partial W_1) \oplus H^2(W_2, \partial W_2) \xrightarrow{f} H^2(W, \partial W) \\ & & \downarrow \rho_1 \oplus \rho_2 \\ & & H^2(W_1) \oplus H^2(W_2), \end{array}$$

where the horizontal row is (the Poincaré dual of) the Mayer–Vietoris sequence. Write

$$i_*: H^2(W_1, \partial W_1) \rightarrow H^2(W, \partial W) \quad \text{and} \quad j_*: H^2(W_2, \partial W_2) \rightarrow H^2(W, \partial W)$$

for the components of f ; then it is not hard to see that

$$\rho_1 \circ i_*: H^2(W_1, \partial W_1) \rightarrow H^2(W_1) \quad \text{and} \quad \rho_2 \circ j_*: H^2(W_2, \partial W_2) \rightarrow H^2(W_2)$$

agree with the maps induced by inclusion, while

$$\rho_2 \circ i_* = 0 \quad \text{and} \quad \rho_1 \circ j_* = 0.$$

From this it is easy to deduce that $f^{-1}(K(W, Y_1)) = K(W_1) \oplus K(W_2)$, from which the lemma follows. \square

Remark 2.8 If W is a cobordism between homology spheres, or more generally if $H^2(W, \partial W) \rightarrow H^2(W)$ is an isomorphism, then there is an identification

$$K(W, Y_1) = \ker[H^2(W) \rightarrow H^2(W_1) \oplus H^2(W_2)],$$

the kernel of the restriction map in the ordinary Mayer–Vietoris sequence in cohomology. In this case if \mathfrak{s}_1 and \mathfrak{s}_2 are spin^c structures on W_1 and W_2 , then $K(W, Y_1)$ parametrizes spin^c structures \mathfrak{s} on W such that $\mathfrak{s}|_{W_i} = \mathfrak{s}_i$ (when that set is nonempty). In the case of a closed 4–manifold X , the module $M_{X,Y}$ of the introduction is simply $\mathbb{Z}[K(W, Y)]$ where W is obtained from X by removing a 4–ball on each side of Y .

When regarding W as a single cobordism the group relevant to twisted coefficient modules is $K(W)$, while if $W = W_1 \cup W_2$ is viewed as a composite the coefficient

modules change by tensor product with the group ring of $K(W, Y_1)$ (in light of the lemma above). By commutativity of the diagram

$$\begin{array}{ccc} H^2(W, \partial W) & \longrightarrow & H^2(W) \\ & \searrow \rho_1 \oplus \rho_2 & \downarrow \\ & & H^2(W_1) \oplus H^2(W_2), \end{array}$$

there is a natural inclusion $\iota: K(W) \rightarrow K(W, Y_1)$. This gives rise to a projection map

$$\Pi: \mathbb{Z}[K(W, Y_1)] \rightarrow \mathbb{Z}[K(W)],$$

namely (cf [15, Section 3.1])

$$\Pi(e^w) = \begin{cases} e^w & \text{if } w = \iota(v) \text{ for some } v \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently, if we think of a group ring $\mathbb{Z}[G]$ as a set of functions $G \rightarrow \mathbb{Z}$, then Π corresponds to the restriction of functions on $K(W, Y_1)$ to $K(W)$.

Thus, if M is a module for $\mathbb{Z}[H^1(Y_0)]$ we obtain a map

$$\Pi_M: M(W_1)(W_2) \rightarrow M(W)$$

by tensor product of the identity with Π under the identifications

$$M(W_1)(W_2) = \bar{M} \otimes_{\mathbb{Z}[H^1(Y_0)]} \mathbb{Z}[K(W, Y_1)]$$

and

$$M(W) = \bar{M} \otimes_{\mathbb{Z}[H^1(Y_0)]} \mathbb{Z}[K(W)].$$

The refined composition law for twisted coefficients can be stated as follows.

Theorem 2.9 [15, Theorem 3.9] *Let $W = W_1 \cup_{Y_1} W_2$ be a composite cobordism as above with spin^c structure \mathfrak{s} . Write $\mathfrak{s}_i = \mathfrak{s}|_{W_i}$. Then there are choices of representatives for the various maps involved such that*

$$[F_{W, \mathfrak{s}}^\circ] = [\Pi_M \circ F_{W_2, \mathfrak{s}_2}^\circ \circ F_{W_1, \mathfrak{s}_1}^\circ].$$

More generally, if $h \in H^1(Y_1)$ then for these choices we have

$$[F_{W, \mathfrak{s} - \delta h}^\circ] = [\Pi_M \circ F_{W_2, \mathfrak{s}_2}^\circ \circ e^h \cdot F_{W_1, \mathfrak{s}_1}^\circ],$$

where δh is the image of h under the Mayer–Vietoris coboundary $H^1(Y_1) \rightarrow H^2(W)$.

We should also remark that for a cobordism $W: Y_1 \rightarrow Y_2$ with spin^c structure \mathfrak{s} the map

$$F_{W,\mathfrak{s}}^\circ: HF^\circ(Y_1, \mathfrak{s}_1; \mathbb{Z}) \rightarrow HF^\circ(Y_2, \mathfrak{s}_2; \mathbb{Z})$$

in untwisted Floer homology can be obtained from the twisted-coefficient map

$$HF^\circ(Y_1, \mathfrak{s}_1; \mathbb{Z}) \rightarrow HF^\circ(Y_2, \mathfrak{s}_2; \mathbb{Z}(W))$$

(here $\mathbb{Z}(W)$ is the module $M(W)$ with $M = \mathbb{Z}$, namely $\mathbb{Z}(W) = \mathbb{Z} \otimes_{\mathbb{Z}[H^1(Y_1)]} \mathbb{Z}[K(W)] = \mathbb{Z}[\ker(H^2(W, Y_2) \rightarrow H^2(W))]$) by composition with the map ϵ_* induced in homology by the homomorphism

$$\epsilon: \mathbb{Z}(W) \rightarrow \mathbb{Z}$$

of coefficient modules that sends each element of $\ker(H^2(W, Y_2) \rightarrow H^2(W))$ to 1.

3 Refined relative gradings

The \mathbb{Z} -coefficient version of Heegaard Floer homology is naturally a relatively cyclically graded theory, in general. This means that if $\mathcal{S} = \{[\mathbf{x}, i] \mid s_z(\mathbf{x}) = \mathfrak{s}\}$ denotes the natural generating set for $CF^\infty(Y, \mathfrak{s}; \mathbb{Z})$ then there is a map

$$\text{gr}: \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{Z}/\mathfrak{d}(\mathfrak{s})\mathbb{Z},$$

where $\mathfrak{d}(\mathfrak{s}) = \text{gcd}\{\langle c_1(\mathfrak{s}), h \rangle \mid h \in H_2(Y; \mathbb{Z})\}$

is the *divisibility* of $c_1(\mathfrak{s})$ (or by abuse of language, of \mathfrak{s} itself). The differential in CF^∞ has degree -1 with respect to this grading, while the endomorphism U has degree -2 .

In the case of fully twisted coefficients (coefficients in $\mathbb{Z}[H^1(Y)]$), Ozsváth and Szabó [11, Section 8] observe that there is a lift of this cyclic grading to a relative \mathbb{Z} -grading. Here we provide an extension of this construction to Floer homology with coefficients in an arbitrary (graded) module M , in which elements of $H^1(Y) \subset R_Y$ are explicitly assigned nontrivial degrees depending on their Chern numbers. That the action of such elements on fully-twisted Floer homology shifts degree by their Chern numbers is implicit in the definition given in [11].

Definition 3.1 Fix a closed, oriented, spin^c 3–manifold (Y, \mathfrak{s}) . Define the \mathfrak{s} -grading of $\mathbb{Z}[H^1(Y)]$ by

$$(5) \quad \text{gr}_{\mathfrak{s}}(x) = -\langle c_1(\mathfrak{s}) \cup x, [Y] \rangle \quad \text{for } x \in H^1(Y).$$

The \mathfrak{s} -grading makes $\mathbb{Z}[H^1(Y)]$ into a graded ring, isomorphic to a multivariable Laurent polynomial ring in which the variables have degrees determined by their negative Chern numbers (5). When thinking of $\mathbb{Z}[H^1(Y)]$ as a graded ring, we write it as $R_{Y,\mathfrak{s}}$ or just R_Y . It is important to recognize that this grading depends on both the spin^c structure \mathfrak{s} and the orientation of Y , though we usually do not include \mathfrak{s} in the notation. In particular, if $-Y$ denotes the 3-manifold Y with its orientation reversed, then although $R_{Y,\mathfrak{s}} = R_{-Y,\mathfrak{s}}$ as sets, the gradings have opposite sign. On the other hand, the conjugation homomorphism $c: r \mapsto \bar{r}$ induces an isomorphism of graded rings $c: R_{Y,\mathfrak{s}} \rightarrow R_{-Y,\mathfrak{s}}$.

Definition 3.2 Let $(\Sigma, \alpha, \beta, z)$ be a marked Heegaard triple describing the 3-manifold Y . Fix a spin^c structure \mathfrak{s} for Y and an additive assignment $\{A_{\mathbf{x},\mathbf{y}}\}$ for the diagram. The *relative \mathbb{Z} grading* between generators $[\mathbf{x}, i]$ and $[\mathbf{y}, j]$ for $CF^\circ(Y, \mathfrak{s}; R_Y)$ is defined by

$$(6) \quad \underline{\text{gr}}([\mathbf{x}, i], [\mathbf{y}, j]) = \mu(\phi) + 2(i - j) - 2n_z(\phi) - \langle c_1(\mathfrak{s}) \cup A_{\mathbf{x},\mathbf{y}}(\phi), [Y] \rangle,$$

where ϕ is any element of $\pi_2(\mathbf{x}, \mathbf{y})$. More generally, if $r_1, r_2 \in R_Y$ are homogeneous elements, then we set

$$\underline{\text{gr}}(r_1 \cdot [\mathbf{x}, i], r_2 \cdot [\mathbf{y}, j]) = \underline{\text{gr}}([\mathbf{x}, i], [\mathbf{y}, j]) + \text{gr}_{\mathfrak{s}}(r_1) - \text{gr}_{\mathfrak{s}}(r_2).$$

It is not hard to check that the expression (6) is independent of the choice of $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, and that the differential in $CF^\infty(Y, \mathfrak{s}; R_Y)$ has relative degree -1 with respect to the above grading.

Now suppose M is a module for R_Y , which is equipped with a grading gr_M satisfying

$$(7) \quad \text{gr}_M(r \cdot m) = \text{gr}_{\mathfrak{s}}(r) + \text{gr}_M(m).$$

(Here we suppose R_Y is equipped with the \mathfrak{s} -grading induced by some $\mathfrak{s} \in \text{Spin}^c(Y)$.) Then the twisted Floer complex $CF(Y, \mathfrak{s}; M) = CF(Y, \mathfrak{s}; R_Y) \otimes_{R_Y} M$ naturally carries a relative \mathbb{Z} grading given by

$$\underline{\text{gr}}(m_1 [\mathbf{x}, i], m_2 [\mathbf{y}, j]) = \underline{\text{gr}}([\mathbf{x}, i], [\mathbf{y}, j]) + \text{gr}_M(m_1) - \text{gr}_M(m_2),$$

inducing a relative \mathbb{Z} grading on the Floer homology with coefficients in M .

More generally, if (7) holds modulo some integer $d \in \mathbb{Z}$, we obtain a relative $\mathbb{Z}/d\mathbb{Z}$ grading on $CF(Y, \mathfrak{s}; M)$. For example, taking $M = \mathbb{Z}$ to be the trivial R_Y -module supported in grading 0, we have for $n \in \mathbb{Z}$ and $r \in H^1(Y)$,

$$\text{gr}_M(r \cdot n) = \text{gr}_M(n) = 0 \quad \text{while} \quad \text{gr}_{\mathfrak{s}}(r) + \text{gr}_M(n) = -\langle c_1(\mathfrak{s}) \cup r, [Y] \rangle.$$

Thus (7) holds modulo the divisibility $\mathfrak{d}(\mathfrak{s})$ of \mathfrak{s} ; in particular, the “untwisted” Floer complex $CF(Y, \mathfrak{s}; \mathbb{Z})$ carries a relative cyclic grading by $\mathbb{Z}/\mathfrak{d}(\mathfrak{s})\mathbb{Z}$.

In light of these observations, henceforth we will assume that all modules for R_Y are graded, and often omit this assumption from statements. In particular, we will suppose that (7) holds at least modulo some $d \in \mathbb{Z}$.

That the homology of $CF^\circ(Y, \mathfrak{s}; M)$ is an invariant of (Y, \mathfrak{s}) follows by verifying that the arguments in [12; 11] respect the grading described here, together with the following.

Lemma 3.3 *The chain complex $CF^\infty(Y, \mathfrak{s}; M)$, equipped with the relative grading defined above, is independent of the choice of additive assignment up to graded chain isomorphism.*

Proof Suppose A_1 and A_2 are two additive assignments satisfying the criteria at the beginning of the previous section, and let $\{\theta_{\mathbf{x}} \in \pi_2(\mathbf{x}_0, \mathbf{x})\}$ be a complete set of paths for the spin^c structure \mathfrak{s} . That is, \mathbf{x}_0 is a fixed intersection point with $s_z(\mathbf{x}_0) = \mathfrak{s}$ and $\theta_{\mathbf{x}}$ is some choice of homotopy class for each $\mathbf{x} \in s_z^{-1}(\mathfrak{s})$. Define a homomorphism $F: CF^\infty(Y, \mathfrak{s}, A_1) \rightarrow CF^\infty(Y, \mathfrak{s}, A_2)$ between the chain complexes constructed with the two choices of assignment, by the formula

$$F([\mathbf{x}, i]) = e^{A_2(\theta_{\mathbf{x}}) - A_1(\theta_{\mathbf{x}})}[\mathbf{x}, i].$$

Then
$$F(\partial^\infty[\mathbf{x}, i]) = \sum_{\mathbf{y}, \phi} \#\widehat{\mathcal{M}}(\phi)[\mathbf{y}, i - n_z(\phi)] e^{A_1(\phi)} e^{A_2(\theta_{\mathbf{y}}) - A_1(\theta_{\mathbf{y}})},$$

while
$$\partial^\infty F([\mathbf{x}, i]) = \sum_{\mathbf{y}, \phi} \#\widehat{\mathcal{M}}(\phi)[\mathbf{y}, i - n_z(\phi)] e^{A_2(\theta_{\mathbf{x}}) - A_1(\theta_{\mathbf{x}})} e^{A_2(\phi)}.$$

Now, for $i = 1, 2$ and given $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, we have $\theta_{\mathbf{x}} * \phi = \theta_{\mathbf{y}} * \mathcal{P}_\phi$ for a periodic domain $\mathcal{P}_\phi \in \pi_2(\mathbf{y}, \mathbf{y})$ (up to addition of a multiple of the sphere class $[S]$, which does not affect the value of the additive assignment). Therefore by additivity

$$(8) \quad A_i(\theta_{\mathbf{x}}) + A_i(\phi) = A_i(\theta_{\mathbf{y}}) + H(\mathcal{P}_\phi)$$

where $H(\mathcal{P}_\phi) \in H^1(Y; \mathbb{Z})$ is the cohomology class corresponding to \mathcal{P}_ϕ . It follows that the group ring elements appearing in the previous expressions are equal, so that F is a chain map. Since F is clearly an isomorphism of R_Y -modules, we get that $CF^\infty(Y, \mathfrak{s}, A_1)$ and $CF^\infty(Y, \mathfrak{s}, A_2)$ are isomorphic as ungraded R_Y -chain complexes.

To see invariance of the relative gradings, we calculate that

$$\begin{aligned} \underline{\text{gr}}_{A_2}(F([\mathbf{x}, i]), F([\mathbf{y}, j])) &= \underline{\text{gr}}_{A_2}([\mathbf{x}, i], [\mathbf{y}, j]) + \text{gr}_{\mathfrak{s}}(A_2(\theta_{\mathbf{x}}) - A_1(\theta_{\mathbf{x}})) \\ &\quad - \text{gr}_{\mathfrak{s}}(A_2(\theta_{\mathbf{y}}) - A_1(\theta_{\mathbf{y}})) \\ &= \mu(\phi) + 2(i - j) - 2n_z(\phi) - \langle c_1(\mathfrak{s}) \cup A_2(\phi), [Y] \rangle \\ &\quad - \langle c_1(\mathfrak{s}) \cup (A_2(\theta_{\mathbf{x}}) - A_1(\theta_{\mathbf{x}})), [Y] \rangle \\ &\quad + \langle c_1(\mathfrak{s}) \cup (A_2(\theta_{\mathbf{y}}) - A_1(\theta_{\mathbf{y}})), [Y] \rangle \end{aligned}$$

Applying the identity (8) twice, this easily reduces to $\underline{\text{gr}}_{A_1}([\mathbf{x}, i], [\mathbf{y}, j])$. \square

We now show that one can always work with relatively \mathbb{Z} -graded Floer homology (rather than groups with a finite cyclic grading) if the coefficient module and spin^c structure are induced by a cobordism from S^3 to Y . To do so we spell out the notion of a conjugate module in the current, graded, context. As usual, if Y is an oriented 3-manifold then $-Y$ denotes the same manifold with the opposite orientation.

Definition 3.4 Suppose M is a graded R_Y -module. The *conjugate module* \bar{M} is the graded R_{-Y} module whose underlying graded group is the same as M , but whose multiplication is given by

$$r \otimes m \mapsto \bar{r} \cdot m, \quad r \in R_{-Y}.$$

It is clear that if (7) is satisfied for M as a graded R_Y -module (modulo d), then the same is true for \bar{M} as a graded R_{-Y} -module.

Proposition 3.5 Suppose $W: S^3 \rightarrow Y$ is an oriented cobordism with spin^c structure \mathfrak{s}_W , and $M(W)$ is the induced module for R_Y . Then $M(W)$ carries a natural grading induced by \mathfrak{s}_W that is compatible with the \mathfrak{s} -grading on R_Y in the sense of (7), where \mathfrak{s} is the restriction of \mathfrak{s}_W to Y . In particular, $HF^\circ(Y, \mathfrak{s}; M(W))$ carries a relative \mathbb{Z} grading.

More generally, if $W: Y_1 \rightarrow Y_2$ is an oriented cobordism with spin^c structure \mathfrak{s}_W whose restrictions to Y_1 and Y_2 are \mathfrak{s}_1 and \mathfrak{s}_2 respectively, and M is a module for R_{Y_1} satisfying (7) modulo d , then the induced module $M(W)$ carries a grading induced by \mathfrak{s}_W also satisfying (7) modulo d .

Proof Observe first that since $\partial W = -Y_1 \sqcup Y_2$, we should most naturally consider $\mathbb{Z}[K(W)]$ as a module for R_{-Y_1} and R_{Y_2} . Recall that $M(W) = \bar{M} \otimes_{R_{-Y_1}} \mathbb{Z}[K(W)]$, where $K(W) = \ker(H^2(W, \partial W) \rightarrow H^2(W))$. Define $\text{gr}_W: K(W) \rightarrow \mathbb{Z}$ by

$$\text{gr}_W(k) = -\langle c_1(\mathfrak{s}_W) \cup k, [W, \partial W] \rangle,$$

and use this to impose a grading gr_W on $\mathbb{Z}[K(W)]$. We claim that this grading respects the action by R_{-Y_1} and R_{Y_2} , where the latter are equipped with gradings coming from the restrictions of \mathfrak{s}_W . To see this, it suffices to note that the actions of R_{-Y_1} and R_{Y_2} on $\mathbb{Z}[K(W)]$ are induced by the coboundary maps $\delta_i: H^1(Y_i) \rightarrow K(W)$, and that

$$c_1(\mathfrak{s}_W) \cup \delta_i r = \delta_i(j_i^* c_1(\mathfrak{s}_W) \cup r) = \delta_i(c_1(\mathfrak{s}_i) \cup r)$$

where $j_i: Y_i \rightarrow W$, $i = 1, 2$ are the inclusions of the boundary components. Hence for $r \in H^1(Y_1)$,

$$\langle c_1(\mathfrak{s}_W) \cup \delta_1 r, [W, \partial W] \rangle = \langle c_1(\mathfrak{s}_1) \cup r, [-Y_1] \rangle$$

and correspondingly for elements of $H^1(Y_2)$. \square

4 Pairings and duality

In [15], Ozsváth and Szabó defined a pairing

$$\langle \cdot, \cdot \rangle: HF^+(Y, \mathfrak{s}; \mathbb{Z}) \otimes HF^-(-Y, \mathfrak{s}; \mathbb{Z}) \rightarrow \mathbb{Z}$$

on Floer homology, with respect to which cobordism-induced maps satisfy a certain duality. Here we extend this pairing to Floer homology with twisted coefficients and prove a corresponding duality; throughout we use the ring R_Y and modules M that are graded via some choice of spin^c structure on Y as in the previous section.

Recall that if $(\Sigma, \alpha, \beta, z)$ is a pointed Heegaard diagram for Y , then $(-\Sigma, \alpha, \beta, z)$ describes the oppositely oriented manifold $-Y$, and the map s_z is invariant under this change of orientation.

Definition 4.1 Define a pairing

$$\langle \cdot, \cdot \rangle: CF^\infty(Y, \mathfrak{s}; R_Y) \otimes_{R_Y} \overline{CF^\infty(-Y, \mathfrak{s}; R_{-Y})} \longrightarrow R_Y$$

as follows: for generators $[\mathbf{x}, i] \in CF^\infty(Y, \mathfrak{s}; R_Y)$ and $[\mathbf{y}, j] \in CF^\infty(-Y, \mathfrak{s}; R_{-Y})$ set

$$\langle [\mathbf{x}, i], [\mathbf{y}, j] \rangle = \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{y} \text{ and } j = -i - 1 \\ 0 & \text{otherwise.} \end{cases}$$

The desired pairing is obtained by extending by R_Y -linearity.

We must check that this definition has the desired properties:

Lemma 4.2 For any $\xi \in CF^\infty(Y, \mathfrak{s}; R_Y)$, $\eta \in CF^\infty(-Y, \mathfrak{s}; R_{-Y})$, we have

$$\begin{aligned} \langle \partial^\infty \xi, \eta \rangle &= \langle \xi, \partial^\infty \eta \rangle \\ \langle U\xi, \eta \rangle &= \langle \xi, U\eta \rangle. \end{aligned}$$

Proof This is much like the proof of the corresponding fact in untwisted Floer homology [15, Lemma 5.1], but we must be more careful with the coefficients. Observe that composition with the reflection $r: D^2 \rightarrow D^2$ across the real axis gives a map $\pi_2(\mathbf{x}, \mathbf{y}) \rightarrow \pi_2(\mathbf{y}, \mathbf{x})$ that exchanges J -holomorphic disks in $\text{Sym}^g(\Sigma)$ with $-J$ -holomorphic disks in $\text{Sym}^g(-\Sigma)$; in other words

$$\mathcal{M}_{-\Sigma}(\phi \circ r) = \mathcal{M}_\Sigma(\phi)$$

for $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$.

Furthermore, if A_Y is an additive assignment for $(\Sigma, \alpha, \beta, z)$ then we can think of A_Y as also giving an additive assignment A_{-Y} for $(-\Sigma, \alpha, \beta, z)$. For $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ we have that $\phi * (\phi \circ r)$ is homotopic to a constant map, from which it follows that

$$A_{-Y}(\phi \circ r) = -A_Y(\phi).$$

Since $n_z^\Sigma(\phi) = n_z^{-\Sigma}(\phi \circ r)$, we have

$$\begin{aligned} \partial^\infty[\mathbf{y}, j] &= \sum_{\phi \in \pi_2(\mathbf{y}, \mathbf{w})} \#\widehat{\mathcal{M}}_{-\Sigma}(\phi)[\mathbf{w}, j - n_z^{-\Sigma}(\phi)] \otimes e^{A_{-Y}(\phi)} \\ &= \sum_{\tilde{\phi} \in \pi_2(\mathbf{w}, \mathbf{y})} \#\widehat{\mathcal{M}}_{-\Sigma}(\tilde{\phi} \circ r)[\mathbf{w}, j - n_z^{-\Sigma}(\tilde{\phi} \circ r)] \otimes e^{A_{-Y}(\tilde{\phi} \circ r)} \\ &= \sum_{\tilde{\phi} \in \pi_2(\mathbf{w}, \mathbf{y})} \#\widehat{\mathcal{M}}_\Sigma(\tilde{\phi})[\mathbf{w}, j - n_z^\Sigma(\tilde{\phi})] \otimes e^{-A_Y(\tilde{\phi})}. \end{aligned}$$

From this it follows (using the conjugate module structure on the second factor) that

$$\begin{aligned} \langle [\mathbf{x}, i], \partial^\infty[\mathbf{y}, j] \rangle &= \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu_\Sigma(\phi) = 1 \\ n_z^\Sigma(\phi) = i + j + 1}} \#\widehat{\mathcal{M}}(\phi) e^{A_Y(\phi)} \\ &= \langle \partial^\infty[\mathbf{x}, i], [\mathbf{y}, j] \rangle. \end{aligned}$$

The first claim of the lemma follows from this, while the second is obvious. □

Thus we obtain a pairing on homology

$$HF^+(Y, \mathfrak{s}; R_Y) \otimes_{R_Y} \overline{HF^-(-Y, \mathfrak{s}; R_{-Y})} \longrightarrow R_Y$$

that descends to the reduced homologies.

More generally, suppose M and N are (graded) modules for R_Y and R_{-Y} , respectively: we can extend the construction above to a pairing between $HF^+(Y, \mathfrak{s}; M)$ and $HF^-(-Y, \mathfrak{s}; N)$. To this end, define

$$\langle \cdot, \cdot \rangle: CF^\infty(Y, \mathfrak{s}; M) \otimes_{R_Y} \overline{CF^\infty(-Y, \mathfrak{s}; N)} \rightarrow M \otimes_{R_Y} \bar{N}$$

on generators by

$$\langle [\mathbf{x}, i] \otimes m, [\mathbf{y}, j] \otimes n \rangle = \langle [\mathbf{x}, i], [\mathbf{y}, j] \rangle \cdot m \otimes n,$$

where the pairing on the right is the universal one just defined. It follows from the calculation above that the pairing descends to homology:

$$HF^+(Y, \mathfrak{s}; M) \otimes_{R_Y} \overline{HF^-(-Y, \mathfrak{s}; N)} \longrightarrow M \otimes_{R_Y} \bar{N}.$$

We can now give the analogue for twisted coefficients of Theorem 3.5 of [15].

Theorem 4.3 (Duality for twisted coefficients) *Let $W: Y_1 \rightarrow Y_2$ be a cobordism and M_1 and M_2 coefficient modules for R_{Y_1} and R_{-Y_2} respectively. Write W' for the manifold W regarded as a cobordism $-Y_2 \rightarrow -Y_1$, and let \mathfrak{s} be a spin^c structure on W with restrictions $\mathfrak{s}_i = \mathfrak{s}|_{Y_i}$. Then for any $\xi \in HF^+(Y_1, \mathfrak{s}_1; M_1)$ and $\eta \in HF^-(-Y_2, \mathfrak{s}_2; M_2)$, we have*

$$\langle F_{W, \mathfrak{s}}^+(\xi), \eta \rangle = \langle \xi, F_{W', \mathfrak{s}}^-(\eta) \rangle.$$

Observe that the two pairings in the theorem above take values in

$$M_1(W) \otimes_{R_{Y_2}} \bar{M}_2 = \bar{M}_1 \otimes_{R_{-Y_1}} \mathbb{Z}[K(W)] \otimes_{R_{Y_2}} \bar{M}_2$$

(for the left hand side) and

$$M_1 \otimes_{R_{Y_1}} \overline{M_2(W)} = M_1 \otimes_{R_{Y_1}} \overline{\mathbb{Z}[K(W)] \otimes_{R_{Y_2}} \bar{M}_2} = M_1 \otimes_{R_{Y_1}} \overline{\mathbb{Z}[K(W)]} \otimes_{R_{-Y_2}} M_2,$$

(for the right). Thus the two target groups are identical (with conjugate module structures), and the equality of the theorem makes sense.

Proof We adapt the proof from [15]. Decompose W into a composition of 1–handle additions, followed by 2–handles and then 3–handles. The verification of duality for

1– and 3–handle cobordisms is unchanged from the untwisted case given in [15], so we omit it here.

Assume, then, that W is a cobordism comprised entirely of 2–handle additions. Let R denote the reflection of the standard 2–simplex Δ that fixes one corner and exchanges the other two. Specifically, if the edges are labeled e_α , e_β and e_γ , we take R to exchange e_β and e_γ while reversing e_α . If A_W is an additive assignment for a Heegaard triple $(\Sigma, \alpha, \beta, \gamma, z)$ associated to W as in Definition 2.5 (using a base triangle ψ_0), then we obtain an additive assignment $A_{W'}$ for W' (described by the triple $(-\Sigma, \alpha, \gamma, \beta, z)$) from the triangle $\psi_0 \circ R$.

More generally, for any (homotopy class of) triangle $\psi \in \pi_2(\mathbf{x}, \mathbf{y}, \mathbf{w})$, precomposition with R gives a triangle $\psi \circ R \in \pi_2(\mathbf{w}, \mathbf{y}, \mathbf{x})$. Moreover, if $\psi = \psi_0 + \phi_{\alpha\beta} + \phi_{\beta\gamma} + \phi_{\alpha\gamma}$ then it is easy to see that

$$\psi \circ R = \psi_0 \circ R + (\phi_{\alpha\gamma} \circ r) + \phi_{\gamma\beta} + (\phi_{\alpha\beta} \circ r),$$

where r is the reflection across the real axis used previously. Therefore

$$A_{W'}(\psi \circ R) = \delta(-A_{-Y_2}(\phi_{\alpha\gamma} \circ r) + A_{-Y_1}(\phi_{\alpha\beta} \circ r)) = A_W(\psi)$$

(cf the proof of Lemma 4.2). Furthermore, just as in the case of disks we have an identification

$$\mathcal{M}_{-\Sigma}(\psi \circ R) = \mathcal{M}_\Sigma(\psi).$$

Thus for $m_i \in M_i$:

$$\begin{aligned} \langle F_{W,s}([\mathbf{x}, i]m_1), [\mathbf{w}, k]m_2 \rangle &= \left\langle \sum_{\substack{\mathbf{v} \in T_\alpha \cap T_\gamma \\ \psi \in \pi_2(\mathbf{x}, \Theta, \mathbf{v})}} \#\mathcal{M}_\Sigma(\psi) \cdot [\mathbf{v}, i - n_z(\psi)]m_1 \otimes e^{A_W(\psi)}, \right. \\ &\quad \left. [\mathbf{w}, k]m_2 \right\rangle \\ &= \sum_{\substack{\psi \in \pi_2(\mathbf{x}, \Theta, \mathbf{w}) \\ n_z^\Sigma(\psi) = i+k+1}} \#\mathcal{M}_\Sigma(\psi) \cdot (m_1 \otimes e^{A_W(\psi)}) \otimes m_2, \end{aligned}$$

an element of $M_1(W) \otimes \bar{M}_2$ (where all sums are over triangles with Maslov index 0).

On the other hand, in $M_1 \otimes \overline{M_2(W)}$.

$$\begin{aligned}
 \langle [x, i]m_1, F_{W',s}([w, k]m_2) \rangle &= \left\langle [x, i]m_1, \right. \\
 &\quad \left. \sum_{\substack{v \in T_\alpha \cap T_\beta \\ \tilde{\psi} \in \pi_2(w, \Theta, v)}} \# \mathcal{M}_{-\Sigma}(\tilde{\psi}) \cdot [v, k - n_z(\tilde{\psi})]m_2 \otimes e^{A_{W'}(\tilde{\psi})} \right\rangle \\
 &= \sum_{\substack{\tilde{\psi} \in \pi_2(w, \Theta, x) \\ n_z^\Sigma(\tilde{\psi}) = i+k+1}} \# \mathcal{M}_{-\Sigma}(\tilde{\psi}) \cdot m_1 \otimes (m_2 \otimes e^{A_{W'}(\tilde{\psi})}) \\
 &= \sum_{\substack{\psi \in \pi_2(x, \Theta, w) \\ n_z^\Sigma(\psi) = i+k+1}} \# \mathcal{M}_{-\Sigma}(\psi \circ R) \cdot m_1 \otimes (m_2 \otimes e^{A_{W'}(\psi \circ R)}) \\
 &= \sum_{\substack{\psi \in \pi_2(x, \Theta, w) \\ n_z^\Sigma(\psi) = i+k+1}} \# \mathcal{M}_\Sigma(\psi) \cdot m_1 \otimes (m_2 \otimes e^{A_W(\psi)}) \quad \square
 \end{aligned}$$

5 Action of first homology

In this section we extend to twisted coefficients an additional aspect of the algebraic structure of Heegaard Floer homology, namely the action of $\Lambda^*(H_1(Y)/\text{tors})$ on $HF^\circ(Y, \mathfrak{s})$. We also discuss the interaction of this structure with cobordism-induced homomorphisms. Much of this section is a straightforward generalization of material from [12; 11; 15], so we omit many of the details.

Proposition 5.1 *Fix an oriented spin^c 3–manifold (Y, \mathfrak{s}) and a module M for $R_Y = \mathbb{Z}[H^1(Y)]$. Then for any $h \in H_1(Y)/\text{tors}$ there is a chain endomorphism \mathcal{A}_h of $CF^\infty(Y, \mathfrak{s}; M)$ of degree -1 , equivariant with respect to U and the R_Y action, with the property that $\mathcal{A}_h \circ \mathcal{A}_h$ is chain homotopic to 0.*

Thus, the collection of maps \mathcal{A}_h provides $HF^\circ(Y, \mathfrak{s}; M)$ with the structure of a module over $R_Y[U] \otimes \Lambda^*(H_1(Y)/\text{tors})$.

Proof For a generator $[x, i] \otimes m \in CF^\infty(Y, \mathfrak{s}; M)$ we set

$$\mathcal{A}_h([x, i] \otimes m) = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\substack{\phi \in \pi_2(x, y) \\ \mu(\phi) = 1}} \# \widehat{\mathcal{M}}(\phi) \langle A(\phi), h \rangle \cdot [y, i - n_z(\phi)] \otimes e^{A(\phi)} \cdot m.$$

Then the proof that \mathcal{A}_h is a chain map whose square is trivial in homology is virtually identical to the proof in the untwisted case (cf Proposition 4.17 of [12]), and it is straightforward to check that the action of \mathcal{A}_h on homology is independent of the choice of additive assignment A (cf the proof of Lemma 3.3). \square

We will omit the map \mathcal{A}_h from the notation and simply write $h.\xi$ for the action of h on the element $\xi \in HF^\circ(Y, \mathfrak{s}; M)$.

Remark 5.2 Though the action of $H_1(Y)/\text{tors}$ is defined for Floer homology with any coefficients, it may be largely trivial. Indeed, suppose M is an R_Y -module, and let $Z_M \subset H^1(Y)$ denote the stabilizer of M : that is, the set of all $\alpha \in H^1(Y)$ such that $\alpha m = m$ for all $m \in M$. Then it can be shown that if $h \in H_1(Y)$ has the property that

$$\langle \alpha, h \rangle = 0 \text{ for all } \alpha \in Z_M,$$

then \mathcal{A}_h is chain homotopic to 0. In particular, this implies that the $H_1(Y)/\text{tors}$ action on the fully twisted homology $HF^\circ(Y, \mathfrak{s}; R_Y)$ is trivial.

Lemma 5.3 *Let (Y, \mathfrak{s}) be as above, and let M and N be modules for R_Y and R_{-Y} respectively. Then for any $h \in H_1(Y)/\text{tors}$, any $\xi \in HF^+(Y, \mathfrak{s}; M)$ and any $\eta \in HF^-(-Y, \mathfrak{s}; N)$ we have*

$$\langle h.\xi, \eta \rangle = -\langle \xi, h.\eta \rangle.$$

Proof This follows from a calculation very similar to the one in Lemma 4.2. Indeed, the only difference is the appearance of the factors $\langle A(\phi), h \rangle$, which change sign under orientation reversal. \square

We now extend the twisted cobordism invariants from the previous section to include the action of first homology. Specifically, for a cobordism $W: Y_0 \rightarrow Y_1$ we wish to define $F_{W, \mathfrak{s}}$ as a map

$$(9) \quad F_{W, \mathfrak{s}}^\circ: HF^\circ(Y_0, \mathfrak{s}_0; M) \otimes \Lambda^* H_1(W)/\text{tors} \longrightarrow HF^\circ(Y_1, \mathfrak{s}_1; M(W)).$$

With the preceding in place the definition runs precisely as in the untwisted case in [15]; we summarize the construction.

Suppose first that $W: Y_0 \rightarrow Y_1$ is a cobordism consisting only of 2-handle additions. Then it is easy to see that the map

$$i_* = i_{0*} - i_{1*}: H_1(Y_0)/\text{tors} \oplus H_1(Y_1)/\text{tors} \rightarrow H_1(W)/\text{tors}$$

is surjective. Fix $h \in H_1(W)/\text{tors}$ and suppose $h = i_*(h_0, h_1)$. For $\xi \in HF^\circ(Y_0, \mathfrak{s}_0; M)$, we set

$$(10) \quad F_{W, \mathfrak{s}}^\circ(\xi \otimes h) = F_{W, \mathfrak{s}}^\circ(h_0 \cdot \xi) - h_1 \cdot F_{W, \mathfrak{s}}^\circ(\xi).$$

Clearly $F_{W, \mathfrak{s}}^\circ((\xi \otimes h) \otimes h) = 0$, so the action extends to $\Lambda^* H_1(W)/\text{tors}$.

In fact, we can define this action using a Heegaard triple $(\Sigma, \alpha, \beta, \gamma, z)$ describing the cobordism, just as in Lemma 2.6 of [15]. It follows as in that proof that the action of pairs (h_0, h_1) in the image of $H_2(W, \partial W, \mathbb{Z})$ is trivial, so the action descends as claimed to $H_1(W)/\text{tors}$.

In general for a cobordism containing 1–, 2– and 3–handles we write the induced homomorphism as a composition $F_W^\circ = E^\circ \circ H^\circ \circ G^\circ$ as in Section 2.2. This composition corresponds to a factorization $W = W_1 \cup W_2 \cup W_3$ where W_i includes only handles of index i . As observed in [15], the inclusion induces an isomorphism $H_1(W_2) \rightarrow H_1(W)$; thus for $\omega \in \Lambda^* H_1(W)/\text{tors}$ we set

$$F_W^\circ(\xi \otimes \omega) = E^\circ(H^\circ(G^\circ(\xi) \otimes \omega))$$

just as in [15].

Many properties of the extended cobordism maps (9) follow from corresponding properties of the original ones. We mention two results here.

Theorem 5.4 *Let $W: Y_0 \rightarrow Y_1$ be a cobordism with spin^c structure \mathfrak{s} and suppose $\omega \in \Lambda^* H_1(W)/\text{tors}$. Write \mathfrak{s}_i for $\mathfrak{s}|_{Y_i}$. Then for modules M and N over R_{Y_0} and R_{-Y_1} respectively, and for any $x \in HF^+(Y_0, \mathfrak{s}_0; M)$ and $y \in HF^-(-Y_1, \mathfrak{s}_1; N)$, we have*

$$\langle F_{W, \mathfrak{s}}^+(x \otimes \omega), y \rangle = \langle x, F_{W', \mathfrak{s}}^-(y \otimes \omega) \rangle.$$

Proof Assume first that W consists of 2–handles only, and suppose $h \in H_1(W)/\text{tors}$ has the expression $h = i_*(h_0, h_1)$ for $h_i \in H_1(Y_i)/\text{tors}$. Then using the duality theorem for twisted coefficients (Theorem 4.3) and Lemma 5.3 we have

$$\begin{aligned} \langle F_{W, \mathfrak{s}}^+(x \otimes h), y \rangle &= \langle F_{W, \mathfrak{s}}^+(h_0 \cdot x) - h_1 \cdot F_{W, \mathfrak{s}}^+(x), y \rangle \\ &= -\langle x, h_0 \cdot F_{W', \mathfrak{s}}^-(y) \rangle + \langle x, F_{W', \mathfrak{s}}^-(h_1 \cdot y) \rangle \\ &= \langle x, F_{W', \mathfrak{s}}^-(y \otimes h) \rangle. \end{aligned}$$

It is a simple matter to extend to general cobordisms and general ω . □

Theorem 5.5 *The composition law (Theorem 2.9) holds for the extended maps (9). More precisely, suppose $W = W_1 \cup_{Y_1} W_2$ is a composite cobordism and write*

$$j_*: \Lambda^*(H_1(W_1)/\text{tors}) \otimes \Lambda^*(H_1(W_2)/\text{tors}) \rightarrow \Lambda^*(H_1(W)/\text{tors})$$

for the surjection induced on exterior algebras by the Mayer–Vietoris map $H_1(W_1) \oplus H_1(W_2) \rightarrow H_1(W)$. Fix $\omega_i \in \Lambda^* H_1(W_i)/\text{tors}$ and write ω for the image of $\omega_1 \otimes \omega_2$ under j_* . Then for any spin^c structure \mathfrak{s} on W , we can find choices of representatives for the maps F° such that for any $\alpha \in H^1(Y_1)$

$$F_{W, \mathfrak{s} - \delta\alpha}^\circ(\xi \otimes \omega) = \Pi_W \left[F_{W_2, \mathfrak{s}|_{W_2}}^\circ(e^\alpha \cdot F_{W_1, \mathfrak{s}|_{W_1}}^\circ(\xi \otimes \omega_1) \otimes \omega_2) \right].$$

Proof This follows from Theorem 2.9 together with the formal properties of the H_1 –action, particularly (10) in the case of 2–handles. (See Ozsváth–Szabó [15], particularly Proposition 4.20. Note that here the strengthened composition law means that summing over spin^c structures is unnecessary.) \square

6 Conjugation and orientation reversal

As in the original Heegaard Floer theory, there are simple relationships between the twisted Heegaard Floer homologies of (Y, \mathfrak{s}) , $(-Y, \mathfrak{s})$, and $(Y, \bar{\mathfrak{s}})$, where $\bar{\mathfrak{s}}$ is the conjugate spin^c structure. To describe the effect of spin^c conjugation, recall that though we normally do not include it in the notation, the ring R_Y depends on \mathfrak{s} through the grading (5), and here we write $R_{Y, \mathfrak{s}}$ to indicate this. Thus $R_{Y, \mathfrak{s}}$ and $R_{Y, \bar{\mathfrak{s}}}$ are identical rings with opposite gradings; in fact $R_{Y, \bar{\mathfrak{s}}} = R_{-Y, \mathfrak{s}}$ as graded rings. In particular, if M is a graded module for $R_{Y, \mathfrak{s}}$, the conjugate module \bar{M} can be considered either as a module for $R_{-Y, \mathfrak{s}}$ or for $R_{Y, \bar{\mathfrak{s}}}$.

Theorem 6.1 *If (Y, \mathfrak{s}) is a closed spin^c 3–manifold and M is a module for $R_{Y, \mathfrak{s}}$, then there is a grading-preserving isomorphism of $R_{Y, \mathfrak{s}}$ –modules*

$$J: HF^\circ(Y, \mathfrak{s}; M) \xrightarrow{\sim} \overline{HF^\circ(Y, \bar{\mathfrak{s}}; \bar{M})}.$$

Proof We mimic the argument in the untwisted case [11, Theorem 2.4]. Recall that if $(\Sigma, \alpha, \beta, z)$ is a Heegaard diagram for Y and $\mathbf{x} \in T_\alpha \cap T_\beta$ has $s_z(\mathbf{x}) = \mathfrak{s}$, then $(-\Sigma, \beta, \alpha, z)$ also describes Y , and in this diagram $s_z(\mathbf{x}) = \bar{\mathfrak{s}}$. If $\{A_{\mathbf{x}, y}\}$ is an additive assignment for $(\Sigma, \alpha, \beta, z)$, then we obtain an assignment A' for $(-\Sigma, \beta, \alpha, z)$ by $A'_{\mathbf{x}, y}(\phi) = -A_{\mathbf{x}, y}(\phi \circ f)$, where f is the reflection across the imaginary axis in \mathbb{C} .

Define a homomorphism $J: CF^\circ(Y, \mathfrak{s}; R_{Y, \mathfrak{s}}) \rightarrow CF^\circ(Y, \bar{\mathfrak{s}}; R_{Y, \bar{\mathfrak{s}}})$ by mapping $[\mathbf{x}, i]$ in the diagram $(\Sigma, \alpha, \beta, z)$ to $[\mathbf{x}, i]$ in the diagram $(-\Sigma, \beta, \alpha, z)$ and extending

by $\mathbb{Z}[H^1(Y)]$ –antilinearity. Then it is a straightforward exercise to check that J is a chain map preserving relative gradings, recalling that $\mathcal{M}_{-\Sigma}(\phi \circ f) = \mathcal{M}_{\Sigma}(\phi)$, $n_z^{-\Sigma}(\phi \circ f) = n_z^{\Sigma}(\phi)$, and $\mu_{-\Sigma}(\phi \circ f) = \mu_{\Sigma}(\phi)$. In general,

$$J: CF^\circ(Y, \mathfrak{s}) \otimes_{R_{Y, \mathfrak{s}}} M \rightarrow CF^\circ(Y, \bar{\mathfrak{s}}) \otimes_{R_{Y, \bar{\mathfrak{s}}}} \bar{M}$$

is given by $[\mathbf{x}, i] \otimes m \mapsto [\mathbf{x}, i] \otimes m$. Since this is an antilinear chain isomorphism, the theorem follows. \square

It is not hard to generalize the naturality of cobordism-induced maps under conjugation to the twisted case.

Before describing the effect of orientation reversal, we pause to spell out our duality conventions. Let M be a graded $R_{Y, \mathfrak{s}}$ –module, and set

$$CF_\circ^*(Y, \mathfrak{s}; M) = \text{Hom}_{R_{Y, \mathfrak{s}}}(CF_*^\circ(Y, \mathfrak{s}), M),$$

made into an $R_{Y, \mathfrak{s}}$ –module in the obvious way. For the grading, suppose $\alpha, \beta \in CF_\circ^*(Y, \mathfrak{s}; M)$ are homogeneous (as homomorphisms between relatively graded $R_{Y, \mathfrak{s}}$ –modules). Set

$$\text{gr}_{CF^*}(\alpha, \beta) = \text{gr}_M(\alpha(f)) - \text{gr}_M(\beta(g)) - \text{gr}_{CF_*}(f, g)$$

for any homogeneous $f, g \in CF_*(Y, \mathfrak{s})$ with $\alpha(f)$ and $\beta(g)$ nonzero in M . Thus, for example, $\text{gr}_{CF^*}(r\alpha, \alpha) = \text{gr}_{\mathfrak{s}}(r)$ for $r \in R_{Y, \mathfrak{s}}$.

Observe that there is a natural generating set for $CF_\infty^*(Y, \mathfrak{s}; R_{Y, \mathfrak{s}})$. Namely, for a generator $[\mathbf{x}, i] \in CF_*^\infty(Y, \mathfrak{s})$, define $[\mathbf{x}, i]^*: CF_*^\infty(Y, \mathfrak{s}) \rightarrow R_{Y, \mathfrak{s}}$ by setting $[\mathbf{x}, i]^*([\mathbf{y}, j] \otimes r) = r$ if $[\mathbf{y}, j] = [\mathbf{x}, i]$, and 0 otherwise. Since $CF_*^\infty(Y, \mathfrak{s})$ is a free complex over $R_{Y, \mathfrak{s}}$, elements of $CF_\infty^*(Y, \mathfrak{s}; M)$ can be expressed as combinations of elements of the form $[\mathbf{x}, i]^* \otimes m$, whose value on $[\mathbf{y}, j]$ is $[\mathbf{x}, i]^*([\mathbf{y}, j]) \cdot m$.

In terms of these generators, the coboundary in CF^* can be expressed explicitly by

$$\delta([\mathbf{x}, i]^* \otimes m) = \sum_{\substack{\phi \in \pi_2(\mathbf{y}, \mathbf{x}) \\ \mu(\phi)=1}} \#\widehat{\mathcal{M}}(\phi)[\mathbf{y}, i + n_z(\phi)]^* \otimes e^{A(\phi)} m.$$

With the grading conventions outlined above, we have

$$\text{gr}_{CF^*(Y)}([\mathbf{x}, i]^*, [\mathbf{y}, j]^*) = -\text{gr}_{CF_*(Y)}([\mathbf{x}, i], [\mathbf{y}, j]).$$

Observe that with these conventions, the codifferential has degree -1 , in other words, $CF_\infty^*(Y, \mathfrak{s}; M)$ is a chain complex rather than a cochain complex. Likewise, the transpose action of U given by $U : [\mathbf{x}, i]^* \mapsto [\mathbf{x}, i + 1]^*$ decreases grading by 2.

Theorem 6.2 For (Y, \mathfrak{s}) a closed spin^c 3–manifold and M a module for $R_{Y, \mathfrak{s}}$, there is a grading-preserving isomorphism of $R_{Y, \mathfrak{s}}$ –modules

$$HF_*^\pm(Y, \mathfrak{s}; M) \cong \overline{HF_{\mp}^*(-Y, \mathfrak{s}; \overline{M})}.$$

Proof Just as in the proof in [11, Proposition 2.5], define a homomorphism

$$CF_*^\circ(Y, \mathfrak{s}; M) \rightarrow CF_\circ^*(-Y, \mathfrak{s}; \overline{M})$$

by $[\mathbf{x}, i] \otimes m \mapsto [\mathbf{x}, -1 - i]^* \otimes m$, where on the right we consider m as an element of \overline{M} . One checks easily that this gives rise to a $\mathbb{Z}[H^1(Y)]$ –antilinear chain isomorphism that preserves relative grading. \square

7 Invariants for 4–manifolds

We briefly recall the definition of Ozsváth–Szabó 4–manifold invariants from [15], and then proceed to discuss their calculation in the context of 4–manifolds obtained by gluing two manifolds with boundary.

Suppose X is a closed 4–manifold having $b^+(X) \geq 2$. Then we can find an *admissible cut* for X : that is, a hypersurface $N \subset X$ separating X into components $X = V_1 \cup_N V_2$ with the following properties:

- (1) For $i = 1, 2$, we have $b^+(V_i) \geq 1$.
- (2) The image of the Mayer–Vietoris map $\delta: H^1(N) \rightarrow H^2(X)$ is trivial.

As observed previously (Remark 2.8), the second condition ensures that spin^c structures on X are determined by their restrictions to V_1 and V_2 .

The first condition is relevant because of the following.

Lemma 7.1 [15] *If W is a cobordism having $b^+(W) \geq 1$ then for any spin^c structure \mathfrak{s} and in any coefficient module, the map $F_{W, \mathfrak{s}}^\infty$ vanishes.*

Recall that for all sufficiently large integers r , the subgroups $\ker(U_-^r) \subset HF^-(Y, \mathfrak{s})$ and $\text{im}(U_+^r) \subset HF^+(Y, \mathfrak{s})$ are independent of r (where U_\pm denotes the action of U on HF^\pm). The *reduced Floer homology groups* are defined by $HF_{\text{red}}^-(Y, \mathfrak{s}) = \ker(U_-^r)$ and $HF_{\text{red}}^+(Y, \mathfrak{s}) = \text{coker}(U_+^r)$. Together with the long exact sequence

$$\dots \longrightarrow HF^\infty(Y, \mathfrak{s}) \longrightarrow HF^+(Y, \mathfrak{s}) \xrightarrow{\tau} HF^-(Y, \mathfrak{s}) \longrightarrow \dots$$

and the fact that U is an isomorphism on HF^∞ , [Lemma 7.1](#) implies that the image of $F_{W,\mathfrak{s}}^-$ for W a cobordism with $b^+(W) \geq 1$ lies in HF_{red}^- , while $F_{W,\mathfrak{s}}^+$ factors through HF_{red}^+ . Note also that the homomorphism τ in the sequence induces an isomorphism

$$\tau: HF_{\text{red}}^+(Y, \mathfrak{s}) \rightarrow HF_{\text{red}}^-(Y, \mathfrak{s}).$$

(All of the above holds in any coefficient system).

Definition 7.2 [15] Let Θ^- denote a top-degree generator of $HF^-(S^3)$. Let N be an admissible cut for a 4–manifold X as above, and fix a spin^c structure \mathfrak{s} on X . The *Ozsváth–Szabó invariant* of (X, \mathfrak{s}) is the integer-valued function

$$\Phi_{X,\mathfrak{s}}: \mathbb{A}(X) := \mathbb{Z}[U] \otimes \Lambda^*(H_1(X)/\text{tors}) \longrightarrow \mathbb{Z}/\pm 1$$

defined by

$$\Phi_{X,\mathfrak{s}}(U^n \otimes \omega) = \langle (F_{V_2}^+ \circ \tau^{-1} \circ F_{V_1}^-)(U^n \cdot \Theta^- \otimes \omega), \Theta^- \rangle.$$

Note that $\Phi_{X,\mathfrak{s}}$ is defined only modulo a sign, due to the sign ambiguity of the maps associated to cobordisms.

Remark 7.3 As a slight abuse of notation, if Z is a 4–manifold with one boundary component Y , and \mathfrak{s} is a spin^c structure on Z , we will denote by $F_{Z,\mathfrak{s}}^\circ$ the homomorphism $HF^\circ(S^3) \rightarrow HF^\circ(Y)$ induced by the cobordism obtained by removing a 4–ball from the interior of Z .

Remark 7.4 It follows from the formula for the degree shift induced by a cobordism that $\Phi_{X,\mathfrak{s}}$ is nonzero only on elements of $\mathbb{A}(X)$ having degree $d(\mathfrak{s})$, where

$$d(\mathfrak{s}) = \frac{1}{4}(c_1^2(\mathfrak{s}) - 2e(X) - 3\sigma(X)).$$

Here $e(X)$ is the Euler characteristic of X and $\sigma(X)$ is the signature, and $\mathbb{A}(X)$ is graded so that U carries degree 2 and elements of $H_1(X)/\text{tors}$ carry degree 1.

Ozsváth and Szabó show that $\Phi_{X,\mathfrak{s}}$ does not depend on the choice of admissible cut N , and therefore gives an invariant of smooth spin^c 4–manifolds with $b^+ \geq 2$. An important property of $\Phi_{X,\mathfrak{s}}$ is that it is nonzero for at most finitely many spin^c structures \mathfrak{s} on X .

In many situations there are convenient decompositions $X = Z_1 \cup_Y Z_2$, in which Y fails to be admissible in the sense above—specifically, condition (2) in the definition of admissibility is violated. Ozsváth and Szabó prove that one can use such a cut to

obtain information about sums of invariants of X (Lemma 8.8 of [15]), but in order to obtain more detailed information we must pass to twisted coefficients.

We express our results in terms of group rings. In the situation of cutting X along a 3-manifold Y satisfying (1) but not (2) in the definition of admissible cut, the relevant group is $K(X, Y) = \ker(H^2(X) \rightarrow H^2(Z_1) \oplus H^2(Z_2))$ (cf Remark 2.8). For a given $\mathfrak{s} \in \text{Spin}^c(X)$ and $\alpha \in \mathbb{A}(X)$, we would like a way to calculate the element

$$(11) \quad \sum_{t \in K(X, Y)} \Phi_{X, \mathfrak{s}+t}(\alpha) \cdot e^t \in \mathbb{Z}[K(X, Y)]$$

in terms of invariants on the manifolds-with-boundary Z_1 and Z_2 . Indeed, the invariants of all spin^c structures on X can be read from the coefficients of the above expressions for various \mathfrak{s} .

Since we need to refer to maps in both twisted and untwisted Floer homology, in this section we will follow the notation of Ozsváth and Szabó and write \underline{F}_W° for the map in twisted coefficients induced by W and F_W° for the untwisted map.

Definition 7.5 Suppose Z is an oriented 4-manifold with connected boundary $\partial Z = Y$ and $\mathfrak{s} \in \text{Spin}^c(Z)$. Define the *relative Ozsváth–Szabó invariant* $\Psi_{Z, \mathfrak{s}}$ of Z to be the function

$$\Psi_{Z, \mathfrak{s}}: \mathbb{A}(Z) \longrightarrow HF^-(Y, \mathfrak{s}|_Y; \mathbb{Z}[K(Z)])/K(Z)$$

given by
$$\Psi_{Z, \mathfrak{s}}(U^n \otimes \omega) = [\underline{F}_{Z, \mathfrak{s}}^-(U^n \cdot \Theta^- \otimes \omega)].$$

Here the brackets indicate equivalence class under the action of $K(Z)$, where $K(Z) = \ker(H^2(Z, Y) \rightarrow H^2(Z))$.

Normally we will think of $\Psi_{Z, \mathfrak{s}}$ as an honest function with values in $HF^-(Y)$ by choosing a representative for the equivalence class (though strictly, the twisted-coefficient map $\underline{F}_{Z, \mathfrak{s}}^-$ is defined only up to sign and the action of $\delta(H^1(\partial Z)) = K(Z)$). Note also that if $b^+(Z) \geq 1$ then $\Psi_{Z, \mathfrak{s}}$ takes values in $HF_{\text{red}}^-(Y)$.

The following result gives the central statement of Theorem 1.5 from the introduction, and shows how to calculate (11) in terms of relative invariants.

Theorem 7.6 *Let X be a closed 4-manifold with $b^+(X) \geq 2$ and $Y \subset X$ a 3-dimensional submanifold separating X into components Z_1 and Z_2 . Let \mathfrak{s} be a spin^c structure on X and write $\mathfrak{s}_i = \mathfrak{s}|_{Z_i}$. Assume that $\Psi_{Z_i, \mathfrak{s}_i}$ takes values in HF_{red}^- for*

$i = 1, 2$, and also that $b^+(Z_i) \geq 1$ for at least one of Z_1, Z_2 . Then for any $\alpha_i \in \mathbb{A}(Z_i)$ we have

$$(12) \quad \sum_{t \in K(X, Y)} \Phi_{X, s+t}(\alpha) \cdot e^t = \langle \tau^{-1}(\Psi_{Z_1, s_1}(\alpha_1)), \Psi_{Z_2, s_2}(\alpha_2) \rangle$$

as elements of $\mathbb{Z}[K(X, Y)]$, up to sign and multiplication by an element of $K(X, Y)$. Here α is the image of $\alpha_1 \otimes \alpha_2$ under the natural map $\mathbb{A}(Z_1) \otimes \mathbb{A}(Z_2) \rightarrow \mathbb{A}(X)$.

In the statement of the theorem, we are implicitly choosing representatives for $\Psi_{Z_i, s_i}(\alpha_i)$ and pairing them using the twisted-coefficient pairing defined earlier. Lemma 2.7 shows that the pairing does indeed take values in $\mathbb{Z}[K(X, Y)]$, and it follows also that different choices of representatives give rise to elements of $\mathbb{Z}[K(X, Y)]$ differing by multiplication by an element of $K(X, Y)$.

The rest of this section is devoted to the proof of Theorem 7.6. For simplicity, we focus on the case $\alpha = 1$ in the following; the general case follows by an entirely analogous argument with Theorem 5.4 and Theorem 5.5 replacing Theorem 4.3 and Theorem 2.9.

We begin with a few easy preparatory lemmas.

Lemma 7.7 Fix a spin^c 3–manifold Y and R_Y –modules M and N . Let $\phi: M \rightarrow N$ be a module homomorphism, and write $\phi_*: HF^\circ(Y; M) \rightarrow HF^\circ(Y; N)$ for the induced map in Floer homology. Then the following diagram commutes:

$$\begin{array}{ccccccc} \longrightarrow & HF^-(Y; M) & \longrightarrow & HF^\infty(Y; M) & \longrightarrow & HF^+(Y; M) & \longrightarrow \\ & \downarrow \phi_* & & \downarrow \phi_* & & \downarrow \phi_* & \\ \longrightarrow & HF^-(Y; N) & \longrightarrow & HF^\infty(Y; N) & \longrightarrow & HF^+(Y; N) & \longrightarrow \end{array}$$

In particular, ϕ_* descends to a map on reduced homology, and commutes with τ (and τ^{-1}).

Proof This is clear. □

Lemma 7.8 For $i = 1, 2$ let M_i and N_i be modules for R_Y and R_{-Y} respectively, and consider homomorphisms $\phi: M_1 \rightarrow M_2$ and $\psi: N_1 \rightarrow N_2$. For any $\xi \in HF^+(Y; M_1)$ and $\eta \in HF^-(Y; N_1)$, we have

$$\langle \phi_*(\xi), \psi_*(\eta) \rangle = \phi \otimes \psi(\langle \xi, \eta \rangle) \in M_2 \otimes_{R_Y} \bar{N}_2.$$

Proof This follows easily from the definitions. □

Lemma 7.9 Suppose $W = W_1 \cup_{Y_1} W_2: Y_0 \rightarrow Y_2$ is a composite cobordism, and \mathfrak{s}_1 and \mathfrak{s}_2 are spin^c structures on W_1 and W_2 with $\mathfrak{s}_1|_{Y_1} = \mathfrak{s}_2|_{Y_1}$. If $\underline{F}_{W_1, \mathfrak{s}_1}^-$ has image in $HF_{\text{red}}^-(Y_1)$ then for any coefficient module M for Y_0 ,

- (1) $\text{im}(\underline{F}_{W_2, \mathfrak{s}_2}^- \circ \underline{F}_{W_1, \mathfrak{s}_1}^-) \subset HF_{\text{red}}^-(Y_2; M(W_1)(W_2))$, and
- (2) $\tau^{-1} \circ \underline{F}_{W_2, \mathfrak{s}_2}^- \circ \underline{F}_{W_1, \mathfrak{s}_1}^- = \underline{F}_{W_2, \mathfrak{s}_2}^+ \circ \tau^{-1} \circ \underline{F}_{W_1, \mathfrak{s}_1}^-$.

Proof Point (1) is clear from the fact that $\underline{F}_{W_2, \mathfrak{s}_2}^-$ maps $HF_{\text{red}}^-(Y_1; M(W_1))$ into $HF_{\text{red}}^-(Y_2; M(W_1)(W_2))$.

Statement (2) follows from the commutative diagram

$$\begin{array}{ccccc}
 & & HF^+(Y_1; M(W_1)) & \xrightarrow{\underline{F}_{W_2, \mathfrak{s}_2}^+} & HF^+(Y_2; M(W_1)(W_2)) \\
 & & \downarrow \tau & & \downarrow \tau \\
 HF^-(Y_0; M) & \xrightarrow{\underline{F}_{W_1, \mathfrak{s}_1}^-} & HF^-(Y_1; M(W_1)) & \xrightarrow{\underline{F}_{W_2, \mathfrak{s}_2}^-} & HF^-(Y_2; M(W_1)(W_2))
 \end{array}$$

together with part (1). □

With these preliminaries in place, we turn our attention to the proof of [Theorem 7.6](#) (with $\alpha_1 = \alpha_2 = \alpha = 1$). Thus, let $X = Z_1 \cup_Y Z_2$ be as in the statement of the theorem, and let us assume $b^+(Z_2) \geq 1$. Then we can find an admissible cut N for X contained in Z_2 (cf the construction in example 8.4 of [\[15\]](#)). Suppose X is decomposed into pieces V_1 and V_2 along N , so that

$$X = V_1 \cup_N V_2 = Z_1 \cup_Y W \cup_N V_2$$

where $W = V_1 \cap Z_2$ is a cobordism $Y \rightarrow N$.

Let us fix a spin^c structure \mathfrak{s} on X . For simplicity we will omit the spin^c structure from the notation for homomorphisms induced by cobordisms, but all relevant cobordisms and their boundaries will be equipped with spin^c structures obtained by restricting \mathfrak{s} .

By definition, we have

$$\begin{aligned}
 \Phi_{X, \mathfrak{s}}(1) &= \langle F_{V_2}^+ \circ \tau^{-1} \circ F_{V_1}^-(\Theta^-), \Theta^- \rangle \\
 &= \langle \tau^{-1} \circ F_{V_1}^-(\Theta^-), F_{V_2}^-(\Theta^-) \rangle \\
 &= \langle \tau^{-1} \circ \epsilon_* \circ \underline{F}_{V_1}^-(\Theta^-), \epsilon_* \circ \underline{F}_{V_2}^-(\Theta^-) \rangle \\
 (13) \quad &= \langle \tau^{-1} \circ \underline{F}_{V_1}^-(\Theta^-), \underline{F}_{V_2}^-(\Theta^-) \rangle.
 \end{aligned}$$

We have passed to twisted coefficients using the remark after [Theorem 2.9](#). The last line uses [Lemma 7.7](#) and the twisted pairing which takes values in $\mathbb{Z}[K(X, N)]$. Since N is admissible the group $K(X, N)$ is trivial and hence the pairing is \mathbb{Z} -valued; the homomorphism $\epsilon_* \otimes \epsilon_*$ arising from [Lemma 7.8](#) is the identity here.

According to [Theorem 2.9](#) we can find representatives for the maps involved that satisfy

$$\underline{F}_{V_1}^- = \Pi_{V_1} \circ \underline{F}_{\mathcal{W}}^- \circ \underline{F}_{Z_1}^-,$$

where Π_{V_1} is the map induced in homology by a projection map $\mathbb{Z}[K(V_1, Y)] \rightarrow \mathbb{Z}[K(V_1)]$, which we also denote by Π_{V_1} . Different choices of representatives for $[\underline{F}_{V_1}^-]$ and the other maps differ by the action of R_N on $\mathbb{Z}[K(X, N)] = \mathbb{Z}$, which is trivial. Hence we can replace [\(13\)](#) with

$$\begin{aligned} \Phi_{X,s}(1) &= \langle \tau^{-1} \circ \Pi_{V_1} \circ \underline{F}_{\mathcal{W}}^- \circ \underline{F}_{Z_1}^-(\Theta^-), \underline{F}_{V_2}^-(\Theta^-) \rangle \\ (14) \qquad &= \Pi_{V_1} \otimes 1 \cdot \langle \tau^{-1} \circ \underline{F}_{\mathcal{W}}^- \circ \underline{F}_{Z_1}^-(\Theta^-), \underline{F}_{V_2}^-(\Theta^-) \rangle. \end{aligned}$$

Lemma 7.10 *Under the isomorphism*

$$\mathbb{Z}[K(V_1, Y)] \otimes_{R_N} \mathbb{Z}[K(V_2)] \cong \mathbb{Z} \left[\frac{K(V_1, Y) \oplus K(V_2)}{H^1(N)} \right],$$

the map $\Pi_{V_1} \otimes 1$ corresponds to the homomorphism $\Pi_{\mathbb{Z}}$ sending an element of a group ring to the coefficient of the identity element.

Proof We have a diagram of identifications:

$$\begin{array}{ccc} \mathbb{Z}[K(V_1, Y)] \otimes_{R_N} \mathbb{Z}[K(V_2)] & \xrightarrow{\Pi_{V_1} \otimes 1} & \mathbb{Z}[K(V_1)] \otimes_{R_N} \mathbb{Z}[K(V_2)] \\ \downarrow = & & \downarrow = \\ \mathbb{Z} \left[\frac{K(V_1, Y) \oplus K(V_2)}{H^1(N)} \right] & \xrightarrow{p} & \mathbb{Z} \left[\frac{K(V_1) \oplus K(V_2)}{H^1(N)} \right] \end{array}$$

Again, since N is admissible

$$\frac{K(V_1) \oplus K(V_2)}{H^1(N)} = \ker(H^2(X) \rightarrow H^2(V_1) \oplus H^2(V_2)) = 0.$$

The projection p is induced by some map

$$\frac{K(V_1) \oplus K(V_2)}{H^1(N)} \rightarrow \frac{K(V_1, Y) \oplus K(V_2)}{H^1(N)},$$

for which there is only one choice since the domain group is trivial. The construction of p from this map proves the claim. \square

Returning with this to Equation (14), we have

$$\begin{aligned}
 \Phi_{X,s}(1) &= \Pi_{\mathbb{Z}} \langle \tau^{-1} \circ \underline{F}_{\mathcal{W}}^- \circ \underline{F}_{Z_1}^-(\Theta^-), \underline{F}_{V_2}^-(\Theta^-) \rangle \\
 &= \Pi_{\mathbb{Z}} \langle \underline{F}_{\mathcal{W}}^+ \circ \tau^{-1} \circ \underline{F}_{Z_1}^-(\Theta^-), \underline{F}_{V_2}^-(\Theta^-) \rangle \\
 (15) \qquad &= \Pi_{\mathbb{Z}} \langle \tau^{-1} \circ \underline{F}_{Z_1}^-(\Theta^-), \underline{F}_{\mathcal{W}}^- \circ \underline{F}_{V_2}^-(\Theta^-) \rangle
 \end{aligned}$$

using Lemma 7.9 and Theorem 4.3. Note that the pairings above can be thought of as taking values in

$$\mathbb{Z} \left[\frac{K(Z_1) \oplus K(W) \oplus K(V_2)}{H^1(Y) \oplus H^1(N)} \right]$$

with appropriate grading.

We would like to apply the composition law in (15) to replace $\underline{F}_{\mathcal{W}}^- \circ \underline{F}_{V_2}^-$ by $\underline{F}_{Z_2}^-$, but we are missing a factor of Π_{Z_2} required by Theorem 2.9. By commutativity of the square (*) in the following diagram, we are free to introduce this factor:

$$\begin{array}{ccc}
 \mathbb{Z}[K(Z_1)] \otimes_{R_Y} \overline{\mathbb{Z}[K(Z_2, N)]} & \xrightarrow{1 \otimes \Pi_{Z_2}} & \mathbb{Z}[K(Z_1)] \otimes_{R_Y} \overline{\mathbb{Z}[K(Z_2)]} \\
 \downarrow = & (*) & \downarrow = \\
 \mathbb{Z} \left[\frac{K(Z_1) \oplus K(Z_2, N)}{H^1(Y)} \right] & \longrightarrow & \mathbb{Z} \left[\frac{K(Z_1) \oplus K(Z_2)}{H^1(Y)} \right] \\
 \downarrow = & & \downarrow \Pi_Z \\
 \mathbb{Z} \left[\frac{K(Z_1) \oplus K(W) \oplus K(V_2)}{H^1(Y) \oplus H^1(N)} \right] & \xrightarrow{\Pi_Z} & \mathbb{Z}
 \end{array}$$

Indeed, it follows that $\Pi_{\mathbb{Z}} = \Pi_{\mathbb{Z}} \circ (1 \otimes \Pi_{Z_2})$ (after identifying the groups in the column on the left). Thus (15) becomes:

$$\begin{aligned}
 \Phi_{X,s}(1) &= \Pi_{\mathbb{Z}} \circ (1 \otimes \Pi_{Z_2}) \cdot \langle \tau^{-1} \circ \underline{F}_{Z_1}^-(\Theta^-), \underline{F}_{\mathcal{W}}^- \circ \underline{F}_{V_2}^-(\Theta^-) \rangle \\
 &= \Pi_{\mathbb{Z}} \langle \tau^{-1} \circ \underline{F}_{Z_1}^-(\Theta^-), \Pi_{Z_2} \circ \underline{F}_{\mathcal{W}}^- \circ \underline{F}_{V_2}^-(\Theta^-) \rangle \\
 &= \Pi_{\mathbb{Z}} \langle \tau^{-1} \circ \underline{F}_{Z_1}^-(\Theta^-), \underline{F}_{Z_2}^-(\Theta^-) \rangle,
 \end{aligned}$$

after possibly translating by an element of R_Y . This verifies the ‘‘constant coefficient’’ of (12). For the general statement, suppose $t = \delta h \in K(X, Y)$. Then since $s + t = s$ when restricted to V_2 we can follow the same steps as above (and using the second

part of [Theorem 2.9](#)) to see

$$\begin{aligned}\Phi_{X,s+t}(1) &= \langle \tau^{-1} \circ F_{V_1,s+t}^-(\Theta^-), F_{V_2,s}^-(\Theta) \rangle \\ &= \langle \tau^{-1} \circ \underline{F}_{V_1,s+t}^-(\Theta^-), \underline{F}_{V_2,s}^-(\Theta^-) \rangle \\ &= \langle \tau^{-1} \circ \Pi_{V_1} \circ \underline{F}_{W,s}^- \circ e^{-h} \cdot \underline{F}_{Z_1,s}^-(\Theta^-), \underline{F}_{V_2,s}^-(\Theta^-) \rangle \\ &= \Pi_{\mathbb{Z}}[e^{-h} \cdot \langle \tau^{-1} \circ \underline{F}_{Z_1,s}^-(\Theta^-), \underline{F}_{Z_2,s}^-(\Theta^-) \rangle]\end{aligned}$$

where we can use the same representatives for $[\underline{F}_{Z_i,s}^-]$ as before. Since the action of R_Y on $\mathbb{Z}[K(X, Y)]$ is via the coboundary, this last expression is exactly the coefficient of $e^{\delta h} = e^t$ in $\langle \tau^{-1} \circ \underline{F}_{Z_1,s}^-(\Theta^-), \underline{F}_{Z_2,s}^-(\Theta^-) \rangle$, completing the proof of [Theorem 7.6](#).

8 Perturbed Heegaard Floer invariants

The utility of [Theorem 7.6](#) is limited in many practical circumstances by the restriction on $b^+(Z_i)$. In particular, if one wishes to split a 4–manifold along the boundary of a tubular neighborhood of a surface of square 0, it is not obvious whether the assumptions of that theorem are satisfied. In this section we show how to remedy this circumstance by making use of Heegaard Floer homology “perturbed” by a 2–dimensional cohomology class $\eta \in H^2(Y; \mathbb{R})$. (A version of this theory was mentioned briefly in [\[12\]](#); here we give a rather fuller treatment.)

8.1 Definitions and basic properties

Definition 8.1 Fix a closed oriented 3–manifold Y and a class $\eta \in H^2(Y; \mathbb{R})$. The Novikov ring associated to (Y, η) is the ring $\mathcal{R}_{Y,\eta} \subset \mathbb{Z}^{H^1(Y; \mathbb{Z})}$ of \mathbb{Z} –valued functions on $H^1(Y; \mathbb{Z})$ defined by the condition that $f \in \mathcal{R}_{Y,\eta}$ if and only if for each $N \in \mathbb{Z}$, the set $\text{supp}(f) \cap \{a \in H^1(Y) \mid \langle a \cup \eta, [Y] \rangle < N\}$ is finite.

More concretely, we can think of $\mathcal{R}_{Y,\eta}$ as the collection of formal series

$$\mathcal{R}_{Y,\eta} = \left\{ \sum_{g \in H^1(Y; \mathbb{Z})} a_g \cdot g \mid a_g \in \mathbb{Z} \right\}$$

subject to the condition that for each $N \in \mathbb{Z}$ the set of $g \in H^1(Y)$ with a_g nonzero and $\langle g \cup \eta, [Y] \rangle < N$ is finite.

The multiplication on $\mathcal{R}_{Y,\eta}$ is the usual convolution product; note that in the case $\eta = 0$ we have $\mathcal{R}_{Y,\eta} = \mathbb{Z}[H^1(Y)]$. Clearly, $\mathcal{R}_{Y,\eta} = \mathcal{R}_{Y,c\eta}$ for any positive constant c . Furthermore, $\mathcal{R}_{Y,\eta}$ depends on the orientation of Y in the sense that $\mathcal{R}_{-Y,\eta} = \mathcal{R}_{Y,-\eta}$.

The above is an instance of a general construction: given an abelian group G and a homomorphism $\eta: G \rightarrow \mathbb{R}$, we can form an associated Novikov ring $\mathcal{N}(G, \eta)$ as a subset of \mathbb{Z}^G . Namely,

$$\mathcal{N}(G, \eta) = \left\{ \sum_{g \in G} a_g \cdot g \mid a_g \in \mathbb{Z} \right\},$$

with the condition that for each $N \in \mathbb{Z}$ the set of $g \in G$ with a_g nonzero and $\eta(g) < N$ is finite. We now pause to remark on the functoriality of this construction.

Suppose (G, η) and (H, ν) are two groups with homomorphisms to \mathbb{R} , and $f: G \rightarrow H$ is a group homomorphism. There are two ways in which one might wish to associate a homomorphism between $\mathcal{N}(G, \eta)$ and $\mathcal{N}(H, \nu)$. First is a “pushforward”

$$\hat{f}_*: \mathcal{N}(G, \eta) \rightarrow \mathcal{N}(H, \nu)$$

defined by

$$\hat{f}_* \left(\sum_{g \in G} a_g \cdot g \right) = \sum_{g \in G} a_g \cdot f(g) = \sum_{h \in H} b_h \cdot h,$$

where b_h is the sum of all a_g for $g \in f^{-1}(h)$. For the above to make sense we must have that each collection $\{g \in f^{-1}(h) \mid a_g \neq 0\}$ is finite, and for the above to define an element of $\mathcal{N}(H, \nu)$ we need the set $\{h \mid h \in \text{im}(f), \nu(h) < N\}$ to be finite for each N . It is easy to see that both of these properties are satisfied if whenever $S \subset H$ is a subset on which ν is bounded, the function η is bounded on $f^{-1}(S)$, which in turn is true if $\eta = \nu \circ f$. In this situation, the expression above gives a well-defined function \hat{f}_* , which is easily seen to be a ring homomorphism.

On the other hand we can define a “pullback” map

$$\hat{f}^*: \mathcal{N}(H, \nu) \rightarrow \mathcal{N}(G, \eta)$$

by

$$\hat{f}^* \left(\sum_{h \in H} b_h \cdot h \right) = \sum_{g \in G} b_{f(g)} \cdot g.$$

The expression above defines an element of $\mathcal{N}(G, \eta)$ if the set $\{g \mid b_{f(g)} \neq 0 \text{ and } \eta(g) < N\}$ is finite for each N . This will happen if whenever η is bounded on a subset $S \subset G$ we have that ν is bounded on $f(S)$, and additionally, $\ker(f)$ is finite. In particular, if $\eta = \nu \circ f$ and f is injective, \hat{f}^* is a well-defined function. In general, \hat{f}^* is not a ring homomorphism. However, we can regard $\mathcal{N}(H, \nu)$ as a module over $\mathcal{N}(G, \eta)$ using the homomorphism \hat{f}_* constructed above, and then one can check that \hat{f}^* is a homomorphism of $\mathcal{N}(G, \eta)$ -modules.

To summarize:

Lemma 8.2 *Suppose G and H are abelian groups and $\eta: G \rightarrow \mathbb{R}$ and $\nu: H \rightarrow \mathbb{R}$ are additive homomorphisms. Let $f: G \rightarrow H$ be a homomorphism with the property that $\eta = \nu \circ f$. Then there is a well-defined homomorphism*

$$\widehat{f}_*: \mathcal{N}(G, \eta) \rightarrow \mathcal{N}(H, \nu)$$

of Novikov rings.

If in addition f is injective, then there is a well-defined pullback or “restriction” map

$$\widehat{f}^*: \mathcal{N}(H, \nu) \rightarrow \mathcal{N}(G, \eta),$$

which is a homomorphism of $\mathcal{N}(G, \eta)$ –modules.

It is easy to see that the above constructions are functorial with respect to compositions of homomorphisms. Furthermore, if f is injective then clearly $\widehat{f}^* \circ \widehat{f}_* = \text{id}_{\mathcal{N}(G, \eta)}$, so in this case \widehat{f}^* is surjective and \widehat{f}_* is injective.

We can now recite the definition of twisted-coefficient Heegaard Floer homology using the Novikov ring $\mathcal{R}_{Y, \eta}$ in place of R_Y .

Definition 8.3 Let (Y, \mathfrak{s}) be a closed oriented spin^c 3–manifold, and let $\eta \in H^2(Y; \mathbb{R})$. Endow $\mathcal{R}_{Y, \eta}$ with the \mathfrak{s} –grading defined by (5). Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a marked Heegaard diagram for Y , and choose an additive assignment A for the diagram. The η –perturbed Heegaard Floer complex is the free $\mathcal{R}_{Y, \eta}$ –module $CF^\infty(Y, \mathfrak{s}; \mathcal{R}_{Y, \eta})$ generated by pairs $[\mathbf{x}, i]$ where $\mathbf{x} \in T_\alpha \cap T_\beta$ is an intersection point with $s_z(\mathbf{x}) = \mathfrak{s}$, equipped with the relative \mathbb{Z} grading defined in (6).

The boundary operator is given as in Definition 2.3, where $e^{A(\phi)}$ is interpreted as an element of $\mathcal{R}_{Y, \eta}$.

If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is strongly \mathfrak{s} –admissible, in the sense of [12, Definition 4.10], then the definition above obviously yields the Heegaard Floer complex for the unperturbed theory with coefficients in the R_Y –module $\mathcal{R}_{Y, \eta}$, ie, the complex $CF^\circ(Y, \mathfrak{s}; R_Y) \otimes_{R_Y} \mathcal{R}_{Y, \eta}$.

In fact, the perturbed complex can be defined with relaxed admissibility hypotheses: if η is generic in the sense that the induced map $H^1(Y; \mathbb{Z}) \rightarrow \mathbb{R}$ is injective, weak admissibility suffices to define $HF^\infty(Y, \mathfrak{s}; \mathcal{R}_{Y, \eta})$ and $HF^-(Y, \mathfrak{s}; \mathcal{R}_{Y, \eta})$, while no admissibility conditions are necessary to define $HF^+(Y, \mathfrak{s}; \mathcal{R}_{Y, \eta})$ or $\widehat{HF}(Y, \mathfrak{s}; \mathcal{R}_{Y, \eta})$. However, we have no need for this generality, and the observation in the previous paragraph suffices to show that the perturbed Heegaard Floer homology is a topological invariant of (Y, \mathfrak{s}, η) .

Note that if M is a (graded) module for R_Y , we can obtain a module for $\mathcal{R}_{Y,\eta}$ by tensor product: $\mathcal{M}_\eta \equiv M \otimes_{R_Y} \mathcal{R}_{Y,\eta}$. Thus we can consider perturbed Heegaard Floer homology with coefficients in the “completed” module \mathcal{M}_η , namely the homology of the complex $CF(Y, \mathfrak{s}; \mathcal{R}_{Y,\eta}) \otimes_{\mathcal{R}_{Y,\eta}} \mathcal{M}_\eta$ (of course, since any module for $\mathcal{R}_{Y,\eta}$ is also a module for R_Y , we see trivially that any $\mathcal{R}_{Y,\eta}$ module is obtained in this way).

Calculation of perturbed Floer homology is facilitated by the following.

Lemma 8.4 *For any $\eta \in H^2(Y; \mathbb{R})$, the ring $\mathcal{R}_{Y,\eta}$ is flat as an R_Y -module.*

Proof Let K denote the kernel of the homomorphism $H^1(Y; \mathbb{Z}) \rightarrow \mathbb{R}$ given by $x \mapsto \langle x \cup \eta, [Y] \rangle$; note that K is a direct summand of $H^1(Y; \mathbb{Z})$. Let $\text{rk}(K) = k$. The ring R_Y can be identified with a Laurent polynomial ring in variables $\{x_1, \dots, x_b\}$, $b = b_1(Y)$, and we can choose the generators x_i such that $\langle x_i \cup \eta, [Y] \rangle = 0$ for $i = 1, \dots, k$, while $\langle x_i \cup \eta, [Y] \rangle > 0$ for $i > k$. The Novikov ring $\mathcal{R}_{Y,\eta}$ can be constructed as follows. First let \mathcal{Z}_η denote the (“partial”) power series ring obtained by completing the ring $\mathcal{Z} = \mathbb{Z}[x_1, \dots, x_b]$ with respect to the ideal generated by x_{k+1}, \dots, x_b . Then if V denotes the multiplicative subset generated by the variables x_1, \dots, x_b , we have that $\mathcal{R}_{Y,\eta} = V^{-1}\mathcal{Z}_\eta$. It is a standard fact that \mathcal{Z}_η is flat over \mathcal{Z} (see, eg, [1, Theorem 7.2]), and it follows easily that $\mathcal{R}_{Y,\eta} = V^{-1}\mathcal{Z}_\eta$ is flat over $R_Y = V^{-1}\mathcal{Z}$. \square

Definition 8.5 Let Y be a closed oriented 3-manifold, $\mathfrak{s} \in \text{Spin}^c(Y)$, and $\eta \in H^2(Y; \mathbb{R})$. We say η is *generic for \mathfrak{s}* if $\ker(c_1(\mathfrak{s})) \not\subset \ker(\eta)$. That is to say, η is generic for \mathfrak{s} if there exists a class $x \in H^1(Y)$ such that

$$\eta \cup x \neq 0 \quad \text{but} \quad c_1(\mathfrak{s}) \cup x = 0.$$

Observe that if $c_1(\mathfrak{s})$ is torsion and η is nonzero then η is automatically generic for \mathfrak{s} , while if $b_1(Y) = 1$ and $c_1(\mathfrak{s})$ is nontorsion, then a generic class η for \mathfrak{s} does not exist. Once $b_1(Y) > 1$, however, any class $\eta \in H^2(Y; \mathbb{R})$ that is “generic” in the sense that $\langle \eta \cup \cdot, [Y] \rangle: H^1(Y; \mathbb{Z}) \rightarrow \mathbb{R}$ is injective, is automatically generic for any spin^c structure \mathfrak{s} .

In Seiberg–Witten theory, once $b_1(Y) > 0$ it is possible to “perturb away” reducible solutions to the Seiberg–Witten equations on Y . The following can be seen as an analog of that statement in Heegaard Floer theory.

Corollary 8.6 *If $\eta \in H^2(Y; \mathbb{R})$ is generic for a spin^c structure \mathfrak{s} on Y , then*

$$HF^\infty(Y, \mathfrak{s}; \mathcal{R}_{Y,\eta}) = 0,$$

and therefore $HF^\infty(Y, \mathfrak{s}; \mathcal{M}) = 0$ for any $\mathcal{R}_{Y,\eta}$ –module \mathcal{M} . In particular, under this assumption, for any R_Y –module M with completion $\mathcal{M}_\eta = M \otimes_{R_Y} \mathcal{R}_{Y,\eta}$ we have isomorphisms

$$HF^\pm(Y, \mathfrak{s}; \mathcal{M}_\eta) = HF_{\text{red}}^\pm(Y, \mathfrak{s}; \mathcal{M}_\eta) \cong HF_{\text{red}}^\pm(Y, \mathfrak{s}; M) \otimes_{R_Y} \mathcal{R}_{Y,\eta}.$$

Proof By the previous lemma, $HF^\infty(Y, \mathfrak{s}; \mathcal{R}_{Y,\eta}) \cong HF^\infty(Y, \mathfrak{s}; R_Y) \otimes_{R_Y} \mathcal{R}_{Y,\eta}$. Ozsváth and Szabó showed [11, Theorem 10.12, Remark 10.13] that for any 3–manifold Y , the fully-twisted Floer homology satisfies

$$HF^\infty(Y, \mathfrak{s}; R_Y) \cong \mathbb{Z}[U, U^{-1}]$$

where an element $x \in H^1(Y; \mathbb{Z})$ having $\langle c_1(\mathfrak{s}) \cup x, [Y] \rangle = 2k$ acts as multiplication by U^k . Take x to be as in the definition of generic above, so that $k = 0$. Without loss of generality we can assume that $\langle \eta \cup x, [Y] \rangle > 0$, so that the element $1 - x$ has an inverse $\sum_{n \geq 0} x^n$ in $\mathcal{R}_{Y,\eta}$. But then $1 - x$ is a unit that acts as 0 on $HF^\infty(Y, \mathfrak{s}; \mathcal{R}_{Y,\eta})$, meaning the latter module must vanish. The remaining statements follow easily from the flatness of $\mathcal{R}_{Y,\eta}$. \square

As noted above, it is not always possible to guarantee the existence of a generic perturbation (namely when $b_1(Y) \leq 1$). Of more concern for our purposes, a similar situation arises when considering cobordisms $W: Y_1 \rightarrow Y_2$, where perturbations and spin^c structures on Y_i are taken to be induced from W . Here if $b_2(W) = 1$, for example, then any class η induced from W must be a multiple of the Chern class of a spin^c structure on Y_2 induced from W , and again we cannot arrange genericity regardless of the value of $b_1(Y)$.

To deal with this situation we make a further completion of Heegaard Floer homology, this time with respect to U .

Definition 8.7 Let $\mathbb{Z}[[U]]$ denote the ring of integer power series in U . The U –completed Heegaard Floer groups for (Y, \mathfrak{s}, η) in a module \mathcal{M} are defined by

$$HF_\bullet^\circ(Y, \mathfrak{s}; \mathcal{M}) = HF^\circ(Y, \mathfrak{s}; \mathcal{M}) \otimes_{\mathbb{Z}[U]} \mathbb{Z}[[U]].$$

Thus the perturbed, completed Floer homology $HF_\bullet^\circ(Y, \mathfrak{s}; \mathcal{M})$ is a module for $\mathcal{R}_{Y,\mathfrak{s}}[[U]]$. Observe that since the action of U is nilpotent on elements of HF^+ , this completion has no effect on the latter group:

$$HF_\bullet^+(Y, \mathfrak{s}; \mathcal{M}) = HF^+(Y, \mathfrak{s}; \mathcal{M}).$$

There is a natural map $HF^\circ(Y, \mathfrak{s}; \mathcal{M}) \rightarrow HF_\bullet^\circ(Y, \mathfrak{s}; \mathcal{M})$ that is typically (when η is generic for \mathfrak{s} , for example) an injection. We will often implicitly make use of this homomorphism when extending previous results to the U -completed setting.

The definition is most useful when the uncompleted group $HF^\circ(Y, \mathfrak{s}; \mathcal{M})$ carries a relative \mathbb{Z} grading (not a cyclic grading). We will generally be interested in coefficient modules \mathcal{M} that arise from cobordisms $S^3 \rightarrow Y$, and in light of [Proposition 3.5](#) we will therefore be in the \mathbb{Z} -graded case.

Corollary 8.8 *Let (Y, \mathfrak{s}) be a closed spin^c 3-manifold and $\eta \in H^2(Y; \mathbb{R})$ a fixed class. If $c_1(\mathfrak{s})$ is torsion, assume that $\eta \neq 0$. Then*

$$HF_\bullet^\infty(Y, \mathfrak{s}; \mathcal{M}_\eta) = 0$$

for any $\mathcal{R}_{Y, \eta}$ module \mathcal{M}_η .

The other conclusions of [Corollary 8.6](#) of course follow as well for the U -completed Floer homology perturbed by a compatible class η . Note that if $c_1(\mathfrak{s})$ is nontorsion, then it suffices to take $\eta = 0$.

Proof It suffices to show the vanishing with coefficients in $\mathcal{R}_{Y, \eta}$; if $c_1(\mathfrak{s})$ is torsion then a nonzero η is necessarily generic for \mathfrak{s} so that [Corollary 8.6](#) applies. Otherwise, we can find $t \in H^1(Y)$ such that $\langle c_1(\mathfrak{s}) \cup t, [Y] \rangle = -2k$ with $k > 0$; then as before t acts as multiplication by U^{-k} on $HF^\infty(Y, \mathfrak{s}; R_Y)$. Hence the element $1 - tU^k$ acts as 0, but the former is a unit in the completed ring $\mathcal{R}_{Y, \eta}[[U]]$. \square

We now wish to extend perturbed Heegaard Floer theory to cobordism-induced homomorphisms. To do so, we again follow the program from the unperturbed case; we need only make sure that the coefficient modules respect the algebraic nature of the Novikov rings.

Definition 8.9 Let $W: Y_1 \rightarrow Y_2$ be an oriented cobordism between 3-manifolds Y_i , and fix $\eta \in H^2(W; \mathbb{R})$ with restrictions $\eta_i = \eta|_{Y_i}$. Let $\mathcal{K}(W, \eta)$ be the Novikov completion of $\mathbb{Z}[K(W)]$ with respect to η , where as usual $K(W) = \text{im}(H^1(\partial W; \mathbb{Z}) \rightarrow H^2(W, \partial W; \mathbb{Z}))$. Concretely, $\mathcal{K}(W, \eta)$ is the ring of formal series

$$\mathcal{K}(W, \eta) = \left\{ \sum_{g \in K(W)} a_g \cdot g \mid a_g \in \mathbb{Z} \right\}$$

subject to the condition that for each $N \in \mathbb{Z}$ the set of $g \in K(W)$ with a_g nonzero and $\langle g \cup \eta, [W, \partial W] \rangle < N$ is finite. Then $\mathcal{K}(W, \eta)$ is a module for both $\mathcal{R}_{-Y_1, \eta_1}$ and $\mathcal{R}_{Y_2, \eta_2}$.

If \mathcal{M} is a (graded) module for $\mathcal{R}_{Y_1, \eta_1}$, the module $\mathcal{M}(W, \eta)$ for $\mathcal{R}_{Y_2, \eta_2}$ induced by (W, η) is defined by

$$\mathcal{M}(W, \eta) = \bar{\mathcal{M}} \otimes_{\mathcal{R}_{-Y_1, \eta_1}} \mathcal{K}(W, \eta).$$

The module structure of $\mathcal{K}(W, \eta)$ over $\mathcal{R}_{-Y_1, \eta_1}$ and $\mathcal{R}_{Y_2, \eta_2}$ is induced by the maps $\delta: H^1(Y_i) \rightarrow H^2(W, \partial W)$, as usual. The fact that these maps induce well-defined homomorphisms between Novikov rings relies on [Lemma 8.2](#), together with the fact that η_i is the restriction of η . Note that $\mathcal{K}(W, \eta)$ can be given an integer grading depending on a choice of spin^c structure just as in [Proposition 3.5](#).

For the conjugate module appearing in the last statement of the definition, observe that the map $x \mapsto -x$ in $H^1(Y)$ induces a conjugation map $\mathcal{R}_{Y, \eta} \rightarrow \mathcal{R}_{-Y, \eta}$. Thus $\bar{\mathcal{M}}$, defined to be the same graded group as \mathcal{M} with conjugate module structure, makes sense as a graded $\mathcal{R}_{-Y, \eta}$ -module.

It is now straightforward to define a homomorphism

$$F_{W, \eta, \mathfrak{s}}^\circ: HF_\bullet^\circ(Y_1, \mathfrak{s}_1, \mathcal{M}) \rightarrow HF_\bullet^\circ(Y_2, \mathfrak{s}_2, \mathcal{M}(W, \eta))$$

associated to a spin^c cobordism (W, \mathfrak{s}) with chosen perturbation η (or similar maps between the groups without the “ \bullet ”), by making the usual formal construction for 1– and 3–handles, and using [Definition 2.6](#) for the 2–handles, where $e^{A_W(\psi)}$ is considered to lie in $\mathcal{K}(W, \eta)$. The proof that the result of this construction is a chain map whose induced map in homology is an invariant of W (up to a sign and the action of $H^1(Y_1; \mathbb{Z}) \oplus H^1(Y_2; \mathbb{Z})$) is identical to the proof in the unperturbed case in [\[15\]](#). Alternatively, one can deduce this fact from the corresponding fact in the unperturbed theory using [Lemma 8.4](#).

Similarly, there is a composition law for perturbed cobordism maps that follows from the usual one given in [Theorem 2.9](#). Indeed, suppose we are given a cobordism $W: Y_1 \rightarrow Y_2$ and a module M for R_Y , along with a class $\eta \in H^2(W; \mathbb{R})$. Write $\mathcal{M} = M \otimes_{R_{Y_1}} \mathcal{R}_{Y_1, \eta}$ for the Novikov completion of M (we do not distinguish in the notation between η and its restrictions to Y_1, Y_2); then \mathcal{M} can also be considered as an R_{Y_1} -module. As such, we obtain an induced R_{Y_2} -module $\mathcal{M}(W) = \bar{\mathcal{M}} \otimes_{R_{-Y_1}} \mathbb{Z}[K(W)]$. It is not hard to see that the $\mathcal{R}_{Y_2, \eta}$ -module induced by (W, η) is then

$$(16) \quad \mathcal{M}(W, \eta) = \mathcal{M}(W) \otimes_{R_{Y_2}} \mathcal{R}_{Y_2, \eta} = \mathcal{R}_{-Y_1, \eta} \otimes_{R_{-Y_1}} M(W) \otimes_{R_{Y_2}} \mathcal{R}_{Y_2, \eta}$$

and we have a commutative diagram (with or without \bullet 's)

$$(17) \quad \begin{array}{ccc} HF_{\bullet}(Y_1, \mathcal{M}) & \xrightarrow{F_W} & HF_{\bullet}(Y_2, \mathcal{M}(W)) \xrightarrow{\cdot \otimes 1} HF_{\bullet}(Y_2, \mathcal{M}(W)) \otimes_{R_{Y_2}} \mathcal{R}_{Y_2, \eta} \\ & \searrow^{F_{W, \eta}} & \downarrow \\ & & HF_{\bullet}(Y_2, \mathcal{M}(W, \eta)) \end{array}$$

where the vertical arrow is an isomorphism according to [Lemma 8.4](#). Combining this observation with the original composition law gives the desired result for perturbed Floer homology.

All of the algebraic constructions earlier introduced for twisted coefficients go through with only minor modifications in the perturbed setup. The action of $H_1(Y; \mathbb{Z})/\text{tors}$ on $HF_{\bullet}^{\circ}(Y, \mathfrak{s}; \mathcal{M})$ for \mathcal{M} an $\mathcal{R}_{Y, \eta}$ -module is defined just as before, as is the extension of cobordism-induced maps to incorporate this action. Likewise the previous definition applies to give a pairing

$$HF_{\bullet}^{+}(Y, \mathfrak{s}; \mathcal{M}) \otimes_{\mathcal{R}_{Y, \eta}} \overline{HF_{\bullet}^{-}(-Y, \mathfrak{s}; \mathcal{N})} \rightarrow \mathcal{M} \otimes_{\mathcal{R}_{Y, \eta}} \overline{\mathcal{N}}$$

for any $\mathcal{R}_{Y, \eta}$ -module \mathcal{M} and $\mathcal{R}_{-Y, \eta}$ -module \mathcal{N} (and similarly without \bullet 's).

8.2 Conjugation and orientation reversal

The perturbed versions of the results of [Section 6](#) are straightforward generalizations, with the caveat that conjugation $\mathcal{R}_{Y, \mathfrak{s}} \rightarrow \mathcal{R}_{Y, \bar{\mathfrak{s}}}$ extends to the setting of Novikov rings only at the cost of reversing the sign of η . Indeed, if $r \in \mathcal{R}_{Y, \mathfrak{s}, \eta}$, we can consider the conjugate \bar{r} to lie either in $\mathcal{R}_{-Y, \mathfrak{s}, \eta}$ or in $\mathcal{R}_{Y, \bar{\mathfrak{s}}, -\eta}$. Hence if \mathcal{M} is a module for $\mathcal{R}_{Y, \mathfrak{s}, \eta}$, we can think of $\bar{\mathcal{M}}$ as a module either for $\mathcal{R}_{-Y, \mathfrak{s}, \eta}$ or for $\mathcal{R}_{Y, \bar{\mathfrak{s}}, -\eta}$.

Theorem 8.10 *If (Y, \mathfrak{s}) is a closed spin^c 3-manifold with class $\eta \in H^2(Y; \mathbb{R})$, and \mathcal{M} is a module for $\mathcal{R}_{Y, \eta}$, then we have an isomorphism of $\mathcal{R}_{Y, \eta}$ -modules*

$$HF_{\bullet}^{\circ}(Y, \mathfrak{s}; \mathcal{M}) \cong \overline{HF_{\bullet}^{\circ}(Y, \bar{\mathfrak{s}}, \bar{\mathcal{M}})}$$

preserving relative gradings, where $\bar{\mathcal{M}}$ is considered as a module for $\mathcal{R}_{Y, \bar{\mathfrak{s}}, -\eta}$. In particular,

$$HF_{\bullet}^{\circ}(Y, \mathfrak{s}; \mathcal{R}_{Y, \mathfrak{s}, \eta}) \cong \overline{HF_{\bullet}^{\circ}(Y, \bar{\mathfrak{s}}, \overline{\mathcal{R}_{Y, \mathfrak{s}, \eta})}} \cong \overline{HF_{\bullet}^{\circ}(Y, \bar{\mathfrak{s}}, \mathcal{R}_{Y, \bar{\mathfrak{s}}, -\eta})}. \quad \square$$

Thus in the perturbed case, there is a natural equivalence between Floer homology for spin^c structure \mathfrak{s} perturbed by a form η , and the homology for $\bar{\mathfrak{s}}$ perturbed by $-\eta$.

Theorem 8.11 For (Y, \mathfrak{s}) a closed oriented spin^c 3–manifold with class $\eta \in H^2(Y; \mathbb{R})$, and \mathcal{M} a module for $\mathcal{R}_{Y, \eta}$, there is an isomorphism

$$HF_{\bullet}^{\pm}(Y, \mathfrak{s}; \mathcal{M}) \cong \overline{HF_{\mp}^{\bullet}(-Y, \mathfrak{s}; \overline{\mathcal{M}})},$$

of relatively graded $\mathcal{R}_{Y, \mathfrak{s}, \eta}$ –modules, where $\overline{\mathcal{M}}$ is a module for $\mathcal{R}_{-Y, \mathfrak{s}, \eta}$. \square

As usual, there are obvious analogues of each of these results before taking U –completions.

8.3 Perturbed 4–manifold invariants

We are now in a position to define invariants for closed 4–manifolds using perturbed Floer homology. If (Z, \mathfrak{s}) is a spin^c 4–manifold with boundary Y and $\eta \in H^2(Z; \mathbb{R})$, we define the *perturbed relative invariant* for (Z, \mathfrak{s}, η) to be the map

$$\Psi_{Z, \mathfrak{s}, \eta}: \mathbb{A}(Z) \rightarrow HF_{\bullet}^{-}(Y, \mathfrak{s}; \mathcal{K}(Z, \eta))$$

given by $\Psi_{Z, \mathfrak{s}, \eta}(U^n \otimes \omega) = [F_{Z, \mathfrak{s}, \eta}^{-}(U^n \cdot \Theta^{-} \otimes \omega)]$, where the brackets indicate the equivalence class under the action of $K(Z)$ as before. Here $\mathcal{K}(Z, \eta)$ is the $\mathcal{R}_{Y, \eta}$ –module induced by Z , thought of as a cobordism $S^3 \rightarrow Y$; in other words $\mathcal{K}(Z, \eta)$ is the Novikov completion of $\mathbb{Z}[\ker(H^2(Z, Y) \rightarrow H^2(Z))]$ with respect to η .

Definition 8.12 Let X be a closed 4–manifold and $\eta \in H^2(X; \mathbb{R})$. An oriented 3–dimensional embedded submanifold $Y \subset X$ is an *allowable cut for η* if Y separates X into two components, $X = Z_1 \cup_Y Z_2$ with $\partial Z_1 = Y = -\partial Z_2$, and at least one of the following conditions are satisfied:

- (1) $\eta|_Y \neq 0$.
- (2) $b^+(Z_i) \geq 1$ for $i = 1, 2$.

Observe that if property (1) of the definition holds, then it follows from [Corollary 8.6](#) that the induced map $F_{Z, \mathfrak{s}, \eta}^{\infty}$ is trivial in perturbed, U –completed Floer homology for any spin^c structure on X . (Indeed, if the restriction of $c_1(\mathfrak{s})$ to Y is a nontorsion class then we need not even assume (1), but of course in this case one can always find a class η satisfying (1), namely the image in real cohomology of $c_1(\mathfrak{s})$. To avoid complicating the statements of results to follow, we ignore this point.)

On the other hand, if $W: Y_1 \rightarrow Y_2$ is a cobordism with $b^+(W) > 0$ and $\eta \in H^2(W; \mathbb{R})$, then since the unperturbed map in HF^{∞} induced by W is trivial, the same is true for the map perturbed by η , whether η vanishes on ∂W or not. Hence the perturbed relative invariant $\Psi_{Z, \mathfrak{s}, \eta}$, for a component of X arising from a cut allowable for η , takes values in the reduced Floer homology in both cases, and the following makes sense.

Definition 8.13 Let X be a closed oriented 4–manifold and \mathfrak{s} a spin^c structure on X . For a pair (Y, η) consisting of an element $\eta \in H^2(X; \mathbb{R})$ and a cut $X = Z_1 \cup_Y Z_2$ of X that is allowable for η , the *perturbed Ozsváth–Szabó invariant* of X associated to (Y, η, \mathfrak{s}) is the linear map $\Phi_{X,Y,\eta,\mathfrak{s}}: \mathbb{A}(X) \rightarrow \mathcal{K}(X, Y, \eta)$ defined by

$$\Phi_{X,Y,\eta,\mathfrak{s}}(\alpha) = \langle \tau^{-1} \Psi_{Z_1,\mathfrak{s},\eta}(\alpha_1), \Psi_{Z_2,\mathfrak{s},\eta}(\alpha_2) \rangle,$$

up to sign and multiplication by an element of $K(X, Y)$. Here α is the image of $\alpha_1 \otimes \alpha_2$ under the natural map $\mathbb{A}(Z_1) \otimes \mathbb{A}(Z_2) \rightarrow \mathbb{A}(X)$.

In this definition, we set

$$\mathcal{K}(X, Y, \eta) = \mathcal{K}(Z_1, \eta) \otimes_{\mathcal{R}_{Y,\eta}} \overline{\mathcal{K}(Z_2, \eta)}.$$

This can be identified with the Novikov completion of the R_Y –module $\mathbb{Z}[K(X, Y)]$ with respect to η as in [Lemma 2.7](#). That is to say, $\mathcal{K}(X, Y, \eta) = \mathbb{Z}[K(X, Y)] \otimes_{R_Y} \mathcal{R}_{Y,\eta}$. Note that $\Phi_{X,Y,\eta,\mathfrak{s}}$ depends on the orientation of Y in the sense that $\Phi_{X,Y,\eta,\mathfrak{s}} = \overline{\Phi_{X,-Y,\eta,\mathfrak{s}}}$. Indeed, it is easy to see that the two are related by the action of the obvious antihomomorphism $\mathcal{K}(Z_1, \eta) \otimes \overline{\mathcal{K}(Z_2, \eta)} \rightarrow \mathcal{K}(Z_2, \eta) \otimes \overline{\mathcal{K}(Z_1, \eta)}$, which in turn corresponds to the conjugation homomorphism $\mathcal{K}(X, Y, \eta) \rightarrow \mathcal{K}(X, -Y, \eta)$.

We will see that when $b^+(X) \geq 2$, the definition above recovers the ordinary Ozsváth–Szabó invariants in the sense of [Theorem 7.6](#): that is, $\Phi_{X,Y,\eta,\mathfrak{s}}$ lies in $\mathbb{Z}[K(X, Y)]$, and the coefficients of this group ring element are the Ozsváth–Szabó invariants of X in the various spin^c structures that have given restrictions to Z_1 and Z_2 . The utility of this definition is that we no longer need to assume that $b^+(Z_i) \geq 1$ or even that $F_{Z_i,\mathfrak{s}}^-$ takes values in the reduced Floer homology, so long as $\eta|_Y \neq 0$.

It should be noted, however, that the existence of a 3–manifold Y separating X and a class $\eta \in H^2(X, \mathbb{R})$ restricting nontrivially to Y implies that X is indefinite; in particular $b^+(X) \geq 1$.

Lemma 8.14 Suppose X is a 4–manifold with $b^+(X) \geq 2$, Y a submanifold splitting X into components Z_1 and Z_2 with $b^+(Z_i) \geq 1$ (or more generally satisfying the hypotheses of [Theorem 7.6](#)), and $\eta \in H^2(X; \mathbb{R})$ a perturbing class on X . Then for any spin^c structure \mathfrak{s} on X , $\Phi_{X,Y,\eta,\mathfrak{s}}$ takes values in $\mathbb{Z}[K(X, Y)] \subset \mathcal{K}(X, Y, \eta)$, and

$$\Phi_{X,Y,\eta,\mathfrak{s}} = \sum_{t \in K(X, Y)} \Phi_{X,\mathfrak{s}+t} \cdot e^t$$

up to multiplication by ± 1 and an element of $K(X, Y)$.

Proof We have a commutative diagram

$$(18) \quad \begin{array}{ccc} HF^-(S^3) & \longrightarrow & HF_{\text{red}}^-(Y, \mathbb{Z}[K(Z_1)]) \\ \downarrow = & & \downarrow i_* \\ HF^-(S^3) & \longrightarrow & HF_{\bullet}^-(Y, \mathcal{K}(Z_1, \eta)) \end{array}$$

where the upper arrow is the unperturbed, twisted-coefficient homomorphism induced by (Z_1, \mathfrak{s}) , and the lower arrow uses the perturbation $\eta|_{Z_1}$. Here i_* is the natural map induced by the homomorphism $\mathbb{Z}[K(Z_1)] \rightarrow \mathcal{K}(Z_1, \eta)$ of a ring to its Novikov completion; commutativity of the diagram is obvious from the definition of cobordism-induced maps. We have a similar diagram for Z_2 with Y replaced by $-Y$.

Likewise, there is a diagram:

$$(19) \quad \begin{array}{ccc} HF_{\text{red}}^-(Y, \mathbb{Z}[K(Z_1)]) \otimes_{R_Y} \overline{HF_{\text{red}}^-(-Y, \mathbb{Z}[K(Z_2)])} & \xrightarrow{\langle \tau^{-1}(\cdot), \cdot \rangle} & \mathbb{Z}[K(X, Y)] \\ \downarrow i_* \otimes i_* & & \downarrow j \\ HF_{\bullet}^-(Y, \mathcal{K}(Z_1, \eta)) \otimes_{\mathcal{R}_{Y, \eta}} \overline{HF_{\bullet}^-(-Y, \mathcal{K}(Z_2, \eta))} & \xrightarrow{\langle \tau^{-1}(\cdot), \cdot \rangle} & \mathcal{K}(X, Y, \eta) \end{array}$$

If $\eta|_Y = 0$, then the maps i_* and j are the identity maps: indeed, it follows from (16) that $\mathcal{K}(Z_i, \eta) = \mathbb{Z}[K(Z_i)]$ and $\mathcal{K}(X, Y, \eta) = \mathbb{Z}[K(X, Y)]$. Thus in this case, the lemma is just a restatement of Theorem 7.6.

Assume that $\eta|_Y \neq 0$. We claim that the map $j: \mathbb{Z}[K(X, Y)] \rightarrow \mathbb{Z}[K(X, Y)] \otimes_{R_Y} \mathcal{R}_{Y, \eta}$ is injective, and this combined with Theorem 7.6 clearly implies the lemma. To see the injectivity, observe that since η is induced from the 4–manifold X , we have a diagram

$$\begin{array}{ccc} H^1(Y; \mathbb{Z}) & \xrightarrow{\delta} & K(X, Y) \\ & \searrow \langle \eta \cup \cdot, [Y] \rangle & \swarrow \langle \eta \cup \cdot, [X] \rangle \\ & & \mathbb{R} \end{array}$$

with δ surjective (recall that $K(X, Y)$ is identified with a subgroup of $H^2(X)$; cf Remark 2.8). From this it follows that the Novikov completion of $\mathbb{Z}[K(X, Y)]$ as an R_Y –module is the same as its Novikov completion $\mathcal{N}(K(X, Y), \eta)$ as a ring with respect to the function $\langle \eta \cup \cdot, [X] \rangle$. It is straightforward to see that the map $\mathbb{Z}[G] \rightarrow \mathcal{N}(G, \eta)$ from a group to its Novikov completion is injective. \square

Remark 8.15 The injectivity of $j: \mathbb{Z}[K(X, Y)] \rightarrow \mathbb{Z}[K(X, Y)] \otimes_{\mathcal{R}_Y} \mathcal{R}_{Y, \eta}$ from $\mathbb{Z}[K(X, Y)]$ to its (module) completion is not automatic, as can be seen in the case of a cut Y that is admissible in the sense of Ozsváth and Szabó: in this case $K(X, Y)$ is trivial, $\mathbb{Z}[K(X, Y)] \cong \mathbb{Z}$, and $\mathbb{Z} \otimes_{\mathcal{R}_Y} \mathcal{R}_{Y, \eta} = 0$ if $\eta \neq 0$.

To handle cases where not both sides of the cut Y have nonvanishing b^+ , we consider the dependence of $\mathbb{D}_{X, Y, \eta, s}$ on Y . For our present purposes, it suffices to assume that we are given a closed 4-manifold X with two disjoint cuts Y_1 and Y_2 , and both cuts are allowable for a class $\eta \in H^2(X; \mathbb{R})$. Assume also that the orientations on Y_1 and Y_2 are compatible, in the sense that in the decomposition $X = Z_1 \cup_{Y_1} Z_2 \cup_{Y_2} Z_3$, we have $\partial Z_1 = Y_1$, $\partial Z_2 = -Y_1 \sqcup Y_2$, and $\partial Z_3 = -Y_2$. We consider the Ozsváth–Szabó invariants $\mathbb{D}_{X, Y_1, \eta, s}$ and $\mathbb{D}_{X, Y_2, \eta, s}$, and for simplicity we calculate in particular their values on $\alpha = 1 \in \mathbb{A}(X)$ (which we omit from the notation in the following).

According to the composition law, we can find representatives for the maps involved such that

$$\begin{aligned} \mathbb{D}_{X, Y_1, \eta, s} &= \langle \tau^{-1} \Psi_{Z_1, \eta}, \Psi_{Z_2 \cup Z_3, \eta} \rangle \\ &= \langle \tau^{-1} F_{Z_1, \eta}^-(\Theta^-), F_{Z_2 \cup Z_3, \eta}^-(\Theta^-) \rangle \\ (20) \quad &= 1 \otimes \Pi \langle \tau^{-1} F_{Z_1, \eta}^-(\Theta^-), F_{Z_2, \eta}^- \circ F_{Z_3, \eta}^-(\Theta^-) \rangle. \end{aligned}$$

Similarly,

$$(21) \quad \mathbb{D}_{X, Y_2, \eta, s} = \Pi \otimes 1 \langle \tau^{-1} F_{Z_2, \eta}^- \circ F_{Z_1, \eta}^-(\Theta^-), F_{Z_3, \eta}^-(\Theta^-) \rangle.$$

Thus, we can consider the quantity

$$\tilde{\mathbb{D}} := \langle \tau^{-1} F_{Z_1, \eta}^-(\Theta^-), F_{Z_2, \eta}^- \circ F_{Z_3, \eta}^-(\Theta^-) \rangle = \langle \tau^{-1} F_{Z_2, \eta}^- \circ F_{Z_1, \eta}^-(\Theta^-), F_{Z_3, \eta}^-(\Theta^-) \rangle,$$

where the second equality uses duality and the analog of [Lemma 7.9](#) in the perturbed case. Here we also note that the pairings above take values in

$$\mathcal{K}(Z_1, \eta) \otimes_{\mathcal{R}_{Y_1, \eta}} \overline{\mathcal{K}(Z_2 \cup Z_3, Y_2, \eta)} \quad \text{and} \quad \mathcal{K}(Z_1 \cup Z_2, Y_1, \eta) \otimes_{\mathcal{R}_{Y_2, \eta}} \overline{\mathcal{K}(Z_3, \eta)},$$

which are mutually isomorphic to the Novikov completion of

$$\mathbb{Z} \left[\frac{K(Z_1) \oplus K(Z_2) \oplus K(Z_3)}{H^1(Y_1) \oplus H^1(Y_2)} \right]$$

with respect to (the linear function on $K(Z_1) \oplus K(Z_2) \oplus K(Z_3)$ induced by) η . Just as in [Lemma 2.7](#), there is an isomorphism

$$\frac{K(Z_1) \oplus K(Z_2) \oplus K(Z_3)}{H^1(Y_1) \oplus H^1(Y_2)} \cong \ker[\rho_1 \oplus \rho_2 \oplus \rho_3: H^2(X) \rightarrow H^2(Z_1) \oplus H^2(Z_2) \oplus H^2(Z_3)],$$

where ρ_i is the restriction $H^2(X) \rightarrow H^2(Z_i)$. We denote the above group by $K(X; Y_1, Y_2)$ and the Novikov completion of $\mathbb{Z}[K(X; Y_1, Y_2)]$ with respect to η by $\mathcal{K}(X; Y_1, Y_2, \eta)$; this is a module for $\mathcal{R}_{Y_1, \eta}$ and $\mathcal{R}_{Y_2, \eta}$ via the coboundary map $H^1(Y_1) \oplus H^1(Y_2) \rightarrow H^2(X)$ in the Mayer–Vietoris sequence arising from the decomposition $X = (Z_1 \sqcup Z_3) \cup_{Y_1 \sqcup Y_2} Z_2$. For this to make sense with the above identifications, we adopt the convention that in this situation the map $H^1(Y_2) \rightarrow H^2(X)$ is signed oppositely from the one that arises when considering the single cut along Y_2 ; this is to account for the conjugate modules appearing above.

To understand the situation more concretely, consider the diagram:

$$\begin{array}{ccc} H^2(X) & \xrightarrow{\rho_1 \oplus \rho_{23}} & H^2(Z_1) \oplus H^2(Z_2 \cup Z_3) \\ & & \downarrow 1 \oplus \rho_2 \oplus \rho_3 \\ & & H^2(Z_1) \oplus H^2(Z_2) \oplus H^2(Z_3) \end{array}$$

The kernel of the composition of the two maps is $K(X; Y_1, Y_2)$, while the kernel of the horizontal map $\rho_1 \oplus \rho_{23}$ is $K(X, Y_1)$: this shows $K(X, Y_1) \subset K(X; Y_1, Y_2)$ (and there is a corresponding diagram showing $K(X, Y_2) \subset K(X; Y_1, Y_2)$). On the other hand, the kernel of the composition is the preimage under $\rho_1 \oplus \rho_{23}$ of the kernel of $1 \oplus \rho_2 \oplus \rho_3$. The latter is clearly the image $\text{im}(\delta: H^1(Y_2) \rightarrow H^2(Z_2 \cup Z_3))$. Let $\mathcal{S} \subset K(X; Y_1, Y_2)$ be a complete set of coset representatives for $K(X, Y_1) \subset K(X; Y_1, Y_2)$. Thus we can think of \mathcal{S} as a collection of lifts to $H^2(X)$ of the members of the image of $H^1(Y_2) \rightarrow H^2(Z_2 \cup Z_3)$ that lie in the range of ρ_{23} . In particular, for a given $s \in \mathcal{S}$ the spin^c structure $\mathfrak{s} + s$ has the same restrictions to Z_1, Z_2 and Z_3 as does \mathfrak{s} , but the restriction to $Z_2 \cup Z_3$ differs from that of \mathfrak{s} .

Now, according to [\(20\)](#) and [\(21\)](#), the invariants $\mathbb{Q}_{X, Y_1, \eta, \mathfrak{s}}$ and $\mathbb{Q}_{X, Y_2, \eta, \mathfrak{s}}$ are the images of $\tilde{\mathbb{Q}}$ under the homomorphisms $1 \otimes \Pi$ and $\Pi \otimes 1$, respectively. The projections Π are the restriction homomorphisms (in the sense of [Lemma 8.2](#)) associated to inclusions $K(Z_1 \cup Z_2) \rightarrow K(Z_1 \cup Z_2, Y_1)$ (cf the discussion before [Theorem 2.9](#)). It is not hard to see that under the identifications above, $1 \otimes \Pi$ is a restriction homomorphism associated to the inclusion $K(X, Y_1) \rightarrow K(X; Y_1, Y_2)$. When thinking of $1 \otimes \Pi$ in

this way, we will write it as $\Pi_1: \mathcal{K}(X; Y_1, Y_2, \eta) \rightarrow \mathcal{K}(X, Y_1, \eta)$; and similarly write $\Pi \otimes 1$ as $\Pi_2: \mathcal{K}(X; Y_1, Y_2, \eta) \rightarrow \mathcal{K}(X, Y_2, \eta)$.

Suppose we are given a spin^c structure \mathfrak{s} and an element $s \in \mathcal{S}$. We can write s as a lift δh of an element $\delta h \in \text{im}(\delta: H^1(Y_2) \rightarrow H^2(Z_2 \cup Z_3))$, and according to [Theorem 2.9](#) and the sign convention on the coboundary operator mentioned previously, we have

$$\begin{aligned} \mathbb{Q}_{X, Y_1, \eta, \mathfrak{s}+s} &= \langle \tau^{-1} F_{Z_1, \eta, \mathfrak{s}}^-(\Theta^-), F_{Z_2 \cup Z_3, \eta, \mathfrak{s}-\delta h}^-(\Theta^-) \rangle \\ &= 1 \otimes \Pi \langle \tau^{-1} F_{Z_1, \eta, \mathfrak{s}}^-(\Theta^-), F_{Z_2, \eta, \mathfrak{s}}^- \circ e^h \cdot F_{Z_3, \eta, \mathfrak{s}}^-(\Theta^-) \rangle \\ &= \Pi_1 \langle \tau^{-1} F_{Z_2, \eta, \mathfrak{s}}^- \circ F_{Z_1, \eta, \mathfrak{s}}^-(\Theta^-), e^h \cdot F_{Z_3, \eta, \mathfrak{s}}^-(\Theta^-) \rangle \\ &= \Pi_1(e^{-h} \cdot \tilde{\mathbb{Q}}) \end{aligned}$$

where we can use the same representatives for the maps involved as in [\(20\)](#).

The above can be interpreted as follows. The ‘‘two-cut invariant’’ $\tilde{\mathbb{Q}}$ is an integer-valued function on the group $K(X; Y_1, Y_2)$, whose values on each coset $K(X, Y_1) + s$ are equal to the values of the one-cut invariant $\mathbb{Q}_{X, Y_1, \eta, \mathfrak{s}+s}$. Indeed, [\(20\)](#) indicates that the restriction of $\tilde{\mathbb{Q}}$ to the subgroup $K(X, Y_1)$ is equal to $\mathbb{Q}_{X, Y_1, \eta, \mathfrak{s}}$, while the above calculation indicates that after translating $\tilde{\mathbb{Q}}$ by $-h$ and restricting, we obtain $\mathbb{Q}_{X, Y_1, \eta, \mathfrak{s}+s}$. Thus, with appropriate choices of representatives,

$$\tilde{\mathbb{Q}} = \sum_{s \in \mathcal{S}} s \cdot \mathbb{Q}_{X, Y_1, \eta, \mathfrak{s}+s},$$

thinking of both sides as members of the Novikov ring $\mathcal{K}(X; Y_1, Y_2)$.

An entirely analogous discussion holds for the one-cut invariant arising from Y_2 , which yields the following.

Lemma 8.16 *Suppose X is given with two disjoint cuts Y_1 and Y_2 that are allowable for a class $\eta \in H^2(X; \mathbb{R})$, and whose orientations are compatible. Then for a spin^c structure \mathfrak{s} on X , $\mathbb{Q}_{X, Y_1, \eta, \mathfrak{s}}$ and $\mathbb{Q}_{X, Y_2, \eta, \mathfrak{s}}$ contain the same information in the following sense.*

Let $K(X; Y_1, Y_2)$ be as above, and let \mathcal{S}_1 and \mathcal{S}_2 be complete sets of coset representatives for $K(X, Y_1)$ and $K(X, Y_2)$ respectively in $K(X; Y_1, Y_2)$. Then there are choices of representatives for the invariants $\mathbb{Q}_{X, Y_i, \eta, \mathfrak{s}+s_i}$ (for $s_i \in \mathcal{S}_i$) such that

$$\sum_{s_1 \in \mathcal{S}_1} s_1 \cdot \mathbb{Q}_{X, Y_1, \eta, \mathfrak{s}+s_1} = \sum_{s_2 \in \mathcal{S}_2} s_2 \cdot \mathbb{Q}_{X, Y_2, \eta, \mathfrak{s}+s_2}.$$

as members of $\mathcal{K}(X; Y_1, Y_2)$. □

This implies, for example, that the collection of values (coefficients) taken by $\Phi_{X, Y_1, \eta, \mathfrak{s}}$ for various \mathfrak{s} ranging through the collection $\{\mathfrak{s}_0 + k \mid k \in K(X; Y_1, Y_2)\}$ is the same as that of $\Phi_{X, Y_2, \eta, \mathfrak{s}}$, though of course the above also means that the identification between the coefficients can be made to be equivariant with respect to the affine structure on $K(X; Y_1, Y_2)$.

Finally we obtain the following, which is a restatement of [Theorem 1.6](#) from the introduction. It should be seen as a generalization of [Theorem 7.6](#) that allows us to calculate Ozsváth–Szabó invariants using essentially any cut Y , if we can find an appropriate perturbation.

Theorem 8.17 *Let X be a closed oriented 4–manifold with $b^+(X) \geq 2$, and $Y \subset X$ a connected submanifold determining a decomposition $X = Z_1 \cup_Y Z_2$, where Z_i are 4–manifolds with boundary. Fix a class $\eta \in H^2(X; \mathbb{R})$, and assume that Y is an allowable cut for η . If $b^+(Z_1)$ and $b^+(Z_2)$ are not both 0, then for any spin^c structure \mathfrak{s} on X and element $\alpha \in \mathbb{A}(X)$,*

$$\sum_{t \in K(X, Y)} \Phi_{X, \mathfrak{s}+t}(\alpha) \cdot e^t = \Phi_{X, Y, \eta, \mathfrak{s}}(\alpha) = \langle \tau^{-1} \Psi_{Z_1, \eta, \mathfrak{s}}(\alpha_1), \Psi_{Z_2, \eta, \mathfrak{s}}(\alpha_2) \rangle$$

up to sign and multiplication by an element of $K(X, Y)$, where $\alpha_1 \otimes \alpha_2 \mapsto \alpha$ as before. If $b^+(Z_1) = b^+(Z_2) = 0$ then the same is true after possibly replacing η by another class $\tilde{\eta}$, where $\tilde{\eta}|_{Z_i} = \eta|_{Z_i}$ for $i = 1, 2$.

Proof If both $b^+(Z_1) \geq 1$ and $b^+(Z_2) \geq 1$ then this follows from [Lemma 8.14](#). Assume, therefore, that $b^+(Z_1) = 0$. We wish to find a cut Y' for X such that (1) Y' is disjoint from Y , and (2) in the decomposition $X = Z'_1 \cup_{Y'} Z'_2$, we have $b^+(Z'_i) \geq 1$ for $i = 1, 2$.

To find Y' , first consider the restriction $\eta|_Y$. Since Y is allowable for η and $b^+(Z_1) = 0$, we must have $\eta|_Y \neq 0 \in H^2(Y; \mathbb{R})$. Hence we can find a surface $\Sigma \subset Y$ such that $\int_{\Sigma} \eta \neq 0$, and since η is defined on X , we infer $[\Sigma]$ is nonvanishing in $H_2(X; \mathbb{Z})$. Clearly $\Sigma \cdot \Sigma = 0$. Let S be an embedded surface in X intersecting Σ transversely in a single point; then $\{[\Sigma], [S]\}$ determine a direct summand of the intersection matrix of X having one positive and one negative eigenvalue. Let N be a tubular neighborhood of S in X ; then Y separates N into two components $N_1 \cup N_2$, with $N_i \subset Z_i$. Let \tilde{Z}_1 be obtained by adding a collar $Y \times [0, \epsilon] \subset Z_2$ to Z_1 , and set $Z'_1 = \tilde{Z}_1 \cup N_2$. Thus $Y' = \partial Z'_1$ is obtained by pushing Y slightly into Z_2 and attaching the boundary of N_2 .

Since $\Sigma \cup S \subset Z'_1$ we see $b^+(Z'_1) = 1$; on the other hand, the complement $Z'_2 = X \setminus Z'_1$ has $b^+(Z'_2) = b^+(Z_2)$. There are several cases to distinguish.

Case 1 ($b^+(Z'_2) \geq 1$) Here we are done, by [Lemma 8.14](#) and [Lemma 8.16](#). Indeed, [Lemma 8.14](#) shows that the invariants calculated from Y' are the Ozsváth–Szabó invariants, while [Lemma 8.16](#) implies that the invariants calculated from Y , taken over all spin^c structures, can be made to correspond in an equivariant way to those calculated from Y' .

Case 2 ($b^+(Z'_2) = 0$, but $\eta|_{Y'} \neq 0$) Then Y' is still an allowable cut for η and disjoint from Y , so [Lemma 8.16](#) applies. We can now run the construction above with Z'_2 playing the role of Z_1 ; the result is a new cut Y'' , disjoint from Y' , with $b^+(Z''_i) \geq 1$ for $i = 1, 2$. [Lemma 8.16](#) implies that the invariants calculated from Y , Y' , and Y'' agree, while [Lemma 8.14](#) shows that the invariants calculated from Y'' are the Ozsváth–Szabó invariants.

Case 3 ($b^+(Z'_2) = 0$ and $\eta|_{Y'} = 0$) Let $\tilde{\eta} = \eta + \text{PD}_X[\Sigma]$, where $\text{PD}_X[\Sigma]$ denotes the image in real cohomology of the Poincaré dual of $[\Sigma]$ in $H^2(X; \mathbb{R})$. Then it is easy to see that $\tilde{\eta}|_{Y'} \neq 0$, so that Y' is an allowable cut for $\tilde{\eta}$. Note that since the classes $\tilde{\eta}$ and η differ by an element in the image of $\delta: H^1(Y; \mathbb{R}) \rightarrow H^2(X; \mathbb{R})$, they agree on Z_1 and Z_2 . Running the preceding proof with $\tilde{\eta}$ in place of η we end in case 2 above, hence the conclusion of the theorem holds with the modified perturbation. \square

Note that in case 3 of the proof, it works just as well to take $\tilde{\eta} = \eta + \epsilon \text{PD}_X[\Sigma]$, where ϵ is an arbitrary nonzero real number. Thus [Theorem 8.17](#) could be rephrased to say that when $b^+(X) \geq 2$, the perturbed Ozsváth–Szabó invariants are equal to the ordinary Ozsváth–Szabó invariants when calculated with respect to a cut Y that is allowable for a “generic” class $\eta \in H^2(X; \mathbb{R})$.

The preceding results provide sufficient understanding of the dependence of $\Phi_{X, Y, \eta, s}$ on Y for our purposes. We do not study the dependence of the perturbed invariants on η here.

9 Heegaard Floer homology of a surface times a circle

From the general considerations of the preceding sections, we turn now to the problem mentioned in the introduction of determining the behavior of Ozsváth–Szabó invariants under fiber sum. Since a fiber sum along surfaces with trivial normal bundle is obtained by gluing two manifolds together along the product of the summing surface Σ with a circle, and the relative invariants of the pieces take values in the Floer homology of the latter manifold, we will need a fairly detailed understanding of that Floer homology.

This section is devoted to the calculation of the perturbed Heegaard Floer homology groups of $\Sigma \times S^1$, for a particular choice of perturbation η . Indeed, our choice of η is

induced by the cobordism $\Sigma_g \times D^2 - D^4$. The main input for this computation comes from [3] where we developed most of the technical tools required. We start this section by elucidating the new phenomena associated with working with twisted coefficients in surgery exact sequences.

9.1 Exact sequences with twisted coefficients

Let K be a nullhomologous knot in a 3–manifold Y . Following typical notation in the subject, we write $Y_\ell = Y_\ell(K)$ for the 3–manifold obtained by ℓ –framed surgery on K . As described in [11], there are exact sequences relating HF^+ (or \widehat{HF}) of the two triples of three 3–manifolds (Y_0, Y, Y_{-n}) and (Y_0, Y_n, Y) with $n > 0$ but otherwise arbitrary:

$$(22) \quad \begin{array}{ccccccc} \cdots & \xrightarrow{G} & HF^+(Y_0, [\mathfrak{s}_k]) & \xrightarrow{H} & HF^+(Y, \mathfrak{s}) & \xrightarrow{F} & HF^+(Y_{-n}, \mathfrak{s}_k) \xrightarrow{G} \cdots \\ \cdots & \xrightarrow{G} & HF^+(Y_0, [\mathfrak{s}_k]) & \xrightarrow{H} & HF^+(Y_n, \mathfrak{s}_k) & \xrightarrow{F} & HF^+(Y, \mathfrak{s}) \xrightarrow{G} \cdots \end{array}$$

By abuse of notation we have labeled the maps appearing in the two sequences by the same letters although they are of course different functions. The map $F: HF^+(Y, \mathfrak{s}) \rightarrow HF^+(Y_{-n}, \mathfrak{s}_k)$ will be of special interest below and we proceed by first providing more details concerning its definition as well as explaining the notation from (22).

Let W_{-n} be the cobordism from Y to Y_{-n} obtained by attaching a $-n$ –framed 2–handle to $Y \times [0, 1]$ along $K \times \{1\}$. Let $\sigma \subset Y$ be a Seifert surface of K and let $S \subset W_{-n}$ be the surface obtained by capping off $\sigma \times \{1\}$ with the core of the attaching 2–handle. Given a spin^c –structure $\mathfrak{s} \in \text{Spin}^c(Y)$ let $\mathfrak{s}_k \in \text{Spin}^c(Y_{-n})$ be the spin^c –structure on Y_{-n} which is spin^c –cobordant to \mathfrak{s} via $(W_{-n}, \mathfrak{r}_{k,0})$ where $\mathfrak{r}_{k,\ell} \in \text{Spin}^c(W_{-n})$ is uniquely determined by $\mathfrak{r}_{k,\ell}|_Y = \mathfrak{s}$, $\langle c_1(\mathfrak{r}_{k,\ell}), [S] \rangle = 2k - (2\ell - 1)n$ and $k \in \{0, 1, \dots, n-1\}$.¹ By $[\mathfrak{s}_k]$ we denote the preimage $Q_{\pm}^{-1}(\mathfrak{s}_k)$ of a surjective map $Q_{\pm}: \text{Spin}^c(Y_0) \rightarrow \text{Spin}^c(Y_{\pm n})$ defined in [11, Theorem 9.19] whose details need not concern us save the fact that when $n \gg 0$ this preimage includes at most a single spin^c –structure whose Floer homology $HF^+(Y_0, \mathfrak{t})$ is nontrivial. By writing $HF^+(Y_0, [\mathfrak{s}_k])$ we mean the direct sum of $HF^+(Y_0, \mathfrak{t})$ over all spin^c –structures $\mathfrak{t} \in [\mathfrak{s}_k]$.

The map F from (22) is a sum

$$(23) \quad F = \sum_{\ell \in \mathbb{Z}} F_\ell, \quad \text{where} \quad F_\ell: HF^+(Y, \mathfrak{s}) \rightarrow HF^+(Y_{-n}, \mathfrak{s}_k)$$

is the homomorphism induced by $(W_{-n}, \mathfrak{r}_{k,\ell})$.

¹Every spin^c –structure $\mathfrak{r} \in \text{Spin}^c(W_{-n})$ with $\mathfrak{r}|_Y = \mathfrak{s}$ and $\mathfrak{r}|_{Y_{-n}} = \mathfrak{s}_k$ is of the form $\mathfrak{r} = \mathfrak{r}_{k,\ell}$ for some $\ell \in \mathbb{Z}$.

Recall that when $c_1(\mathfrak{s})$ is torsion both $HF^+(Y, \mathfrak{s})$ and $HF^+(Y_{-n}, \mathfrak{s}_k)$ come equipped with an absolute \mathbb{Q} -grading $\widetilde{\text{gr}}$ lifting the relative \mathbb{Z} -grading gr (cf [15; 9]). With respect to the absolute grading $\widetilde{\text{gr}}$ the degrees of the maps F_ℓ on homogeneous elements are

$$(24) \quad \deg F_\ell = \frac{1}{4} \left(1 - \frac{(2k - (2\ell - 1)n)^2}{n} \right)$$

This function attains its maximum at $\ell = \frac{1}{2} - \frac{k}{n}$, though of course ℓ is constrained to be an integer. When $k \neq 0$ there is therefore a unique value of ℓ corresponding to the maximal degree shift, while for $k = 0$ the maximum is attained for both $\ell = 0, 1$ (granted that $n > 1$).

To state the version of the sequence for twisted coefficients we first introduce some more notation. Write $W_0: Y \rightarrow Y_0$ for the surgery cobordism; then it is a simple matter to calculate that $K(W) \subset H^2(W, \partial W; \mathbb{Z})$ is infinite cyclic, generated by the Poincaré dual of the capped-off Seifert surface $S_0 \subset W_0$. The group ring $\mathbb{Z}[K(W)]$ is then isomorphic to the ring $L(t)$ of Laurent polynomials in a single variable t , with R_{Y_0} -module structure induced by the natural homomorphism $H^1(Y_0) \rightarrow K(W) = H^1(Y_0)/H^1(W)$. With the choice of Seifert surface σ fixed, we can think of $\mathbb{Z}[K(W)] = L(t)$ more concretely as follows. Let $\widehat{\sigma} \subset Y_0$ be the surface obtained by capping off σ with the surgery disk, and take $t = \text{PD}([\widehat{\sigma}]) \in H^1(Y_0; \mathbb{Z})$. Then let $L(t) = \mathbb{Z}[t, t^{-1}]$ be the ring of Laurent polynomials in t ; equivalently $L(t)$ is the group ring on the subgroup of $H^1(Y_0)$ generated by t . There is a natural homomorphism $\mathbb{Z}[H^1(Y_0)] \rightarrow L(t)$ induced by the map $\alpha \mapsto \langle \alpha, [K] \rangle \cdot t$, or in multiplicative notation $\alpha \mapsto t^{\langle \alpha, [K] \rangle}$, corresponding to the map $R_{Y_0} \rightarrow \mathbb{Z}[K(W)]$ above. (Here $[K]$ indicates the homology class in $H_1(Y_0)$ coming from the core of the surgery torus.) Observe that if $\mathfrak{t} \in \text{Spin}^c(Y_0)$ is a spin^c structure whose Chern class is dual to a multiple of $[\widehat{\sigma}]$ then as in Proposition 3.5, $L(t)$ is naturally a graded module for $R_{Y_0, \mathfrak{t}}$ with $\text{gr}_{\mathfrak{t}}(t) = -\langle c_1(\mathfrak{t}) \cup t, [Y_0] \rangle$.

More generally, suppose $\mathfrak{s} \in \text{Spin}^c(Y)$ is a spin^c structure on the original 3-manifold, and M is a graded module for $R_{Y, \mathfrak{s}}$. Then the surgery cobordism $Y \rightarrow Y_0$, equipped with some spin^c structure, induces a graded module $\overline{M} \otimes_{R_{Y, \mathfrak{s}}} \mathbb{Z}[K(W)]$ for $R_{Y_0, \mathfrak{t}}$ that we denote by $M[t^{\pm 1}]$, where $H^1(Y)$ acts trivially on $L(t)$.

With this understood, the next theorem is a slight generalization of Theorem 9.21 from Section 9.6 in [11]. Indeed, the proof of that theorem goes through verbatim with only notational modifications to yield:

Theorem 9.1 *Let Y be a three manifold, K a nullhomologous knot in Y and M an R_Y –module. Then for any $n > 0$ there are long exact sequences of $R_{Y_0} \otimes \mathbb{Z}[U]$ –modules for the Heegaard Floer homology groups with twisted coefficients:*

$$\begin{aligned} \dots \xrightarrow{G} HF^+(Y_0, [\mathfrak{s}_k]; M[t^{\pm 1}]) \xrightarrow{H} HF^+(Y, \mathfrak{s}; M)[t^{\pm 1}] & \xrightarrow{F} HF^+(Y_{-n}, \mathfrak{s}_k; M)[t^{\pm 1}] \xrightarrow{G} \dots \\ \dots \xrightarrow{G} HF^+(Y_0, [\mathfrak{s}_k]; M[t^{\pm 1}]) \xrightarrow{H} HF^+(Y_n, \mathfrak{s}; M)[t^{\pm 1}] & \xrightarrow{F} HF^+(Y, \mathfrak{s}_k; M)[t^{\pm 1}] \xrightarrow{G} \dots \end{aligned}$$

The analogous sequences for \widehat{HF} are also exact.

We shall refer to the above sequences as the *negative n* and *positive n surgery sequences* respectively. As in (22) we abuse notation by labelling the maps in both sequences by the same letters. It is worthwhile to single out a case of special interest later on, namely the choice of $M = \mathbb{Z}$ with trivial R_Y –module structure. In this case the negative n sequence becomes

$$(25) \quad \dots \xrightarrow{G} HF^+(Y_0, [\mathfrak{s}_k]; L(t)) \xrightarrow{H} HF^+(Y, \mathfrak{s})[t^{\pm 1}] \xrightarrow{F} HF^+(Y_{-n}, \mathfrak{s}_k)[t^{\pm 1}] \xrightarrow{G} \dots$$

The fact that the Novikov ring $\mathcal{R}_{Y_0, \eta}$ is flat over R_{Y_0} implies the following, by tensor product of (25) with $\mathcal{R}_{Y_0, \eta}$.

Corollary 9.2 *Suppose $\eta_0 \in H^2(W_0; \mathbb{R})$ is given, and the image η of η_0 under the map $H^2(W_0; \mathbb{R}) \rightarrow H^2(Y_0; \mathbb{R})$ satisfies $\langle \eta \cup t, [Y_0] \rangle > 0$. Then for any $n > 0$ there is a long exact sequence*

$$\begin{aligned} \dots \xrightarrow{G} HF^+(Y_0, [\mathfrak{s}_k]; \mathcal{L}(t)) \xrightarrow{H \otimes 1} HF^+(Y, \mathfrak{s})[t^{\pm 1}] \otimes_{R_{Y_0}} \mathcal{R}_{Y_0, \eta} & \xrightarrow{F \otimes 1} HF^+(Y_{-n}, \mathfrak{s}_k)[t^{\pm 1}] \otimes_{R_{Y_0}} \mathcal{R}_{Y_0, \eta} \xrightarrow{G \otimes 1} \dots, \end{aligned}$$

where $\mathcal{L}(t)$ denotes the ring of Laurent series in t .

Indeed, it is not hard to see using Lemma 8.2 that the map $H^1(Y_0) \rightarrow K(W_0)$ induces a homomorphism of Novikov rings $\mathcal{R}_{Y_0, \eta} \rightarrow \mathcal{K}(W_0, \eta)$ under the hypotheses above, and that in fact $\mathcal{K}(W_0, \eta) \cong \mathcal{L}(t)$.

There is a straightforward relationship between the exact sequences (22) and (25).

Proposition 9.3 *Let K be a nullhomologous knot in Y and let $F: HF^+(Y, \mathfrak{s}) \rightarrow HF^+(Y_{-n}, s_k)$ and $\underline{F}: HF^+(Y, \mathfrak{s})[t^{\pm 1}] \rightarrow HF^+(Y_{-n}, s_k)[t^{\pm 1}]$ be the maps appearing in the exact sequences (22) and (25) respectively. Let F_i be the components of F as in (23). Then*

$$\underline{F} = \sum_{\ell \in \mathbb{Z}} F_\ell \otimes t^\ell$$

up to sign and overall multiplication by a power of t .

Moreover, when \mathfrak{s} is torsion and for a given fixed number $d_0 \in \mathbb{Q}$, the only nonzero terms in the restriction of \underline{F} to $HF_{\leq d_0}^+(Y, \mathfrak{s})$ are those for which $\ell = 0$ or 1 , for any sufficiently large n .

Proof The homomorphisms in both sequences are defined by counts of holomorphic triangles in appropriate Heegaard triple-diagrams, and the stated relationship between \underline{F} and F follows from elementary considerations in these diagrams. Indeed, in notation from [11] (also [16]), the map in the twisted sequence can be written as

$$\underline{F}([\mathbf{x}, i]) = \sum_{\psi \in \pi_2(\mathbf{x}, \Theta, \mathbf{y})} \#\mathcal{M}(\psi)[\mathbf{y}, i - n_z(\psi)] \cdot t^{n_\gamma(\psi)},$$

where the sum is over homotopy classes of triangles ψ in a diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ describing the natural cobordism $Y \rightarrow Y_{-n}$. In this situation we are using twisted coefficients on Y_{-n} constructed by fixing a reference point τ lying on the surgery circle γ_g , such that the boundary operator in the twisted chain complex for Y_{-n} records (in the power of t) the intersection of the γ -component of a holomorphic disk with the subvariety $V = \gamma_1 \times \cdots \times \gamma_{g-1} \times \{\tau\} \subset T_\gamma \subset \text{Sym}^g(\Sigma)$. (This formal device induces trivially twisted coefficients on Y_{-n} .) In the formula above, the power $n_\gamma(\psi)$ is similarly the intersection of the γ -component of the boundary of ψ with V .

The first claim of the proposition amounts to the fact that the power of t appearing above determines and is determined by the value $\langle c_1(s_z(\psi)), H(\mathcal{P}) \rangle$, where $H(\mathcal{P})$ is the 2-dimensional homology class of the triply periodic domain \mathcal{P} corresponding to the generator of the 2-dimensional homology of the surgery cobordism. This in turn follows easily from inspection of the Heegaard triple itself, together with the expression for $\langle c_1(s_z(\psi)), H(\mathcal{P}) \rangle$ in terms of data on the Heegaard diagram obtained by Ozsváth and Szabó (Proposition 6.3 of [15]).

To see the remaining claim, recall that the homomorphism F_ℓ (corresponding to the spin^c structure on the cobordism with $\langle c_1(\tau_{k,\ell}), [S] \rangle = 2k - (2\ell - 1)n$) induces a shift

in degree given by (24). In particular we have

$$\begin{aligned} \deg F_\ell - \deg F_0 &= 2k\ell - \ell(\ell - 1)n \\ \deg F_\ell - \deg F_1 &= 2k(\ell - 1) - \ell(\ell - 1)n, \end{aligned}$$

and so for $\ell \neq 0, 1$, the degree shift of F_ℓ can be made arbitrarily large by choosing n large. The result follows since the grading of HF^+ is bounded below. \square

As an example, let us consider the Floer homology of the 3–torus T^3 , in the torsion spin^c structure. We can realize T^3 as the result of 0–surgery on each component of the Borromean rings (Figure 1 below), and if we take K to be one component of that link, then $T^3 = Y_0(K)$ where $Y = \#^2 S^1 \times S^2$. It is well-known that

$$HF^+(\#^2 S^1 \times S^2, \mathfrak{s}) \cong \Lambda^* H^1(\#^2 S^1 \times S^2; \mathbb{Z}) \otimes \mathcal{T}_{-1},$$

where \mathcal{T}_n is the $\mathbb{Z}[U]$ –module $\mathbb{Z}[U, U^{-1}]/U \cdot \mathbb{Z}[U]$, graded so that the homogeneous factor of lowest degree is in dimension n . It can also be shown that for sufficiently large n there is an isomorphism

$$HF_d^+(Y_{-n}, \mathfrak{s}_0) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & d \geq \tau \\ 0 & d < \tau \end{cases}$$

where $\tau = \frac{1}{4}(1 - n)$ (this follows from results in [10], cf also the discussion in the next subsection).

Hence the surgery sequence (25) appears as:

$$\begin{array}{ccc} HF^+(\#^2 S^1 \times S^2, \mathfrak{s})[t^{\pm 1}] & HF^+(Y_{-n}, \mathfrak{s}_0)[t^{\pm 1}] & HF^+(T^3, \mathfrak{s}_0, L(t)) \\ \hline \mathbb{Z}^2[t^{\pm 1}] & \mathbb{Z}^2[t^{\pm 1}] & \mathbb{Z}^2 \\ \mathbb{Z}^2[t^{\pm 1}] & \mathbb{Z}^2[t^{\pm 1}] & \mathbb{Z}^2 \\ \mathbb{Z}^2[t^{\pm 1}] & \mathbb{Z}^2[t^{\pm 1}] & \mathbb{Z}^2 \oplus \mathbb{Z}[t^{\pm 1}] \\ \mathbb{Z}[t^{\pm 1}] & 0 & 0 \end{array}$$

Here we have arranged the columns “typewriter style,” so that the top-degree components of the homomorphisms between the groups map horizontally, with the map $T^3 \rightarrow Y_{-n}$ moving to the next row of the array. Strictly, the groups along the row second from the bottom appear in degrees $0, \tau$, and $-\frac{1}{2}$, respectively. Furthermore, we have chosen n large enough that only F_0 and F_1 are relevant for the displayed portion of the sequence (which continues upward indefinitely, repeating the top row above). The structure of the Floer homology of T^3 in large degrees can be deduced from the universal coefficients spectral sequence, or by using the description of F_0 and F_1 given in the following subsection. The U –equivariance of the sequence together

with some easy homological algebra then shows that $HF_{-1/2}^+(T^3)$ has the indicated form. Finally, we point out for future reference that the map

$$HF_{-1/2}^+(T^3, \mathfrak{s}_0; L(t)) \rightarrow HF_{-1}^+(\#^2 S^1 \times S^2, \mathfrak{s})[t^{\pm 1}]$$

appearing above is induced by the surgery cobordism equipped with the torsion spin^c structure, and corresponds to the projection $\mathbb{Z}^2 \oplus L(t) \rightarrow L(t)$.

9.2 A surface cross a circle: partially twisted coefficients

In this section we apply the general discussion from the previous section to the case of $Y = \#^{2g}(S^1 \times S^2)$ and $K = K_g = \#^g B(0, 0)$ with $B(m, n)$ defined in [Figure 1](#).

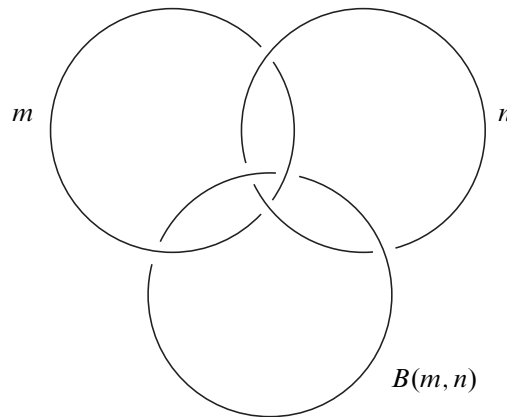


Figure 1: The knot $B(m, n) \subset L(m, 1) \# L(n, 1)$ with $m, n \in \mathbb{Z}$. (Note that $L(0, 1)$ is just $S^1 \times S^2$.)

Let σ_1 be the genus-1 Seifert surface for K_1 obtained from the obvious disk bounded by $B(0, 0)$ in [Figure 1](#) by adding a handle where one of the other two components of the Borromean rings intersects that disk. Let $\sigma = \sigma_g = \natural^g \sigma_1$ be the choice of Seifert surface for K_g . It is then not hard to see that $Y_0 = \Sigma_g \times S^1$ and t becomes the Poincaré dual of $[\Sigma_g]$. For the rest of this subsection and the next, we assume $g \geq 2$.

Let \mathfrak{s} be the unique torsion spin^c structure on Y . Since HF^+ is nonzero only for a finite set of spin^c -structures, when $n \gg 0$ there is only one spin^c structure in the set $[\mathfrak{s}_k]$ having nontrivial HF^+ , namely the unique spin^c -structure $\mathfrak{s}_k \in \text{Spin}^c(\Sigma_g \times S^1)$ with $c_1(\mathfrak{s}_k) = 2k \text{ PD}([S^1])$. We shall take as perturbation the class

$$\eta = \text{PD}([S^1]),$$

which is easily seen to be the restriction of a class on the surgery cobordism $W_0: Y \rightarrow Y_0$.

With these choices of Y, K, σ the maps $F_0, F_1: HF^+(Y, \mathfrak{s}) \rightarrow HF^+(Y_{-n}, \mathfrak{s}_k)$ appearing in Proposition 9.3 have been made completely explicit by the results from [3; 10]. Before proceeding we describe these maps.

Let us use the shorthand Λ^k and Λ^* to denote $\Lambda^k H^1(\Sigma_g; \mathbb{Z})$ and $\Lambda^* H^1(\Sigma_g; \mathbb{Z})$ respectively. The reader familiar with knot Floer homology [10; 18] may recall that Λ^{g+j} is just the knot Floer homology $\widehat{HF\bar{K}}(\#^g B(0, 0), j)$, which is the homology of the associated graded complex arising from a filtration on the Heegaard Floer chain groups \widehat{CF} . Such a filtration is induced by any nullhomologous knot K in a 3–manifold Y ; here we specialize to the case $K = \#^g B(0, 0)$. In fact, K induces a filtration on all the Heegaard Floer chain complexes, in particular on CF^∞ ; the filtration level j of a generator $[x, i]$ for CF^∞ can be calculated directly from a Heegaard diagram for Y (suitably adapted to K), and is recorded in the notation $[x, i, j]$. The function $[x, i, j] \mapsto i + j$ gives rise to another filtration on CF^∞ , and thence a spectral sequence converging to $HF^\infty(Y)$ whose E^1 term is $\widehat{HF\bar{K}}(Y, K) \otimes \mathbb{Z}[U, U^{-1}]$ (cf [10]). It is this object that is currently under consideration in the case $K = \#^g B(0, 0)$; in the case at hand (ie, $K = \#^g B(0, 0)$, with integral coefficients in all Floer homology), there are no further differentials in any of the spectral sequences.

We place the $\mathbb{Z}[U]$ –module $\Lambda^* \otimes \mathbb{Z}[U, U^{-1}]$ in a 2–dimensional coordinate system by assigning $\Lambda^k \otimes U^\ell$ the coordinates $(-\ell, k - g - \ell)$ (see Figure 2 below). We equip the coordinate plane with a \mathbb{Q} –grading $\widetilde{\text{gr}}: \mathbb{Z}^2 \rightarrow \mathbb{Q}$ by setting

$$(26) \quad \widetilde{\text{gr}}(i, j) = i + j.$$

In this description, the action of U can be thought of as translation by $(-1, -1)$ (see again Figure 2); as usual it decreases grading by two. Following [10], we shall write

$$H\{\text{condition on } (i, j)\}$$

to denote the various $\mathbb{Z}[U]$ submodules and quotient modules of $\Lambda^* \otimes \mathbb{Z}[U, U^{-1}]$ obtained as a direct sum over all the terms in the coordinate system which reside at coordinates (i, j) subject to the stated conditions. For example $H\{i < 0\}$ is the submodule $\Lambda^* \otimes (U \cdot \mathbb{Z}[U])$ and $H\{i \geq 0\}$ is the quotient module $\Lambda^* \otimes \mathcal{T}$, where $\mathcal{T} = \mathbb{Z}[U, U^{-1}]/U \cdot \mathbb{Z}[U]$. For $0 \leq d \leq g - 1$ we define the $\mathbb{Z}[U]$ –module $X(g, d)$ as

$$X(g, d) = \bigoplus_{i=0}^d \left(\Lambda^i \otimes_{\mathbb{Z}} \frac{\mathcal{T}}{U^{i-d-1} \cdot \mathbb{Z}[U^{-1}]} \right).$$

While our definition of $X(g, d)$ differs slightly from that used by Ozsváth and Szabó in [10], it is not hard to see that they agree up to an isomorphism of relatively graded

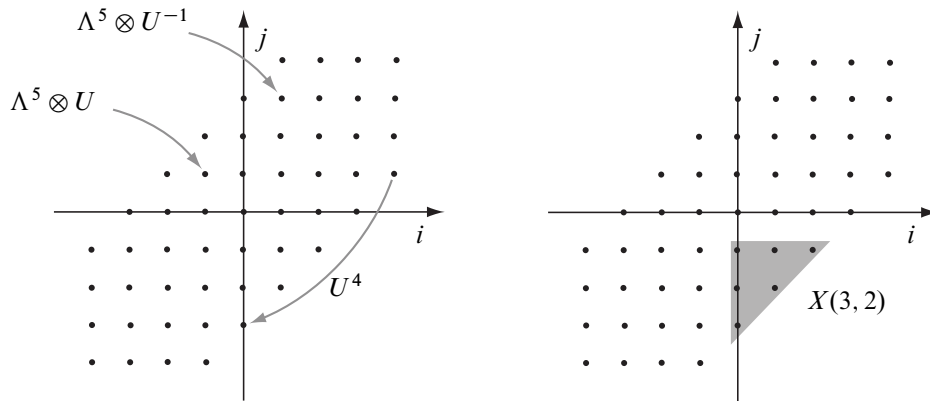


Figure 2: A visual representation of $\Lambda^* \otimes \mathbb{Z}[U, U^{-1}]$ in the case of $g = 3$. The dot at coordinate (x, y) represents the group $\Lambda^{g-x+y} \otimes U^{-x}$, two examples are indicated. The action of U^ℓ in this description of $\Lambda^* \otimes \mathbb{Z}[U, U^{-1}]$ is simply translation by $(-\ell, -\ell)$. The direct sum of the groups represented by the dots in the shaded triangle is $X(3, 2)$.

$\mathbb{Z}[U]$ -modules. In fact, as was noted in [10], $X(g, d)$ is isomorphic as a relatively graded group to the homology of $\text{Sym}^d(\Sigma_g)$, the d -fold symmetric power of a genus g surface. Indeed, the action of U on $X(g, d)$ corresponds to the cap product with the Poincaré dual of $pt \times \text{Sym}^{d-1}(\Sigma_g)$ on $H_*(\text{Sym}^d(\Sigma_g))$.

In the notation above, $X(g, d)$ is isomorphic to

$$(27) \quad X(g, d) \cong H\{i \geq 0 \text{ and } j < d + 1 - g\}$$

as a $\mathbb{Z}[U]$ -module; see Figure 2 for a visual representation. We shall refer to this identification as the *standard embedding* of $X(g, d)$ into $\Lambda^* \otimes \mathbb{Z}[U, U^{-1}]$. We shall encounter “nonstandard embeddings” of $X(g, d)$ as well; see Theorem 9.4 below.

It was shown in [10, Section 9] (as an instance of a general result on the relationship between the knot Floer homology of K and the Floer homology of surgeries along K) that for $Y = \#^{2g} S^1 \times S^2$ and $K = \#^g B(0, 0)$ as above, and $n \gg 0$, there are $\Lambda^* H_1(\Sigma_g; \mathbb{Z}) \otimes \mathbb{Z}[U]$ -module isomorphisms

$$(28) \quad HF^+(Y, \mathfrak{s}) \cong H\{i \geq 0\} \quad \text{and} \quad HF^+(Y_{-n}, \mathfrak{s}_k) \cong H\{i \geq 0 \text{ and } j \geq k\}$$

where the action of $\Lambda^* H_1(\Sigma_g; \mathbb{Z}) \cong \Lambda^* H_1(Y; \mathbb{Z})$ on $\Lambda^k \otimes U^\ell$ is given by

$$(29) \quad \gamma \cap (\alpha \otimes U^\ell) = \iota_\gamma \alpha \otimes U^\ell + (\text{PD}(\gamma) \wedge \alpha) \otimes U^{\ell+1}$$

for $\gamma \in H_1(\Sigma_g; \mathbb{Z})$ and $\alpha \in \Lambda^k$. Here ι_γ is contraction with γ and $\text{PD}(\gamma)$ is the Poincaré dual of γ taken on Σ_g . By virtue of (27) this action induces an action on $X(g, d)$. We shall refer to (29) as the *standard action*, and use the cap product notation \cap to distinguish it from actions of first homology on Floer homology that need not be “standard” (we use the “dot” notation for the latter: $h \cdot \xi$ for $h \in H_1, \xi \in HF^\circ$).

To describe the maps $F_0, F_1: H\{i \geq 0\} \rightarrow H\{i \geq 0 \text{ and } j \geq k\}$ (under the identifications (28)) we need a bit more notation. Let e_1, \dots, e_{2g} be a symplectic basis for $H_1(\Sigma_g; \mathbb{Z})$ and set $\omega = e_1 \wedge e_2 + \dots + e_{2g-1} \wedge e_{2g} \in \Lambda^2 H_1(\Sigma_g; \mathbb{Z})$. For a given $\beta \in \Lambda^1$ define $\beta \lrcorner: \Lambda^k \rightarrow \Lambda^{k-1}$ as contraction associated to ω , ie

$$\beta \lrcorner (\alpha_1 \wedge \dots \wedge \alpha_k) = \sum_{\ell=1}^k (-1)^\ell \omega(\alpha_\ell, \beta) \alpha_1 \wedge \dots \wedge \hat{\alpha}_\ell \wedge \dots \wedge \alpha_k$$

where $\omega(\alpha_\ell, \beta)$ refers to the natural pairing $\Lambda^2 H_1(\Sigma_g) \otimes \Lambda^2 H^1(\Sigma_g) \rightarrow \mathbb{Z}$. The contraction \lrcorner defined this way extends readily to a contraction $\lrcorner: \Lambda^m \otimes \Lambda^k \rightarrow \Lambda^{k-m}$ given by $(\beta_1 \wedge \dots \wedge \beta_m) \lrcorner (\alpha_1 \wedge \dots \wedge \alpha_k) = \beta_1 \lrcorner (\beta_2 \lrcorner (\dots (\beta_m \lrcorner (\alpha_1 \wedge \dots \wedge \alpha_k) \dots))$). Let $\tilde{\star}: \Lambda^k \rightarrow \Lambda^{2g-k}$ be the “Hodge–Lefschetz star operator” associated to ω and defined as

$$\tilde{\star} \alpha = \frac{1}{g!} \alpha \lrcorner \omega^g$$

where we have by abuse of notation used ω to also denote $e^1 \wedge e^2 + \dots + e^{2g-1} \wedge e^{2g} \in \Lambda^2$, which is the dual of the symplectic form ω from earlier. Here $e^i \in H^1(\Sigma_g; \mathbb{Z})$, $i = 1, \dots, 2g$ is the dual basis (with respect to the Kronecker pairing) of $e_i \in H_1(\Sigma_g; \mathbb{Z})$, ie $e^i(e_j) = \delta_{ij}$. Let

$$\begin{aligned} \pi_k &: H\{i \geq 0\} \rightarrow H\{i \geq 0 \text{ and } j \geq k\} \\ \pi_{i \geq 0} &: \Lambda^* \otimes \mathbb{Z}[U, U^{-1}] \rightarrow H\{i \geq 0\} \\ \pi_{j \geq 0} &: \Lambda^* \otimes \mathbb{Z}[U, U^{-1}] \rightarrow H\{j \geq 0\} \end{aligned}$$

be the natural projection maps and let $J: H\{i \geq 0\} \rightarrow H\{j \geq 0\}$ be the map

$$J(xU^\ell) = \pi_{j \geq 0} \left((-1)^{k+g-1} \exp(2\omega U) \lrcorner (\tilde{\star} x) U^{g+\ell-k} \right) \quad \text{when } x \in \Lambda^k,$$

where we omit the tensor product signs and, by convention, contraction with U^n is taken to mean multiplication by U^{-n} . This map was studied in detail in [3]; see Figure 3 for a pictorial explanation of J .

The results from [3] show that $F_0, F_1: H\{i \geq 0\} \rightarrow H\{i \geq 0 \text{ and } j \geq k\}$ are given by

$$F_0 = \begin{cases} \pi_k & ; k \leq 0 \\ \pi_k \circ (U^{-k} J) & ; k > 0 \end{cases} \quad \text{and} \quad F_1 = \begin{cases} \pi_k \circ (U^{-k} J) & ; k \leq 0 \\ \pi_k & ; k > 0. \end{cases}$$

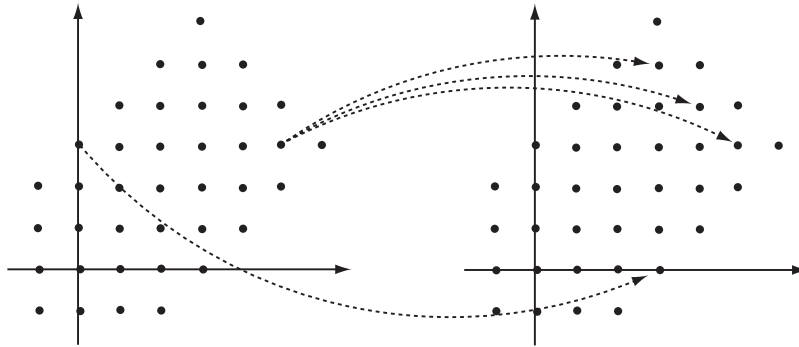


Figure 3: A visual explanation of the map $J: H\{i \geq 0\} \rightarrow H\{j \geq 0\}$ (with $g = 3$), which is perhaps best understood as a perturbation of the star operator $\tilde{\star}$. The latter maps an element at coordinate (x, y) to an element at coordinate (y, x) . The action of $\exp(2\omega U)\angle \cdot$, thought of as a formal sum, then “smears out” such an element along the line with slope -1 and passing through (y, x) , in such a way that the only nonzero terms are those lying to the right and below (y, x) . Thus, for example when $g = 3$, an element at coordinate $(0, 3)$ is mapped by J to an element at coordinate $(3, 0)$ while an element at coordinate $(5, 3)$ is mapped to a sum of terms living at coordinates $(5, 3)$, $(4, 4)$ and $(3, 5)$.

With all these preliminaries out of the way and with our notation in place, we now turn to the actual calculations of the twisted Heegaard Floer groups of $\Sigma_g \times S^1$. The adjunction inequality implies that for any spin^c -structure \mathfrak{s} on $\Sigma_g \times S^1$ which is not among the \mathfrak{s}_k , the associated Heegaard Floer groups $HF^+(\Sigma_g \times S^1, \mathfrak{s}; M)$ vanish (for any coefficient module M); the same is true for $\mathfrak{s} = \mathfrak{s}_k$ when $|k| \geq g$. The remaining spin^c -structures \mathfrak{s}_k with $|k| \leq g - 1$ give rise to nontrivial Heegaard Floer groups as the next theorem explains.

Theorem 9.4 *Fix an integer k with $|k| \leq g - 1$. If $k \neq 0$ choose Λ to be either $L(t)$ or $\mathcal{L}(t)$ and if $k = 0$ choose $\Lambda = \mathcal{L}(t)$. Then the Heegaard Floer homology groups $HF^+(\Sigma_g \times S^1, \mathfrak{s}_k; \Lambda)$ are isomorphic to*

$$HF^+(\Sigma_g \times S^1, \mathfrak{s}_k; \Lambda) \cong X(g, d) \otimes \Lambda \quad \text{with} \quad d = g - 1 - |k|$$

as $\mathbb{Z}[H^1(\Sigma_g \times S^1; \mathbb{Z})] \otimes_{\mathbb{Z}} \mathbb{Z}[U]$ -modules. The action of $\Lambda^* H_1(\Sigma_g; \mathbb{Z}) \subset \Lambda^* H_1(\Sigma_g \times S^1; \mathbb{Z})$ on $X(g, d)$ is induced by the standard action (29) under the nonstandard

embedding of $X(g, d)$ into $\Lambda^* H^1(\Sigma_g; \mathbb{Z}) \otimes \mathbb{Z}[U, U^{-1}] \otimes \Lambda$ given by

$$x \mapsto x + \pi_{-|k|} \left(\sum_{\ell \geq 1} (-t U^{|\ell|} J)^\ell x \right).$$

Proof The proof of the theorem follows slightly different arguments depending on whether $k \neq 0$ or $k = 0$. We first address the former.

Case of $k \neq 0$ For concreteness let us assume $k < 0$. Choose $\Lambda = L(t)$ for now and consider the negative surgery exact sequence in twisted coefficients (25). By Proposition 9.3 the map \underline{F} equals $F_0 + tF_1$ after restricting to $HF_{\leq d_0}^+(Y)$ for suitably large d_0 , once n is chosen sufficiently large (which we assume tacitly throughout). It follows from the degree shift formula (24) that $\deg F_0 = \deg F_1 - 2k$ and thus $\deg F_0 > \deg F_1$ when $k < 0$. From the description of F_0 and F_1 above, F_0 is clearly surjective. Since $\deg F_1 < \deg F_0$ we see that \underline{F} is also surjective. Moreover, we claim that the kernel of \underline{F} is generated by elements of the form

$$\ker(\underline{F}) = \left\langle x + \pi_k \left(\sum_{\ell \geq 1} (-t U^{-k} J)^\ell x \right) \middle| x \in H\{i \geq 0 \text{ and } j < k\} \right\rangle$$

Indeed, suppose $\xi = \xi_0 + \xi_1 + \dots$ is in $\ker(\underline{F})$, where ξ_j are the homogeneous terms in order of decreasing degree. Then certainly $\xi_0 \in \ker(F_0) = \ker(\pi_k) = H\{i \geq 0 \text{ and } j < k\}$. On the other hand, if $x \in H\{i \geq 0 \text{ and } j < k\}$, we can “build” an element of $\ker(\underline{F})$ from x by observing that $F_0(x) = 0$, while $tF_1(x) = t\pi_k(U^{-k}J(x))$. Thus replacing x by $x - t\pi_k(U^{-k}J(x))$, we obtain an element still annihilated by F_0 , whose image under tF_1 is $(-t(U^{-k}J))^2x$. Continuing in this way, and using that since $k < 0$ the action of U^{-k} is nilpotent on any element of $\Lambda^* \otimes \mathbb{Z}[U, U^{-1}]$, we eventually find an element killed by $F_0 + tF_1 = \underline{F}$. This gives the identification above, and then projection onto the homogeneous term of highest degree establishes the isomorphism $\ker \underline{F} \cong X(g, d)$ with $d = g - 1 - |k|$. Since the sequence (25) is equivariant with respect to the $H_1(\Sigma_g; \mathbb{Z})$ -action, the proposition (for the case $k < 0$ and $\Lambda = L(t)$) follows. The results with $\Lambda = \mathcal{L}(t)$ follows from the result for $L(t)$ by tensoring with $\mathcal{R}_{Y_0, \eta}$ and using the flatness of $\mathcal{R}_{Y_0, \eta}$ (cf Corollary 9.2). The case of $k > 0$ can be proved analogously, or by appeal to conjugation invariance (Theorem 8.10).

We note that the argument above shows that \underline{F} is surjective, and that if $\xi \in \ker(\underline{F})$ then the largest degree of a homogeneous term in ξ is bounded above (by $g - 2 - 2|k|$), independent of n . Thus the restriction of \underline{F} to $HF_{\leq d_0}^+$ above is permissible.

Case of $k = 0$ Consider once more the sequence (25) and again note that upon restriction to a given grading we may take $\underline{F} = F_0 + tF_1$. The key difference from the case of $k \neq 0$ is that the degrees of F_0 and F_1 are now equal.

We begin by showing that \underline{F} is again surjective: for a given $y \in H\{i \geq 0 \text{ and } j \geq 0\}$ let $x_y \in H\{i \geq 0\} \otimes \Lambda_\eta$ be $x_y = \pi_{i \geq 0}(\sum_{\ell \geq 0} (-tJ)^\ell y)$. Then

$$\underline{F}(x_y) = \pi_0(\text{id} + tJ) \left(\pi_{i \geq 0} \sum_{\ell \geq 0} (-tJ)^\ell y \right) = \pi_0(\text{id} + tJ) \left(\sum_{\ell \geq 0} (-tJ)^\ell y \right) = \pi_0(y) = y.$$

This shows $F_0 + tF_1$ is surjective; the remaining terms in \underline{F} have strictly lower degree, hence \underline{F} is also surjective.

To determine the kernel of \underline{F} pick a kernel element $\xi = \xi_0 + \xi_1 t + \xi_2 t^2 + \dots \in \ker(\underline{F})$. Such an element ξ is then subject to the infinite system of equations (coming from the requirement $(F_0 + tF_1)(\xi) = 0$):

$$\begin{aligned} \pi_0(\xi_0) &= 0 \\ \pi_0(\xi_1 + J(\xi_0)) &= 0 \\ &\vdots \\ \pi_0(\xi_k + J(\xi_{k-1})) &= 0 \\ &\vdots \end{aligned} \tag{30}$$

The equation $\pi_0(\xi_0) = 0$ implies $\xi_0 \in H\{i \geq 0 \text{ and } j < 0\}$. The second equation determines the $H\{i \geq 0 \text{ and } j \geq 0\}$ -component of ξ_1 uniquely but imposes no condition on the $H\{i \geq 0 \text{ and } j < 0\}$ -component of ξ_1 . The same holds true for all ξ_k , $k \geq 1$:

- The $H\{i \geq 0 \text{ and } j \geq 0\}$ -component of ξ_k is determined by ξ_{k-1} .
- The $H\{i \geq 0 \text{ and } j < 0\}$ -component of ξ_k can be chosen arbitrarily.

In particular, as before there is an *a priori* upper bound on the degree of a kernel element and therefore it suffices to assume that $\underline{F} = F_0 + tF_1$.

This immediately shows that the kernel of the map \underline{F} is isomorphic (but *not* equal!) to $H\{i \geq 0 \text{ and } j < 0\} \otimes \mathcal{L}(t)$. As the above system shows, the isomorphism from $H\{i \geq 0 \text{ and } j < 0\} \otimes \mathcal{L}(t)$ to $\ker(\underline{F}) \subset H\{i \geq 0\} \otimes \mathcal{L}(t)$ is given by the $\mathcal{L}(t)$ -equivariant map

$$\xi \mapsto \xi + \pi_0 \left(\sum_{\ell \geq 1} (-tJ)^\ell \xi \right). \quad \square$$

Remark 9.5 Consider the embedding $X(g, d) \hookrightarrow \Lambda^* H^1(\Sigma_g; \mathbb{Z}) \otimes \mathbb{Z}[U, U^{-1}] \otimes \Lambda$ from [Theorem 9.4](#):

$$x \mapsto x - \pi_k \left(tU^{-k} Jx - (tU^{-k} J)^2 x + (tU^{-k} J)^3 x - \dots \right)$$

It follows from this expression that the induced action by $H_1(\Sigma_g)$ on $X(g, d)$ is standard in the lowest power of t but typically has nonzero “correction terms” involving higher powers of t . However, when $3|k| > g - 2$ then all of the terms $(tU^{-k} J)^\ell$ for $\ell \geq 1$ lie in the kernel of π_k showing that in that range the $\Lambda^* H_1(\Sigma_g; \mathbb{Z})$ –action has no correction terms. This was already observed by Ozsváth–Szabó [[10](#), [Theorem 9.3](#)] in the case of \mathbb{Z} coefficients (see also [[3](#)]).

Remark 9.6 The isomorphism $HF^+(\Sigma_g \times S^1, \mathfrak{s}_k; \Lambda) \cong X(g, d) \otimes \Lambda$ from [Theorem 9.4](#) does not extend to the case of $k = 0$ and $\Lambda = L(t)$. With $k = 0$ and $\Lambda = L(t)$ the infinite system ([30](#)) becomes a finite system which terminates with the equation $\pi_0(J(\xi_m)) = 0$ for some choice of $m \in \mathbb{N}$. This equation breaks the symmetry of the system and imposes additional constraints not satisfied by all elements of the form $\pi_0 \sum_{\ell \geq 0} (-tJ)^\ell x$ with $x \in X(g, g - 1) \otimes L(t)$.

Remark 9.7 From the proof of the theorem, the homomorphism $F: HF^+(Y, \mathfrak{s}; \Lambda) \rightarrow HF^+(Y_{-n}, \mathfrak{s}_k; \Lambda)$ is surjective in all cases, so that $HF^+(\Sigma \times S^1, \mathfrak{s}_k; \Lambda)$ can be thought of as a submodule of $HF^+(Y, \mathfrak{s}; \Lambda) = H\{i \geq 0\} \otimes \Lambda$. The latter carries a grading with respect to which $t \in \Lambda$ ($= L(t)$ or $\mathcal{L}(t)$) carries degree 0, so we can use this to impose a similar grading on $HF^+(\Sigma \times S^1, \mathfrak{s}_k; \Lambda) = X(g, d) \otimes \Lambda$. Equivalently, we grade the latter group by lifting the natural grading on $X(g, d)$, induced by the standard embedding. This grading lifts the relative cyclic grading obtained by forgetting the grading on R_{Y_0} in the definition of twisted-coefficient Floer homology, and has the property, for example, that homogeneous summands are R_{Y_0} –submodules. However, it is no longer the case that the action by $H_1(Y_0)$ decreases degree by 1, or is even homogeneous. We will refer to this alternative grading as the *height* in $HF^+(\Sigma \times S^1, \mathfrak{s}_k; \Lambda)$.

In the next section we will have occasion to consider the relative Ozsváth–Szabó invariant of the 4–manifold $\Sigma \times D^2$, for which the following result is central.

Theorem 9.8 Consider the cobordism W from $\Sigma_g \times S^1$ to S^3 obtained by removing a small 4–ball from $\Sigma_g \times D^2$. For $|k| \leq g - 1$ let $\mathfrak{v}_k \in \text{Spin}^c(W)$ be the unique spin^c –structure on W which restricts to \mathfrak{s}_k on $\Sigma_g \times S^1$. If $k = 0$ let $\Lambda = \mathcal{L}(t)$ and if $k \neq 0$ choose Λ to be either $L(t)$ or $\mathcal{L}(t)$. Then the component of

$$\mathcal{F}_k = F_{W, \mathfrak{v}_k}: HF^+(\Sigma_g \times S^1, \mathfrak{s}_k; \Lambda) \rightarrow HF^+(S^3) \otimes \Lambda$$

mapping into the lowest-degree part $HF_0^+(S^3) \otimes \Lambda$ is given by projection onto the summand of lowest height, corresponding to $H\{(0, -g)\} \otimes \Lambda \cong \Lambda^0 H^1(\Sigma) \otimes U^0 \otimes \Lambda \subset X(g, d) \otimes \Lambda$.

Proof We decompose the cobordism W as $W = W_0 \cup W_1 \cup \dots \cup W_{2g}$ where W_0 is the cobordism from $\Sigma_g \times S^1$ to $Y = \#^{2g}(S^1 \times S^2)$ obtained by attaching a 0-framed 2-handle to the latter along the knot K_g . The orientation on W_0 is the one that induces the orientations $\partial W_0 = -(\Sigma_g \times S^1) \sqcup Y$ on its boundary components. The cobordisms W_i , $i = 1, \dots, 2g$ are obtained by the obvious 3-handle additions corresponding to the $2g$ 1-handles of Y .

As explained in Section 2.2, the map \mathcal{F}_k can be calculated by separately calculating the contribution from each of the maps F_{W_i} induced by W_i (the spin^c -structure on W_i is the restriction of $\tau_k|_{W_i}$ which we omit from the notation for simplicity).

The map F_{W_0} is just the map \underline{H} from the sequence (25); it maps $HF^+(\Sigma_g \times S^1, \mathfrak{s}_k; \Lambda)$ isomorphically onto the kernel of \underline{F} . This kernel was explicitly identified in the proof of Theorem 9.4 and equals the image of the embedding of $H\{i \geq 0 \text{ and } j < -|k|\} \otimes \Lambda \hookrightarrow \Lambda^* \otimes \mathbb{Z}[U^{-1}] \otimes \Lambda$ given by

$$x \mapsto x + \pi_k \left(\sum_{\ell \geq 1} (-tU^{-k} J)^\ell x \right)$$

Under the identification of $HF^+(\Sigma_g \times S^1, \mathfrak{s}_k; \Lambda)$ with $H\{i \geq 0 \text{ and } j < -|k|\} \otimes \Lambda$ (Theorem 9.4 and (27)), this embedding precisely corresponds to the map F_{W_0} .

It is a simple matter to see that the homomorphism in Floer homology induced by the composition of 3-handle cobordisms $\#^{2g} S^1 \times S^2 \rightarrow S^3$ is given by projection onto the lowest-degree factor (and shifting degree up by g). The result follows from the above description of the image of $HF^+(\Sigma \times S^1, \mathfrak{s}_k; \Lambda)$ in $HF^+(\#^{2g} S^1 \times S^2)$: indeed, the only elements of $HF^+(\Sigma_g \times S^1, \mathfrak{s}_k; \Lambda)$ having any component in the lowest-degree part of $HF^+(\#^{2g} S^1 \times S^2)$ are those in lowest height. \square

9.3 A surface cross a circle: universally twisted coefficients

It will be useful to have a small amount of information on the Floer homology of $\Sigma \times S^1$ with “universal” coefficients, ie, coefficients in the group ring $R_{\Sigma \times S^1}$. Strictly, the material in this section is not necessary for the fiber sum theorems to follow, but we find it conceptually helpful to understand the Floer groups in which relative invariants take values. The result we need, Theorem 9.12 below, states that the Floer homology $HF^+(\Sigma \times S^1, \mathfrak{s}_{\pm(g-1)}; R_{\Sigma \times S^1})$ is a free module of rank 1, ie, it is isomorphic to

$R_{\Sigma \times S^1}$. This statement is in fact a consequence of work of Ozsváth and Szabó [14], who show essentially that if Y is a 3–manifold that fibers over the circle with fiber genus > 1 , then there is an admissible Heegaard diagram for Y containing just a single generator corresponding to the “extremal” spin^c structure. We give a different argument here based on knot Floer homology in the universally twisted case, which is useful in other situations (eg [6]).

Continuing our notation from the last section, let Y be the manifold $Y = \#^{2g}(S^1 \times S^2)$. An easy application of Theorem 9.1 (for surgery on the unknot in S^3 and with $n = 1$) and the connected sum formula for HF^+ and \widehat{HF} (cf [11]) yields

$$\widehat{HF}(\#^{2g}(S^1 \times S^2), \mathfrak{s}_0; R_Y) \cong \mathbb{Z}_{(-g)} \quad \text{and} \quad HF^+(\#^{2g}(S^1 \times S^2), \mathfrak{s}_0; R_Y) \cong \mathcal{T}_{-g}$$

where $\mathcal{T}_n = \mathbb{Z}[U, U^{-1}]/U \cdot \mathbb{Z}[U]$ as before, graded such that the summand of lowest degree lies in degree n . For all spin^c –structures $\mathfrak{s} \neq \mathfrak{s}_0$, $HF^+(\#^{2g}(S^1 \times S^2), \mathfrak{s}; R_Y)$ and $\widehat{HF}(\#^{2g}(S^1 \times S^2), \mathfrak{s}; R_Y)$ are zero, so we shall drop the spin^c –structure from our notation. Also, we shall drop the 3–manifold from our notation for the knot Floer homology groups whenever there is not risk of confusion.

Lemma 9.9 *Let $g \geq 1$, set $Y = \#^{2g}(S^1 \times S^2)$ and let K_g be the nullhomologous knot $K_g = \#^g B(0, 0) \subset Y$. Then for $j \in \{-g, \dots, g\}$, the twisted knot Floer homology $\widehat{HFK}(K_g, j; R_Y)$ is a free module over R_Y having rank $\binom{2g}{g+j}$ and supported in degree j , and is zero for all other values of j .*

The proof of this lemma relies on a filtered version of Theorem 9.1.

Theorem 9.10 (Ozsváth–Szabó [10]) *Let $K, L \subset Z$ be two nullhomologous knots with linking number 0. Let K_0, K_1 and K_{-1} be the knots in $Z_0(L), Z_1(L)$ and $Z_{-1}(L)$ induced by K where $Z_\ell(L)$ is the result of ℓ –framed surgery on L . Then for any $\mathfrak{s} \in \text{Spin}^c(Z)$ and any R_Z –module M there are exact sequences of $R_{Z_0(L)}$ –modules: for each $j \in \mathbb{Z}$.*

$$\begin{aligned} \dots \rightarrow \widehat{HFK}(K, \mathfrak{s}, j; M)[t^{\pm 1}] &\rightarrow \widehat{HFK}(K_0, [\mathfrak{s}_k], j; M[t^{\pm 1}]) \\ &\rightarrow \widehat{HFK}(K_1, \mathfrak{s}_k, j; M)[t^{\pm 1}] \rightarrow \dots \\ \dots \rightarrow \widehat{HFK}(K, \mathfrak{s}, j; M)[t^{\pm 1}] &\rightarrow \widehat{HFK}(K_{-1}, [\mathfrak{s}_k], j; M[t^{\pm 1}]) \\ &\rightarrow \widehat{HFK}(K_0, \mathfrak{s}_k, j; M)[t^{\pm 1}] \rightarrow \dots \quad \square \end{aligned}$$

Proof of Lemma 9.9 Lemma 9.9 follows from repeated applications of Theorem 9.10 to various triples of knots; our proof is a straightforward adaptation of the \mathbb{Z} –coefficient proof first obtained by Ozsváth and Szabó in [10]. We first consider the case of $g = 1$.

The three knots $B(\infty, 1) \rightarrow B(0, 1) \rightarrow B(1, 1)$ fit into the positive n surgery sequence from [Theorem 9.10](#). It is easy to see that $B(\infty, 1)$ is the unknot in S^3 while $B(1, 1)$ is the right-handed trefoil. Thus

$$\widehat{HFK}(B(\infty, 1), j) \cong \begin{cases} \mathbb{Z}_{(0)} & (j = 0) \\ 0 & (j \neq 0) \end{cases}$$

and

$$\widehat{HFK}(B(1, 1), j) \cong \begin{cases} \mathbb{Z}_{(0)} & (j = 1) \\ \mathbb{Z}_{(-1)} & (j = 0) \\ \mathbb{Z}_{(-2)} & (j = -1) \\ 0 & (j \neq 0, \pm 1) \end{cases}$$

where a subscript (n) indicates that the corresponding module is supported in degree n .

Using these in the surgery sequence leads to

$$\widehat{HFK}(B(0, 1), j; R_{S^1 \times S^2}) \cong \begin{cases} (R_{S^1 \times S^2})_{(\frac{3}{2})} & (j = 1) \\ (R_{S^1 \times S^2}^2)_{(\frac{1}{2})} & (j = 0) \\ (R_{S^1 \times S^2})_{(-\frac{1}{2})} & (j = -1) \\ 0 & (j \neq 0, \pm 1). \end{cases}$$

In a similar vein using the negative n surgery sequence from [Theorem 9.10](#) for the triple $B(\infty, -1) \rightarrow B(-1, -1) \rightarrow B(0, -1)$ (and observing that $B(-1, -1)$ is the left-handed trefoil) leads to

$$\widehat{HFK}(B(0, -1), j; R_{S^1 \times S^2}) \cong \begin{cases} (R_{S^1 \times S^2})_{(\frac{1}{2})} & (j = 1) \\ (R_{S^1 \times S^2}^2)_{(-\frac{1}{2})} & (j = 0) \\ (R_{S^1 \times S^2})_{(-\frac{3}{2})} & (j = -1) \\ 0 & (j \neq 0, \pm 1). \end{cases}$$

For our next set of surgery sequences note that $B(0, \infty)$ is the unknot in $S^1 \times S^2$ and therefore

$$\widehat{HFK}(B(0, \infty), j; R_{S^1 \times S^2}) \cong \begin{cases} \mathbb{Z}_{(-\frac{1}{2})} & (j = 0) \\ 0 & (j \neq 0) \end{cases}$$

where \mathbb{Z} is the trivial $R_{S^1 \times S^2}$ -module. Using the negative n surgery sequence on the triple $B(0, \infty) \rightarrow B(0, -1) \rightarrow B(0, 0)$ for $j = 0$ shows that

$$\widehat{HFK}_{(-1)}(B(0, 0), 0; R_{\#^2(S^1 \times S^2)}) = 0.$$

The positive n surgery sequence for the triple $B(0, \infty) \rightarrow B(0, 0) \rightarrow B(0, 1)$ for $j = \pm 1$ leads to

$$\widehat{HFK}(B(0, 0), j; R_{\#^2(S^1 \times S^2)}) \cong (R_{\#^2(S^1 \times S^2)})_{(j)} \quad (j = \pm 1)$$

while $j = 0$ yields the sequence

$$0 \rightarrow \widehat{HFK}_{(0)}(B(0, 0), 0; R_{\#^2(S^1 \times S^2)}) \rightarrow (R_{\#^2(S^1 \times S^2)}^2)_{(-\frac{1}{2})} \rightarrow (R_{S^1 \times S^2})_{(-\frac{1}{2})} \rightarrow 0.$$

Let s_1, s_2 be two generators of $H^1(\#^2(S^1 \times S^2); \mathbb{Z})$, then we can write the above as

$$0 \rightarrow \widehat{HFK}_{(0)}(B(0, 0), 0; R_{\#^2(S^1 \times S^2)}) \rightarrow L(s_1, s_2)_{(-\frac{1}{2})}^2 \xrightarrow{f} L(s_2)_{(-\frac{1}{2})} \rightarrow 0,$$

where $L(s_1, s_2)$ is the ring of Laurent polynomials in s_1 and s_2 .

Notice that since s_1 acts trivially on the factor $L(s_2)$ above, f factors through a quotient module:

$$\begin{array}{ccc} L(s_1, s_2)^2 & \xrightarrow{f} & L(s_2) \\ \downarrow \pi & \nearrow \tilde{f} & \\ \frac{L(s_1, s_2)^2}{(s_1 - 1) \cdot L(s_1, s_2)^2} & \cong & L(s_2)^2 \end{array}$$

Here π is the map which sends a pair of polynomials $(p(s_1, s_2), q(s_1, s_2)) \in L(s_1, s_2)^2$ to $(p(1, s_2), q(1, s_2)) \in L(s_2)^2$. Consider \tilde{f} as a homomorphism between $L(s_2)$ modules. Since f surjects onto the free module $L(s_2)$, there is a splitting of the domain of \tilde{f} as $L(s_2)^2 \cong \ker(\tilde{f}) \oplus L(s_2)$, as an $L(s_2)$ –module (and incidentally, $\ker(\tilde{f}) \cong L(s_2)$). Let a and b be $L(s_2)$ –module generators for the two factors in this splitting; then under the standard decomposition of $L(s_2)^2$ we have that $a = (a_1, a_2)$ and $b = (b_1, b_2)$, where a_i, b_i are Laurent polynomials in s_2 . There is a canonical lift of a and b to $\tilde{a}, \tilde{b} \in L(s_1, s_2)^2$, by considering a_i and b_i to be polynomials in both s_1 and s_2 . Since a and b generate $L(s_2)^2$ as an $L(s_2)$ module, it is easy to see that \tilde{a} and \tilde{b} generate $L(s_1, s_2)^2$ as an $L(s_1, s_2)$ module.

Put another way, we have arranged the splitting $L(s_1, s_2)^2 \cong L(s_1, s_2)\tilde{a} \oplus L(s_1, s_2)\tilde{b}$ in such a way that f appears as the map $(p(s_1, s_2)\tilde{a}, q(s_1, s_2)\tilde{b}) \mapsto q(1, s_2)$. Therefore

$$\ker(f) = L(s_1, s_2) \oplus (s_1 - 1)L(s_1, s_2) \cong L(s_1, s_2)^2 \cong (R_{\#^2 S^1 \times S^2})^2,$$

completing the proof of [Lemma 9.9](#) when $g = 1$. The case of $g > 1$ follows from this and the connected sum formula for knot Floer homology [\[10\]](#). \square

The results of [Lemma 9.9](#) can be rewritten in a more concise way as follows: let M be the free R –module of rank $2g$ (with R still denoting $R_Y = R_{\#^{2g} S^1 \times S^2}$). Then for $j = -g, \dots, g$ we have $\widehat{HFK}(K_g, j; R) \cong \Lambda^{g+j} M$ supported entirely in grading j . Recall that there is a spectral sequence associated to the knot filtration, whose E^1 term is $\widehat{HFK}(K_g, j; R)$, and whose E^∞ term is $\widehat{HF}(\#^{2g}(S^1 \times S^2); R) \cong \mathbb{Z}_{(-g)}$.

Since the grading of any term in $\widehat{HFK}(K_g, j; R)$ equals j , it follows that this spectral sequence collapses at the E^2 stage, in particular the only nonzero differentials in the spectral sequence are those on the E^1 level:

$$\partial_v: \Lambda^{\ell+1} M \rightarrow \Lambda^\ell M \quad \ell = 0, \dots, 2g-1.$$

In particular, we infer that the chain complex

$$(31) \quad 0 \rightarrow \Lambda^{2g} M \xrightarrow{\partial_v} \Lambda^{2g-1} M \xrightarrow{\partial_v} \dots \xrightarrow{\partial_v} \Lambda^1 M \xrightarrow{\partial_v} \Lambda^0 M \rightarrow 0$$

which appears at the E^1 stage of the spectral sequence has homology equal to $\widehat{HF}(\#^{2g} S^1 \times S^2; R) = \mathbb{Z}_{(-g)}$. In other words, the complex above is a (minimal) free resolution of \mathbb{Z} in the category of R -modules, which we will be a useful observation for us momentarily.

As before, knowledge of the knot Floer homology of K_g allows calculation of the Floer homology of large integer surgeries along K_g . We will need only the following result; the twisted Floer homology of circle bundles is studied in greater depth in [6].

Lemma 9.11 *Fix $g \geq 2$ and $n \gg 0$. There is an isomorphism of $R_{Y_n} \otimes \mathbb{Z}[U]$ -modules*

$$HF^+(Y_n, \mathfrak{s}_{\pm(g-1)}; R_{Y_n}) \cong R_{Y_n} \oplus \mathcal{T}_{-g-\tau},$$

where $\tau = (n - (2g - 2 - n)^2)/(4n)$, and R_{Y_n} is supported in degree $g - 2 - \tau$ (with trivial action by U).

Observe that there is an identification $H^1(Y_n; \mathbb{Z}) = H^1(Y; \mathbb{Z})$, so that $R_{Y_n} = R_Y$.

Proof By conjugation symmetry, we may consider only the case of \mathfrak{s}_{g-1} . We have

$$HF^+(Y_n, \mathfrak{s}_{g-1}; R_{Y_n}) \cong H\{i \geq 0 \text{ or } j \geq g-1\}.$$

To calculate this homology, we use the filtration induced by $i + j$ as before to obtain a spectral sequence with E^1 term $\widehat{HFK}(\#^{2g} B(0, 0), R_Y) \otimes \mathbb{Z}[U, U^{-1}]$ (or, the portion of this group lying in the region $\{i \geq 0 \text{ or } j \geq g-1\}$). From the work above, we can express this as $E_{i,j}^1 = \Lambda^{g+j-i} M \otimes U^{-i}$, and for dimensional reasons only the d_1 differential can be nontrivial in the spectral sequence. Since d_1 splits as horizontal and vertical parts as before, we can consider E^1 as a double complex and use a second spectral sequence to calculate its homology. Specifically: we use the spectral sequence that arises from the filtration of E^1 by the subcomplexes $\{i \leq i_0\}$.

The first stage in this secondary spectral sequence calculates the homology of the vertical differential. In each column $i = c$ with $c \geq 0$, we see a shifted copy of the complex (31) above, thus the homology of such a column is \mathbb{Z} supported in degree

$c - g$. The only other nonzero column is $i = -1$, which contains the single term $\Lambda^{2g} M \cong R_Y$ in degree $g - 2$ (and hence is equal to its homology). Further differentials in the secondary spectral sequence must be trivial: the only possible differential would map \mathbb{Z} into $\Lambda^{2g} M$, but there are no nontrivial R_Y -module maps of the torsion module \mathbb{Z} into the free module $\Lambda^{2g} M$. Hence the lemma follows, recalling that the identification of $HF^+(Y_n, \mathfrak{s}_{g-1})$ with $H\{i \geq 0 \text{ or } j \geq g - 1\}$ shifts degree by τ . \square

Theorem 9.12 *Let $g \geq 2$. The Floer homology of $Y_0 = \Sigma_g \times S^1$ in the spin^c structure $\mathfrak{t}_{\pm(g-1)}$ characterized by $c_1(\mathfrak{t}_{\pm(g-1)}) = \pm(2g - 2) \text{PD}[S^1]$ is given as an R_{Y_0} -module by*

$$HF^+(Y_0, \mathfrak{t}_{\pm(g-1)}; R_{Y_0}) \cong R_{Y_0},$$

with trivial action by U .

Proof As usual, we consider \mathfrak{t}_{g-1} only. We consider the surgery sequence for large positive surgeries, which gives rise to an exact triangle:

$$\begin{array}{ccc} HF^+(Y_n, \mathfrak{s}_{g-1}, R_Y)[t^{\pm 1}] & \xrightarrow{F} & HF^+(Y, \mathfrak{s}, R_Y)[t^{\pm 1}] \\ & \searrow & \swarrow \\ & HF^+(Y_0, \mathfrak{t}_{g-1}, R_{Y_0}) & \end{array}$$

Here we have chosen n large enough that \mathfrak{t}_{g-1} is the only spin^c structure in the equivalence class $[\mathfrak{s}_{g-1}]$ having nontrivial Floer homology, and of course \mathfrak{s} is the torsion spin^c structure on $Y = \#^{2g} S^1 \times S^2$.

The highest-degree component of F is the t -linear extension of the natural projection $H\{i \geq 0 \text{ or } j \geq g - 1\} \rightarrow H\{i \geq 0\}$, which is an isomorphism in all degrees $d \geq g - 1 - \tau$, with τ as in the preceding lemma. It follows that F itself is an isomorphism in those degrees. By U -equivariance, we have that F restricts to the factor $\mathcal{T}_{-g-\tau} \subset HF^+(Y_n, \mathfrak{s}_{g-1}; R_Y)$ as an isomorphism onto $\mathcal{T}_{-g} = HF^+(Y, \mathfrak{s}; R_Y)$, whence $HF^+(Y_0, \mathfrak{t}_{g-1}; R_{Y_0}) \cong \ker(F)$. We claim that $\ker(F) \cong R_Y[t^{\pm 1}] \cong R_{Y_0}$.

To see this it suffices to find the kernel of F restricted to some bounded range of degrees, where we know F can be identified with $F_0 + tF_1$ (continuing notation from previously). Furthermore, results of Ozsváth and Szabó [16] show that F_1 is identified (up to chain isomorphism) with the projection

$$H\{i \geq 0 \text{ or } j \geq g - 1\} \rightarrow H\{j \geq g - 1\},$$

and in particular is surjective in all high degrees. Then U -equivariance of maps induced by cobordisms implies F_1 (and also F_0) must vanish on $HF_{\text{red}}^+(Y_n, \mathfrak{s}_{g-1}, R_Y)[t^{\pm 1}] = R_Y[t^{\pm 1}]$, since Y has no reduced Floer homology. \square

10 Product formulae

We can now piece together the ingredients of the preceding sections to deduce the results stated in the introduction. The conceptual plan is reasonably straightforward: if $X = M_1 \#_{\Sigma} M_2$ is a fiber sum as in the introduction, then the Ozsváth–Szabó invariants for X are given by a pairing between relative invariants for $Z_i = M_i \setminus (\Sigma \times D^2)$, after perturbing by a class $\eta \in H^2(X; \mathbb{R})$ that restricts nontrivially to $\Sigma \times S^1$, according to [Theorem 8.17](#).

To determine the relative invariants $\Psi_{Z_i, \eta}$ of the pieces and obtain a formula for OS_X in terms of OS_{M_i} , we observe that the Ozsváth–Szabó invariants of the M_i are themselves determined by the pairing between $\Psi_{Z_i, \eta}$ and the relative invariant of $\Sigma \times D^2$, again using the perturbed version of Floer theory since, even if $b^+(Z_i) > 0$, we have $b^+(\Sigma \times D^2) = 0$ and [Theorem 7.6](#) need not apply. Hence we need to understand the perturbed relative invariant $\Psi_{\Sigma \times D^2, \eta}$, as well as the relevant pairing on Floer homology.

Now, it is easy to see that the coefficient module for $\Sigma \times S^1$ induced by $\Sigma \times D^2$ (with a 4-ball removed) is $\mathbb{Z}[K(\Sigma \times D^2)] = L(t)$, where $t \in H^1(\Sigma \times S^1; \mathbb{Z})$ is a generator Poincaré dual to $\Sigma \times \text{pt}$. There is little choice in the perturbation η on $\Sigma \times D^2$; namely we take η to be (a positive multiple of the) Poincaré dual to the relative class $\text{pt} \times D^2$, which has $\langle t \cup \eta, [\Sigma \times S^1] \rangle = 1$. Thus the Novikov completion of $L(t)$ with respect to η is the ring $\mathcal{L}(t)$ of Laurent series in t .

Assuming $[\Sigma]$ to be (rationally) nontrivial in M_1 and M_2 , we can extend η to M_i , and consider the relative invariants of the complements Z_i . In particular, if $\mathcal{K}(Z_i, \eta)$ is the module for $\mathcal{R}_{\Sigma \times S^1, \eta}$ induced by (Z_i, η) , we are interested in the pairing

$$HF_{\bullet}^-(\Sigma \times S^1, \mathfrak{s}; \mathcal{K}(Z_i, \eta)) \otimes \overline{HF_{\bullet}^-(-\Sigma \times S^1, \mathfrak{s}; \mathcal{L}(t))} \rightarrow \mathcal{K}(M_i, \Sigma \times S^1, \eta)$$

$$\xi_1 \otimes \xi_2 \mapsto \langle \tau^{-1}(\xi_1), \xi_2 \rangle$$

between the perturbed Floer homologies. In fact, more specifically we are interested in the homomorphism $HF_{\bullet}^-(\Sigma \times S^1, \mathfrak{s}; \mathcal{K}(Z_i, \eta)) \rightarrow \mathcal{K}(M_i, \Sigma \times S^1, \eta)$ induced by the pairing above when ξ_2 is equal to $\Psi_{\Sigma \times D^2, \eta}$, the relative invariant for $\Sigma \times D^2$.

Now, it is a simple exercise to see that $K(M_i, \Sigma \times S^1)$ is cyclic, generated by the Poincaré dual of $[\Sigma]$ in $H^2(M_i)$. Since Σ is assumed to represent a nontorsion class

in each of M_1 and M_2 , then, we have $\mathcal{K}(M_i, \Sigma \times S^1, \eta) \cong \mathcal{L}(t)$. There is a natural surjection

$$(32) \quad \rho: K(Z_i) \cong \frac{H^1(\Sigma \times S^1)}{H^1(Z_i)} \rightarrow \frac{H^1(\Sigma \times S^1)}{H^1(Z_i) + H^1(\Sigma \times D^2)} \cong K(M_i, \Sigma \times S^1),$$

inducing a surjection $\rho: \mathcal{K}(Z_i, \eta) \rightarrow \mathcal{L}(t)$ (cf Lemma 8.2), and a commutative diagram

$$(33) \quad \begin{array}{ccc} HF_{\bullet}^{-}(\Sigma \times S^1; \mathcal{K}(Z_i, \eta)) & \xrightarrow{\langle \tau^{-1}(\cdot), \Psi_{\Sigma \times D^2, \eta} \rangle} & \mathcal{K}(M_i, \Sigma \times S^1, \eta) \cong \mathcal{L}(t) \\ \downarrow \rho_* & & \downarrow \text{id} \\ HF_{\bullet}^{-}(\Sigma \times S^1; \mathcal{L}(t)) & \xrightarrow{\langle \tau^{-1}(\cdot), \Psi_{\Sigma \times D^2, \eta} \rangle} & \mathcal{L}(t) \end{array}$$

(cf Lemma 7.7 and Lemma 7.8). We will see that the arrow on the bottom of this diagram is determined essentially uniquely by algebraic considerations. Hence, determining the pairing mentioned above is equivalent to understanding the change-of-coefficient map ρ_* , and the relative invariant $\Psi_{\Sigma \times D^2, \eta}$.

Naturally, we cannot hope to do better than determining these objects up to a unit in $\mathcal{L}(t)$. Since units abound in a power series ring, this is not necessarily sufficient. However, we know that all the algebra we must use in the context of Novikov rings is induced from corresponding algebra over ordinary group rings: that is, the perturbed case is an obvious Novikov completion of the unperturbed case. Since $L(t)$ has many fewer units than $\mathcal{L}(t)$, this is a useful observation: we work initially in twisted, but unperturbed, coefficients.

10.1 Relative invariants in case $g = 1$

Let M be a closed 4–manifold with $b^+(M) \geq 1$ containing a smoothly embedded torus $T \hookrightarrow M$ with trivial normal bundle. We assume that $[T]$ is an element of infinite order in $H_2(M)$. Write $T \times D^2$ for a tubular neighborhood of T , and let $Z = M \setminus (T \times D^2)$ be the complement of this neighborhood. We wish to understand the relationship between the Ozsváth–Szabó invariants of M and the relative invariant of Z . If $b^+(M) = 1$, we will be interested in the invariant $\mathbb{Q}_{M, T \times S^1, \eta}$, where $\eta \in H^2(M; \mathbb{R})$ is a class that restricts to $T \times D^2$ as a nonzero multiple of the Poincaré dual of the relative class $[\text{pt} \times D^2]$.

Recall the following result of Ozsváth and Szabó.

Theorem 10.1 (Ozsváth–Szabó [9, Proposition 8.5]) *The twisted Heegaard Floer homology $HF^+(T^3, \mathfrak{s}; R_{T^3})$ is trivial unless \mathfrak{s} is equal to the unique spin^c structure \mathfrak{s}_0 with $c_1(\mathfrak{s}_0) = 0$. In this case, there is an isomorphism*

$$HF^+(T^3, \mathfrak{s}_0; R_{T^3}) = \mathcal{T}_{1/2} \oplus \ker(\varepsilon),$$

where $\mathcal{T}_{1/2}$ is the $\mathbb{Z}[U]$ -module $\mathbb{Z}[U, U^{-1}]/U \cdot \mathbb{Z}[U]$, graded so that its homogeneous summand of least degree lies in dimension $1/2$, and

$$\varepsilon: R_{T^3} = \mathbb{Z}[H^1(T^3)] \rightarrow \mathbb{Z}$$

is the augmentation homomorphism that maps each element of $H^1(T^3)$ to 1. In the above, $\ker(\varepsilon)$ lies in degree $-1/2$.

In particular, the reduced Floer homology $HF_{\text{red}}^+(T^3, \mathfrak{s}_0; R_{T^3})$ in the fully-twisted case is equal to $\ker(\varepsilon)$, lying entirely in degree $-1/2$.

Proposition 10.2 *Let $Z = M \setminus (T \times D^2)$ be the complement of an essentially embedded torus in a 4-manifold as above, and let $K(Z) = \ker(H^2(Z, \partial Z) \rightarrow H^2(Z))$ as usual, so that $K(Z) \cong H^1(T^3)/H^1(Z)$. Then $HF_k^+(T^3, \mathfrak{s}_0; \mathbb{Z}[K(Z)]) = 0$ if $k < -1/2$, and there is an isomorphism*

$$HF_{-1/2}^+(T^3, \mathfrak{s}_0; \mathbb{Z}[K(Z)]) \cong \ker(\varepsilon) \otimes_{R_{T^3}} \mathbb{Z}[K(Z)].$$

The change-of-coefficient map $\rho_*: HF^+(T^3, \mathfrak{s}_0; \mathbb{Z}[K(Z)]) \rightarrow HF^+(T^3, \mathfrak{s}_0; L(t))$ is given by the natural map

$$\text{id} \otimes \rho: \ker(\varepsilon) \otimes_{R_{T^3}} \mathbb{Z}[K(Z)] \rightarrow \ker(\varepsilon) \otimes_{R_{T^3}} L(t)$$

Proof For an R_{T^3} -module M , there is a “first quadrant” universal coefficients spectral sequence converging to $HF^+(T^3, \mathfrak{s}_0; M)$, which has E_2 term equal to $\text{Tor}^j(HF_i^+(T^3; R_{T^3}), M)$. In particular, the group in lowest total degree in the E_2 term is

$$\text{Tor}^0(HF_{-1/2}^+(T^3; R_{T^3}), M) = HF_{-1/2}^+(T^3; R_{T^3}) \otimes M = \ker(\varepsilon) \otimes M.$$

From the structure of the differentials in the spectral sequence this group must survive as the lowest-degree part of E_∞ , which proves the first statement.

If $\rho: M \rightarrow M'$ is a module homomorphism then we get a corresponding map of spectral sequences, for which the map on the E_2 term is $\text{id} \otimes \rho$ on the $j = 0$ row. The second statement of the theorem follows as before, since we consider only the bottom-degree groups. \square

Note that although the fully-twisted Floer homology for T^3 in dimension $-1/2$ is precisely equal to the reduced Floer homology, the same is not true in other coefficient systems (indeed, with untwisted coefficients, the reduced Floer homology is trivial). For example, in [Section 9.1](#) we found that there is an isomorphism

$$HF_{-1/2}^+(T^3, \mathfrak{s}_0; L(t)) = \ker(\varepsilon) \otimes L(t) = \mathbb{Z} \oplus \mathbb{Z} \oplus L(t),$$

and in this decomposition only the $L(t)$ factor lies in the reduced Floer homology. However, there is a natural projection $HF^+ \rightarrow HF_{\text{red}}^+$ for any 3–manifold and any coefficient module. When we apply ρ_* we often implicitly compose with this projection without including it in the notation, and hope this will not cause confusion; note that this problem disappears when we pass to perturbed Floer homology.

In the special case $K(Z) = H^1(T^3)$ (equivalently, the restriction $H^1(Z) \rightarrow H^1(T^3)$ is trivial), we can identify the change-of-coefficient map (in reduced Floer homology) with the surjection

$$H: \ker(\varepsilon) \rightarrow L(t)$$

$$a(r, s, t) \mapsto \frac{a(1, 1, t)}{t-1}.$$

Indeed, thinking of $\mathbb{Z}[K(Z)] = R_{T^3}$ as the ring of Laurent polynomials in three variables r, s, t , we have that $\ker(\varepsilon)$ is the ideal generated by the three elements $r-1$, $s-1$, and $t-1$. In particular, if $a(r, s, t) \in \ker(\varepsilon)$, then $a(1, 1, t)$ is divisible by $t-1$; the given map is uniquely determined up to units in R_{T^3} .

With this understanding of the change of coefficients in the unperturbed case, we can now introduce a perturbation η . As before, we choose any $\eta \in H^2(M; \mathbb{R})$ such that the restriction of η to $T^2 \times D^2$ is Poincaré dual to $\text{pt} \times D^2$. To understand the relative invariant $\Psi_{T^2 \times D^2, \eta}$, observe first that $HF^+(T^3, \mathfrak{s}; \mathcal{M})$ is trivial for any \mathcal{R}_{T^3} –module \mathcal{M} unless $\mathfrak{s} = \mathfrak{s}_0$, the unique spin^c structure with $c_1(\mathfrak{s}_0) = 0$. Thus from now on we consider only spin^c structures on M that restrict to \mathfrak{s}_0 on T^3 , which means also that $c_1(\mathfrak{s})|_{T^2 \times D^2} = 0$. It is straightforward to see that in the fully-twisted case, $HF^+(T^3, \mathfrak{s}_0; \mathcal{R}_{T^3, \eta}) \cong \mathcal{R}_{T^3, \eta}$, using [Lemma 8.4](#) and [Theorem 10.1](#).

Consider the complement $Z = M \setminus (T^2 \times D^2)$ as a cobordism $S^3 \rightarrow T^3$ by removing a 4–ball (we still use the symbol Z for this cobordism). In this situation the diagram

(17) becomes

$$\begin{array}{ccc}
 HF^-(S^3) & \xrightarrow{F_{\mathbb{Z}}} & HF^-(T^3; \mathbb{Z}[K(Z)]) \\
 & \searrow^{F_{Z,\eta}} & \downarrow i_* \\
 & & HF^-(T^3; \mathcal{K}(Z, \eta)),
 \end{array}$$

where i_* is the homomorphism induced by the natural map $i: \mathbb{Z}[K(Z)] \rightarrow \mathcal{K}(Z, \eta)$. If $K(Z) = H^1(T^3)$ (eg, if the complement of T in M is simply connected), then in the lowest nontrivial degree, i_* is a map $\ker(\varepsilon) \rightarrow \mathcal{R}_{T^3}$. This is induced by the natural homomorphisms

$$\begin{array}{ccc}
 HF^-(T^3; \mathcal{R}_{T^3}) & \rightarrow & HF^-(T^3; \mathcal{R}_{T^3}) \otimes \mathcal{R}_{T^3, \eta} \\
 & & \rightarrow HF^-(T^3; \mathcal{R}_{T^3} \otimes \mathcal{R}_{T^3, \eta}) \rightarrow HF^-(T^3; \mathcal{R}_{T^3, \eta}) \\
 [x] & \longrightarrow & [x] \otimes 1 \\
 & & \longrightarrow [x \otimes 1]
 \end{array}$$

so we can think of i_* as the homomorphism mapping a Laurent polynomial $a(r, s, t) \in \ker(\varepsilon)$ into the Laurent series ring $\mathcal{R}_{T^3, \eta}$ by the natural inclusion.

Continuing to assume that $K(Z) = H^1(T^3)$, we consider the diagram

$$(34) \quad \begin{array}{ccc}
 HF_{-1/2}^+(T^3; \mathcal{R}_{T^3}) & \longrightarrow & L(t) \\
 \downarrow i_* & & \downarrow \\
 HF_{-1/2}^+(T^3; \mathcal{R}_{T^3, \eta}) & \longrightarrow & \mathcal{L}(t) \\
 \downarrow \rho_* & \nearrow (\cdot, \Psi_{T^2 \times D^2, \eta}) & \\
 HF_{-1/2}^+(T^3; \mathcal{L}(t)) & &
 \end{array}$$

where the upper arrow is $a \mapsto \langle a, F_{T^2 \times D^2}^-(\Theta^-) \rangle$, and the middle arrow is $b \mapsto \langle b, \Psi_{T^2 \times D^2, \eta} \rangle$. In the unperturbed case, we have

$$\langle a, F_{T^2 \times D^2}^-(\Theta^-) \rangle = \langle \rho_*(a), F_{T^2 \times D^2}^-(\Theta^-) \rangle = \langle F_{T^2 \times D^2}^+(\rho_*(a)), \Theta^- \rangle$$

where ρ_* is the projection to $L(t)$ coefficients. We know that in $L(t)$ coefficients, $F_{T^2 \times D^2}^+$ induces a surjection to $L(t)$ (see the discussion at the end of Section 9.1).

It follows from (the unperturbed analog of) (33) and Proposition 10.2 that the upper arrow in (34) is given by

$$a \mapsto \langle a, F_{T^2 \times D^2}^-(\Theta^-) \rangle = H(a),$$

where $H: \ker(\varepsilon) \rightarrow L(t)$ is the homomorphism introduced above.

On the other hand, if we think of $a \in \ker(\varepsilon)$ as a Laurent polynomial $a(r, s, t)$ then $i_*(a) = a$. The coefficient-change map $\rho_*: HF^+(T^3; \mathcal{R}_{T^3, \eta}) \cong \mathcal{R}_{T^3, \eta} \rightarrow HF^+(T^3; \mathcal{L}(t)) \cong \mathcal{L}(t)$ is necessarily the reduction $b(r, s, t) \mapsto b(1, 1, t)$, so that $\rho_*(i_*(a)) = a(1, 1, t)$.

Thus, $\Psi_{T^2 \times D^2, \eta}(1)$ is a generator of $HF_{-1/2}^-(T^3; \mathcal{L}(t))$, satisfying the property that for $a(r, s, t) \in \ker(\varepsilon)$,

$$\langle a, \Psi_{T^2 \times D^2, \eta}(1) \rangle = H(a) = \frac{a(1, 1, t)}{t - 1},$$

up to a unit in $L(t)$. Identifying $HF_{-1/2}^\pm(T^3; \mathcal{L}(t)) = \mathcal{L}(t)$, the diagonal arrow in the preceding diagram can be taken to be multiplication by $\Psi_{T^2 \times D^2, \eta}(1)$. We conclude:

Proposition 10.3 *The relative invariant $\Psi_{T^2 \times D^2, \eta}$ can be identified with the map $\mathbb{A}(T^2 \times D^2) \rightarrow HF^-(T^3, \mathfrak{s}_0; \mathcal{L}(t)) \cong \mathcal{L}(t)$ whose value on the element 1 is*

$$(35) \quad \Psi_{T^2 \times D^2, \eta}(1) = \frac{1}{t - 1}$$

up to multiplication by $\pm t^n$, and which vanishes on elements of $\mathbb{A}(T^2 \times D^2)$ having nonzero degree.

Let M be a closed 4–manifold containing an embedded torus T with trivial normal bundle, and $\eta \in H^2(M; \mathbb{R})$ a class with $\int_T \eta > 0$. Let \mathfrak{s} be a spin^c structure on M , and write $Z = M \setminus (T \times D^2)$. If $\langle c_1(\mathfrak{s}), [T] \rangle \neq 0$, then the relative invariant $\Psi_{Z, \eta, \mathfrak{s}}$ and the closed invariant $\mathbb{D}_{M, T \times S^1, \eta, \mathfrak{s}}$ both vanish.

If $\langle c_1(\mathfrak{s}), [T] \rangle = 0$, then the relative invariant $\Psi_{Z, \eta, \mathfrak{s}}$ takes values in the Novikov ring $\mathcal{K}(Z, \eta)$. Furthermore, the value of the perturbed Ozsváth–Szabó invariant on a class $\alpha_1 \otimes \alpha_2$ with $\alpha_1 \in \mathbb{A}(Z)$, $\alpha_2 \in \mathbb{A}(T^2 \times D^2)$, is given up to multiplication by $\pm t^n$ by

$$\begin{aligned} \mathbb{D}_{M, T^3, \eta, \mathfrak{s}}(\alpha_1 \otimes \alpha_2) &= \langle \tau^{-1} \Psi_{Z, \eta, \mathfrak{s}}(\alpha_1), \Psi_{T^2 \times D^2, \eta, \mathfrak{s}}(\alpha_2) \rangle \\ &= \begin{cases} \frac{1}{t - 1} \rho(\Psi_{Z, \eta, \mathfrak{s}}(\alpha_1)) & \text{if } \alpha_2 = 1 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where ρ is the natural map $\mathcal{K}(Z, \eta) \rightarrow \mathcal{L}(t)$ induced by the projection $K(Z) \rightarrow K(M, T^3) \cong \mathbb{Z}$.

Note that since η is defined on all of M , the fact that the induced homomorphism ρ is well-defined follows from [Lemma 8.2](#).

Proof First, observe that since the Floer homology $HF_{\bullet}^{-}(T^3, \mathfrak{s}_0; \mathcal{M})$ is supported entirely in degree $-1/2$ for any $\mathcal{R}_{T^3, \eta}$ -module \mathcal{M} , the elements of $\mathbb{A}(T^2 \times D^2)$ having nonzero degree must act trivially. Thus, only $\Psi_{T^2 \times D^2, \eta}(1)$ can be nontrivial.

If $H^1(Z) \rightarrow H^1(T^3)$ is trivial, then $\mathcal{K}(Z, \eta) = \mathcal{R}_{T^3, \eta}$ and the result follows from the preceding discussion. In the general case, observe that since $HF_{\bullet}^{-}(T^3, \mathfrak{s}_0; \mathcal{R}_{T^3, \eta}) \cong \mathcal{R}_{T^3, \eta}$ is free over $\mathcal{R}_{T^3, \eta}$, we have

$$HF_{\bullet}^{-}(T^3, \mathfrak{s}_0; \mathcal{K}(Z, \eta)) \cong \mathcal{R}_{T^3, \eta} \otimes_{\mathcal{R}_{T^3, \eta}} \mathcal{K}(Z, \eta) = \mathcal{K}(Z, \eta).$$

Thus the last statement of the proposition follows from [\(35\)](#) and [\(34\)](#). \square

10.2 Relative invariants in case $g > 1$

We follow an outline similar to the previous subsection; as before, we begin with twisted but unperturbed coefficients.

Let $\Sigma \hookrightarrow M$ be an embedded surface of square 0 and genus g and $\mathfrak{s} \in \text{Spin}^c(M)$ a spin^c structure. We have seen that unless $c_1(\mathfrak{s})|_{\Sigma \times S^1}$ is Poincaré dual to $2k[\text{pt} \times S^1]$ with $|k| \leq g-1$, the Floer homology $HF^+(\Sigma \times S^1, \mathfrak{s}; N) = 0$ for any $R_{\Sigma \times S^1}$ -module N (and similarly after perturbation), forcing $\mathbb{D}_{M, \Sigma \times S^1, \eta, \mathfrak{s}} = 0$ for such \mathfrak{s} . Thus we suppose that the restriction of \mathfrak{s} has the indicated form; we write \mathfrak{s}_k for the spin^c structure on $\Sigma \times S^1$ with $c_1(\mathfrak{s}_k) = 2k \text{PD}[S^1]$.

The following is an easy consequence of [Theorem 9.12](#), together with the universal coefficients spectral sequence.

Lemma 10.4 *Let N be a module for $R_{\Sigma \times S^1}$. Then when $|k| = g-1$, we have an isomorphism*

$$HF_{\text{red}}^+(\Sigma \times S^1, \mathfrak{s}_k; N) = HF^+(\Sigma \times S^1, \mathfrak{s}_k; N) = N.$$

If $Z = M \setminus (\Sigma \times D^2)$ is the complement of a surface representing a class of infinite order in $H_2(M; \mathbb{Z})$, then the homomorphism

$$\rho_*: HF_{\text{red}}^+(\Sigma \times S^1, \mathfrak{s}_k; \mathbb{Z}[K(Z)]) \rightarrow HF_{\text{red}}^+(\Sigma \times S^1, \mathfrak{s}_k; L(t))$$

is equal to the projection $\rho: \mathbb{Z}[K(Z)] \rightarrow \mathbb{Z}[K(M, \Sigma \times S^1)] = L(t)$.

Applying a perturbation $\eta \in H^2(M; \mathbb{R})$ restricting as a positive multiple of the Poincaré dual of $[\text{pt} \times D^2]$ on $\Sigma \times D^2$ as before, we obtain the following, which describes the relative invariant of $\Sigma \times D^2$ for general k , and gives the analog of [Proposition 10.3](#) in higher genus for $|k| = g - 1$.

Proposition 10.5 *For the spin^c structure \mathfrak{s}_k on $\Sigma \times D^2$ with $c_1(\mathfrak{s}_k)$ Poincaré dual to $2k[\text{pt} \times D^2]$, the relative invariant $\Psi_{\Sigma \times D^2, \eta, \mathfrak{s}_k}$ is a linear map*

$$\mathbb{A}(\Sigma \times D^2) \rightarrow HF_{\bullet}^-(\Sigma \times S^1, \mathfrak{s}_k; \mathcal{L}(t))$$

whose value $\Psi_{\Sigma \times D^2, \eta, \mathfrak{s}_k}(1)$ lies in the summand of maximal height (cf [Remark 9.7](#)). Furthermore, there is a natural identification $HF_{\bullet, \text{top}}^-(\Sigma \times S^1, \mathfrak{s}_k; \mathcal{L}(t)) \cong \mathcal{L}(t)$ such that

$$(36) \quad \Psi_{\Sigma \times D^2, \eta, \mathfrak{s}_k}(1) = 1$$

up to multiplication by $\pm t^n$. More generally, if $\alpha \in \mathbb{A}(\Sigma \times D^2)$ then $\Psi_{\Sigma \times D^2, \eta, \mathfrak{s}_k}(\alpha) = \alpha \cdot 1$, where the right hand side makes use of the action of $\Lambda^* H_1(\Sigma \times S^1; \mathbb{Z}) \otimes \mathbb{Z}[U]$ on $HF_{\bullet}^-(\Sigma \times S^1, \mathfrak{s}_k; \mathcal{L}(t))$.

Let $Z: S^3 \rightarrow \Sigma \times S^1$ be a cobordism and $\eta \in H^2(Z; \mathbb{R})$ a class restricting to $\Sigma \times S^1$ as a positive multiple of the Poincaré dual of $\text{pt} \times S^1$. Let \mathfrak{s} be a spin^c structure on Z restricting to the spin^c structure \mathfrak{s}_k on $\Sigma \times S^1$, where $|k| = g - 1$. Then $\Psi_{Z, \eta, \mathfrak{s}}$ takes values in the Novikov ring $\mathcal{K}(Z, \eta)$.

Finally, let M be the closed manifold obtained by gluing $\Sigma \times D^2$ to Z , and filling in the other boundary component of Z by a 4–ball. Extend η across $\Sigma \times D^2$ and B^4 to give a class $\eta \in H^2(M; \mathbb{R})$. For a spin^c structure \mathfrak{s} with $c_1(\mathfrak{s})|_{\Sigma \times D^2}$ Poincaré dual to $\pm(2g - 2)[\text{pt} \times D^2]$, the value of the perturbed Ozsváth–Szabó invariant $\Phi_{M, \Sigma \times S^1, \eta, \mathfrak{s}}$ on $\alpha_1 \otimes \alpha_2 \in \mathbb{A}(Z) \otimes \mathbb{A}(T^2 \times D^2)$ is given by

$$\begin{aligned} \Phi_{M, \Sigma \times S^1, \eta, \mathfrak{s}}(\alpha_1 \otimes \alpha_2) &= \langle \tau^{-1} \Psi_{Z, \eta, \mathfrak{s}}(\alpha_1), \Psi_{\Sigma \times D^2, \eta, \mathfrak{s}}(\alpha_2) \rangle \\ &= \begin{cases} \rho(\Psi_{Z, \eta, \mathfrak{s}}(\alpha_1)) & \text{if } \alpha_2 = 1 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where ρ is the natural map $\mathcal{K}(Z, \eta) \rightarrow \mathcal{L}(t)$ induced by the projection $K(Z) \rightarrow K(M, \Sigma \times S^1) \cong \mathbb{Z}$.

Proof We have seen that the homomorphism $F_{\Sigma \times D^2, \mathfrak{s}_k}^+ : HF^+(\Sigma \times D^2, \mathfrak{s}_k; \mathcal{L}(t)) \rightarrow HF^+(S^3; \mathcal{L}(t))$ can be identified with the projection of $X(g, d) \otimes L(t)$ onto the

summand having minimal height. Combining this with the preceding lemmas, we have that for $x \in HF^+(\Sigma \times S^1, \mathfrak{s}_k; \mathcal{K}(Z, \eta))$,

$$\langle x, F_{\Sigma \times D^2, \eta, \mathfrak{s}_k}^-(\Theta^-) \rangle = \langle F_{\Sigma \times D^2, \eta, \mathfrak{s}_k}^+(x), \Theta^- \rangle = \begin{cases} \rho(x) & \text{if } \rho(x) \text{ has minimal height} \\ 0 & \text{otherwise,} \end{cases}$$

The analog of diagram (34) with $\Sigma \times S^1$ in place of T^3 implies that $\Psi_{\Sigma \times D^2, \eta, \mathfrak{s}_k}(1) = F_{\Sigma \times D^2, \eta, \mathfrak{s}_k}^-(\Theta^-)$ is a class pairing with x to give $\rho(x)$ when the latter has minimal height, and 0 otherwise. It is not hard to see that the pairing between HF^+ and HF^- can be nontrivial only on elements of complementary height, and we have seen (Theorem 9.4) that the summands in minimal and maximal height in the Floer homology are $HF_{\text{bot}}^+(\Sigma \times S^1, \mathfrak{s}_k; \mathcal{L}(t)) \cong \mathcal{L}(t)$ and $HF_{\text{top}}^-(\Sigma \times S^1, \mathfrak{s}_k; \mathcal{L}(t)) \cong \mathcal{L}(t)$. Hence the $\mathcal{L}(t)$ -valued pairing between these groups must be given by multiplication of Laurent series (up to a unit in $L(t)$, since the pairing is induced from the unperturbed situation). The above can therefore be interpreted as the statement that $\Psi_{\Sigma \times D^2, \eta, \mathfrak{s}_k}(1) = 1$.

Since $\mathbb{A}(\Sigma \times S^1) \rightarrow \mathbb{A}(\Sigma \times D^2)$ is surjective, the statement $\Psi_{\Sigma \times D^2, \eta, \mathfrak{s}_k}(\alpha) = \alpha \cdot 1$ holds by the naturality of cobordism-induced homomorphisms under the action of H_1 .

The remaining statements follow from the preceding lemma, together with (36). \square

The expression of the relative invariant $\Psi_{Z, \eta, \mathfrak{s}}$ where $c_1(\mathfrak{s})|_{\Sigma \times S^1} = \text{PD}(2k[\text{pt} \times S^1])$ with $|k| < g - 1$ is somewhat more complicated; in principle it may take as a value any element of the Floer homology $HF_{\bullet}^-(\Sigma \times S^1, \mathfrak{s}_k; \mathcal{K}(Z, \eta))$, which in the case at hand is not a cyclic module. However, it is still possible to express the relative invariant for Z (after applying ρ) in terms of the absolute invariants for M . To do so, we make use of the structure of the Floer homology of $\Sigma \times S^1$ deduced previously.

First, recall that the graded group $X(g, d)$ is equipped with a ‘‘standard’’ action of $\Lambda^* H_1(\Sigma) \otimes \mathbb{Z}[U]$. In fact, suppose $\mathcal{B}_{g, d} = \{\beta\} \subset X(g, d)$ is a basis for $X(g, d)$ as a free abelian group, with each β a homogeneous element. Then it is easy to see that there is a uniquely determined collection of elements $\{\tilde{\beta}\} \subset \Lambda^* H_1(\Sigma) \otimes \mathbb{Z}[U]$ (lying in degrees $\leq 2d$) with the property that $\tilde{\beta} \cap \beta' = \delta_{\beta\beta'} \cdot 1$ for $\beta, \beta' \in \mathcal{B}_{g, d}$, where \cap is the standard action, $\delta_{\beta\beta'}$ is the Kronecker delta, and 1 denotes a fixed generator in lowest degree for $X(g, d)$.

We have seen in Theorem 9.4 that there is an isomorphism $HF_{\bullet}^-(\Sigma \times S^1, \mathfrak{s}_k; \mathcal{L}(t)) \cong X(g, d) \otimes \mathcal{L}(t)$, and furthermore that the action of $\Lambda^* H_1(\Sigma \times S^1) \otimes \mathbb{Z}[U]$ agrees with the $\mathcal{L}(t)$ -linear extension of the standard action to leading order in t (and that the class $\text{pt} \times S^1$ acts trivially). Suppose first that $k \neq 0$, so that the variable t carries a nonzero degree. The pairing $\langle \cdot, \cdot \rangle$ between HF_{\bullet}^+ and HF_{\bullet}^- is nontrivial only on elements of complementary height, and is induced from the untwisted pairing since

in our situation the twisting is trivial. Writing Ξ for the generator in maximal height given by $\Psi_{\Sigma \times D^2, \eta, \mathfrak{s}_k}(1)$, it follows for dimensional reasons that if the degree of $\tilde{\beta}$ is equal to the height of $x \in X(g, d)$, then

$$\langle \tau^{-1}(\tilde{\beta}.x), \Xi \rangle = \langle \tau^{-1}(\tilde{\beta} \cap x), \Xi \rangle$$

(identifying x with $x \otimes 1 \in X(g, d) \otimes \mathcal{L}(t) = HF^-(\Sigma \times S^1, \mathfrak{s}_k; \mathcal{L}(t))$). Indeed, it follows from [Theorem 9.4](#) and [Remark 9.5](#) that we can write

$$\tilde{\beta}.x = \tilde{\beta} \cap x + \sum_{n \geq 1} t^{nk} \rho_n(\tilde{\beta}, x)$$

for some elements $\rho_n(\tilde{\beta}, x) \in X(g, d)$. Since $\tilde{\beta}.x$ is homogeneous (with respect to the \mathbb{Z} grading in which t carries degree $-2k$), we infer that the $\rho_n(\tilde{\beta}, x)$ lie in degrees of $X(g, d)$ other than that of $\tilde{\beta} \cap x$. That is, these elements carry different height from $\tilde{\beta} \cap x$, and hence pair trivially with Ξ .

On the other hand, if $k = 0$ then corrections to the H_1 action $\tilde{\beta}.x$ appear in the same height as $\tilde{\beta} \cap x$. Since the action is standard to leading order in t , however, we can say that for basis elements β, γ as previously the action satisfies $\tilde{\beta}.\gamma = \delta_{\beta\gamma} u_\beta(t) \cdot 1$, where $u_\beta \in \mathcal{L}(t)$ is monic (in the sense that the coefficient of the lowest power of t is 1, cf [Remark 9.5](#)) and hence a unit in $\mathcal{L}(t)$ (here we observe that if $\tilde{\beta} \cap \gamma = 0$ then also $\tilde{\beta}.\gamma = 0$). When $k = 0$, then, the above becomes

$$\langle \tau^{-1}(\tilde{\beta}.x), \Xi \rangle = u_\beta \langle \tau^{-1}(\tilde{\beta} \cap x), \Xi \rangle.$$

Now suppose ξ is an element of a given height in $HF_\bullet^-(\Sigma \times S^1, \mathfrak{s}_k; \mathcal{L}(t))$. Then fixing the basis $\mathcal{B}_{g,d} = \{\beta\}$ as previously, we can express ξ in terms of $\{\beta\}$ by

$$\xi = \begin{cases} \sum_{\beta} \langle \tau^{-1}(\tilde{\beta}.\xi), \Xi \rangle \cdot \beta & \text{if } k \neq 0 \\ \sum_{\beta} \langle \tau^{-1}(\tilde{\beta}.\xi), \Xi \rangle \cdot u_\beta^{-1} \beta & \text{if } k = 0, \end{cases}$$

where the sum is over basis elements β having the degree equal to the height of ξ , and we have chosen the lowest-height generator 1 for $X(g, d)$ such that $\langle \tau^{-1}(1), \Xi \rangle = 1$.

Applying this idea to the case $\xi = \rho(\Psi_{Z, \eta, \mathfrak{s}}(\alpha_1))$ leads to the following.

Proposition 10.6 *Let M be a closed 4–manifold containing an embedded surface Σ of genus $g > 1$ and trivial normal bundle, and $\eta \in H^2(M; \mathbb{R})$ a class with $\int_{\Sigma} \eta > 0$. Write $Z = M \setminus (\Sigma \times D^2)$, and let \mathfrak{s} be a spin^c structure on M restricting to \mathfrak{s}_k on $\Sigma \times S^1$.*

If $0 < |k| < g - 1$, then for an element $\alpha \in \mathbb{A}(Z)$, the reduced relative invariant $\rho(\Psi_{Z,\eta,s}(\alpha))$ is given in terms of a basis $\{\beta\}$ for $HF_{\bullet}^{-}(\Sigma \times S^1, \mathfrak{s}_k; \mathcal{L}(t))$ by

$$\rho(\Psi_{Z,\eta,s}(\alpha)) = \sum_{\beta} \Phi_{M,\Sigma \times S^1,\eta,s}(\alpha \otimes \tilde{\beta}) \cdot \beta,$$

where $\{\tilde{\beta}\} \subset \mathbb{A}(\Sigma \times D^2)$ are elements dual in the above sense to the basis $\{\beta\}$. If $k = 0$, then

$$\rho(\Psi_{Z,\eta,s}(\alpha)) = \sum_{\beta} \Phi_{M,\Sigma \times S^1,\eta,s}(\alpha \otimes \tilde{\beta}) \cdot u_{\beta}^{-1} \beta,$$

where $u_{\beta} \in \mathcal{L}(t)$ are units depending only on the basis $\{\beta\}$.

In the expressions above, $\alpha \otimes \tilde{\beta}$ is shorthand for the image of that element under the natural map $\mathbb{A}(Z) \otimes \mathbb{A}(\Sigma \times D^2) \rightarrow \mathbb{A}(M)$.

Proof When $k \neq 0$, we expand $\rho(\Psi_{Z,\eta,s}(\alpha))$ in the basis β as indicated previously:

$$\begin{aligned} \rho(\Psi_{Z,\eta,s}(\alpha)) &= \sum_{\beta} \langle \tau^{-1}(\tilde{\beta} \cdot \rho(\Psi_{Z,\eta,s}(\alpha))), \Xi \rangle \cdot \beta \\ &= \sum_{\beta} (-1)^{\deg(\tilde{\beta})} \langle \tau^{-1}(\rho(\Psi_{Z,\eta,s}(\alpha))), \tilde{\beta} \cdot \Xi \rangle \cdot \beta \\ &= \sum_{\beta} (-1)^{\deg(\tilde{\beta})} \langle \tau^{-1}(\rho(\Psi_{Z,\eta,s}(\alpha))), \Psi_{\Sigma \times D^2,\eta,s_k}(\tilde{\beta}) \rangle \cdot \beta \\ &= \sum_{\beta} \Phi_{M,\Sigma \times S^1,\eta,s}(\alpha \otimes \tilde{\beta}) \cdot \beta, \end{aligned}$$

up to an overall sign and translation by a power of t , where we sum only over those β whose degree is equal to the height of $\rho(\Psi_{Z,\eta,s}(\alpha))$. The case $k = 0$ is identical except for the introduction of the elements u_{β}^{-1} . \square

10.3 Fiber sum formulas

A minor technicality in deducing the formulae in the introduction is the presence of the orientation-reversing gluing map in the fiber sum construction. We will have occasion to refer to the map in Floer homology induced by this diffeomorphism, so we make a few basic observations.

First, if $f: Y \rightarrow Y'$ is an orientation-preserving diffeomorphism between 3-manifolds, we can construct the mapping cylinder $C_f = (Y \times [0, 1]) \cup_f (Y' \times \{1\})$ in the usual way, which we can view as a smooth cobordism $Y \rightarrow Y'$ of oriented manifolds. The action

of f on Floer homology is by definition the homomorphism $F_{C_f}^\circ$ in Floer homology induced by C_f . We will normally write this action as f_* .

It is easy to see from basic properties of the cobordism maps that if $h \in H_1(Y)$ then for $x \in HF^\circ(Y)$ we have $f_*(h.x) = f_*(h).f_*(x)$. In twisted coefficients, there is an isomorphism $f_* = (f^{-1})^*: \mathbb{Z}[H^1(Y; \mathbb{Z})] \rightarrow \mathbb{Z}[H^1(Y'; \mathbb{Z})]$, and for $\alpha \in \mathbb{Z}[H^1(Y; \mathbb{Z})]$ we have $f_*(\alpha x) = f_*(\alpha)f_*(x)$. A similar statement holds in the perturbed case, if Y is equipped with a class $\eta_1 \in H^2(Y; \mathbb{R})$ and we take $\eta_2 = (f^{-1})^*\eta_1$.

In the case of a fiber sum, we are given two closed 4–manifolds M_1, M_2 with embedded surfaces Σ_1, Σ_2 of genus g and square 0. Write $Z_i = M_i \setminus (\Sigma_i \times D^2)$, so that $\partial Z_i = \Sigma \times S^1$. Then we choose an orientation-preserving diffeomorphism between Σ_1 and Σ_2 , extending it to $\Sigma_i \times S^1$ by conjugation in the S^1 factor. The result is an orientation-reversing diffeomorphism $f: \partial Z_2 \rightarrow \partial Z_1$, and the fiber sum is defined to be $X = Z_1 \cup_f Z_2$. To make the gluing f more explicit, replace Z_2 by $W_2 = Z_2 \cup_{\text{id}} C_f$. Then $X = Z_1 \cup_{\text{id}} W_2$, and the relative invariants of W_2 and Z_2 are related by $\Psi_{W_2} = f_*\Psi_{Z_2}$, according to the composition law. Thus both Ψ_{Z_1} and Ψ_{Z_2} naturally take values in $HF_\bullet^-(\Sigma \times S^1)$ (with appropriate coefficients), while Ψ_{W_2} takes values in $HF_\bullet^-(-\Sigma \times S^1)$.

Note that in certain situations, the above observations are sufficient to determine the action of f_* . For example, if the genus of Σ is 1, then the reduced part of $HF^-(\Sigma \times S^1; L(t))$ is isomorphic to $L(t)$, where t is Poincaré dual to the torus Σ . Since the action of f in cohomology reverses the sign of the latter class, linearity of the induced map in Floer homology forces $f_*: HF_{\text{red}}^-(\Sigma \times S^1; L(t)) \rightarrow HF_{\text{red}}^-(-\Sigma \times S^1; L(t))$ to be the conjugation map $L(t) \rightarrow \overline{L(t)}$, up to multiplication by $\pm t^n$. Hence, the same conclusion follows in perturbed Floer homology, using a class $\eta \in H^2(\Sigma \times S^1; \mathbb{R})$ fixed by f^* , eg, the Poincaré dual to $[S^1]$. A similar conclusion holds when considering the action of the gluing map in higher genus, if we restrict attention to the highest (or lowest) nontrivial heights in the perturbed Floer homology.

The fiber sum formula in the genus 1 case is as follows.

Theorem 10.7 *Let $X = M_1 \#_{T_1=T_2} M_2$ be the fiber sum of two 4–manifolds M_1, M_2 along tori T_1, T_2 of square 0. Assume that there exist classes $\eta_i \in H^2(M_i; \mathbb{R})$, $i = 1, 2$, such that the restrictions of η_i to $T_i \times S^1 \subset M_i$ correspond under the gluing diffeomorphism $f: T_2 \times S^1 \rightarrow T_1 \times S^1$, and assume that $\int_{T_i} \eta_i > 0$. Let $\eta \in H^2(X; \mathbb{R})$ be a class whose restrictions to $Z_i = M_i \setminus (T_i \times D^2)$ agree with those of η_i , and choose spin^c structures $\mathfrak{s}_i \in \text{Spin}^c(M_i)$, $\mathfrak{s} \in \text{Spin}^c(X)$ whose restrictions correspond similarly. Then for any $\alpha \in \mathbb{A}(X)$, the image of $\alpha_1 \otimes \alpha_2$ under the map $\mathbb{A}(Z_1) \otimes \mathbb{A}(Z_2) \rightarrow \mathbb{A}(X)$,*

we have

$$\rho(\mathbb{Q}_{X,T \times S^1, \eta, \mathfrak{s}}(\alpha)) = (t^{1/2} - t^{-1/2})^2 \mathbb{Q}_{M_1, T_1 \times S^1, \eta_1, \mathfrak{s}_1}(\alpha_1) \cdot \mathbb{Q}_{M_2, T_2 \times S^1, \eta_2, \mathfrak{s}_2}(\alpha_2)$$

up to multiplication by $\pm t^n$.

Note that the closed invariants $\mathbb{Q}_{M_i, T_i \times S^1, \eta_i, \mathfrak{s}_i}$ each take values in $\mathcal{L}(t)$, where t is the appropriate generator of $K(M_i, T_i \times S^1)$, and the multiplication takes place in that Laurent series ring.

Proof By definition,

$$\mathbb{Q}_{X, T \times S^1, \eta, \mathfrak{s}}(\alpha) = \langle \tau^{-1}(\Psi_{Z_1, \eta_1, \mathfrak{s}_1}(\alpha_1)), \Psi_{W_2, \eta_2, \mathfrak{s}_2}(\alpha_2) \rangle,$$

where $W_2 = Z_2 \cup C_f$ as in the remarks above. Applying ρ to each side, we see

$$\rho(\mathbb{Q}_{X, T \times S^1, \eta, \mathfrak{s}}(\alpha)) = \langle \tau^{-1} \rho(\Psi_{Z_1, \eta_1, \mathfrak{s}_1}(\alpha_1)), f_* \rho(\Psi_{Z_2, \eta_2, \mathfrak{s}_2}(\alpha_2)) \rangle.$$

Since $f_* \rho(\Psi_{Z_2, \eta_2, \mathfrak{s}_2}(\alpha_2)) = \overline{\rho(\Psi_{Z_2, \eta_2, \mathfrak{s}_2}(\alpha_2))}$, the theorem follows quickly from this, [Proposition 10.3](#), and antilinearity of $\langle \cdot, \cdot \rangle$. Observe that the pairing in Floer homology of $T \times S^1$ can only be multiplication, up to $\pm t^n$, since the modules are cyclic. \square

The higher-genus case is similarly easy, after some preparatory remarks. Recall that given a (homogeneous) basis $\{\beta_i\}$ for $X(g, d)$, we obtain a “dual” collection $\{\tilde{\beta}_i\}$ of elements of $\mathbb{A}(\Sigma) = \Lambda^* H_1(\Sigma) \otimes \mathbb{Z}[U]$. This dual basis is defined by the condition that $\tilde{\beta}_i \cap \beta_j = \delta_{ij}$, where \cap is the standard action of $\mathbb{A}(\Sigma)$ on $X(g, d)$. The dual basis satisfies $\langle \tau^{-1}(\tilde{\beta}_j \cdot \beta_i), \Xi \rangle = \delta_{ij} u_i$, where Ξ is the usual topmost generator of $HF^-(\Sigma \times S^1, \mathfrak{s}_k; \mathcal{L}(t))$, $\tilde{\beta}_j \cdot \beta_i$ denotes the action of $\mathbb{A}(\Sigma)$ on Floer homology, and u_i is a unit in $\mathcal{L}(t)$ that equals 1 unless $k = 0$. Furthermore, the basis $\{\tilde{\beta}_i\}$ is unique if we specify that it is contained in the subgroup $\tilde{X}(g, d) = \bigoplus_{i=0}^d \Lambda^i H_1(\Sigma) \otimes \mathbb{Z}[U]/U^{d-i+1} \subset \mathbb{A}(\Sigma)$.

If $\{\tilde{\beta}_i\}$ is the “Kronecker dual” basis, we can find a “Poincaré dual” basis $\{\beta_i^\circ\}$ for $X(g, d)$, namely $\beta_i^\circ = \tilde{\beta}_i \cap (1 \otimes U^{-d})$, where we think of $1 \otimes U^{-d}$ as a topmost generator for the Floer homology $X(g, d) \otimes \mathcal{L}(t) = HF_\bullet^-(\Sigma \times S^1, \mathfrak{s}_k; \mathcal{L}(t))$. We could also say that β_i° is the leading order part (in t) of $\tilde{\beta}_i \cdot \Xi$, except that in our conventions, Ξ is a generator for $HF_\bullet^-(\Sigma \times S^1, \mathfrak{s}_k; \mathcal{L}(t))$.

Associated to the basis $\{\beta_i^\circ\}$, of course, there is a dual $\{\tilde{\beta}_i^\circ\}$, generating a subset of $\mathbb{A}(\Sigma)$. This set satisfies $\tilde{\beta}_i^\circ \cap \beta_j^\circ = \tilde{\beta}_i \cap \tilde{\beta}_j \cap (1 \otimes U^{-d}) = \delta_{ij}$.

With these conventions in mind, we have the following.

Theorem 10.8 Let $X = M_1 \#_{\Sigma_1 = \Sigma_2} M_2$ be the fiber sum of two 4–manifolds M_1, M_2 along surfaces Σ_1, Σ_2 of genus g and square 0. Let η_1, η_2, η be 2–dimensional cohomology classes satisfying conditions analogous to those in the previous theorem, and choose spin^c structures $\mathfrak{s}_1, \mathfrak{s}_2$, and \mathfrak{s} restricting compatibly as before. If the Chern classes of each spin^c structure restrict to $\Sigma \times S^1$ as a class other than $2k \text{PD}[S^1]$ with $|k| \leq g - 1$ then the Ozsváth–Szabó invariants of all manifolds involved vanish. Otherwise, writing f for the gluing map $\Sigma_2 \times S^1 \rightarrow \Sigma_1 \times S^1$, we have

$$\begin{aligned} \rho(\mathbb{Q}_{X, \Sigma \times S^1, \eta, \mathfrak{s}}(\alpha)) &= \sum_{\beta} \mathbb{Q}_{M_1, \Sigma_1 \times S^1, \eta_1, \mathfrak{s}_1}(\alpha_1 \otimes \tilde{\beta}) \cdot \mathbb{Q}_{M_2, \Sigma_2 \times S^1, \eta_2, \mathfrak{s}_2}(\alpha_2 \otimes f_*^{-1}(\tilde{\beta}^\circ)) \cdot u_{\beta} \end{aligned}$$

up to multiplication by $\pm t^n$. Here $\{\beta\}$ is a basis for $HF_{\bullet}^{-}(\Sigma \times S^1, \mathfrak{s}_k; \mathcal{L}(t))$ associated to a basis for $X(g, d)$, $d = g - 1 - |k|$, and $\{\tilde{\beta}\}$ and $\{\tilde{\beta}^\circ\}$ are the dual elements of $\mathbb{A}(\Sigma)$ described above. The elements $u_{\beta} \in \mathcal{L}(t)$ are units that are equal to 1 unless $k = 0$.

Proof As in the previous theorem,

$$\rho(\mathbb{Q}_{X, \Sigma \times S^1, \eta, \mathfrak{s}}(\alpha)) = \langle \tau^{-1} \rho(\Psi_{Z_1, \eta_1, \mathfrak{s}_1}(\alpha_1)), f_* \rho(\Psi_{Z_2, \eta_2, \mathfrak{s}_2}(\alpha_2)) \rangle.$$

Applying [Proposition 10.5](#), this is

$$\left\langle \tau^{-1} \sum_i \mathbb{Q}_{M_1, \Sigma_1 \times S^1, \eta_1, \mathfrak{s}_1}(\alpha_1 \otimes \tilde{\beta}_i) \cdot \beta_i, f_* \sum_j \mathbb{Q}_{M_2, \Sigma_2 \times S^1, \eta_2, \mathfrak{s}_2}(\alpha_2 \otimes \tilde{\gamma}_j) \cdot \gamma_j \right\rangle$$

for bases $\{\beta_i\}$ and $\{\gamma_j\}$ whose relationship will be determined momentarily. As before, f_* is conjugate-linear in $\mathcal{L}(t)$ and the pairing is also conjugate-linear in the second variable. Hence the above is equal to

$$\sum_{i, j} \mathbb{Q}_{M_1, \Sigma_1 \times S^1, \eta_1, \mathfrak{s}_1}(\alpha_1 \otimes \tilde{\beta}_i) \mathbb{Q}_{M_2, \Sigma_2 \times S^1, \eta_2, \mathfrak{s}_2}(\alpha_2 \otimes \tilde{\gamma}_j) \langle \tau^{-1}(\beta_i), f_*(\gamma_j) \rangle.$$

Choose the basis $\{\gamma_j\}$ by setting $\gamma_j = f_*^{-1}(\beta_j^\circ)$; from this it is easy to see that $\langle \tau^{-1}(\beta_i), f_*(\gamma_j) \rangle = \delta_{ij} u_{\beta_i}$, and $\tilde{\gamma}_j = f_*^{-1}(\tilde{\beta}_j^\circ)$. The result follows immediately. \square

Suppose now that each of M_1, M_2 , and X have $b^+ \geq 2$, so that [Theorem 8.17](#) applies to identify the perturbed invariants \mathbb{Q} with the usual Ozsváth–Szabó invariants Φ . Assume also that $\mathfrak{s} \in \text{Spin}^c(X)$ restricts to a nonzero multiple of $\text{PD}[S^1]$, ie, $k \neq 0$ in the theorem above. The coefficient change ρ sums the coefficients of $\mathbb{Q}_{X, \Sigma \times S^1}$

corresponding to spin^c structures differing by rim tori, so since $k \neq 0$, [Theorem 10.8](#) translates to the equation

$$\sum_n \Phi_{X, s+nt}^{\text{Rim}}(\alpha) t^n = \sum_{\beta, n_1, n_2} \Phi_{M_1, s+n_1 t_1}(\alpha \otimes \tilde{\beta}) \Phi_{M_2, s_2+n_2 t_2}(\alpha_2 \otimes f_*^{-1}(\tilde{\beta}^\circ)) t^{n_1+n_2},$$

where t_i is Poincaré dual to the class of Σ_i in M_i , and t is simultaneously the dual of Σ in X and the formal variable in $\mathcal{L}(t)$. The above holds after possibly a multiplication by a power of t ; thus equating coefficients yields the formula

$$\Phi_{X, s}^{\text{Rim}}(\alpha) = \sum_{\substack{\beta \\ n_1+n_2=s_0}} \Phi_{M_1, s+n_1 t_1}(\alpha \otimes \tilde{\beta}) \Phi_{M_2, s_2+n_2 t_2}(\alpha_2 \otimes f_*^{-1}(\tilde{\beta}^\circ))$$

for some fixed integer n_0 .

Now, if $\{\tilde{\beta}\}$ is a basis of homogeneous elements, it is not hard to see that $\{\tilde{\beta}^\circ\}$ are likewise homogeneous of complementary degree. Specifically, if $\text{deg}(\tilde{\beta}) = m$ in $\tilde{X}(g, d) \subset \mathbb{A}(\Sigma)$ then $\text{deg}(\tilde{\beta}^\circ) = 2d - m$ (as usual, $d = g - 1 - |k|$). Thus in the above formula, we have

$$\text{deg}(\alpha_1 \otimes \tilde{\beta}) + \text{deg}(\alpha_2 \otimes f_*^{-1}(\tilde{\beta}^\circ)) = \text{deg}(\alpha) + 2g - 2 - 2|k|.$$

On the other hand, if a spin^c 4-manifold (N, ν) has $\Phi_{N, \nu}(\xi) \neq 0$ for $\xi \in \mathbb{A}(N)$ then we must have $\text{deg}(\xi) = d(\nu)$. Substituting this in the above and using $\sigma(X) = \sigma(M_1) + \sigma(M_2)$ and $e(X) = e(M_1) + e(M_2) + 4g - 4$ gives

$$(37) \quad c_1^2(s) = c_1^2(s_1 + n_1 t_1) + c_1^2(s_2 + n_2 t_2) + 8|k|.$$

When $k = 0$, of course, changing s_i by multiples of t_i does not affect the self-intersection so that (37) holds in that case as well.

This observation motivates the following “patching” construction producing elements of $H^2(X; \mathbb{Z})$ (modulo rim tori) from certain pairs of elements in $H^2(M_i; \mathbb{Z})$. We find it easiest to describe this construction in homology rather than cohomology; the cohomological version is obtained by Poincaré duality. Suppose, then, that $x_1 \in H_2(M_1)$ and $x_2 \in H_2(M_2)$ are integral homology classes, represented by embedded surfaces also denoted x_1, x_2 , and assume that $x_i \cdot \Sigma_i = m$ for $i = 1, 2$. Let $\rho: H_2(M_i) \rightarrow H_2(Z_i, \partial Z_i)$ denote the composition of the natural map $H_2(M_i) \rightarrow H_2(M_i, \Sigma_i \times D^2)$ followed by the excision isomorphism of the latter group with $H_2(Z_i, \partial Z_i)$ where $Z_i = M_i \setminus \text{int}(\Sigma_i \times D^2)$. Consider the long exact sequence for $X = Z_1 \cup_{\partial} Z_2$:

$$\cdots \rightarrow H_2(\Sigma \times S^1) \rightarrow H_2(X) \rightarrow H_2(Z_1, \partial Z_1) \oplus H_2(Z_2, \partial Z_2) \rightarrow H_1(\Sigma \times S^1) \rightarrow \cdots$$

The condition on $x_i \cdot \Sigma_i$ and the fact that the $\rho(x_i)$ are restrictions of classes on the closed manifolds M_i imply that there exists a lift $x \in H_2(X)$ of $(\rho(x_1), \rho(x_2))$, uniquely determined up to the image of $H_2(\Sigma \times S^1)$.

Choose the surfaces x_i to intersect $\Sigma_i \times D^2$ in a collection of normal disks; at the expense of increasing the genus of the x_i we may assume that there are exactly $|m|$ such disks. Then removing $\Sigma_i \times D^2$ from each of M_1, M_2 and gluing we can obtain a smooth surface representing the lifted class x . It is clear that $x \cdot \Sigma = m$, and furthermore by using pushoffs of the x_i that are disjoint from the normal disks in $\Sigma_i \times D^2$ we see that the self-intersection of x satisfies $x^2 = x_1^2 + x_2^2$.

Now let $x_1 * x_2 = x + 2\varepsilon\Sigma$, where ε is the sign of m . Then the self-intersection of $x_1 * x_2$ is

$$(x_1 * x_2)^2 = x_1^2 + x_2^2 + 4|m|,$$

and moreover the class $x_1 * x_2$ is determined by this condition up to addition of elements of $H_2(\Sigma \times S^1)/[\Sigma]$, in other words, up to rim tori.

The multiplication in [Theorem 1.1](#) is the Poincaré dual of this patching construction; the proof of that theorem is immediate from [Theorem 10.8](#) and the remarks leading to [\(37\)](#). [Theorem 1.4](#) follows similarly from [Theorem 10.7](#).

11 Manifolds of simple type

[Corollary 1.2](#) is an easy consequence of the fiber sum formula. Indeed, if M_1 and M_2 have simple type, then the only contributions to the right hand side of [\(1\)](#) are those in which $\alpha_1, \alpha_2, \beta$ and β° have degree zero. Hence $\alpha = \alpha_1 \otimes \alpha_2$ also has degree 0, showing that $\rho(\mathbb{Q}_{X,s}(\alpha)) = 0$ unless $\deg(\alpha) = 0$, which is the first statement of the corollary. Furthermore, since β and β° have complementary degree in $\tilde{X}(g, d)$, their degrees can both be 0 only if $|k| = g - 1$, which gives [\(2\)](#).

In the case of a 4–manifold containing a torus of square 0, we have the following analog of a result of Morgan, Mrowka and Szabó [\[7\]](#) in Seiberg–Witten theory. Recall that a 4–manifold X containing a surface Σ is said to have $\mathbb{A}(\Sigma)$ –simple type if all Ozsváth–Szabó invariants of X vanish on elements of $\mathbb{A}(X)$ lying in the ideal generated by U and $H_1(\Sigma)$.

Proposition 11.1 *Suppose X is a closed 4–manifold with $b^+(X) \geq 2$ containing a torus $T \subset X$ of self-intersection 0 representing a class of infinite order in $H_2(X)$. Then X has $\mathbb{A}(T)$ –simple type.*

Proof By [Theorem 8.17](#) we can write

$$\mathbb{Q}_{X,T^3,\eta,\mathfrak{s}}(\alpha) = \langle \tau^{-1}\Psi_{Z,\eta,\mathfrak{s}}(\alpha_1), \Psi_{T^2 \times D^2,\eta,\mathfrak{s}}(\alpha_2) \rangle,$$

for suitable perturbation η , and we may assume that $\mathfrak{s}|_{T^2 \times D^2} = \mathfrak{s}_0$, the torsion spin^c structure. If $\alpha = \alpha' \cdot \alpha_T$ is in the ideal generated by $\mathbb{A}(T)$, where $\alpha' \in \mathbb{A}(X)$ and $\alpha_T \in \mathbb{A}(T)$, then we can take $\alpha_1 = \alpha'$ and $\alpha_2 = \alpha_T$. But $\Psi_{T^2 \times D^2,\eta,\mathfrak{s}_0}(\alpha_T) = \alpha_T \cdot \Psi_{T^2 \times D^2,\eta,\mathfrak{s}_0}(1)$, and the relative invariant $\Psi_{T^2 \times D^2,\eta,\mathfrak{s}_0}(1)$ lies in the only nontrivial degree of $HF^-(T^3, \mathfrak{s}_0; \mathcal{L}(t))$. Hence if $\deg \alpha_T > 0$ we have $\alpha_T \cdot \Psi_{T^2 \times D^2,\eta,\mathfrak{s}_0}(1) = 0$ and the result follows. \square

Note that the proof applies also to 4-manifolds with $b^+(X) = 1$, if we consider only the perturbed invariant $\mathbb{Q}_{X,T^3,\eta,\mathfrak{s}}$ relative to the decomposition of X along the boundary of a tubular neighborhood of the torus.

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