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Modifying surfaces in 4–manifolds by twist spinning

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In this paper, given a knot K, for any integer m we construct a new surface $\Sigma_K(m)$ from a smoothly embedded surface Σ in a smooth 4–manifold X by performing a surgery on Σ. This surgery is based on a modification of the 'rim surgery' which was introduced by Fintushel and Stern, by doing additional twist spinning. We investigate the diffeomorphism type and the homeomorphism type of (X, Σ) after the surgery. One of the main results is that for certain pairs (X, Σ), the smooth type of $\Sigma_K(m)$ can be easily distinguished by the Alexander polynomial of the knot K and the homeomorphism type depends on the number of twist and the knot. In particular, we get new examples of knotted surfaces in $\mathbb{C}\mathbf{P}^2$, not isotopic to complex curves, but which are topologically unknotted.

57R57; 14J80, 57R95

1 Introduction

Let X be a smooth 4–manifold and Σ be an embedded positive genus surface and nonnegative self-intersection. In [3], Fintushel and Stern introduced a technique, called 'rim surgery', of modifying Σ without changing the ambient space X. This surgery on Σ may change the diffeomorphism type of the embedding Σ_K but the topological embedding is preserved when $\pi_1(X-\Sigma)$ is trivial. Rim surgery is determined by a knotted arc $K_+ \in B^3$, and may be described as follows. Choose a curve α in Σ, which has a neighborhood $S^1 \times B^3$ meeting Σ on an annulus $S^1 \times I$. Replacing the pair $(S^1 \times B^3, S^1 \times I)$ by $(S^1 \times B^3, S^1 \times K_+)$ gives a new surface Σ_K in X.

In [17], Zeeman described the process of twist-spinning an n–knot to obtain an $(n+1)$–knot. Here an *n–knot* is a locally flat pair (S^{n+2}, K) with $K \cong S^n$. Then here is the description for the process of twist-spinning to obtain a knot in dimension 4: Suppose we have a knotted arc K_+ in the half 3–space $\mathbb{R}^3_+$, with its end points in $\mathbb{R}^2 = \partial\mathbb{R}^3_+$. Spinning $\mathbb{R}^3_+$ about $\mathbb{R}^2$ generates $\mathbb{R}^4$, the arc K_+ generates a knotted 2–sphere in $\mathbb{R}^4$, called a *spun knot*. During the spinning process we spin the arc K_+ m times keeping its end points within $\mathbb{R}^3_+$, obtaining again a 2–sphere $K(m)$ in $\mathbb{R}^4$. A more explicit definition is the following.

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For any 1–knot (S^3, K), let (B^3, K_+) be its ball pair with the knotted arc K_+. Let τ be the diffeomorphism of (B^3, K_+), called 'twist map' defined in Section 2. Then for any integer m this induces a 2–knot called the *m–twist spun knot*

$$(S^4, K(m)) = \partial(B^3, K_+) \times B^2 \cup_\partial (B^3, K_+) \times_{\tau^m} \partial B^2$$

where $(B^3, K_+) \times_{\tau^m} \partial B^2$ means that $(B^3, K_+) \times [0,1]/(x,0) = (\tau^m x, 1)$.

In this paper, using these two ideas — rim surgery and spun knot — we will construct a new surface, denoted by $\Sigma_K(m)$, from the embedded surface in X without changing its ambient space. Our technique may be called a 'twist rim surgery'. We will see later (in Section 3 and Section 4) that the smooth and topological type of $\Sigma_K(m)$ obtained by twist rim surgery depends on m, K, and Σ. For a precise definition of the surgery, we will give two descriptions of $\Sigma_K(m)$. One is provided by using the twist map τ in the construction of Zeeman's twist spun knot. The other one can be obtained by performing the same operation which Fintushel and Stern introduced in [4] as it corresponds to doing a surgery on a homologically essential torus in X. In [4], they constructed exotic manifolds X_K according to a knot K and also showed that the Alexander polynomial $\Delta_K(t)$ of K can detect the smooth type of X_K.

In our circumstance, we consider a pair (X, Σ), where X is a smooth simply connected 4–manifold and Σ is an embedded genus g surface with self-intersection $n \geq 0$ such that the homology class $[\Sigma] = d \cdot \beta$, where β is a primitive element in $H_2(X)$ and $\pi_1(X - \Sigma) = \mathbb{Z}/d$. Then in Section 3, we will study the smooth type of $\Sigma_K(m)$ obtained by performing twist rim surgery on Σ. In fact, using the result in [3], we conclude that the Alexander polynomial $\Delta_K(t)$ of K can distinguish the smooth type of $\Sigma_K(m)$. In particular, applying this result to $\mathbb{C}\mathbf{P}^2$ we can get new examples of knotted surfaces in $\mathbb{C}\mathbf{P}^2$, not isotopic to complex curves. This solves, for an algebraic curve of degree ≥ 3, Problem 4.110 in the Kirby list [9]. Note that $d = 1, 2$ which are the only degrees where the curve is a sphere, are still open.

In Section 4, we will study topological conditions under which $(X, \Sigma_K(m))$ is pairwise homeomorphic to (X, Σ). This problem is also related to the knot type of K and the relation between d and m. In particular, if $d \not\equiv \pm 1 \pmod{m}$ then computing the fundamental group of the exterior of surfaces in X we easily distinguish $(X, \Sigma_K(m))$ and (X, Σ) for some nontrivial knot K. But when $d \equiv \pm 1 \pmod{m}$, it turns out that the fundamental group $\pi_1(X - \Sigma_K(m))$ is same as $\pi_1(X - \Sigma) = \mathbb{Z}/d$. So, in the case $d \equiv \pm 1 \pmod{m}$ we show that if K is a ribbon knot and the d–fold cover of the knot complement $S^3 - K$ is a homology circle then (X, Σ) and $(X, \Sigma_K(m))$ are topologically equivalent. This means that there is a pairwise homeomorphism $(X, \Sigma) \longrightarrow (X, \Sigma_K(m))$.

2 Definitions

Let X be a smooth 4–manifold and let Σ be an embedded surface of positive genus g. Given a knot K in S^3, let $E(K)$ be the exterior $\mathrm{cl}(S^3 - K \times D^2)$ of K. First we need to consider a certain diffeomorphism τ on (S^3, K) which will be used to define our surgery. Take a tubular neighborhood of the knot and then using a suitable trivialization with 0–framing, let $\partial E(K) \times I = K \times \partial D^2 \times I$ be a collar of $\partial E(K)$ in $E(K)$ with $\partial E(K)$ identified with $\partial E(K) \times \{0\}$. Define $\tau\colon (S^3, K) \longrightarrow (S^3, K)$ by

$$\tau(x \times e^{i\theta} \times t) = x \times e^{i(\theta + 2\pi t)} \times t \quad \text{for} \quad x \times e^{i\theta} \times t \in K \times \partial D^2 \times I \tag{1}$$

and $\tau(y) = y$ for $y \notin K \times \partial D^2 \times I$.

Note that τ is not the identity on the collar $\partial E(K) \times I = K \times \partial D^2 \times I$. However, it is the identity on the exterior $\mathrm{cl}(S^3 - K \times \partial D^2 \times I)$ of the collar. If we restrict τ to the exterior of the knot K then τ is isotopic to the identity although the isotopy is not the identity on the boundary of the knot complement. Explicitly, the isotopy can be given as the following. For any $s \in [0,1]$,

$$\tau_s(x \times e^{i\theta} \times t) = x \times e^{i\theta + 2\pi t(1-s) + 2\pi s} \times t.$$

We will refer to this diffeomorphism τ as a *twist map*.

Now take a non-separating curve α in Σ. Then choose a trivialization of the normal bundle $\nu(\Sigma)|_\alpha$ in X, $\alpha \times I \times D^2 = \alpha \times B^3 \longrightarrow \nu(\Sigma)|_\alpha$ where $\alpha \times I$ corresponds to the normal bundle $\nu(\alpha)$ in Σ. For any trivialization of the tubular neighborhood of α we can construct a new surface from Σ using the chosen curve. We will choose a specific framing of α later in Section 3 to study the diffeomorphism type of the new surface constructed in the way discussed now. Identifying α with S^1, two descriptions of the construction of $(X, \Sigma_K(m))$ called *m–twist rim surgery* follow.

Definition 2.1 Define for any integer m,

$$(X, \Sigma_K(m)) = (X, \Sigma) - S^1 \times (B^3, I) \cup_\partial S^1 \times_{\tau^m} (B^3, K_+).$$

Note that for $m = 0$, $\Sigma_K(m)$ is the surface obtained by rim surgery. In [3], its smooth type was studied when $\pi_1(X - \Sigma)$ is trivial. As in the paper [3], we will consider the smooth type of the new surface obtained by m–twist surgery in the extended case where $\pi_1(X - \Sigma)$ is cyclic.

If α is a trivial curve, that is it bounds a disk in Σ, we can simply write $(X, \Sigma_K(m))$ as the following.

Lemma 2.2 *If α is a trivial curve in Σ, then $(X, \Sigma_K(m))$ is the connected sum (X, Σ) with the m–twist spun knot $(S^4, K(m))$ of (S^3, K).*

Proof Considering the decomposition of $(X, \Sigma_K(m))$ in Definition 2.1.

$$(X, \Sigma_K(m)) = (X, \Sigma) - S^1 \times (B^3, I) \cup_\partial S^1 \times_{\tau^m} (B^3, K_+),$$

we write the boundary of the ball (B^3, I) in the definition as

$$\partial(B^3, I) = (S^2, \{N, S\}) = (D^2_+, \{N\}) \cup (D^2_-, \{S\})$$

where D^2_+, D^2_- are 2–disks and N, S are north and south poles respectively. Also recall that we identified α as S^1 in the definition and by the choice of α, let's denote the disk bounded by α as B^2 in Σ. Then we can rewrite

$$(X, \Sigma_K(m)) =$$
$$\big((X, \Sigma) - (S^1 \times (B^3, I) \cup B^2 \times (D^2_+, \{N\}))\big) \cup \big(B^2 \times (D^2_+, \{N\}) \cup S^1 \times_{\tau^m} (B^3, K_+)\big).$$

Note that the first component of this decomposition is

$$(X, \Sigma) - S^1 \times (B^3, I) \cup_{\partial B^2 \times D^2_+} B^2 \times (D^2_+, \{N\}) = (X, \Sigma) - (B^4, B^2).$$

In the second component

$$B^2 \times (D^2_+, \{N\}) \cup_{\partial B^2 \times D^2_+} S^1 \times_{\tau^m} (B^3, K_+),$$

gluing $B^2 \times (D^2_-, \{S\})$ to $B^2 \times (D^2_+, \{N\})$ along $B^2 \times \partial D^2_+$ and then taking it out later again we can write

$$\big(B^2 \times (D^2_+, \{N\})\big) \cup_{B^2 \times \partial D^2_+} \big(B^2 \times (D^2_-, \{S\})\big) \cup_\partial \big(S^1 \times_{\tau^m} (B^3, K_+)\big) - \big(B^2 \times (D^2_-, \{S\})\big)$$
$$= \big(B^2 \times \partial(B^3, K_+)\big) \cup_\partial \big(S^1 \times_{\tau^m} (B^3, K_+)\big) - \big(B^2 \times (D^2_-, \{S\})\big).$$

Considering the definition of twist spun knot in Section 1 we can realize this is

$$\big(S^4, K(m)\big) - \big(B^2 \times (D^2_-, \{S\})\big).$$

So,

$$(X, \Sigma_K(m)) = \big((X, \Sigma) - (B^4, B^2)\big) \cup \big((S^4, K(m)) - B^2 \times (D^2_-, \{S\})\big)$$

where the union is taken along the boundary. □

Let's move on to another description of $(X, \Sigma_K(m))$ which is useful in distinguishing the diffeomorphism types of $\Sigma_K(m)$. For a non-separating curve α in Σ, after a trivialization, the normal bundle α in X is of the form $\alpha \times I \times D^2 = \alpha \times B^3$ where

$\alpha\times I$ in Σ. Consider $\alpha\times\gamma\subset\alpha\times I\times D^2$ where γ is a pushed-in copy of the meridian circle $\{0\}\times\partial D^2\subset I\times D^2$. Under our trivialization, $\alpha\times\gamma$ is diffeomorphic to a torus T in $X-\Sigma$, called a *rim torus* by Fintushel and Stern. Note that this torus T is nullhomologous in X. Let $N(\gamma)$ be a tubular neighborhood of γ in $B^3=I\times D^2$ and γ' be the curve γ pushed off into $\partial N(\gamma)$. Then we will identify $\alpha\times N(\gamma)$ as a neighborhood $N(T)$ of T under the trivialization so that $\alpha\times N(\gamma)\subset\nu(\Sigma)|_\alpha\subset\nu(\Sigma)$. For a knot K in S^3, let's denote by μ_K the meridian and λ_K the longitude of the knot. Now consider the following manifold

$$\alpha\times(B^3-N(\gamma))\cup_\varphi(S^1\times E(K))$$

where the gluing map φ is the diffeomorphism determined by $\varphi_*(\alpha)=m\mu_K+S^1$, $\varphi_*(\gamma')=\mu_K$, and $\varphi_*(\partial D^2)=\lambda_K$.

Definition 2.3 Suppose that $T\cong\alpha\times\gamma$ is the smooth torus in X as above. Define

$$(X,\Sigma_K(m))=(X-N(T),\Sigma)\cup_\varphi(E(K)\times S^1,\varnothing).$$

This description means that performing a surgery on a smooth torus T in X, we obtain X again but Σ might be changed. Now we need to check those two descriptions are the same definitions for our construction.

Lemma 2.4 *Definition 2.1 and Definition 2.3 are equivalent.*

Proof Given a knot K, recall that knotting the arc $I=I\times\{0\}\subset B^3=I\times B^2$ can be achieved by a cut-paste operation on the complement. Let γ be an unknot which is the meridian of the arc I in B^3, $E(K)$ be the exterior of the knot K in S^3 and $N(\gamma)$ be the tubular neighborhood of γ in B^3. If we replace the tubular neighborhood $N(\gamma)$ by $E(K)$ then we get B^3 with the knotted arc K_+ instead of the trivial arc I. More precisely, note that $(B^3,K_+)=(\nu(\partial B^3\cup K_+),K_+)\cup E(K)$ where $\nu(\partial B^3\cup K_+)$ is the normal bundle in B^3 (see Figure 1). Let γ' be the push off of γ onto $\partial N(\gamma)$.

Then there is a diffeomorphism $(B^3-N(\gamma),I)\to(\nu(\partial B^3\cup K_+),K_+)$ mapping γ' to μ_K which induces a diffeomorphism

$$h\colon (B^3-N(\gamma),I)\cup_f E(K)\longrightarrow(\nu(\partial B^3\cup K_+),K_+)\cup E(K)=(B^3,K_+),$$

where $f\colon\partial N(\gamma)\longrightarrow\partial E(K)$ is a diffeomorphism determined by identifying γ' to μ_K. Note that the diffeomorphism h has $h(I)=K_+$ and $h|_{E(K)}=\mathrm{id}$.

Recalling the map τ defined in (1), we note that h is the identity on $E(K)$ but τ is not, whereas on the outside of $E(K)$, τ is the identity but h is not. This implies

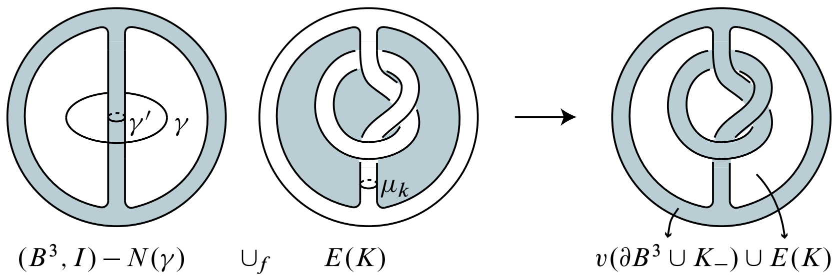

Figure 1: Diffeomorphism h: $(B^3 - N(\gamma), I) \cup_f E(K) \to (B^3, K_+)$

that τ is equivariant with respect to h, $\tau \circ h = h \circ \tau$. This τ induces a well-defined diffeomorphism mapping $[x,t]$ to $[h(x),t]$

$$\big(((B^3, I) - N(\gamma)) \cup_f E(K)\big) \times_{\tau^m} S^1 \longrightarrow (B^3, K_+) \times_{\tau^m} S^1.$$

Since τ^m is the identity on $(B^3, I) - N(\gamma)$, $(((B^3, I) - N(\gamma)) \cup_f E(K)) \times_{\tau^m} S^1$ is the same as $((B^3, I) - N(\gamma)) \times S^1 \cup_{f \times 1_{S^1}} (E(K) \times_{\tau^m} S^1)$ and thus we have

$$((B^3, I) - N(\gamma)) \times S^1 \cup_{f \times 1_{S^1}} (E(K) \times_{\tau^m} S^1) \longrightarrow (B^3, K_+) \times_{\tau^m} S^1.$$

Extending by the identity gives a diffeomorphism

$$((X, \Sigma) - (B^3, I) \times S^1) \cup_\partial ((B^3, I) - N(\gamma)) \times S^1 \cup_{f \times 1_{S^1}} (E(K) \times_{\tau^m} S^1) \longrightarrow \\ ((X, \Sigma) - (B^3, I) \times S^1) \cup_\partial (B^3, K_+) \times_{\tau^m} S^1.$$

Rewriting

$$\begin{aligned} ((X, \Sigma) - (B^3, I) \times S^1) \cup_\partial ((B^3, I) - N(\gamma)) \times S^1 &\cup_{f \times 1_{S^1}} (E(K) \times_{\tau^m} S^1) \\ &= X - N(\gamma) \times S^1 \cup_{f \times 1_{S^1}} (E(K) \times_{\tau^m} S^1) \\ &= X - \gamma \times D^2 \times S^1 \cup_{f \times 1_{S^1}} (E(K) \times_{\tau^m} S^1), \end{aligned}$$

we get a diffeomorphism

$$X - \gamma \times D^2 \times S^1 \cup_{f \times 1_{S^1}} (E(K) \times_{\tau^m} S^1) \to ((X, \Sigma) - (B^3, I) \times S^1) \cup_\partial (B^3, K_+) \times_{\tau^m} S^1.$$

Note that here the gluing map $f \times 1_{S^1}$ sends α to S^1, γ' to μ_K and ∂D^2 to λ_K where μ_K and λ_K are the meridian and the longitude of the knot K. Since τ^m is isotopic to identity, the isotopy induces a diffeomorphism $E(K) \times S^1 \longrightarrow E(K) \times_{\tau^m} S^1$. Again extending by the identity gives a diffeomorphism

$$X - \gamma \times D^2 \times S^1 \cup_{f \times 1_{S^1}} (E(K) \times_{\tau^m} S^1) \to (X - \gamma \times D^2 \times S^1) \cup_\varphi (E(K) \times S^1),$$

where φ is given by

$$\begin{aligned} \alpha &\longleftrightarrow S^1 + m\mu_K \\ \gamma' &\longleftrightarrow \mu_K \\ \partial D^2 &\longleftrightarrow \lambda_K. \end{aligned}$$

Therefore the result follows. □

3 Diffeomorphism types

Now let X be a smooth simply connected 4–manifold and Σ an embedded genus g surface with self-intersection $n \geq 0$ and homology class $[\Sigma] = d \cdot \beta$, where β is a primitive element in $H_2(X)$ and $\pi_1(X - \Sigma) = \mathbb{Z}/d$. Since Σ is diffeomorphic to $T^2 \# \cdots \# T^2$, let's choose a curve α whose image is the curve $\{pt\} \times S^1$ in the first $T^2 = S^1 \times S^1$. As we discussed in the previous section, a neighborhood of α in X is of the form $\alpha \times I \times D^2 = \alpha \times B^3$, where $\alpha \times I$ is in Σ. But we need to choose a certain trivialization of the normal bundle $\nu(\alpha \times I)$ in X which will be used in Section 4 when we compute some topological invariants to identify the homeomorphism type of $\Sigma_K(m)$. It is possible to choose a trivialization σ of $\nu(\alpha \times I)$ with the property that for some point $p \in \partial D^2$, $\sigma|\alpha \times \{0\} \times p$ is trivial in $H_1(X - \Sigma)$; we arbitrarily choose one trivialization $\sigma\colon \alpha \times I \times D^2 \longrightarrow \nu(\alpha \times I)$ and let α' be $\sigma|\alpha \times \{0\} \times p$ for some $p \in \partial D^2$. By composing σ with a self diffeomorphism of $\alpha \times I \times D^2$ sending the element $(e^{i\theta}, t, z)$ to $(e^{i\theta}, t, e^{ik\theta} z)$ for an appropriate integer k, we can arrange α' to be the zero homology element in $H_1(X - \Sigma) \cong \mathbb{Z}/d$, that is generated by the meridian $\sigma(pt \times \partial D^2)$ of Σ.

For a given d, the relation between $\Sigma_K(m)$ and Σ depends somewhat on m. For example, if $d \not\equiv \pm 1 \pmod{m}$ then for a nontrivial knot K, the surface $\Sigma_K(m)$ can be distinguished (even up to homeomorphism) from Σ by considering the fundamental group $\pi_1(X - \Sigma_K(m))$. First, we need to understand the explicit expression of this group.

In this paper, we will denote by $(X, Y)^d$ a d–fold covering of X branched along Y.

Lemma 3.1 *Let μ be the meridian of the knotted arc K_+ and let the base point $*$ be in $\partial E(K) = K \times \partial D^2 \times \{0\}$. Then*

$$\pi_1(X - \Sigma_K(m)) = \langle \pi_1(B^3 - K_+, *) \mid \mu^d = 1, \beta = \tau_*^m(\beta), \text{ for all } \beta \in \pi_1(B^3 - K_+, *) \rangle.$$

Proof Considering the definition of $(X, \Sigma_K(m))$, we have that the complement of $\Sigma_K(m)$ in X, $X - \Sigma_K(m)$, is $(X - S^1 \times B^3 - \Sigma) \cup S^1 \times_{\tau^m} (B^3 - K_+)$. Then we get that the intersection of the two components in the decomposition is

$$(X - S^1 \times B^3 - \Sigma) \cap S^1 \times_{\tau^m} (B^3 - K_+) = S^1 \times (\partial B^3 - \{\text{two points}\}).$$

Here we need to note that the action of τ on $\partial B^3 - \{$ two points $\}$ is trivial. Then using Van Kampen's theorem for this decomposition, we have the following diagram:

$$\begin{tikzcd}
\pi_1(S^1 \times (\partial B^3 - \{\text{two points}\})) \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \pi_1(X - S^1 \times B^3 - \Sigma) \arrow[d, "\psi_1"] \\
\pi_1(S^1 \times_{\tau^m} (B^3 - K_+)) \arrow[r, "\psi_2"'] & \pi_1(X - \Sigma_K(m))
\end{tikzcd}$$

Note that $X - S^1 \times B^3 - \Sigma$ is homotopy equivalent to $X - \Sigma$ and $\pi_1(X - \Sigma) \cong \mathbb{Z}/d$ is generated by the meridian γ of Σ. We also know that $\pi_1(S^1 \times (\partial B^3 - \{\text{two points}\}))$ is generated by $[S^1]$, which is identified with the class of the curve α' pushed off along a given trivialization of neighborhood of α, and by μ. Since the meridian μ of the knot is identified with γ, φ_1 is onto and so ψ_2 is also onto. Moreover, $\ker \psi_2 = \langle \varphi_2(\ker \varphi_1) \rangle$. Since $\ker \varphi_1 = \langle \alpha', \mu^d \rangle$ and

$$\pi_1(S^1 \times_{\tau^m} (B^3, K_+)) = \Big\langle \pi_1(B^3 - K_+), \alpha' \mid {\alpha'}^{-1} \beta \alpha' = \tau_*^m(\beta) \text{ for all } \beta \in \pi_1(B^3 - K_+) \Big\rangle,$$

it follows that

$$\begin{aligned}
&\pi_1(X - \Sigma_K(m)) \\
&= \Big\langle \pi_1(B^3 - K_+), \alpha' \mid \alpha' = 1, \mu^d = 1, {\alpha'}^{-1} \beta \alpha' = \tau_*^m(\beta) \text{ for all } \beta \in \pi_1(B^3 - K_+) \Big\rangle \\
&= \Big\langle \pi_1(B^3 - K_+) \mid \mu^d = 1, \beta = \tau_*^m(\beta) \text{ for all } \beta \in \pi_1(B^3 - K_+) \Big\rangle.
\end{aligned}$$

which completes the proof. □

The following example shows that we can distinguish $\Sigma_K(m)$ using π_1.

Example 3.2 For any nontrivial knot K, let $d = 2$, ie $\pi_1(X - \Sigma) = \mathbb{Z}/2$, and let m be any even number. If we consider the fundamental group $\pi_1(X - \Sigma_K(m))$, then by Lemma 3.1,

$$\pi_1(X - \Sigma_K(m)) = \Big\langle \pi_1(B^3 - K_+, *) \mid \mu^d = 1, \beta = \tau_*^m(\beta) \text{ for all } \beta \in \pi_1(B^3 - K_+, *) \Big\rangle,$$

where μ is the meridian of the knotted arc K_+ and the base point $*$ is in $\partial E(K) = K \times \partial D^2 \times \{0\}$.

Recall the group of the knot $\pi_1(B^3 - K_+, *)$ has the Wirtinger presentation

$$\langle g_1, g_2, \ldots, g_n \mid r_1, r_2, \ldots, r_n \rangle,$$

where $g_1 = \mu$ and other generators g_i represent the loop that, starting from a base point, goes straight to the i^{th} over-passing arc in the knot diagram, encircles it and returns to the base point.

Note that $\tau_*^m(g_1) = g_1$ and $\tau_*^m(g_i) = g_1^{-m} g_i g_1^m$ for other generators g_i by the definition of τ. Since $d = 2$ ie $g_1^2 = 1$ and m is an even number, $\tau_*^m(g_i) = g_1^{-m} g_i g_1^m$ is always g_i and thus we get

$$\pi_1(X - \Sigma_K(m)) = \pi_1(B^3 - K_+)/\mu^2 = \pi_1(S^3 - K)/\mu^2.$$

If we take a 2–fold branched cover $(S^3, K)^2$ along the knot K then the fundamental group $\pi_1((S^3, K)^2)$ is same as the group $\pi_1((S^3 - K)^2)/\widetilde{\mu}$, where $(S^3 - K)^2$ is the 2–fold unbranched cover and $\widetilde{\mu}$ is a lift of μ. So $\pi_1(S^3 - K)/\mu^2$ has $\pi_1((S^3, K)^2)$ as an index 2 subgroup. The Smith conjecture [12] states that for any $d \geq 1$, the fundamental group of a d–fold branched cover $\pi_1((S^3, K)^d)$ is nontrivial unless K is a trivial knot. Hence $\pi_1(X - \Sigma_K(m))$ has a nontrivial index 2 subgroup and so $\pi_1(X - \Sigma_K(m)) \ncong \mathbb{Z}/2$. This proves that there is no homeomorphism $(X - \Sigma) \to (X - \Sigma_K(m))$.

A more interesting case is when π_1 does not distinguish the embedding of $\Sigma_K(m)$, so that we have to use other means to show that Σ is not diffeomorphic to $\Sigma_K(m)$. In particular, for the case $d \equiv \pm 1 \pmod{m}$, we have:

Proposition 3.3 *If $d \equiv \pm 1 \pmod{m}$ then $\pi_1(X - \Sigma) = \pi_1(X - \Sigma_K(m)) = \mathbb{Z}/d$.*

Proof If $d = 1$ then by Lemma 3.1, $\pi_1(X - \Sigma) = \pi_1(X - \Sigma_K(m)) = \{1\}$. So, we assume $d > 1$. To express $\pi_1(X - \Sigma)$ more explicitly, in a Wirtinger presentation of the knot group $\pi_1(B^3 - K_+, *)$, choose meridians g_j conjugate to the meridian $g_1 = \mu$ of the knot K for each $j = 2, ..., n$ as generators of $\pi_1(B^3 - K_+)$. Then with Lemma 3.1, we represent $\pi_1(X - \Sigma_K(m))$ by

$$\langle g_1, g_2, \ldots, g_n \mid g_1^d = 1, r_1, \ldots, r_n, \beta = \tau_*^m(\beta) \text{ for all } \beta \in \pi_1(B^3 - K_+) \rangle$$

where $r_1, \ldots, r_n$ are relations of $\pi_1(B^3 - K_+)$.

Considering the definition of τ, $\tau_*(\mu) = \mu$ and $\tau_*(g_j) = \mu^{-1} g_j \mu$ for each $j = 2, \dots, n$ so that we rewrite

$$\begin{aligned}&\pi_1(X - \Sigma_K(m)) \\ &\quad = \langle g_1, g_2, \dots, g_n \mid g_1^d = 1, r_1, \dots, r_n, g_j = g_1^{-m} g_j g_1^m \text{ for } j = 2, \dots, n \rangle.\end{aligned}$$

Now we claim that this is equal to $\langle g_1, g_2, \dots, g_n \mid g_1^d, r_1, \dots, r_n, g_1 = g_1^{-1} g_j g_1$ for $j = 2, \dots, n \rangle$.

Since $d \equiv \pm 1 \pmod{m}$, we can write $d = mk \pm 1$ for some integer k. Let $l = d - m$. Then $l = d - m = mk \pm 1 - m = m(k-1) \pm 1$.

$$\begin{aligned} g_j = g_1^{-m} g_j g_1^m &\Longrightarrow g_1^{-l} g_j g_1^l = g_1^{-l}(g_1^{-m} g_j g_1^m) g_1^l \\ &\Longrightarrow g_1^{-l} g_j g_1^l = g_1^{-(l+m)} g_j g_1^{(l+m)} = g_j && (l + m = d) \\ &\Longrightarrow g_1^{-m(k-1)\mp 1} g_j g_1^{m(k-1)\pm 1} = g_j && (l = m(k-1) \pm 1) \\ &\Longrightarrow g_1^{\mp 1}(g_1^{-m(k-1)} g_j g_1^{m(k-1)}) g_1^{\pm 1} = g_j \dots && (*) \end{aligned}$$

We claim that $g_1^{-m(k-1)} g_j g_1^{m(k-1)} = g_j$; if $k - 1 = 0$ or 1 then it is clearly true. Let's assume that it is true for $k - 1 = i$. For $k - 1 = i + 1$, by induction

$$g_1^{-m(i+1)} g_j g_1^{m(i+1)} = g_1^{-mi}(g_1^{-m} g_j g_1^m) g_1^{mi} = g_1^{-mi} g_j g_1^{mi} = g_j.$$

This implies that $(*)$ becomes $g_1^{\mp 1} g_j g_1^{\pm 1} = g_j$ and so we now get

$$\begin{aligned}&\pi_1(X - \Sigma_K(m)) \\ &\quad = \langle g_1, g_2, \dots, g_n \mid g_1^d = 1, r_1, \dots, r_n, [g_1, g_j] = 1 \text{ for } j = 2, \dots, n \rangle.\end{aligned}$$

If we consider the Wirtinger presentation of the knot group then we can show $g_1 = g_2 = ... = g_n$ with the relations $r_1, .., r_n$ and $[g_1, g_j]$; corresponding to the following crossing, the relator gives $g_2 g_s = g_s g_1$ or $g_s g_2 = g_1 g_s$.

Figure 2: Wirtinger presentation of the knot group

So, $g_1 = g_2$. By an induction argument, we can conclude that $g_1 = g_2 = \ldots = g_n$. This proves that

$$\pi_1(X - \Sigma_K(m)) \;=\; \langle \mu \mid \mu^d = 1 \rangle \;\cong\; \mathbb{Z}/d. \qquad \square$$

Remark The same technique works for many other cases, for example if $d = 2$ and m is an odd integer.

We can also distinguish some $\Sigma_K(m)$ smoothly by using relative Seiberg–Witten (SW) theory, following the technique of Fintushel and Stern [2]. In [4], they introduced a method called 'knot surgery' modifying a 4–manifold while preserving its homotopy type by using a knot in S^3 and also gave a formula for the SW-invariant of the new manifold to detect the diffeomorphism type under suitable circumstances.

Let X be a smooth 4–manifold and T in X be an imbedded 2–torus with trivial normal bundle. (In [14], C Taubes showed the 'c–embedded' condition on the torus in the original paper [4] to be unnecessary.) Then the knot surgery may be described as follows.

Let K be a knot in S^3, and $K \times D^2$ be the trivialization of its open tubular neighborhood given by the 0–framing. Let $\varphi\colon \partial(T \times D^2) \longrightarrow \partial(K \times D^2) \times S^1$ be any diffeomorphism with $\varphi(p \times \partial D^2) = K \times q$ where $p \in T$, $q \in \partial D^2 \times S^1$ are points. Define

$$X_K = (X - T \times D^2) \cup_\varphi E(K) \times S^1.$$

In our situation, the surgical construction of $\Sigma_K(m)$ is performing a surgery on a torus T in X called a 'rim torus'. Recall the torus T has the form $\gamma \times \alpha$ where γ is the meridian of Σ and α is a curve in Σ (see Lemma 2.4). In other words, we remove a neighborhood of the torus and sew in $E(K) \times S^1$ along the gluing map given in Definition 2.3. Considering this identification, we can observe that the pair $(X, \Sigma_K(m))$ is obtained by a knot surgery.

Fintushel and Stern wrote a note to fill a gap in the proof of the main theorem in [3]. In the note [2], they explained the effect of rim surgery on the relative Seiberg–Witten invariant of $X - \Sigma$. The m–twist rim surgery on $X - \Sigma$ affects its relative Seiberg–Witten invariant exactly same as rim surgery. So we will refer to the note [2] to distinguish the pairs (X, Σ) and $(X, \Sigma_K(m))$ smoothly.

If the self-intersection $\Sigma \cdot \Sigma = n \geq 0$, blow up X n times to get a pair (X_n, Σ_n) and reduce the self intersection to zero. For simplicity, we may assume that $\Sigma \cdot \Sigma = 0$. In general, the relative Seiberg–Witten invariant $SW_{X,\Sigma}$ is an element in the Floer homology of the boundary $\Sigma \times S^1$ [10]. We restrict $SW_{X,\Sigma}$ to the set $\mathcal{T}$ which is the collection of spin^c–structures τ on $X-N(\Sigma)$ whose restriction to $\partial N(\Sigma)$ is the spin^c–structure $\pm s_{g-1}$ corresponding to the element $(g-1,0)$ of $H^2(\Sigma\times S^1)\cong\mathbb{Z}\oplus H^1(\Sigma)$. Then we obtain a well-defined integer-valued Seiberg–Witten invariant $SW^{\mathcal{T}}_{X,\Sigma}$ and so get a Laurent polynomial $SW^{\mathcal{T}}_{X,\Sigma}$ with variables in

$$A=\{\alpha\in H^2(X-\Sigma)|\alpha|_{\Sigma\times S^1}=\pm s_{g-1}\}.$$

If there is a diffeomorphism $f\colon (X,\Sigma)\to(X',\Sigma')$ then it induces a map $f^*\colon A'\to A$ sending $SW^{\mathcal{T}}_{X',\Sigma'}$ to $SW^{\mathcal{T}}_{X,\Sigma}$.

Theorem 3.4 *Suppose the relative Seiberg–Witten invariant $SW^{\mathcal{T}}_{X,\Sigma}$ is nontrivial. If there is a diffeomorphism $(X,\Sigma_K(m))\longrightarrow(X,\Sigma_J(m))$ then the set of coefficients (with multiplicity) of $\Delta_K(t)$ is equal to that of $\Delta_J(t)$, where $\Delta_K(t)$ and $\Delta_J(t)$ are the Alexander polynomials of K and J respectively.*

Proof If there is a pairwise diffeomorphism $(X,\Sigma_K(m))\longrightarrow(X,\Sigma_J(m))$ then it induces a diffeomorphism $(X_n,\Sigma_{n,K}(m))\longrightarrow(X_n,\Sigma_{n,J}(m))$. So, we now may assume that $\Sigma\cdot\Sigma=0$.

According to the note [2], the proof of the knot surgery theorem [4] works in the relative case to show that

$$SW^{\mathcal{T}}_{(X-\Sigma)_K}=SW^{\mathcal{T}}_{X,\Sigma}\cdot\Delta_K(r^2)$$

where $r=[T]$ is the element of R, the subgroup of $H^2(X-\Sigma)$ generated by the rim torus T of Σ. Note that the rim torus T is homologically essential in $X-\Sigma$.

Since the relative Seiberg–Witten invariant $SW^{\mathcal{T}}_{X,\Sigma_K(m)}=SW^{\mathcal{T}}_{(X-\Sigma)_K}$, applying the knot surgery theorem to the m–twist rim surgery we also get that the coefficients of $SW^{\mathcal{T}}_{X,\Sigma}\cdot\Delta_K(r^2)$ must be equal to those of $SW^{\mathcal{T}}_{X,\Sigma}\cdot\Delta_J(r'^2)$. □

Remark (1) The theorem implies that for $\Delta_K(t)\neq 1$, (X,Σ) is not pairwise diffeomorphic to $(X,\Sigma_K(m))$.

(2) In [3] standard pairs (Y_g,S_g) were defined where Y_g is a simply connected Kähler surface, S_g is a primitively embedded genus $g\geq 1$ Riemann surface in Y_g with $S_g\cdot S_g=0$. According to the note [2], the hypothesis $SW_{X\#_{\Sigma=S_g}Y_g}\neq 1$ of [3] implies $SW^{\mathcal{T}}_{X,\Sigma}\neq 1$ by the gluing formula [10].

(3) $SW_{X\#_{\Sigma=S_g}Y_g}$ is nontrivial when Σ is a complex curve in a complex surface.

The case of curves in $\mathbb{C}\mathbf{P}^2$ is particularly interesting. By applying Theorem 3.4, we obtain the following corollary.

Corollary 3.5 *For $d > 2$ with $d \equiv \pm 1 \pmod m$, if Σ is a degree d–curve in $\mathbb{C}\mathbf{P}^2$ then $(\mathbb{C}\mathbf{P}^2, \Sigma)$ is not pairwise diffeomorphic to $(\mathbb{C}\mathbf{P}^2, \Sigma_K(m))$ for any knot K with $\Delta_K(t) \neq 1$, but $\pi_1(\mathbb{C}\mathbf{P}^2 - \Sigma_K(m)) \cong \mathbb{Z}/d$.*

Proof Note that Σ is a symplectically embedded surface with positive genus $g = \frac{1}{2}(d-1)(d-2)$. Under the construction in [3], S_g is also symplectically embedded in Y_g since S_g is a complex submanifold of the Kähler manifold Y_g. Since the group $\pi_1(C\mathbf{P}^2 - \Sigma) = \mathbb{Z}/d$, note that $\pi_1(C\mathbf{P}^2 - \Sigma_K(m)) = \mathbb{Z}/d$ by Proposition 3.3.

Let us denote by $CP^2_{d^2}$ the manifold obtained by blowing up d^2 times CP^2. Then $CP^2_{d^2}\#_{\Sigma_{d^2}=S_g} Y_g$ is also a symplectic manifold by Gompf [7]. So (see Taubes [13]),

$$SW_{CP^2_{d^2}\#_{\Sigma_{d^2}=S_g} Y_g} \neq 0.$$

By Theorem 3.4, the result follows. □

This means that for any $d \geq 3$, there are infinitely many smooth oriented closed surfaces Σ in $\mathbb{C}\mathbf{P}^2$ representing the class $dh \in H_2(\mathbb{C}\mathbf{P}^2)$, where h is a generator of $H_2(\mathbb{C}\mathbf{P}^2)$, having genus$(\Sigma) = \frac{1}{2}(d-1)(d-2)$ and $\pi_1(\mathbb{C}\mathbf{P}^2 - \Sigma) \cong \mathbb{Z}/d$, such that the pairs $(\mathbb{C}\mathbf{P}^2, \Sigma)$ are pairwise smoothly non-equivalent. Such examples, for $d \geq 5$, were known by the work of Finashin which we describe in order to contrast it with our construction. In [1], he constructed a new surface by knotting a standard one along a suitable annulus membrane.

More precisely, let X be a 4–manifold and Σ be a smoothly embedded surface. Suppose that there is a smoothly embedded surface M in X, called a 'membrane', such that $M \cong S^1 \times I$, $M \cap \Sigma = \partial M$ and M meets to Σ normally along ∂M. By adjusting a trivialization of its regular neighborhood U, we can assume that $U(\cong S^1 \times D^3) \cap \Sigma = S^1 \times f$, where $f = I_0 \sqcup I_1 = I \times \partial I$ is a disjoint union of two unknotted segments of a part of the boundary of a band $b = I \times I$ in D^3. Here the band $b = I \times I$ is trivially embedded in D^3 and the intersection $I \times I \cap \partial D^3 = \partial I \times I$ (see Figure 3).

Then given a knot K in S^3, we can get a new surface $\Sigma_{K,F}$ by knotting f along K in D^3 (see Figure 4).

In [1], Finashin showed that we can find such a membrane M in $\mathbb{C}\mathbf{P}^2$ and proved that $(\mathbb{C}\mathbf{P}^2, \Sigma_{K,F})$ is pairwise non-equivalent to $(\mathbb{C}\mathbf{P}^2, \Sigma)$ for an algebraic curve Σ of degree $d \geq 5$. In particular, for an even degree he showed that the double cover branched along $\Sigma_{K,F}$ is diffeomorphic to the 4–manifold obtained from the double cover branched along Σ by knot surgery along the torus, which is the pre-image of the membrane M in the covering, via the knot $K\#K$. So, the knot surgery theorem in [4] distinguishes the branch covers by comparing their SW-invariants. For odd cases, one

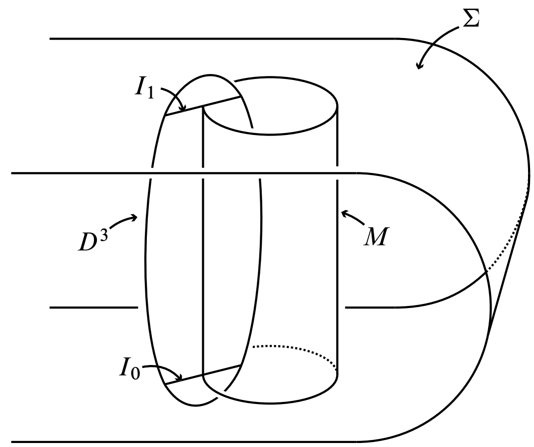

Figure 3: $(\mathbb{C}\mathbf{P}^2, \Sigma)$

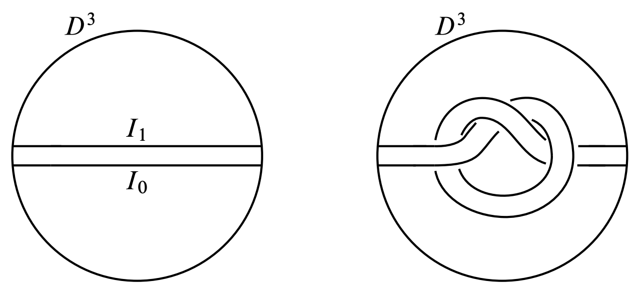

Figure 4: $(D^3, I \times I)$ and $(D^3, K_+ \times I)$

can use the same argument using d–fold coverings to show smooth non-equivalence of embeddings.

Our examples constructed by twist spinning are different from Finashin's for a degree $d \geq 5$. To see this, we compute the SW-invariant of the branched cover of $(\mathbb{C}\mathbf{P}^2, \Sigma_K(m))$. Let Y be a d–fold branch cover along Σ and $Y_{K,m}$ be a d–fold branch cover along $\Sigma_K(m)$. Let's consider the description for the branch cover $Y_{K,m}$. We write $Y_{K,m}$ as the union of two d–fold branched covers:

$$(Y_{K,m}, \Sigma_K(m)) = (X - S^1 \times B^3, \Sigma - S^1 \times I)^d \cup_\partial (S^1 \times_{\tau^m} (B^3, K_+))^d$$

Since the homology group $H_1(X - S^1 \times B^3 - \Sigma) \cong H_1(X - \Sigma) \cong \mathbb{Z}/d$, the branch cover $(X - S^1 \times B^3, \Sigma - S^1 \times I)^d$ is unique and is the same as $Y - S^1 \times B^3$. We also need to note that $(S^1 \times_{\tau^m} (B^3, K_+))^d = S^1 \times_{\tilde{\tau}^m} (B^3, K_+)^d$ for some lift $\tilde{\tau}^m$ of τ^m which is referred to in the proof for Proposition 4.3. So we rewrite

$$(Y_{K,m}, \Sigma_K(m)) = ((Y, \Sigma) - S^1 \times (B^3, I)) \cup_{S^1 \times S^2} (S^1 \times_{\tilde{\tau}^m} (B^3, K_+)^d).$$

If K is any knot with the homology $H_1((S^3 - K)^d) \cong \mathbb{Z}$ then $S^1 \times_{\tilde{\tau}^m} (B^3, K_+)^d$ is homologically equivalent to $S^1 \times B^3$. We may look at knots, introduced in Section 4, having the property that their d–fold covers are homology circles. An extension of the result of Vidussi in [16] shows

$$SW_{Y_{K,m}} = SW_Y.$$

But the SW-invariant of branched cover along the surface $\Sigma_{K,F}$ constructed by Finashin is not standard as we saw above. Our examples also cover the case of degree $d = 3$ and 4 which were not treated in his paper.

Remark By the same argument in Fintushel and Stern [3], we can also say that if X is a simply connected symplectic 4–manifold and Σ is a symplectically embedded surface then $\Sigma_K(m)$ is not smoothly ambient isotopic to a symplectic submanifold of X for $\Delta_K(t) \neq 1$. Using Taubes' result in [13], we can easily get a proof of this (see [3] for more detail).

4 Homeomorphism types

In this section, we shall investigate when $\Sigma_K(m)$ is topologically equivalent to Σ. As we saw in the previous section, in the case $d \equiv \pm 1 \pmod m$ their complements in X have the same fundamental group. So, for this case one would like to show that they are pairwise homeomorphic under a certain condition by constructing an explicit s–cobordism. Note that it is not known if Finashin's examples are topologically unknotted [1, Remark, p50]. Recall that the s–cobordism theorem gives a way for showing manifolds are homeomorphic.

Let W be a compact n–manifold with the boundary being the disjoint union of manifolds M_0 and M_1. Then the original s–cobordism theorem states that for $n \geq 6$, W is diffeomorphic to $M_0 \times [0,1]$ exactly when the inclusions of M_0 and M_1 in W are homotopy equivalences and the Whitehead torsion $\tau(W, M_0)$ in $\mathrm{Wh}(\pi_1(W))$ is zero. By the work of M Freedman [6], the s–cobordism theorem is known to hold topologically in the case $n = 5$ when $\pi_1(W)$ is poly-(finite or cyclic). A relative s–cobordism theorem also holds.

To make use of those theorems we shall construct a relative h–cobordism from $X - \nu(\Sigma)$ to $X - \nu(\Sigma_K(m))$ and then apply the relative s–cobordism theorem.

First consider the following situation. Let K be a ribbon knot in S^3 so that $(S^3, K) = \partial(B^4, \Delta)$ for some ribbon disc Δ in B^4. By Lemma 3.1 in [8], $\pi_1(S^3 - K) \longrightarrow \pi_1(B^4 - \Delta)$ is surjective. Take out a 4–ball $(B', B' \cap \Delta)$ from the interior of (B^4, Δ) such that $B' \cap \Delta$ is an unknotted disk (see Figure 5).

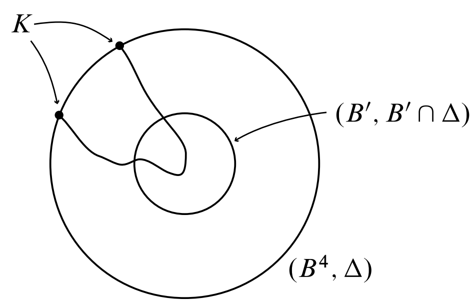

Figure 5: Ribbon disk in B^4

Let $A = \Delta - (B' \cap \Delta)$ then we can easily note that A is a concordance between K and an unknot O. Let $K = K_+ \cup K_-$ where K_+ is a knotted arc and K_- is a trivial arc diffeomorphic to I. Write $S^3 = B^3_+ \cup B^3_-$ where B^3_+, B^3_- are 3–balls. Let's assume that $B^3_- \times I \subset S^3 \times I$ with $(B^3_- \times I, B^3_- \times I \cap A) = (B^3_- \times I, I \times I)$ and $(B^3_- \times 1, B^3_- \times 1 \cap A) = (B^3_- \times 1, K_-)$.

If we take out $B^3_- \times I$ from $S^3 \times I$ then we are left with $(S^3 \times I, A) - (B^3_- \times I, I \times I) = (B^3_+ \times I, A - I \times I)$. Denoting $A - I \times I$ by A_+, we have $B^3_+ \times 1 \cap A_+ = K_+$ and $B^3_+ \times 0 \cap A_+ = O_+$ where O_+ is a trivial arc of O (see Figure 6).

We will define a self diffeomorphism on $(S^3 \times I, A)$ in the same way that we defined the *twist map* in Section 2. Recall $(S^3 \times I, A) - (B^3_- \times I, I \times I) = (B^3_+ \times I, A_+)$. Note the normal bundle $\nu(A)$ in $S^3 \times I$ is $A \times D^2$ and let $E(A)$ be the exterior $\mathrm{cl}(S^3 \times I - A \times D^2)$ of A in $S^3 \times I$. Then $E(A)$ coincides (up to isotopy), with $\mathrm{cl}(B^3_+ \times I - A_+ \times D^2)$. Thus, $\partial E(A) = A \times \partial D^2$ is $\partial(\mathrm{cl}(B^3_+ \times I - A_+ \times D^2)) \cong T \times I$ where T is a torus. Let $A \times \partial D^2 \times I$ be the collar of $\partial E(A)$ in $E(A)$. Define $\tau\colon (S^3 \times I, A) \longrightarrow (S^3 \times I, A)$ by

$$\tau(x \times e^{i\theta} \times t) = x \times e^{i(\theta + 2\pi t)} \times t \quad \text{for} \quad x \times e^{i\theta} \times t \in A \times \partial D^2 \times I$$

and $\tau(y) = y$ for $y \notin A \times \partial D^2 \times I$.

Then note that τ is the identity on a neighborhood of A_+ and that $\tau|_{B^3_+ \times 0 - O_+}$ and $\tau|_{B^3_+ \times 1 - K_+}$ are the twist maps induced by the unknot O and the knot K defined in

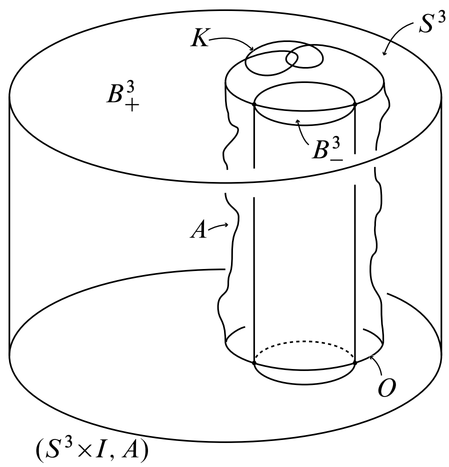

Figure 6: A concordance between K and unknot

(1). Denote those maps by τ_O and τ_K respectively. Using this diffeomorphism τ, we can also construct a new submanifold $(\Sigma \times I)_A(m)$ from an embedded manifold $\Sigma \times I$ to $X \times I$ in the way to construct a new surface $\Sigma_K(m)$.

Definition 4.1 Under the above notation, define

$$(X \times I, (\Sigma \times I)_A(m)) = X \times I - S^1 \times (B^3 \times I, I \times I) \cup S^1 \times_{\tau^m} (B^3 \times I, A_+).$$

Then we can easily note that

$$X \times 1 = X - S^1 \times (B^3 \times 1, I \times 1) \cup S^1 \times_{\tau_K^m} (B^3 \times 1, K_+) = (X, \Sigma_K(m)),$$
$$X \times 0 = X - S^1 \times (B^3 \times 0, I \times 0) \cup S^1 \times_{\tau_O^m} (B^3 \times 0, O_+) = (X, \Sigma)$$

and so the complement $X \times I - (\Sigma \times I)_A(m)$ gives a concordance between $X - \Sigma$ and $X - \Sigma_K(m)$ (See Figure 7). We will denote this concordance by W and will later show this W is a h–cobordism under certain conditions. Here we note that the cobordism W is a product near the boundary. To see what conditions are needed, consider several other properties first.

Recall for any pair (X, Y), we denote by X^d a d–fold cover of X and $(X, Y)^d$ a d–fold cover of X branched along Y. We know $H_*(S^3 - K) \to H_*(B^4 - \Delta)$ is an isomorphism but generally, $H_*((S^3 - K)^d) \to H_*((B^4 - \Delta)^d)$ is not. It is true when K is a ribbon knot:

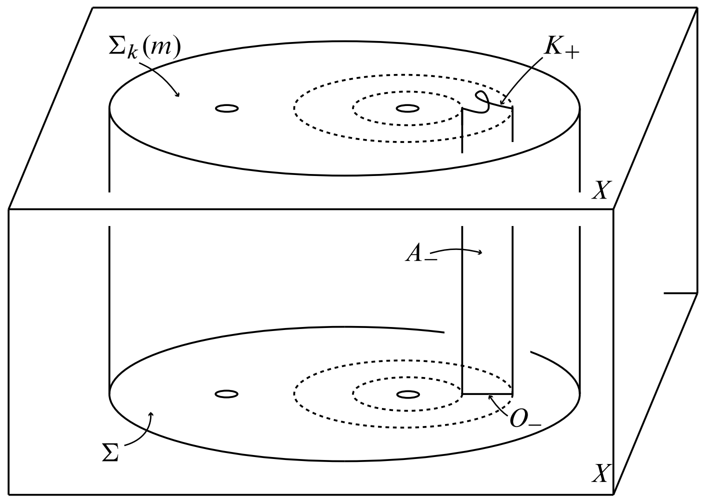

Figure 7: A cobordism between (X, Σ) and $(X, \Sigma_K(m))$

Lemma 4.2 *If K is a ribbon knot and the homology of d–fold cover of $S^3 - K$, $H_1((S^3-K)^d)$ is isomorphic to $\mathbb{Z}$ then the d–fold cover $(B^4-\Delta)^d$ of $B^4-\Delta$ is a homology circle.*

Proof Let $(S^3-K)^d$ and $(B^4-\Delta)^d$ be the d–fold covers of (S^3-K) and $(B^4-\Delta)$ according to the following homomorphisms φ_1, φ_2:

$$\begin{tikzcd}
\pi_1(S^3-K) \arrow[r, "\varphi_1"] \arrow[d, "i_*"', "\text{surj}"] & H_1(S^3-K) \arrow[r] \arrow[d, "\cong"] & \mathbb{Z}/d \\
\pi_1(B^4-\Delta) \arrow[r, "\varphi_2"'] & H_1(B^4-\Delta) \arrow[r] & \mathbb{Z}/d
\end{tikzcd}$$

Since K is a ribbon knot, $i_*\colon \pi_1(S^3-K) \to \pi_1(B^4-\Delta)$ is surjective. It follows that the map $H_1((S^3-K)^d) \to H_1((B^4-\Delta)^d)$ between the d–fold coverings is surjective since $i_*(\ker\varphi_1)$ maps to the trivial element of $\mathbb{Z}/d$ under φ_2. Since $H_1((S^3-K)^d)$ is isomorphic to $\mathbb{Z}$, so is $H_1((B^4-\Delta)^d)$. To show $H_*((B^4-\Delta)^d) = 0$ for $* > 1$, we consider the long exact sequence of the pair $((B^4-\Delta)^d, \partial(B^4-\Delta)^d)$.

$$\begin{aligned}
&H_4((B^4-\Delta)^d, \partial(B^4-\Delta)^d) \xrightarrow{\partial_4} H_3(\partial(B^4-\Delta)^d) \xrightarrow{i_3} H_3(B^4-\Delta)^d \\
&\xrightarrow{j_3} H_3((B^4-\Delta)^d, \partial(B^4-\Delta)^d) \xrightarrow{\partial_3} H_2(\partial(B^4-\Delta)^d) \xrightarrow{i_2} H_2(B^4-\Delta)^d \to \cdots
\end{aligned}$$

Since ∂_4 is an isomorphism, j_3 is injective so that $H_3((B^4-\Delta)^d)$ is isomorphic to $\operatorname{im} j_3 = \ker \partial_3$. Our claim is that $\partial_3\colon H_3((B^4-\Delta)^d, \partial(B^4-\Delta)^d) \to H_2(\partial(B^4-\Delta)^d)$ is an isomorphism. Observe that $\partial(B^4-\Delta)^d = (S^3-K)^d \cup \widetilde{\Delta} \times \partial D^2$ where $\widetilde{\Delta}$ is the lifted disk of Δ in the d–fold cover of B^4. By Poincaré Duality and the Universal Coefficient Theorem,

$$H_3((B^4-\Delta)^d, \partial(B^4-\Delta)^d) \cong H^1((B^4-\Delta)^d) \cong \mathrm{Hom}(H_1((B^4-\Delta)^d), \mathbb{Z})$$

and

$$\begin{aligned} H_2((S^3-K)^d \cup \widetilde{\Delta} \times \partial D^2) &\cong H^1((S^3-K)^d \cup \widetilde{\Delta} \times \partial D^2) \\ &\cong \mathrm{Hom}(H_1((S^3-K)^d \cup \widetilde{\Delta} \times \partial D^2), \mathbb{Z}). \end{aligned}$$

Since $H_1((B^4-\Delta)^d)$ and $H_1((S^3-K)^d)$ are isomorphic to the group $\mathbb{Z}$ generated by the lifted meridian $\widetilde{\mu}$ of K in S^3,

$$H_3((B^4-\Delta)^d, \partial(B^4-\Delta)^d) \cong H_2((S^3-K)^d \cup \widetilde{\Delta} \times \partial D^2) \cong \mathbb{Z}$$

and moreover the boundary map ∂_3 induced by the restriction map from $(B^4-\Delta)^d$ to $(S^3-K)^d$. Hence ∂_3 is an isomorphism and so this proves that $H_3((B^4-\Delta)^d) = 0$ and also $H_4((B^4-\Delta)^d) = 0$.

Considering that the Euler characteristic of $(B^4-\Delta)^d$ is $\chi(B^4-\Delta)^d = d \cdot \chi(B^4-\Delta)$ and $H_*(S^3-K) \to H_*(B^4-\Delta)$ is an isomorphism, we get $H_2((B^4-\Delta)^d) = 0$. □

Remark We may look at Example 4.6 to see infinitely many knots whose d–fold covers satisfy the condition in Lemma 4.2.

In the following Proposition, we will show that W in Definition 4.1 is a homology cobordism. The condition that K is a ribbon knot allows us to show that it is in fact a relative h–cobordism.

Proposition 4.3 *If K is a ribbon knot and the homology of d–fold cover $(S^3-K)^d$ of S^3-K, $H_1((S^3-K)^d) \cong \mathbb{Z}$ with $d \equiv \pm 1 \pmod m$ then there exists an h–cobordism W between $M_0 = X - \Sigma$ and $M_1 = X - \Sigma_K(m)$ rel ∂.*

Proof Keeping the previous notation in mind, let's denote $W = X \times I - (\Sigma \times I)_A(m)$, $M_0 = X - \Sigma$ and $M_1 = X - \Sigma_K(m)$. To show that W is H_*–cobordism rel ∂, we'll prove $H_*(W, M_1) = H_*(W, M_0) = 0$.

First, we need to describe W and M_1 as follows; if we take a neighborhood of the curve α in Σ as $S^1 \times B^3$ meeting Σ on $S^1 \times I$ then denoting the complement of $S^1 \times I$ in Σ by Σ_0, we may write

$$W = (X - S^1 \times B^3 - \Sigma_0) \times I \cup S^1 \times_{\tau^m} (B^3 \times I - A_+) \tag{2}$$

and

$$M_1 = (X - S^1 \times B^3 - \Sigma_0) \cup S^1 \times_{\tau_K^m} (B^3 - K_+) \tag{3}$$

Then considering the above description, the relative Mayer–Vietoris sequence shows

$$H_*(W, M_1) \cong H_*(S^1 \times_{\tau^m} (B^3 \times I - A_+), S^1 \times_{\tau_K^m} (B^3 - K_+)).$$

By the Alexander Duality, this relative homology group is same as

$$H_*(S^1 \times_{\tau^m} (B^3 \times I, A_+), S^1 \times_{\tau_K^m} (B^3, K_+))$$

which is trivial. Similarly, we can show that $H_*(W, M_0)$ is trivial as well.

A similar argument shows that $H_*(V, \partial M_0)$ is trivial and hence we have shown that W is a homology cobordism from M_0 to M_1 rel ∂. To assert that W is a relative h–cobordism, we need to show that $\pi_1(W) = \pi_1(X \times I - \nu(\Sigma \times I)_A(m)) \cong \mathbb{Z}/d$.

For simplicity let us denote $U = X - S^1 \times B^3 - \Sigma_0$ and $V = S^1 \times_{\tau^m} (B^3 - K_+)$ in the decomposition $(X - S^1 \times B^3 - \Sigma_0) \cup S^1 \times_{\tau^m} (B^3 - K_+)$ of $X - \Sigma_K(m)$. Then $U \cap V = S^1 \times (\partial B^3 - \{\text{two points}\})$. Denoting $V' = S^1 \times_{\tau^m} (B^3 \times I - A_+)$, we also rewrite

$$W = (X - S^1 \times B^3 - \Sigma_0) \times I \cup S^1 \times_{\tau^m} (B^3 \times I - A_+) = U \times I \cup V'.$$

Then the intersection $U \times I \cap V'$ is $S^1 \times (\partial B^3 - \{\text{ two points }\}) \times I = (U \cap V) \times I$.

Applying Van Kampen's theorem for these decompositions of M_1 and W, we have the two commutative diagrams:

$$\begin{tikzcd}
\pi_1(U \cap V) \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \pi_1(U) \arrow[d, "\psi_2"] \\
\pi_1(S^1 \times_{\tau^m} (B^3 - K_+)) \arrow[r, "\psi_2"'] & \pi_1(X - \Sigma_K(m))
\end{tikzcd}$$

and

$$\begin{tikzcd}
\pi_1((U \cap V) \times I) \arrow[r, "\varphi_1'"] \arrow[d, "\varphi_2'"'] & \pi_1(U \times I) \arrow[d, "\psi_2'"] \\
\pi_1(S^1 \times_{\tau^m} (B^3 \times I - A_+)) \arrow[r, "\psi_2'"'] & \pi_1(X \times I - (\Sigma \times I)_A(m))
\end{tikzcd}$$

Let

$$\begin{aligned}
&i_1\colon \pi_1(U \cap V) \to \pi_1((U \cap V) \times I)\\
&i_2\colon \pi_1(U) \to \pi_1(U \times I)\\
&i_3\colon \pi_1(S^1 \times_{\tau^m} (B^3 - K_+)) \to \pi_1(S^1 \times_{\tau^m} (B^3 \times I - A_+))
\end{aligned}$$

be the maps induced by inclusions. Then clearly i_1 and i_2 are isomorphisms. To show that i_3 is surjective, let's consider the fundamental group of mapping cylinders $S^1 \times_{\tau^m} (B^3 - K_+)$ and $S^1 \times_{\tau^m} (B^3 \times I - A_+)$. Then representing the element $[S^1]$ in the fundamental group as α', we present

$$\begin{aligned}
&\pi_1(S^1 \times_{\tau^m} (B^3 - K_+)) =\\
&\qquad \langle \pi_1(B^3 - K_+), \alpha' \mid \alpha'^{-1}\beta\alpha' = \tau^m_{K*}(\beta) \text{ for all } \beta \in \pi_1(B^3 - K_+)\rangle.
\end{aligned}$$

and

$$\begin{aligned}
&\pi_1(S^1 \times_{\tau^m} (B^3 \times I - A_+)) =\\
&\qquad \langle \pi_1(B^3 \times I - A_+), \alpha' \mid \alpha'^{-1}\beta'\alpha' = \tau^m_{*}(\beta') \text{ for all } \beta' \in \pi_1(B^3 \times I - A_+)\rangle.
\end{aligned}$$

Since K is a ribbon knot, $\pi_1(S^3 - K) \longrightarrow \pi_1(S^3 \times I - A)$ is surjective. So is i_3. Then by chasing the diagram, we have a surjective map

$$\pi_1(X - \Sigma_K(m)) \to \pi_1(X \times I - (\Sigma \times I)_A(m)).$$

By Proposition 3.3, $\pi_1(X - \Sigma_K(m)) = \mathbb{Z}/d$. Since W is an H_*–cobordism by the above argument, $H_1(X \times I - (\Sigma \times I)_A(m)) = \mathbb{Z}/d$ so that $\pi_1(X \times I - (\Sigma \times I)_A(m)) = \mathbb{Z}/d$.

Now let us prove that the inclusion $i\colon M_1 \longrightarrow W$ is a homotopy equivalence. The above work shows that the induced map $i_*\colon \pi_1 M_1 \longrightarrow \pi_1 W \cong \mathbb{Z}/d$ is an isomorphism. So, the d–fold covers W^d and ${M_1}^d$ of W and M_1 become universal covers and so we denote $\widetilde{W} = W^d$, $\widetilde{M}_1 = {M_1}^d$. Then we claim that the inclusion $\widetilde{M}_1 \to \widetilde{W}$ induces an isomorphism in homology. Considering the decompositions of W and M_1 in (2) and (3), we can express their d–fold covers as the d–fold covers of subcomponents associated to their inclusion maps to $H_1(W) \cong \mathbb{Z}/d$:

$$\begin{aligned}
\widetilde{W} &= (X \times I - (\Sigma \times I)_A(m))^d\\
&= ((X - S^1 \times B^3 - \Sigma_0) \times I)^d \cup (S^1 \times_{\tau^m} (B^3 \times I - A_+))^d
\end{aligned}$$

and

$$\widetilde{M}_1 = (X - \Sigma_K(m)))^d = (X - S^1 \times B^3 - \Sigma_0)^d \cup (S^1 \times_{\tau^m_K} (B^3 - K_+))^d.$$

In the inclusion-induced map $j\colon H_1(S^1 \times_{\tau^m} (B^3 \times I - A_+)) \longrightarrow H_1(W) \cong \mathbb{Z}/d$, from our choice of the curve α in Σ mentioned in the beginning of Section 3, we can easily check that in the Mayer–Vietoris sequence, the homology element $[S^1 \times pt \times 0]$ with $pt \in (\partial B^3 - \text{two points})$ maps under j to a trivial element in $H_1(W)$. The Mayer–Vietoris sequence for the decomposition of W follows.

$$\cdots \longrightarrow H_1(S^1 \times (\partial B^3 - \{\text{two points}\}) \times I) \xrightarrow{\varphi}$$
$$\xrightarrow{\varphi} H_1((X - S^1 \times B^3 - \Sigma_0) \times I) \oplus H_1(S^1 \times_{\tau^m} (B^3 \times I - A_+)) \xrightarrow{\psi}$$
$$\xrightarrow{\psi} H_1(W) \longrightarrow 0.$$

First we note the image of a generator $[S^1 \times pt \times 0] \in H_1(S^1 \times (\partial B^3 - \{\text{two points}\}) \times I)$ under φ is $(0, [S^1 \times pt \times 0]) \in H_1((X - S^1 \times B^3 - \Sigma_0) \times I) \oplus H_1(S^1 \times_{\tau^m} (B^3 \times I - A_+))$ since the pushed-off curve of α along a trivialization is zero in $H_1((X - S^1 \times B^3 - \Sigma_0) \cong H_1(X - \Sigma)$ by adjusting the framing of the curve α.

So, since $(0, [S^1 \times pt \times 0])$ is in the kernel of ψ, $[S^1 \times pt \times 0]$ maps to the trivial element in $H_1(W)$. Then we know the d–fold cover of $S^1 \times_{\tau^m} (B^3 \times I - A_+)$ has the form

$$S^1 \times_{\widetilde{\tau}^m} (B^3 \times I - A_+)^d$$

for a proper lifted map $\widetilde{\tau}^m$ of τ^m and by the same reason, the d–fold cover of

$$S^1 \times_{\tau_K^m} (B^3 - K_+)$$

is also of the form

$$S^1 \times_{\widetilde{\tau}_K^m} (B^3 - K_+)^d$$

for some lift $\widetilde{\tau}_K^m$ of τ_K^m.

Then we have a simple form of the relative homology of the pair (W, M_1),

$$H_*(\widetilde{W}, \widetilde{M}_1) \cong H_*(S^1 \times_{\widetilde{\tau}^m} (B^3 \times I - A_+)^d, S^1 \times_{\widetilde{\tau}_K^m} (B^3 - K_+)^d).$$

Since K is a ribbon knot and $H_1((S^3 - K)^d) \cong \mathbb{Z}$, it follows by Lemma 4.2 that $H_*\big((B^3 \times I - A_+)^d, (B^3 - K_+)^d\big) = 0$. So, the homology $H_*(\widetilde{W}, \widetilde{M}_1)$ is trivial. By the Whitehead theorem, we get $\pi_n \widetilde{M}_1 \cong \pi_n \widetilde{W}$ for $n > 1$. Since $\pi_n \widetilde{M}_1 \cong \pi_n M_1$ and $\pi_n \widetilde{W} \cong \pi_n W$, it follows that $i_*\colon \pi_n M_1 \to \pi_n W$ is an isomorphism. Therefore, again by Whitehead's theorem, $i\colon M_1 \longrightarrow W$ is a homotopy equivalence. $\square$

Now we need to recall the definition of torsion, as given in [11] or [15] to show the Whitehead torsion of the pair (W, M_0) constructed above is zero.

Let Λ be an associative ring with unit such that for any $r \neq s \in \mathbb{N}$, Λ^r and Λ^s are not isomorphic as Λ–modules. Consider an acyclic chain complex C of length m over Λ whose chain groups are finite free Λ–modules with a preferred basis c_i for each chain complex C_i. Then the torsion of the chain complex C — written $\tau(C)$ — is defined as follows.

Let $GL(\Lambda) = \bigcup_{n \geq 0} GL(n, \Lambda)$ be the infinite general group. The torsion $\tau(C)$ will be an element of the abelianization of $GL(\Lambda)$, denoted by $K_1(\Lambda)$. Pick ordered bases b_i of $B_i = \operatorname{Im} \partial_i$ and combine them to bases $b_i b_{i-1}$ of C_i. For the distinguished basis c_i of C_i, let $(b_i b_{i-1}/c_i)$ is the transition matrix over Λ. Denoting the corresponding element of $K_1(\Lambda)$ by $[b_i b_{i-1}/c_i]$, define the torsion

$$\tau(C) = \prod_{i=0}^{m} [b_i b_{i-1}/c_i]^{(-1)^{i+1}} \in K_1(\Lambda).$$

In particular, if (K, L) is a pair of finite, connected CW complexes such that L is a deformation retract of K then consider the universal covering complexes $\widetilde{K} \supset \widetilde{L}$ of K and L. Let's denote π by the fundamental group of K. Then we obtain an acyclic free chain $\mathbb{Z}[\pi]$–complex $C(\widetilde{K}, \widetilde{L})$. So we have a well defined torsion $\tau = \tau(K, L)$ in the Whitehead group $\mathrm{Wh}(\pi) = K_1(\mathbb{Z}[\pi])/\pm\pi$, the so-called 'Whitehead torsion'.

The h–cobordism we have constructed is built out of several pieces, and so our strategy is to compute the Whitehead torsion in terms of those pieces. The pieces may not be h–cobordisms, so they don't have a well-defined Whitehead torsion. However, they do have a more general kind of torsion, the Reidemeister–Franz torsion, which we briefly outline. It will turn out that the Reidemeister–Franz torsion of the pieces determines the Whitehead torsion of the h–cobordism. Moreover, the Reidemeister–Franz torsion satisfies gluing laws which will be able us to compute its value in terms of the pieces.

The 'Reidemeister–Franz torsion' is defined as follows. Consider the pair (K, L) of finite, connected CW-complexes but not requiring that L is a deformation retract of K. Then keeping the notation above, the cellular chain group $C_i(\widetilde{K}, \widetilde{L})$ is a free $\mathbb{Z}[\pi]$–module as before. Let Λ be an associative ring with unit with the above property. Given a ring homomorphism $\varphi\colon \mathbb{Z}[\pi] \longrightarrow \Lambda$, consider a free chain complex

$$C^{\varphi}(K, L) = \Lambda \otimes_{\varphi} C(\widetilde{K}, \widetilde{L}).$$

If C^{φ} is acyclic, the torsion corresponding the chain complex C^{φ} is well defined. We will denote $\tau^{\varphi}(K, L) \in K_1(\Lambda)/\pm\varphi(\pi)$. If Λ is a field then $K_1(\Lambda) = \Lambda^*$ so that $\tau^{\varphi}(K, L) \in \Lambda^*/\pm\varphi(\pi)$.

If the original complex C is acyclic then the new complex C^{φ} is also acyclic and so when the Whitehead torsion of (K, L) is defined, the Reidemeister torsion of (K, L)

is also defined associated to the identity homomorphism id: $\mathbb{Z}[\pi] \longrightarrow \mathbb{Z}[\pi]$. However the relation

$$\tau^{\varphi}(K,L) = \varphi_* \tau(K,L)$$

shows that if the Reidemeister torsion associated to the identity is trivial then the Whitehead torsion is zero. We also need to know some formulas to compute torsion. Suppose $K = K_1 \cup K_2$, $K_0 = K_1 \cap K_2$, $L = L_1 \cup L_2$, $L_0 = L_1 \cap L_2$ and that $i\colon L \longrightarrow K$ is the inclusion which is restricted to homotopy equivalences $i_\alpha\colon L_\alpha \longrightarrow K_\alpha$ (for $\alpha = 0, 1, 2$). Then i is a homotopy equivalence and we have a formula called the 'sum theorem' in Whitehead torsion (see [15])

$$\tau(K,L) = i_{1*}\tau(K_1, L_1) + i_{2*}\tau(K_2, L_2) - i_{0*}\tau(K_0, L_0).$$

Using the multiplicativity of the torsion and the Mayer–Vietoris sequence we obtain a similar one called the 'gluing formula' in the Reidemeister torsion (see [15]).

Given subcomplexes X_1 and X_2 of X such that $X = X_1 \cup X_2$ and $X_1 \cap X_2 = Y$, let $\varphi\colon \mathbb{Z}[H_1(X)] \longrightarrow \Lambda$ be a ring morphism where Λ is a ring as above. Let $i\colon \mathbb{Z}[H_1(Y)] \longrightarrow \mathbb{Z}[H_1(X)]$ and $i_\alpha\colon \mathbb{Z}[H_1(X_\alpha)] \longrightarrow \mathbb{Z}[H_1(X)]$ (for $\alpha = 1, 2$) denote the inclusion-induced morphisms. If $\tau^{\varphi\circ i}(Y) \neq 0$ then we have the gluing formula

$$\tau^{\varphi}(X) \cdot \tau^{\varphi\circ i}(Y) = \tau^{\varphi\circ i_1}(X_1) \cdot \tau^{\varphi\circ i_2}(X_2).$$

Now considering our situation, we have shown that W is a relative h–cobordism from M_0 to M_1 with $\pi_1(W) \cong \mathbb{Z}/d$ and so the Whitehead torsion $\tau(W, M_0) \in Wh(\mathbb{Z}/d)$ is defined. Recall that the decomposition of the pair

$$(W, M_0) = (X \times I - (\Sigma \times I)_A(m), X - \Sigma)$$

in (2) and (3) is

$$((X - S^1 \times B^3 - \Sigma_0) \times I \cup S^1 \times_{\tau^m} (B^3 \times I - A_+), X - S^1 \times B^3 - \Sigma_0 \cup S^1 \times (B^3 - I)).$$

If we rewrite this as

$$((X - S^1 \times B^3 - \Sigma_0) \times I, X - S^1 \times B^3 - \Sigma_0) \cup (S^1 \times_{\tau^m} (B^3 \times I - A_+), S^1 \times (B^3 - I)),$$

then we can observe that the Whitehead torsion of the first component pair

$$((X - S^1 \times B^3 - \Sigma_0) \times I, X - S^1 \times B^3 - \Sigma_0)$$

is zero and so we would like to attempt to use the sum theorem for this decomposition. But in the second pair, $S^1 \times_{\tau^m} (B^3 \times I - A_+)$ is just a homology cobordism which means $S^1 \times (B^3 - I)$ may not be a deformation retract of $S^1 \times_{\tau^m} (B^3 \times I - A_+)$. Then the Whitehead torsion $\tau(S^1 \times_{\tau^m} (B^3 \times I - A_+), S^1 \times (B^3 - I))$ is not defined and thus we can not apply the sum theorem in order to show the Whitehead torsion $\tau(W, M_0) = 0$.

But we will show later that $\tau(S^1 \times_{\tau^m} (B^3 \times I - A_+), S^1 \times (B^3 - I))$ is well defined under an additional assumption to make the complex of the d–fold cover of the pair, $C((S^1 \times_{\tau^m} (B^3 \times I - A_+))^d, (S^1 \times (B^3 - I))^d)$, acyclic with $\mathbb{Z}[\mathbb{Z}/d]$ coefficient. So instead of computing the Whitehead torsion, we will show that the Reidemeister torsion $\tau^{\mathrm{id}}(W, M_0)$, denoted simply by $\tau(W, M_0)$, according to the coefficient $\mathbb{Z}[\mathbb{Z}/d]$ is trivial. Applying the gluing formula to the above decomposition instead of the sum theorem, we can obtain a simpler method to compute the Reidemeister torsion for the pair (W, M_0).

Now we first need to consider the torsion of certain fibration over a circle with a homologically trivial fiber.

A relative fiber bundle

$$(F, F_0) \hookrightarrow (X, Y) \xrightarrow{\pi} S^1$$

means that $F \hookrightarrow X \xrightarrow{\pi} S^1$ is a fiber bundle with a trivialization $\{\varphi_\alpha, U_\alpha\}$ satisfying that for an open cover $U_\alpha \subset S^1$, $(\pi^{-1}(U), Y \cap \pi^{-1}(U)) \cong U \times (F, F_0)$ and the diagram

$$\begin{tikzcd}
(\pi^{-1}(U), Y \cap \pi^{-1}(U)) \arrow[rr, "\varphi_\alpha"] \arrow[dr] & & U \times (F, F_0) \arrow[dl] \\
& U &
\end{tikzcd}$$

commutes. We will now prove the following result.

Proposition 4.4 *Let $(F, F_0) \hookrightarrow (X, Y) \longrightarrow S^1$ be a smooth, relative fiber bundle over S^1 such that the fiber pair (F, F_0) is homologically trivial. Suppose that G is a group and ρ: $H_1(X) \longrightarrow G$ is a group homomorphism such that the image under ρ of the homology class $[S^1] \in H_1(X)$ of the base space in the fibration has finite order in G. Let $(\widetilde{F}, \widetilde{F_0})$ be the cover of (F, F_0) associated to the homomorphism*

$$H_1(F) \hookrightarrow H_1(X) \xrightarrow{\rho} G$$

and denote again by ρ the induced map $\mathbb{Z}[H_1(X)] \longrightarrow \mathbb{Z}[G]$. If the cover $(\widetilde{F}, \widetilde{F_0})$ is homologically trivial, that is $H_*(F, F_0; \mathbb{Z}[G]) = 0$ then the torsion $\tau^\rho(X, Y) \in K_1(\mathbb{Z}[G])/\pm G$ is trivial.

Proof We may assume that X is a mapping torus $X = S^1 \times_\varphi F$ with the monodromy map φ of the fibration. Let $(\widetilde{X}, \widetilde{Y})$ be the cover of (X, Y) associated to ρ. Then $\widetilde{X}$ is also a mapping torus since the homology image $\rho([S^1])$ is of finite order in G. So, let us say $\widetilde{X} = S^1 \times_{\widetilde{\varphi}} \widetilde{F}$ where $\widetilde{F}$ is the cover associated to $H_1(F) \hookrightarrow H_1(X) \xrightarrow{\rho} G$

and $\widetilde{\varphi}$ is a lift of φ in $\widetilde{X}$. Similarly, we also say $\widetilde{Y} = S^1 \times_{\widetilde{\varphi}} \widetilde{F}_0$. Considering the Wang exact homology sequence and the Five Lemma we have

$$\begin{array}{ccccccccc}
H_*(\widetilde{F_0}) & \xrightarrow{\widetilde{\varphi}_*^{-1}} & H_*(\widetilde{F_0}) & \longrightarrow & H_*(S^1 \times_{\widetilde{\varphi}} \widetilde{F_0}) & \longrightarrow & H_{*-1}(\widetilde{F_0}) & \xrightarrow{\widetilde{\varphi}_*^{-1}} & H_{*-1}(\widetilde{F_0}) \\
\cong\downarrow & & \cong\downarrow & & \downarrow & & \cong\downarrow & & \cong\downarrow \\
H_*(\widetilde{F}) & \xrightarrow{\widetilde{\varphi}_*^{-1}} & H_*(\widetilde{F}) & \longrightarrow & H_*(S^1 \times_{\widetilde{\varphi}} \widetilde{F}) & \longrightarrow & H_{*-1}(\widetilde{F}) & \xrightarrow{\widetilde{\varphi}_*^{-1}} & H_{*-1}(\widetilde{F})
\end{array}$$

and we get an acyclic complex $C_*(S^1 \times_{\widetilde{\varphi}} \widetilde{F}, S^1 \times_{\widetilde{\varphi}} \widetilde{F_0})$ since $H_*(\widetilde{F_0}) \longrightarrow H_*(\widetilde{F})$ is an isomorphism. Thus, the associated torsion $\tau^\rho(X, Y)$ is defined.

Now we consider the Mayer–Vietoris sequence for $(\widetilde{X}, \widetilde{Y}) = (S^1 \times_{\widetilde{\varphi}} \widetilde{F}, S^1 \times_{\widetilde{\varphi}} \widetilde{F_0})$. Let us consider closed manifold pairs $(X_1, Y_1) = ([0, \frac{1}{2}] \times \widetilde{F}, [0, \frac{1}{2}] \times \widetilde{F_0})$ and $(X_2, Y_2) = ([\frac{1}{2}, 1] \times \widetilde{F}, [\frac{1}{2}, 1] \times \widetilde{F_0})$. Define a map f of a subspace $A := \{0\} \times \widetilde{F} \cup \{\frac{1}{2}\} \times \widetilde{F}$ of X_1 into X_2 by

$$f|_{\{0\}\times\widetilde{F}} = \widetilde{\varphi} \times \{1\},\ f|_{\{1/2\}\times\widetilde{F}} = 1_{\{1/2\}\times\widetilde{F}}.$$

Then letting $B := \{0\} \times \widetilde{F_0} \cup \{\frac{1}{2}\} \times \widetilde{F_0} \subset A$, we can consider $(S^1 \times_{\widetilde{\varphi}} \widetilde{F}, S^1 \times_{\widetilde{\varphi}} \widetilde{F_0})$ as the adjunction space $(X_1 \cup_f X_2, Y_1 \cup_f Y_2)$ of the system $(X_1, Y_1) \supset (A, B) \xrightarrow{f} (X_2, Y_2)$. There is a short exact sequence

$$0 \longrightarrow C_*(X_1 \cap X_2, Y_1 \cap Y_2) \longrightarrow C_*(X_1, Y_1) \oplus C_*(X_2, Y_2) \\ \longrightarrow C_*(X_1 \cup_f X_2, Y_1 \cup_f Y_2) \longrightarrow 0.$$

If we rewrite this then we have

$$0 \longrightarrow C_*(\widetilde{F}, \widetilde{F_0}) \oplus C_*(\widetilde{F}, \widetilde{F_0})) \\ \longrightarrow C_*([0, 1/2] \times \widetilde{F}, [0, 1/2] \times \widetilde{F_0}) \oplus C_*([1/2, 1] \times \widetilde{F}, [1/2, 1] \times \widetilde{F_0}) \\ \longrightarrow C_*(S^1 \times_{\widetilde{\varphi}} \widetilde{F}, S^1 \times_{\widetilde{\varphi}} \widetilde{F_0}) \longrightarrow 0.$$

If $(\widetilde{F}, \widetilde{F_0})$ is homologically trivial, it follows that if $j\colon \mathbb{Z}[H_1(F)] \longrightarrow \mathbb{Z}[H_1(X)]$ denotes the morphism induced by inclusion then the torsion $\tau^{\rho\circ j}(F, F_0)$ is defined. From the above short exact sequence and the multiplicativity of the torsion we deduce that

$$\tau^{\rho\circ j}(F, F_0) \cdot \tau^{\rho\circ j}(F, F_0) = (\tau^{\rho\circ j}(F, F_0) \cdot \tau^{\rho\circ j}(F, F_0)) \cdot \tau^{\rho}(S^1 \times_\varphi F, S^1 \times_\varphi F_0).$$

This implies that $\tau^\rho(S^1 \times_\varphi F, S^1 \times_\varphi F_0) = \tau^\rho(X, Y) \in K_1(\mathbb{Z}[G])/\pm G$ is trivial. □

Using the proposition above, we get topological equivalence classes of $(X, \Sigma_K(m))$ under the following condition.

Theorem 4.5 *If K is a ribbon knot and the homology of d–fold cover $(S^3-K)^d$ of S^3-K, $H_1((S^3-K)^d)\cong\mathbb{Z}$ with $d\equiv\pm1 \pmod m$ then (X,Σ) is pairwise homeomorphic to $(X,\Sigma_K(m))$.*

Proof Under these assumptions, we have a relative h–cobordism W from $M_0 = X-\Sigma$ to $M_1 = X-\Sigma_K(m)$ by Proposition 4.3. As we discussed before, in order to show the Whitehead torsion $\tau(W,M_0)=0\in Wh(\mathbb{Z}/d)$, it is sufficient to show that the Reidemeister torsion $\tau(W,M_0)\in Wh(\mathbb{Z}/d)$ associated to the identity map id: $\mathbb{Z}[\mathbb{Z}/d]\longrightarrow\mathbb{Z}[\mathbb{Z}/d]$ is trivial.

Consider the decomposition of the pair (W,M_0),

$$((X-S^1\times B^3-\Sigma_0)\times I, X-S^1\times B^3-\Sigma_0)\cup(S^1\times_{\tau^m}(B^3\times I-A_+), S^1\times(B^3-I)).$$

To apply the gluing formula of the Reidemeister torsion for this decomposition, we need to check the torsion of each component is defined.

First, the torsion $\tau((X-S^1\times B^3-\Sigma_0)\times I, X-S^1\times B^3-\Sigma_0)$ is clearly defined and trivial. To check the torsion of the second component, we will show the relative chain complex $C((S^1\times_{\tau^m}(B^3\times I-A_+))^d,(S^1\times(B^3-I))^d)$ of d–fold covers is acyclic.

The same argument in the proof of Proposition 4.3 shows that the d–fold cover

$$(S^1\times_{\tau^m}(B^3\times I-A_+))^d$$

associated to the inclusion-induced map

$$j\colon H_1(S^1\times_{\tau^m}(B^3\times I-A_+))\longrightarrow H_1(W)\cong\mathbb{Z}/d$$

is a mapping torus with the d–fold cover of $B^3\times I-A_+$ that is $S^1\times_{\tilde\tau^m}(B^3\times I-A_+)^d$. Similarly, the d–fold cover $(S^1\times_{\tau^m}(B^3-I))^d$ is $S^1\times_{\tilde\tau^m}(B^3-I)^d$.

Observing the proof of Lemma 4.2, we have an isomorphism $H_*((S^3-K)^d)\longrightarrow H_*((B^4-\Delta)^d)$ when K is a ribbon knot and $H_1((S^3-K)^d)\cong\mathbb{Z}$. In other words, $H_*((B^4-\Delta)^d,(S^3-K)^d)=0$. Excision argument shows that this is isomorphic to

$$H_*((B^3\times I-A_+)^d,(B^3-K_+)^d)=0\cong H_*((B^3\times I-A_+)^d,(B^3-I)^d).$$

This gives that

$$\begin{aligned}&H_*((S^1\times_{\tau^m}(B^3\times I-A_+))^d,(S^1\times_{\tau^m}(B^3-I))^d)\\&\quad= H_*(S^1\times_{\tilde\tau^m}(B^3\times I-A_+)^d, S^1\times_{\tilde\tau^m}(B^3-I)^d)=0.\end{aligned}$$

Then the torsion $\tau^j(S^1 \times_{\tau^m}(B^3 \times I - A_+), S^1 \times_{\tau^m}(B^3 - I))$ associated to the induced ring homomorphism $j\colon \mathbb{Z}[H_1(S^1 \times_{\tau^m}(B^3 \times I - A_+))] \longrightarrow \mathbb{Z}[H_1(W)] \cong \mathbb{Z}[\mathbb{Z}/d]$ is defined.

Now applying the gluing formula of the Reidemeister torsion for the decomposition of (W, M_0), we have

$$\begin{aligned}
&\tau(W, M_0) \cdot \tau(\partial(X - S^1 \times B^3 - \Sigma_0) \times I, \partial(X - S^1 \times B^3 - \Sigma_0)) \\
&\quad = \tau((X - S^1 \times B^3 - \Sigma_0) \times I, X - S^1 \times B^3 - \Sigma_0) \cdot \\
&\qquad\qquad \tau(S^1 \times_{\tau^m}(B^3 \times I - A_+), S^1 \times (B^3 - I)).
\end{aligned}$$

Hence,

$$\tau(W, M_0) = \tau(S^1 \times_{\tau^m}(B^3 \times I - A_+), S^1 \times (B^3 - I)).$$

To compute $\tau(S^1 \times_{\tau^m}(B^3 \times I - A_+), S^1 \times (B^3 - I))$, we note that

$$(B^3 \times I - A_+, B^3 - I) \hookrightarrow (S^1 \times_{\tau^m}(B^3 \times I - A_+), S^1 \times (B^3 - I)) \longrightarrow S^1$$

is a smooth fiber bundle over S^1 with the fiber $(B^3 \times I - A_+, B^3 - I)$. Clearly the fiber $(B^3 \times I - A_+, B^3 - I)$ is homologically trivial and by the above argument, the d–fold cover $((B^3 \times I - A_+)^d, (B^3 - I)^d)$ associated to j is also homologically trivial. Thus, by Proposition 4.4 the torsion $\tau(S^1 \times_{\tau^m}(B^3 \times I - A_+), S^1 \times (B^3 - I))$ is trivial and thus the Whitehead torsion $\tau(W, M_0) = 0$. Then by Freedman's work [6], the h–cobordism W is topologically trivial and so the complements $X - \Sigma$ and $X - \Sigma_K(m)$ are homeomorphic. The homeomorphism $\partial\nu(\Sigma) \longrightarrow \partial\nu(\Sigma_K(m))$ extends to a homeomorphism $(X, \Sigma) \longrightarrow (X, \Sigma_K(m))$. □

Example 4.6 Let's consider examples $(X, \Sigma_K(m))$ which are smoothly knotted but topologically standard. Let J be a torus knot $T_{p,q}$ in S^3 such that p and q are coprime positive integers. Then we have a ribbon knot $K = J\#{-J}$ with its Alexander polynomial $\Delta_K(t) = (\Delta_J(t))^2$ where

$$\Delta_J(t) = \frac{(1-t)(1-t^{pq})}{(1-t^p)(1-t^q)}.$$

Note that the d–fold cover of S^3 branched along the torus knot $J = T_{p,q}$ is the Brieskorn manifold $\Sigma(p,q,d)$, and that this manifold is a homology sphere if p,q and d are pairwise relatively prime. Since $(S^3, K)^d$ is $\Sigma(p,q,d)\#\Sigma(p,q,d)$, $(S^3, K)^d$ is an integral homology 3–sphere. We might obtain a direct proof for this by computing the order of $H_1((S^3, K)^d)$ of d–fold cover $(S^3, K)^d$ of S^3 branched over K. In fact, Fox [5] proved that

$$|H_1((S^3, K)^d)| = \prod_{i=0}^{d-1} \Delta_K(\zeta^i)$$

where ζ is a primitive dth root of unity. And it's easy to show that

$$\prod_{i=0}^{d-1} \Delta_K(\zeta^i) = 1.$$

So, we obtain a ribbon knot K with $\Delta_K(t) \neq 1$ and the d–fold branch cover $(S^3, K)^d$ is a homology 3–sphere when $(p, d) = 1$ and $(q, d) = 1$. Then by Theorem 3.4 and Theorem 4.5, we have infinitely many pairs $(X, \Sigma_K(m))$ which are smoothly knotted but not topologically.

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