On the moduli space of
positive Ricci curvature metrics on homotopy spheres

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We show that the moduli space of Ricci positive metrics on a certain family of homotopy spheres has infinitely many components.

1 Introduction

A fundamental problem in Riemannian geometry is to understand metrics which display certain types of curvature characteristics. Of particular importance among curvature characteristics are the various forms of positive curvature, such as positive sectional curvature, positive Ricci curvature and positive scalar curvature. Existence issues for positive scalar curvature metrics are reasonably well understood. The situation for positive Ricci and positive sectional curvature metrics is somewhat less clear.

For the most part the existence question, that is, whether or not a particular manifold or family of manifolds admits metrics of a given type, has been a primary focus of research. However, there is an equally intruiging secondary question. If a manifold or family of manifolds admit a given type of metric, how are such metrics distributed among all possible Riemannian metrics on these objects? For example are they rare or common?

A particularly natural question concerns connectedness: given two metrics of a given type on a particular manifold, is it possible to smoothly deform one metric into the other *through metrics of the same type*? If the answer to this question is always yes, this shows that the set of metrics having the given type form a connected subset of the space of all metrics (at least if the latter is equipped with the smooth topology). Thus it makes sense to ask about the connectedness of the subset of "nice" metrics in whatever context one is working. If this set is not connected, how many connected components are there? Of course, one could also seek to uncover more subtle aspects of the topology, such as the fundamental or higher homotopy groups. For some recent work in this direction, see Botvinnik, Hanke, Schick and Walsh [4].

In this paper we will be interested in the subset of metrics with positive scalar and positive Ricci curvature. If a given manifold admits a positive Ricci curvature metric, then this metric also has positive scalar curvature. It makes sense then to ask about the *relative* distribution of the positive scalar versus positive Ricci curvature metrics. For example, does every connected component of positive scalar curvature metrics (assuming there is more than one) contain a positive Ricci curvature metric?

Instead of working with spaces of metrics, we might instead choose to work with *moduli spaces* of metrics. For any smooth manifold M, we can consider the group $\mathrm{Diff}(M)$ of all diffeomorphisms of M. Let $\mathrm{Riem}(M)$ denote the space of all Riemannian metrics on M with the smooth topology. Then $\mathrm{Diff}(M)$ acts on $\mathrm{Riem}(M)$ by pulling back metrics. The quotient of $\mathrm{Riem}(M)$ by this action is a moduli space of metrics. Denoting the space of Ricci positive metrics on M by $\mathrm{Riem}^{\mathrm{Ric}+}(M)$ and the space of positive scalar metrics by $\mathrm{Riem}^{\mathrm{scal}+}(M)$, we can also form the moduli spaces of positive Ricci and positive scalar curvature metrics in the same way: $\mathrm{Riem}^{\mathrm{Ric}+}(M)/\mathrm{Diff}(M)$, $\mathrm{Riem}^{\mathrm{scal}+}(M)/\mathrm{Diff}(M)$. Note that a lower bound on the number of components in a moduli space is also a lower bound on the number of components for the corresponding space of metrics.

Possibly the earliest result on the connectedness of spaces of metrics displaying some form of positive curvature was due to Hitchin [9]. He proved that if a closed spin manifold admits a positive scalar curvature metric and has dimension a multiple of eight, then its space of positive scalar curvature metrics has more than one path component and has nontrivial fundamental group.

A few years later, R Carr [7] proved that the space of positive scalar curvature metrics on the spheres S^{4k-1} has *infinitely* many path-components. This result demonstrated for the first time how complex the distribution of positive scalar curvature metrics can be. Of course spheres also admit metrics with positive Ricci and positive sectional curvature. Thus it is natural to ask whether similar results hold if scalar curvature is replaced by Ricci or sectional curvature in Carr's result.

The objects we will study in this paper are homotopy spheres. These are smooth manifolds having the same homotopy type as some (standard) sphere S^n. Homotopy spheres are either standard or *exotic*, that is, homeomorphic but not diffeomorphic to the standard sphere. The set of all homotopy spheres has a particularly nice and very large subfamily: the homotopy spheres which bound parallelisable manifolds. The author showed in [18] that these manifolds all admit Ricci positive metrics. (See also Boyer, Galicki and Nakamaye [5]. For a general reference about the construction and curvature of homotopy spheres, see Joachim and Wraith [10].) Our main result is as follows:

Theorem A   *Let Σ^{4k-1} be any homotopy sphere bounding a parallelisable manifold, where $k > 1$. Let $\mathrm{Riem}(\Sigma)$ denote the space of all smooth Riemannian metrics on Σ equipped with the C^∞ topology, and denote by $\mathrm{Riem}^{\mathrm{Ric}+}(\Sigma)$ the subset consisting of all Ricci positive metrics. Give $\mathrm{Riem}^{\mathrm{Ric}+}(\Sigma)$ the induced topology from $\mathrm{Riem}(\Sigma)$. Then the moduli space $\mathrm{Riem}^{\mathrm{Ric}+}(\Sigma)/\mathrm{Diff}(\Sigma)$ has infinitely many components.*

As an immediate corollary we see that $\mathrm{Riem}^{\mathrm{Ric}+}(\Sigma)$ must also have infinitely many components. Note that it does *not* follow from this that either $\mathrm{Riem}^{\mathrm{scal}+}(\Sigma)$ or the moduli space $\mathrm{Riem}^{\mathrm{scal}+}(\Sigma)/\mathrm{Diff}(\Sigma)$ must have infinitely many components, as to deduce this would require further information about the distribution of positive Ricci curvature metrics among the positive scalar curvature metrics. In fact, both of these positive scalar curvature spaces *do* have infinitely many components. For example it is not difficult to see (though not pointed out in [7]) that Carr's arguments apply equally well – with the same conclusion – to any exotic sphere in these dimensions which bounds a parallelisable manifold.

Kreck and Stolz [13] strengthened Carr's result considerably. They showed that for any closed spin manifold M^{4k-1} ($k > 1$) which admits a positive scalar curvature metric and for which $H^1(M; \mathbb{Z}/2) = 0$, the moduli space of positive scalar curvature metrics has infinitely many components. They then used this result to prove that the moduli space of Ricci positive metrics on certain homogeneous 7–manifolds with $SU(3) \times SU(2) \times U(1)$ symmetry has infinitely many components. To the best of our knowledge, apart from Theorem A above, these are the only examples of Ricci positive manifolds in the literature for which the moduli space of Ricci positive metrics has been shown to have infinitely many components. As the Kreck–Stolz examples all occur in dimension 7, Theorem A also demonstrates for the first time that the following corollary is true:

Theorem B   *There are manifolds in infinitely many dimensions for which the moduli space of Ricci positive metrics has infinitely many components.*

Other results about spaces or moduli spaces of metrics displaying some kind of positive curvature are few and far between in the literature. Shortly after [13], Botvinnik and Gilkey [3] showed that in some cases, the space of positive scalar curvature metrics for odd dimensional spin manifolds with nontrivial finite fundamental group can also have infinitely many path components. More recent results due to Kapovitch, Petrunin and Tuschmann [11] show that there are examples of noncompact manifolds in every dimension ≥ 22 for which the moduli space of nonnegatively curved metrics has infinitely many components. Let us also mention that in [13], examples are given of Wallach spaces for which the moduli space of nonnegatively curved metrics is not

connected. It is observed however in [11] that the $SU(3) \times SU(2) \times U(1)$–homogeneous spaces studied in [13] actually have moduli spaces of nonnegatively curved metrics with infinitely many components. Most recently, in [4] it is shown that for many degrees in a stable range, the homotopy groups of the moduli space of positive scalar curvature metrics on S^n and other manifolds are nontrivial.

It is interesting to compare and contrast the techniques of Carr and examples of Kreck and Stolz with those of this paper.

Carr regards the homotopy spheres in question as the boundaries of "plumbed" manifolds. Plumbing is a construction whereby disk bundles are glued together to create a new manifold with boundary. The basic idea is as follows. Consider two disc bundles, and for each bundle choose a locally trivial neighbourhood over a disc in the base. We glue the bundles by identifying these neighbourhoods. To do this we use a diffeomorphism which identifies the base disc of one neighbourhood with the fibre disc of the other, and vice versa. (Of course, this can only work if the dimensions match.) The resulting object is the so-called plumbing of the disc bundles, and can be made differentiable by simply straightening out the angles. By plumbing certain sequences of disc bundles over spheres together, one can construct every homotopy sphere which bounds a parallelisable manifold (and some others too). The key point is that a given homotopy sphere of this type can be realised as the boundary of infinitely many different plumbed manifolds. (For the basics of plumbing and applications to homotopy sphere construction see Browder [6, V.2] or Wraith [18].)

Carr proves that for any regular neighbourhood of the union of base spheres in the plumbing construction, there is a metric of positive scalar curvature which is a product near the boundary, and thus the boundary itself has positive scalar curvature. If two boundary metrics arising in this way from different plumbing descriptions of the same homotopy sphere belong to the same path component of positive scalar curvature metrics, it is not difficult to see that the two plumbed manifolds can be glued together to give a smooth positive scalar curvature manifold. However, the existence of such a metric on this new object can be ruled out using simple index theory. Thus the metrics arising in this way from different plumbing descriptions must all lie in different components of the space of positive scalar curvature metrics.

The same approach fails to work for positive Ricci curvature as the boundary of the regular neighbourhood will almost certainly not have positive Ricci curvature. Instead, one needs to *begin* with a positive Ricci curvature metric on the boundary, and hope to extend it inwards over the whole plumbed manifold in (at least) a positive scalar fashion so that a neighbourhood of the boundary is a product. If one could do this, then the rest of Carr's argument would still apply, showing that the space of positive Ricci

curvature metrics has infinitely many components. However, such a construction must necessarily depend on the precise form of the Ricci positive metric. A significant part of this paper is dedicated to performing this kind of metric extension.

Of course, our main result concerns moduli spaces of Ricci positive metrics, as opposed to just the set of such metrics. To obtain results about moduli spaces we use the Kreck–Stolz techniques, for which the existence of our metric extension is in fact a prerequisite. Kreck and Stolz define a $\mathbb{Q}$–valued quantity s for $(4n-1)$–dimensional spin manifolds with positive scalar curvature and vanishing real Pontrjagin classes, with the property that $|s|$ is an invariant of the moduli space of positive scalar curvature metrics. The computation of this invariant is difficult in general. However, for the examples we are interested in, we exploit the fact that our homotopy spheres bound parallelisable manifold to simplify the calculation. In a similar fashion to the Carr approach, a prerequisite for this method to work is that the boundary metric should be extendable over the interior of the plumbed manifold in (at least) a positive scalar fashion, in such a way that neighbourhood of the boundary is a product. Again, different plumbing descriptions of the same homotopy sphere lead to metrics with different s–invariants. Thus the metrics must all lie in different components of the moduli space of positive scalar curvature metrics, and hence in different components of the moduli space of positive Ricci curvature metrics.

There is a problem, however. The (reasonably explicit) Ricci positive metrics on homotopy spheres constructed in [18; 19] do not easily extend over the bounding manifold in the desired manner. Our solution is to deform these boundary metrics within positive scalar curvature to a metric which admits the required extension.

The metric extensions are constructed in pieces, corresponding to the various disc bundles being plumbed together. On the boundary, each of these plumbings is equivalent to a surgery, and thus each plumbing can be viewed as the addition of a handlebody. The metrics on our homotopy spheres are constructed directly on the boundary of this construction, surgery by surgery. Extending the metric in the appropriate way over the corresponding handlebody is a delicate issue, which we will explore in detail.

This paper is organised as follows. In Section 2 we discuss the construction of homotopy spheres as boundaries of plumbed manifolds and introduce the s–invariant of Kreck and Stolz. In Section 3 we look at surgery on Ricci positive manifolds. In Section 4 we show how to deform a positive Ricci curvature metric on an exotic sphere through positive scalar curvature metrics into a form suitable for extension across the corresponding bounding manifold. This extension is addressed in Section 5. In Section 6 we consider paths of positive scalar curvature metrics, concluding with the proof of Theorem A.

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2 Homotopy spheres, plumbing and the s-invariant

The diffeomorphism classes of homotopy spheres bounding parallelisable manifolds of dimension n form an abelian group under the connected sum operation. This group is denoted bP_n. Kervaire and Milnor [12] showed that for n odd, $bP_n = 0$, for $n = 4k + 2$, bP_n is either 0 or $\mathbb{Z}_2$, and for $n = 4k$, bP_n is cyclic.

Let $\Sigma \in bP_{4k}$ for some $k > 1$. It was shown in [18] that Σ arises as the boundary of a plumbed manifold: in fact Σ arises as the boundary of a plumbed manifold in infinitely many different ways. For this plumbed manifold, the disc-bundles involved are all D^{2k}-bundles over S^{2k}. To understand the details of the construction, we can represent the plumbing by a schematic diagram in the following way. For each bundle we draw a dot, which should be labelled with the appropriate element of the group $\pi_{2k}BSO(2k)$ which classifies such bundles. Each time we plumb two of the bundles together, join the appropriate dots with a line. In this way we construct a graph. Consider the E_8 graph as a plumbing graph, where each dot represent the tangent disc bundle over S^{2k}.
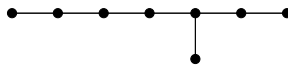

Figure 1: The E_8 graph

The boundary of the manifold resulting from this plumbing can be shown to represent a generator of bP_{4k} (see [6]). We obtain the n-th multiple of this generator by starting with the E_8 graph and successively attaching $n-1$ copies of the following "building block" to the right-hand end.
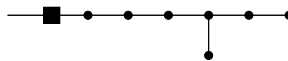

Figure 2: The "building block"

Note that all circular dots here again represent tangent disc bundles. If $\tau \in \pi_{2k}BSO(2k)$ represents the tangent disc bundle, then the square dot represents the bundle 3τ. Plumbing according to the resulting graph yields the desired homotopy sphere as its boundary. For example, twice a generator (2τ) is obtained from the following picture.
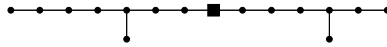
Figure 3: $2\tau \in \pi_{2k}BSO(2k)$

Note that for each $\Sigma \in bP_{4k}$, [18] establishes the existence (as opposed to the form) of corresponding simply connected plumbing diagrams with vertices representing D^{2k}–bundles over S^{2k}. It is easily checked (for example following [6, V.2]) that the graphs shown above have the required properties. However, to the best of the author's knowledge, the explicit form of these graphs has not appeared in the literature before.

As bP_n is a cyclic group, it is clear that we can construct the homotopy sphere representing q times the generator corresponding to the E_8 graph by plumbing according to the graph consisting of E_8 and $(q - 1 + p|bP_{4k}|)$ copies of the "building block", for any integer $p \geq 0$. In other words, for any fixed homotopy sphere $\Sigma \in bP_{4k}$, we can construct infinitely many different bounding manifolds via the plumbing construction. The technique of plumbing was originally introduced as a means of constructing manifolds with a given intersection form on middle-dimension cohomology. In our situation, it is not difficult to see that for a homotopy sphere $\Sigma \in bP_{4k}$, the bounding manifold discussed above has unimodular intersection form and signature $8(p|bP_{4k}| + q)$ (see [18, Propositions 1.4 and 1.5]). Let us denote such a plumbed bounding manifold by X_p.

In [13], Kreck and Stolz introduced the s–invariant of a spin manifold M^{4k-1} which admits a positive scalar curvature metric g and has vanishing real Pontrjagin classes. This is a $\mathbb{Q}$–valued invariant with the property that $|s|$ is constant on each component of the moduli space of positive scalar curvature metrics, provided $H^1(M; \mathbb{Z}/2) = 0$ (see [13, Proposition 2.14]). The actual definition of the s–invariant is somewhat complicated and we refer the reader to [13] for details. However, for the manifolds under consideration in this paper the s–invariant takes a simple form, as described in the following Lemma.

Lemma 2.1 *With (M, g) as above, suppose that W is a parallelisable manifold with $\partial W = M$, equipped with a positive scalar curvature metric $\overline{g}$ which takes the form $dt^2 + g$ in a neighbourhood of the boundary. Then the s–invariant of (M, g) is given by*

$$s(M, g) = \frac{1}{2^{2k+1}(2^{2k-1} - 1)} \sigma(W),$$

where $\sigma(W)$ is the signature of W.

Proof The parallelisability of W means in particular that W is spin, so the following formula [13, Proposition 2.13] applies:

$$s(M, g) = \text{ind } D^+(W, \overline{g}) + t(W).$$

The positive scalar curvature metric on W means that the term ind $D^+(W, \overline{g})$ vanishes (see [13, Remark 2.2(ii)]). The parallelisability of W also means that the Pontrjagin classes $p_1(W), \ldots, p_n(W)$ vanish, and by [13, Equation (2.11)] this means that the term $t(W)$ is equal to $1/(2^{2k+1}(2^{2k-1} - 1))\sigma(W)$. □

Note that the homotopy sphere $\Sigma \in bP_{4k}$ is a spin manifold with vanishing real Pontrjagin classes, and that the bounding manifold X_p is shown to be parallelisable in [18].

3 Surgery

Surgery is an operation which alters the topology of a manifold in a precise way, and was introduced as a tool for use in problems concerning the classification of manifolds. A good introduction to the subject is Ranicki [15]. We describe the basic idea. Begin with a manifold M^{n+m} and a smooth embedding $\iota_0 \colon S^{n-1} \to M^{n+m}$. Assume the normal bundle of $\iota_0(S^{n-1})$ is trivial. This means we can extend ι_0 to an embedding $\iota \colon S^{n-1} \times D^{m+1} \longrightarrow M$. Performing surgery involves cutting something out – the interior of the image of the embedding ι – and stitching something in, namely a copy of $D^n \times S^m$. Note that the boundary of $S^{n-1} \times D^{m+1}$ is $S^n \times S^{m-1}$, which agrees with the boundary of $D^n \times S^m$. This gluing-in is performed in the obvious manner determined by ι. We say that the dimension of such a surgery is $n-1$, and the codimension is $m+1$. There is a subtlety in the above construction however: the extended embedding ι is not unique. Viewing $S^{n-1} \times D^{m+1}$ as a bundle with base S^{n-1} and fibre D^{m+1}, we can compose ι with a bundle isomorphism $S^{n-1} \times D^{m+1} \to S^{n-1} \times D^{m+1}$ to obtain a new embedding ϕ. In general, performing surgery using this new embedding will result in a manifold which is different from that obtained using ι. Therefore when performing surgery, it is important to specify the *trivialisation of the normal bundle*, that is, the choice of embedding $\phi \colon S^{n-1} \times D^{m+1} \to M^{n+m}$ being used.

As noted in Section 2, any homotopy sphere of dimension $4k - 1$ which bounds a parallelisable manifold arises as the boundary of a plumbed manifold, where the disc bundles being plumbed are D^{2k}–bundles over S^{2k}. (In fact, the same is true for homotopy spheres in dimensions $4k + 1$ which bound a parallelisable manifold; see Lawson and Michelsohn [14, page 162].)

It is an elementary observation that the effect on the boundary of plumbing two such bundles E_1 and E_2 together is equivalent to performing surgery on a fibre $(2k-1)$–sphere in ∂E_1, using a trivialisation of the normal bundle determined by E_2 (see [18, page 645]). Thus the homotopy spheres described in Section 2 can be constructed,

starting with an S^{2k-1}–bundle over S^{2k}, by performing successive surgeries. After performing a single surgery, the global S^{2k-1}–bundle structure will be destroyed. However, *locally* this structure will be retained. Each of the successive surgeries required to construct the homotopy sphere is a surgery on a "fibre" S^{2k-1} arising from such a local bundle structure. In fact, this will always mean performing surgeries on the site of previous surgeries in an iterative manner.

An approach to constructing Ricci positive metrics on homotopy spheres is to look for circumstances under which the surgery operation can preserve Ricci positivity. The first major result in this direction was due to Sha and Yang [16]. They assume that a tubular neighbourhood of the sphere on which surgery is to be performed is isometric to the product of a round metric on the sphere and a round normal disk. Using this isometry as the trivialisation of the normal bundle, they show that the manifold resulting from the surgery has positive Ricci curvature provided the dimension of the surgery is at least one, the codimension at least three, and the ratio of the radius of the sphere to the radius of the disk is sufficiently small. This result was developed further in [19] where it was shown that under the same metric assumptions, Ricci positivity can be preserved if different trivialisations of the normal bundle are used. This generalisation comes at the expense of tighter dimensional requirements, but being able to work with different trivialisations of the normal bundle is vital for applying the results to the construction of homotopy spheres.

The proof of these Ricci positive surgery results involves the construction of a Ricci positive metric on $D^n \times S^m$ which glues smoothly with the metric on $M - \phi(S^{n-1} \times D^{m+1})$ when the surgery is completed. The metric on $D^n \times S^m$ is a *submersion metric*. The basic existence result for submersion metrics is due to Vilms [17]; see also Besse [2, Theorem 9.59].

Construction 3.1 Let G be a Lie group and $P \to B$ a principal G–bundle. Let F be a manifold with a smooth (left) G–action and let

$$E = P \times_G F \longrightarrow B$$

be the associated fibre bundle over B with fibre F. Suppose we have

(i) a metric $\hat{g}$ on F which is equivariant with respect to the G–action;
(ii) a metric $\check{g}$ on B;
(iii) a principal connection ∇ on P;
(iv) a smooth function $\mu \colon B \to \mathbb{R}^+$.

Then there exists a unique submersion metric on E with horizontal distribution is associated to the connection, and given $b \in B$, (F_b, g_b) is (noncanonically) isometric to $(F, \mu^2 \hat{g})$.

In our situation, $E = D^n \times S^m$, which we will view as an S^m–bundle over D^m, associated to the trivial $SO(m+1)$–principal bundle over D^n. The metric $\hat{g}$ on S^m is just the round metric ds_m^2. The metric $\check{g}$ is a warped product metric $dr^2 + h^2(r)\, ds_{n-1}^2$ where r is the radial parameter of the disc, and $h(r)$ is a smooth function. The fibre-scaling function $\mu \colon B \to \mathbb{R}^+$ will be a function $f(r)$. We will denote the resulting submersion metric on E by $g(f, h, \nabla)$. The choice of connection ∇ will depend on the particular normal bundle trivialisation being used in the surgery. In the case studied by Sha and Yang, a flat connection sufficed, however, for homotopy sphere construction nonflat connections must be considered. Note that if ∇ is flat, $g(f, h, \nabla)$ is simply isometric to the warped product metric $dr^2 + h^2(r)\, ds_{n-1}^2 + ds_m^2$. We will denote such a warped product by $g(f, h)$. The principal connection ∇ induces a horizontal distribution in E. We will view ∇ from now on as such a horizontal distribution, to avoid having to invoke the principal bundle associated to E.

In the remainder of this section, we summarise the Ricci positive surgery results of [19]. The statements we give below are slightly different to those presented in [19], so as to emphasize the metric features which we will need in the sequel.

Theorem 3.2 Given $E = D^n \times S^m$, a constant $\Delta \in (0, 1)$ and a choice of connection ∇, there is a constant $\rho_0 \in (0, 1)$, $\rho_0 = \rho_0(n, m, \Delta, \nabla)$, and a constant $R' \in (0, 1/4)$ independent of all parameters, such that for any choice of $\rho' \leq \rho_0$ there exists a number $a = a(n, m, \Delta, \rho') > 2$ depending smoothly on ρ', with $a \to \infty$ as $\rho' \to 0$, and functions $f, h \colon [0, a] \to \mathbb{R}^+$ depending smoothly on ρ' with the following properties:

(1) $f(r) \equiv \rho'$ for all $r \in [0, R']$;

(2) $f'(r) \geq 0$ and $f''(r) \geq 0$ for all r;

(3) $f'(r) = \Delta$ in a small neighbourhood of $r = a$;

(4) $h(r) = \sin r$ for all $r \in [0, R']$;

(5) $h(r)$ is independent of all parameters (other than r) for $r \in [0, 1/2]$;

(6) $h(r) > 0$ for $r > 0$, $h'(r) \geq 0$ and $h''(r) \leq 0$ for all r;

(7) $h'(r) \equiv 0$ in a neighbourhood of $r = a$;

(8) $\sup_{r \in [0,a]} |h''(r)| = \sup_{r \in [0,1/2]} |h''(r)|$;

(9) $h(a)/f(a) \to 0$ as $\rho' \to 0$;

such that the submersion metric $g(f, h, \nabla)$ on E has nonnegative Ricci and positive scalar curvature, and strictly positive Ricci curvature for r small.
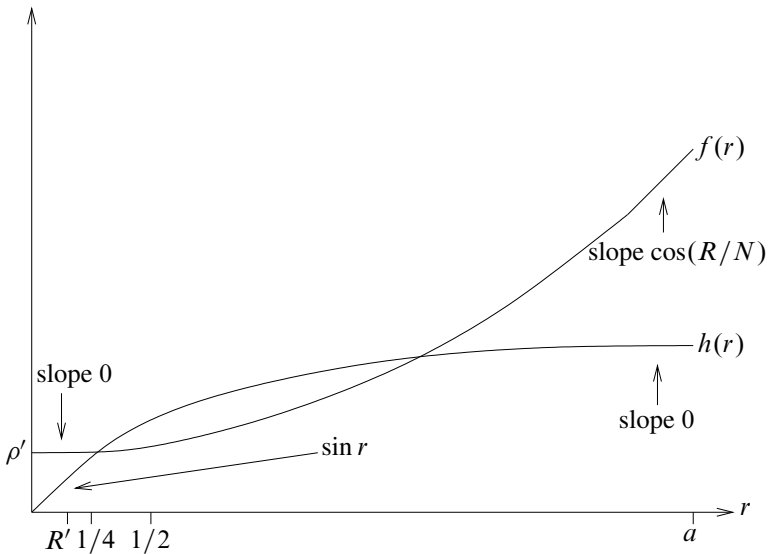
Figure 4

The functions f and h in this theorem have profiles as illustrated in Figure 4.

For the reader wishing to compare the above Theorem with the paper [19], a few words of explanation are in order here. The smooth dependence of f, h and a on ρ' might suggest that they are all uniquely determined by ρ' (given n, m and ∇). However this is not the case, as certain choices have to be made during the construction of these objects. On the other hand, it is clear from the construction details in [19] that the relevant choices can be made so that the variation with respect to ρ' is smooth.

Property (8) does not appear in [19] as it is not needed there. In fact, one can make choices in the construction of h so that (8) fails, but all other properties hold. Equally, one can make choices so that (8) holds. The argument is as follows: $h(r)$ is fixed for $r \in [0, 1/2]$, and for $r > 1/2$ the function is based on the solution to a certain ODE initial value problem, [19, Definition 2.2 and Lemma 2.5]. The only difference between $h(r)$ for $r > 1/2$ and this ODE solution occurs near $r = a$. Here, the ODE solution has to be deformed in a concave down manner to satisfy property (7) above (see [19, Lemma 2.15]). This deformation takes place when $f'(r) \equiv \Delta$, and as a result it is clear from the Ricci curvature formulas [19, Equations (1.1)–(1.3)] that this deformation can take place over an interval of arbitrary length. By choosing this interval to be suitably large (depending on the size of $\sup_{r \in [0, 1/2]} |h''(r)|$), we can guarantee that (8) holds. This places a lower bound on the size of a, but has no other implications.

The other properties of f and h listed in Theorem 3.2 are all evident from the construction of these functions in [19].

The above existence result can easily be adapted for Ricci positive surgery. Suppose we have the following:

(i) a Riemannian manifold (M^{n+m}, g) with positive Ricci curvature.

(ii) An isometric embedding $\iota\colon S^{n-1}(\rho) \times D_R^{m+1}(N) \to M$ where $D_R^{m+1}(N)$ denotes a geodesic ball of radius R in the $(m+1)$–sphere with the round metric of radius N. $S^{n-1}(\rho)$ is the $(n-1)$–sphere with the round metric of radius ρ.

(iii) A smooth map $T\colon S^{n-1} \to SO(m+1)$. This induces a map

$$\widetilde{T}\colon S^{n-1} \times D^{m+1} \to S^{n-1} \times D^{m+1},$$
$$(x, r, y) \mapsto (x, r, T(x)y),$$

where we are using polar coordinates (r, y) in D^{m+1}, with $r \in [0, R]$ and $y \in S^m$. In turn, this induces a map $\phi\colon S^{n-1} \times D^{m+1} \to M$ where $\phi := \widetilde{T} \circ \iota$.

Define the manifold $\widehat{M}$ to be

$$\widehat{M} = (M \setminus \phi(S^{n-1}(\rho) \times D_R^{m+1}(N)) \sqcup (D^n \times S^m)/\sim$$

where $\sim$ indicates the identification of boundaries using ϕ in the obvious manner.

We now come to the main surgery result in [19]. To state this, we will denote by a_0 the value of a in Theorem 3.2 corresponding to $\rho' = \rho_0$. Similarly, the corresponding functions f and h will be denoted f_0, h_0.

Theorem 3.3 *Suppose we have (M^{n+m}, g), ρ, R, N and ϕ as specified in* (i)–(iii) *above. Suppose further that $n \geq m + 1 \geq 3$. Set $\Delta = \cos(R/N)$, and choose $\delta \in (0, R')$ where R' is the universal constant appearing in Theorem 3.2. Then there is a connection ∇ on $E = D^n \times S^m$ which is flat outside the region $r \in (\delta/3, 2\delta/3)$, so that provided*

$$(\dagger) \qquad\qquad \rho < N \sin(R/N) \frac{h_0(a_0)}{f_0(a_0)}$$

is satisfied, for any choice of $\rho' \leq \rho_0$, the Ricci nonnegative submersion metric $g(f, h, \nabla)$ of Theorem 3.2 can be made to fit smoothly into

$$M \setminus \phi(D_R^{m+1}(N) \times S^{n-1}(\rho))$$

by

(a) *globally rescaling $g(f, h, \nabla)$ by a factor of $N \sin(R/N)/f(a)$, and then*

(b) *deforming the rescaled $f(r)$ to join smoothly at $r = a$ with the function $N \sin((R + r - a)/N)$.*

Moreover, this deformation can be performed in such a way that the resulting metric on $\widehat{M}$ has nonnegative Ricci curvature.

Remark 3.4 The metric on $\widehat{M}$ in Theorem 3.3 has strictly positive Ricci curvature on $M - \phi(D^{m+1} \times S^{n-1}) \subset \widehat{M}$, as well as in the central region of the "glue-in" piece E. The fact that the metric elsewhere might only be Ricci nonnegative is, however, not a concern. By a result of Ehrlich [8] (but see also Aubin [1]), if a Ricci nonnegative manifold has strictly positive Ricci curvature at some point, the positive curvature can be spread out from that point via a smooth metric deformation to cover the whole manifold. We are therefore justified in viewing Theorem 3.3 as a Ricci positive surgery theorem: the Ricci nonnegative Riemannian manifolds we create using the Theorem can, as a final step, be smoothly deformed to exhibit Ricci positivity. (In fact, it is not difficult to see how to obtain such a Ricci positive metric explicitly: simply adjust the functions $f(r)$ and $h(r)$ by an arbitrarily small amount in neighbourhoods with some zero Ricci curvatures, so as to create strict concavity at all values of r.)

Suppose we wish to perform the kind of metric surgery described by Theorem 3.3 on the fibre sphere of a sphere bundle over a Ricci positive base manifold. It is well-known (see Besse [2, 9.70]) that the submersion metric on such a bundle constructed by using the given Ricci positive base metric, scaled round metrics $\rho \, ds^2$ for some constant $\rho > 0$ on the fibres, and any choice of connection, will have positive Ricci curvature provided ρ is chosen sufficiently small. If we assume that the base metric in a neighbourhood of the surgery sphere is isometric to $D_R^{m+1}(N)$ for some R and N (and we can always arrange for this to be the case via a deformation of the base metric), then Theorem 3.3 applies. For any choice of trivialisation ϕ we obtain a Ricci positive metric on the manifold $\widehat{M}$ resulting from the surgery, as it is clear that we can choose ρ as small as we like, and so can guarantee that condition (†) in Theorem 3.3 is satisfied.

Properties (1) and (4) in Theorem 3.2, together with the fact that we are free in Theorem 3.2 to choose ρ' to be arbitrarily small, means that the metric on $\widehat{M}$ resulting from Theorem 3.3 is of the correct form to attempt further surgeries using Theorem 3.3 on the S^m–fibres of the glue-in piece $E = D^n \times S^m$. As noted above, an iterative sequence of such surgeries are required to construct the homotopy spheres under consideration in this paper. In particular, repeated application of Theorem 3.3 shows that all homotopy spheres which bound parallelisable manifolds admit Ricci positive metrics.

Convention 3.5 In order to use Theorem 3.3 to perform a metric surgery on the site of a previous surgery (as discussed in the above paragraph), we will always set $R = R'/4$, $N = 1$ and $\rho = \rho'$, where ρ' is the quantity arising from the previous surgery.

The significance of this convention will be made clear Section 4.

Notation 3.6 Let Σ^{4k-1} be a homotopy sphere which bounds a parallelisable manifold. As described in Section 2, view Σ as the boundary of the plumbed manifold X_p. Denote by g_p a Ricci positive metric on Σ obtained from a Ricci positive submersion metric on the tangent sphere bundle of S^{2k} by repeated Ricci positive surgeries of the form described in Theorem 3.3, performed according to the plumbing diagram for X_p.

4 A metric deformation

As described in Section 1, given a homotopy sphere Σ which bounds a parallelisable manifold equipped with a Ricci positive "plumbed" metric g_p (see Notation 3.6), to prove Theorem A we need to show that g_p can be extended to a positive scalar curvature metric on the plumbed bounding manifold X_p (see Section 2) in such a way that it is a product near the boundary. Performing any kind of positive scalar curvature extension of g_p across X_p turns out to be quite difficult. The situation can be made more tractable, however, by first *deforming* g_p smoothly through positive scalar curvature metrics on Σ to a new metric $g_{p,\infty}$. This metric does not have positive Ricci curvature, but it belongs to the same component of the space of positive scalar curvature metrics as g_p, and this is sufficient for our purposes. Because of the form of $g_{p,\infty}$, it is straightforward to extend to a positive scalar curvature metric on X_p. This extension is carried out in Section 5. Unfortunately, this extended metric is not a product in a neighbourhood of the boundary. However, because of the form of $g_{p,\infty}$, this is easy to correct via a metric deformation within positive scalar curvature. This is carried out in Section 6. Thus the Kreck–Stolz invariant for $(\Sigma, g_{p,\infty})$ can be computed using Lemma 2.1 in terms of the signature of X_p, leading to a proof of Theorem A.

In the current section, we show how to deform g_p to $g_{p,\infty}$.

The metric g_p naturally splits into a number of "pieces" arising from the various surgeries used in its construction. Apart from the initial tangent sphere bundle submersion metric, these pieces are all formed from the metrics $g(f, h, \nabla)$ in Theorem 3.2 by global rescale and boundary adjustment (see Theorem 3.3). Our first goal is to describe a deformation of one of these pieces, and our starting point is to consider the simplest case of a metric $g(f, h, \nabla)$: the case where the connection ∇ is flat. In this case, $g(f, h, \nabla)$ is isometric to the warped product $g(f, h) = dr^2 + h^2(r)\,ds_n^2 + f^2(r)\,ds_m^2$.

Lemma 4.1 Consider the metric $g(f, h) = dr^2 + h^2(r)\,ds_n^2 + f^2(r)\,ds_m^2$ on $D^n \times S^m$. There exists a constant $c_0 = c_0(n)$ such that for all $0 < c < c_0$ the metric $g(f, h)$ can be deformed smoothly through positive scalar curvature metrics to a metric $g(f, h_\infty)$, where $h_\infty(r)$ is a concave down function satisfying $h_\infty(r) = \sin r$ for $r \in [0, \delta]$ where

$\delta = \delta(c) < R'$, and $h_\infty(r) \equiv c$ for $r > R'$. (See Figure 5 for an illustration of this function, and Theorem 3.2 for the definition of R'). *Moreover, this deformation can be performed keeping the metric constant for $r \in [0, \delta]$.*

Proof The deformation will be parameterised using a "time" parameter $\tau \in [0, 1]$. Set $h_\tau := (1 - \tau)h + \tau h_\infty$. Notice that h_τ is a concave down function for each τ. We need to study the conditions under which the family of metrics $g(f, h_\tau)$ have positive scalar curvature for each τ.

An elementary calculation shows that the scalar curvature of this metric is given by

$$-2(n-1)\frac{h_\tau''}{h_\tau} - 2m\frac{f''}{f} + (n-1)(n-2)\frac{1 - (h_\tau')^2}{h_\tau^2} + m(m-1)\frac{1 - (f')^2}{f^2} - 2(n-1)m\frac{h_\tau' f'}{h_\tau f},$$

where all the derivatives are taken with respect to r.

First note that there is nothing to prove for $r \in [0, R']$ as $f(r)$ is constant at these values of r, and by concavity, the sum of the terms involving only h_τ and its derivatives is strictly positive for all r and for each $\tau \in [0, 1]$. From now on we therefore assume that $r \geq R'$.

Next, we claim that the term $h_\tau' f'/h_\tau f$ is decreasing with respect to τ for each r. As f is independent to τ, it suffices to consider the behaviour of

$$\frac{h_\tau'}{h_\tau} = \frac{(1 - \tau)h' + \tau h_\infty'}{(1 - \tau)h + \tau h_\infty}$$

$$= \frac{h'}{h + (\tau/(1 - \tau))c}$$

for these values of r. Now $\tau/(1 - \tau)$ is strictly increasing for $\tau \in [0, 1)$, and this establishes the claim.

To complete the proof of Lemma 4.1, it therefore suffices to check that $-2h_\tau''/h_\tau + (n - 2)(1 - (h_\tau')^2)/h_\tau^2$ is nondecreasing with τ for all $r \in [R', a]$. To this end, a straightforward calculation shows that the derivative of this expression with respect to τ is equal to

$$\frac{2}{((1 - \tau)h + \tau c)^3}\Big(ch''((1 - \tau)h + \tau c) + (n - 2)(h - c) + (n - 2)c(1 - \tau)(h')^2\Big).$$

As $2((1 - \tau)h + \tau c)^{-3} > 0$, we only need check the positivity of the term

$$ch''((1 - \tau)h + \tau c) + (n - 2)(h - c) + (n - 2)c(1 - \tau)(h')^2.$$

By Theorem 3.2 (6) we see that the last term in this expression is always nonnegative, the middle term is strictly positive if $c < h(R') = \sin(R')$ as h is increasing, but the
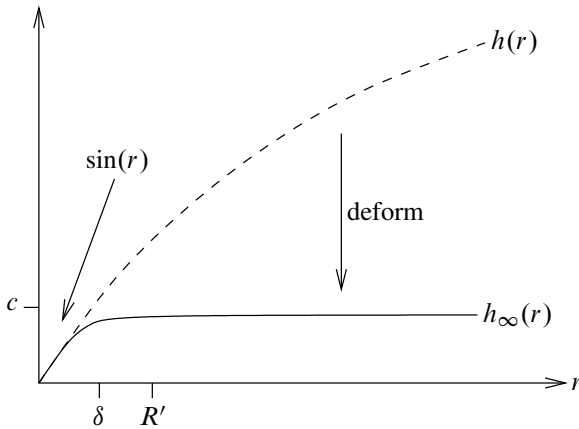
Figure 5

first term is nonpositive as $h'' \leq 0$. By Theorem 3.2 (8) we have that the absolute value of the first term is bounded above by $c(h(r) + \tau c) \sup_{r \in [0, 1/2]} |h''(r)|$. For simplicity, let us set $\lambda := \sup_{r \in [0, 1/2]} |h''(r)|$. Thus it suffices to show that

$$c\lambda\big((1 - \tau)h(r) + \tau c\big) < (n - 2)(h(r) - c)$$

for all $r \in [R', a]$. Rearranging this we obtain

$$(n - 2)c + \tau c^2 \lambda < h(r)\big(n - 2 - c\lambda(1 - \tau)\big).$$

As h is increasing and $\tau \in [0, 1]$ it suffices to establish

$$(n - 2)cf + c^2\lambda < \sin(R')(n - 2 - c\lambda),$$

which is clearly true for all c sufficiently small. Moreover, the upper bound c_0 for the values of c which satisfy this inequality clearly depends only on n, λ and R'. But λ and R' are constants independent of any other quantities, so we are therefore justified in writing $c_0 = c_0(n)$.

Having fixed a value for c, we finally set δ to be any number consistent with the requirements for h_∞ specified in the statement of the Lemma. $\square$

So far we have only considered deformation of the warped product metric $g(f, h)$. It is now time to bring connections back into the picture.

Proposition 4.2 *Suppose we have (M^{n+m}, g), ρ, R, N, ϕ and Δ as in Theorem 3.3. Fix a value of $c < c_0(n)$, where $c_0(n)$ is the constant appearing in Lemma 4.1. Set $\delta = \delta(c) \in (0, R')$ as in Lemma 4.1. Then there is a connection ∇ on $E = D^n \times S^m$ which is flat outside the region $r \in (\delta/3, 2\delta/3)$, so that provided condition (†) in*

Theorem 3.3 is satisfied, the Ricci nonnegative submersion metric $g(f, h, \nabla)$ admits a smooth deformation through positive scalar curvature metrics to $g(f, h_\infty, \nabla)$, with h_∞ as in Lemma 4.1. Moreover, these metrics (after rescale and boundary adjustment) give a smooth family of positive scalar curvature metrics on $\widehat{M}$, the manifold resulting from the surgery.

Proof It suffices to note that since $(\delta/3, 2\delta/3) \subset [0, R']$, in the region where the connection is nonflat the path of metrics is fixed independently of τ. On the other hand, for $r > R'$, the metric $g(f, h, \nabla)$ is isometric to the warped product $g(f, h)$, for which the result was established in Lemma 4.1. The final statement is obvious. □

We immediately deduce:

Corollary 4.3 *For any collection of Ricci positive surgeries of the form described by Theorem 3.3, uniform values for c and δ can be chosen in advance.*

Convention 4.4 For any sequence of metric surgeries we wish to perform, we will assume, in accordance with Corollary 4.3, that we use the same fixed values for c and δ for all surgeries. Moreover, we will also assume (as we did in Proposition 4.2) that the value of δ in Theorem 3.3 (used to bound any region of nonflatness for the connection) is precisely $\delta(c)$.

The significance of Convention 4.4 is that by constraining the nonflat part of the connection to lie in the region $r \in (\delta(c)/3, 2\delta(c)/3)$, the metric deformations being described in this section take place where the metric is a warped product, as do subsequent surgeries, which by Convention 3.5 take place at $r \leq R'/4 (< \delta(c)/3)$.

Having discussed deforming an individual surgery "piece" through positive scalar curvature metrics, we now want to show how this procedure can be iterated to deform the metric g_p to the desired metric $g_{p,\infty}$.

For a given homotopy sphere $\Sigma \in bP_{4k}$, consider the corresponding plumbing diagram, for example one of those illustrated in Figures 1 or 3.

To interpret this diagram as a template for surgery, let us read it from right to left, thus the right-most vertex will represent the initial sphere bundle, and each of the vertices to its left represents a successive surgery. All such graphs bifurcate in a number of places, as illustrated in Figure 6.

In the following we will assume that both the original metrics and the subsequent deformation operations we perform on the objects represented by the vertex to the
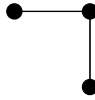
Figure 6

left and the vertex below the bifurcation point are identical. As a result, it suffices to consider *linear* plumbing graphs.

Suppose we have a linear graph with $i_0 + 1$ vertices, and wish to use this as a surgery template. Assume the right-hand vertex is a sphere bundle over a sphere, equipped with a Ricci positive submersion metric having round fibres of radius ρ and a round base sphere of radius N. Without loss of generality, we can assume that the connection on this bundle is chosen to be flat in a neighbourhood of the surgery, and that the base disc removed has radius R. Assume further that the dimension of each surgery is $\geq$ codimension ≥ 3, so Theorem 3.3 applies.

Let us introduce some notation to help clarify later constructions. Let ρ (as before) be the radius of the fibre sphere for the right-hand vertex, and let ρ'_i be the value of ρ' from Theorem 3.3 for the i–th surgery, that is, for the $(i+1)$–st vertex from the right. Label all other quantities associated with this surgery by a subscript i. Let M be the manifold resulting from the surgery process, and let g be the Ricci nonnegative metric on M obtained from successively applying Theorem 3.3.

Lemma 4.5 *For each $t \in [0, 1]$ there is a Ricci nonnegative metric g_t on the manifold M (as defined above) constructed via metric surgery in the same way as g, such that the ρ' value for the left-most vertex is $t\rho'_{i_0}$, and the family g_t depends smoothly on t.*

Proof We proceed by induction on the number of vertices. Consider first the case of a single surgery. The corresponding graph consists of two vertices, as shown in Figure 7.

Figure 7

As discussed above, the right-hand vertex represents a sphere bundle with Ricci positive submersion metric. The left-hand vertex represents a certain surgery on the right hand vertex, where the metric on the glue-in piece is the metric from Theorem 3.3. As pointed out in Theorem 3.3, this metric will differ from the metric $g_1(f_1, h_1, \nabla_1)$ by a global rescale and a concave down adjustment to the (rescaled) function f_1 near the boundary.

Let us consider the metric $g_1(f_1, h_1, \nabla_1)$. Replacing ρ'_1 by $t\rho'_1$ for decreasing values of $t < 1$ results in increasing values for a_1, with the dependence on t being smooth (see Theorem 3.2). In turn, this forces a smoothly decreasing global rescale factor, $\lambda_1(t)$ say, in order to complete the surgery. The rescale is necessary in order for f_1 to join smoothly with the function $N_1 \sin((R_1 - a_1 + r)/N_1)$ at $r = a_1$. However, the smaller rescale factor also has the effect of lowering the (rescaled) value of $h_1(a_1)$ below ρ. This is not a problem as we are free to smoothly reduce ρ accordingly, that is, to shrink the radii of the sphere bundle fibres. This completes the initial step of the induction argument.

Now suppose the result is true for a linear graph of $i_0 - 1$ vertices. Consider the i_0–th vertex and the metric $g_{i_0}(f_{i_0}, h_{i_0}, \nabla_{i_0})$. Replacing ρ'_{i_0} by $t\rho'_{i_0}$ for decreasing values of $t < 1$ forces a smaller global rescale factor $\lambda_{i_0}(t)$ for $g_{i_0}(f_{i_0}, h_{i_0}, \nabla_{i_0})$. Just as for the initial case, this necessitates correspondingly smaller values for $\rho'_{i_0 - 1}$. But by our inductive assumption, we can rechoose $\rho'_{i_0 - 1}$ in this way while inducing smoothly varying Ricci nonnegative metrics on the manifold corresponding to the first $i_0 - 1$ vertices of the graph. This completes he inductive step. □

Lemma 4.6 *Consider again (M, g) as in Lemma 4.5. For each $\tau \in [0, 1]$, there is a positive scalar curvature metric g_τ on M constructed via metric surgery in the same way as g, such that the metric on the leftmost vertex corresponds to $g_{i_0}(f_{i_0}, (h_{i_0})_\tau, \nabla_{i_0})$ and the family g_τ depends smoothly on τ.*

Proof Consider the metric $g_{i_0}(f_{i_0}, h_{i_0}, \nabla_{i_0})$. By rechoosing the connection ∇_{i_0} if necessary so as to be flat outside the region $r \in (\delta/3, 2\delta/3)$, where δ is determined by any fixed choice of $c \in (0, c_0)$ as in Lemma 4.1, we see from Proposition 4.2 that the family of metrics $g_{i_0}(f_{i_0}, (h_{i_0})_\tau, \nabla_{i_0})$ has positive scalar curvature. Each of these metrics will glue smoothly with the metric to its "right" in the graph provided $\rho'_{i_0 - 1}$ is rechosen to agree with the final (rescaled) value of $(h_{i_0})_\tau$. See Figure 8 below. The result now follows from Lemma 4.5. □

Observation 4.7 The deformation of Lemma 4.5 can be applied to any of the surgery "pieces" which make up M, as it has no effect on that part of the manifold represented by vertices to the left.

Corollary 4.8 *The deformations can be performed successively from the left of the diagram to create an overall deformation from g_p to a new positive scalar curvature metric $g_{p,\infty}$.*
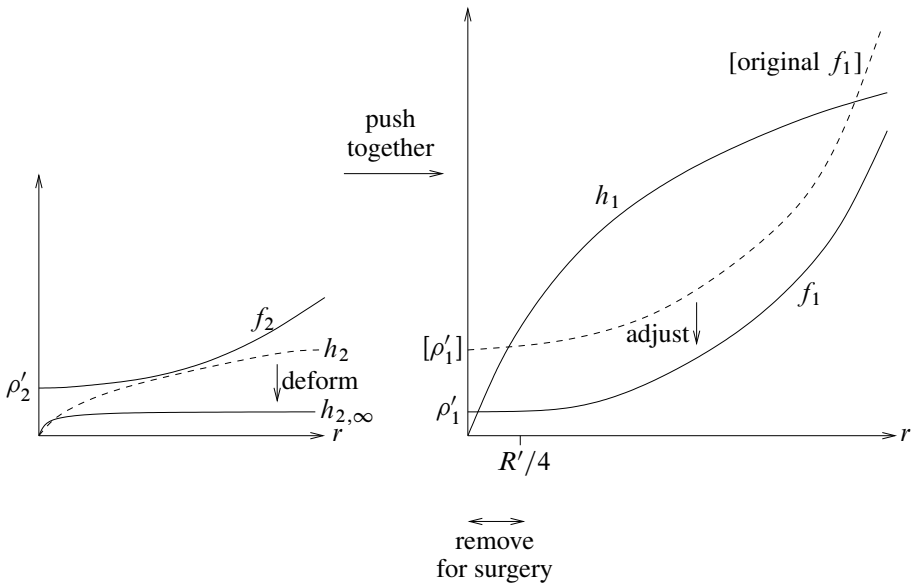
Figure 8: Metric deformation in the second surgery "piece"

5 Handles

Recall from Section 3 that given an embedding $\phi\colon S^{n-1} \times D^{m+1} \to M$, we can perform surgery on M, which results in a new manifold $\widehat{M}$ given by

$$\widehat{M} = (M \setminus \phi(S^{n-1}(\rho) \times D_R^{m+1}(N)) \sqcup_\phi (D^n \times S^m).$$

Now suppose that $M = \partial \overline{M}$. We can perform a related construction on $\overline{M}$ using ϕ. Note that

$$\partial(D^n \times D^{m+1}) = (S^{n-1} \times D^{m+1}) \cup (D^n \times S^m).$$

Using the map ϕ on the first "half" of this boundary, we can glue $D^n \times D^{m+1}$ onto $\overline{M}$, to yield a new manifold with boundary $(D^n \times D^{m+1}) \cup_\phi \overline{M}$. Notice that on the boundary, this construction is just the original surgery. Here, $D^n \times D^{m+1}$ is called a *handle* (or handlebody), the process just described is called *handle addition*.

Notice that the plumbing of two disc-bundles over spheres can be viewed as attaching a handle to either of the bundles. This means, for example, that the plumbed manifolds described in Section 2 having homotopy spheres as boundary can be viewed as constructed from a disc bundle over a sphere by successively adding handles.

Our main aim in this section is to prove a result about extending surgery metrics across the corresponding handles (Proposition 5.2). However, we first need a lemma

concerning metrics on disc bundles. We will say that a smooth Riemannian manifold with boundary has positive scalar (respectively positive Ricci) curvature when the metrics on the interior and the boundary both have positive scalar (respectively Ricci) curvature.

Lemma 5.1 *Suppose that E is the total space of a bundle with base M, a manifold with a fixed Ricci positive metric, fibre S^{n-1}, and structural group $SO(n)$. Let $\theta(s)$, $s \geq 0$, be any function satisfying: θ is odd at $s = 0$, $\theta'(0) = 1$, $\theta''(s) < 0$ for all $s > 0$, $\theta'(s) > 0$ for all s, and $\lim_{s \to 0+} \theta''(s)/\theta(s) < 0$. For a given choice of principal connection ∇ on the principal bundle associated to E, there exists $R_0 = R_0(\nabla, \theta)$ such that for all $\rho \leq R_0$, the submersion metric on E with fibres isometric to $\theta^2(\rho)\,ds_{n-1}^2$, and the submersion metric on the corresponding D^n–bundle $\bar{E}$ with fibres isometric to $ds^2 + \theta^2(s)\,ds_{n-1}^2$ where $s \in [0, \rho]$, both have positive Ricci curvature.*

Proof We view the metric on $\bar{E}$ as $ds^2 + g_s$, where g_s is the induced submersion metric on E with fibre radius $\theta(s)$. The following Ricci curvature formulas for $ds^2 + g_s$ are easily computed (for example by adapting the formulas in [20, Proposition 4.2]):

$$\mathrm{Ric}(\partial/\partial s, \partial/\partial s) = -(n-1)\frac{\theta''}{\theta},$$

$$\mathrm{Ric}(X_i, X_i) = \mathrm{Ric}_M(X_i, X_i) - 2\theta^2 \langle A_{X_i}, A_{X_i} \rangle,$$

$$\mathrm{Ric}(U_j, U_j) = \frac{(n-2)}{\theta^2}(1 - \theta'^2) - \frac{\theta''}{\theta} + \theta^2 \langle AV_j, AV_j \rangle,$$

$$\mathrm{Ric}(X_i, U_j) = -\theta \langle (\delta \check{A})X_i, V_j \rangle.$$

Here, $\{X_i\}$ are (the lifts to E of) orthonormal vector fields tangent to the base M; $\{U_j\}$ are orthonormal vector fields tangent to the distance spheres $S^{n-1} \subset D^n$, and the $\{V_j\}$ are the corresponding vector fields which would be orthonormal with respect to the metric ds_{n-1}^2; the A–tensor expressions appearing here are those for the unit sphere bundle diffeomorphic to E (see [2, Section 9C] for definitions), and so are independent of s.

The concavity of θ ensures that $\mathrm{Ric}(\partial/\partial s, \partial/\partial s) > 0$. The Ricci positivity of M ensures $\mathrm{Ric}(X_i, X_i) > 0$ provided s (and hence $\theta(s)$) is suitably small. Similarly, the positivity of $\mathrm{Ric}(U_j, U_j)$ is clear.

It remains to check the influence of the "mixed" term $\mathrm{Ric}(X_i, U_j)$. (Note that the other mixed terms $\mathrm{Ric}(\partial/\partial s, X_i)$ and $\mathrm{Ric}(\partial/\partial s, U_j)$ vanish.) To ensure that this term does not upset Ricci positivity we need to check that

$$\mathrm{Ric}(X_i, X_i)\,\mathrm{Ric}(U_j, U_j) > (\mathrm{Ric}(X_i, U_j))^2.$$

Using the above expressions and considering both sides of this inequality as $s \to 0$, we arrive at the limiting inequality

$$\mathrm{Ric}_M(X_i, X_i)\left(\lim_{s \to 0^+} \frac{(n-2)}{\theta^2(s)}(1 - \theta'^2(s)) - \frac{\theta''(s)}{\theta(s)}\right) > 0,$$

which is true. Thus the bundle metric on $\overline{E}$ is Ricci positive for ρ suitably small. It is also clear (for example by [2, Section 9G]) that the metrics g_s on E have positive Ricci curvature for all suitably small s. □

Proposition 5.2 *Suppose we have (M^{n+m}, g), ρ, R, N, ϕ and Δ as in Theorem 3.3. Suppose further that $M = \partial \overline{M}$, and that the metric g extends to a positive scalar curvature metric $\overline{g}$ on $\overline{M}$. Using the outward normal parameter to the boundary, extend ϕ to an embedding $\overline{\phi}: [-\epsilon, 0] \times S^{n-1} \times D^{m+1} \to \overline{M}$. Assume that the metric on $\overline{M}$ pulled back to this product takes the form*

$$ds^2 + \zeta^2(s)\, ds_{n-1}^2 + dt^2 + N^2 \sin^2(t/N)\, ds_m^2,$$

where s parametrises $[-\epsilon, 0]$ and $t \in [0, R]$ is the radial parameter of D^{m+1}. In particular, this means that $\zeta(0) = \rho$. Suppose further that $\zeta'(s) > 0$ and $\zeta''(s) < 0$ for all $s \in [-\epsilon, 0]$. Then there is a connection ∇ on $E = D^n \times S^m$ which is flat outside the region $r \in (\delta/3, 2\delta/3)$, so that assuming condition (†) in Theorem 3.3 holds, the positive scalar curvature metric $g(f, h_\infty, \nabla)$ on E from Proposition 4.2, after globally rescaling and a suitable adjustment to f near the boundary, extends to a positive scalar curvature metric on $\overline{E} = D^n \times D^{m+1}$, which glues smoothly via ϕ to give a positive scalar curvature metric on $\overline{M} \cup_\phi \overline{E}$.

Proof We will first consider the situation where the connection ∇ can be chosen to be flat. In this case, the metric $g(f, h_\infty, \nabla) = g(f, h_\infty)$ on E is a warped product metric. Now $g(f, h_\infty)$ has to be globally rescaled and then f adjusted near the boundary of E in order to use it for positive scalar curvature surgery on M (see the comments in Theorem 3.3).

Note that if the global rescale factor is λ^2, then the rescaled metric is

$$\lambda^2\, dr^2 + \lambda^2 h_\infty^2(r/\lambda)\, ds_{n-1}^2 + \lambda^2 f^2(r/\lambda)\, ds_m^2$$

for $r \in [0, a]$. By a parameter change $\widetilde{r} := \lambda r$, this is equivalent to

$$d\widetilde{r}^2 + \lambda^2 h_\infty^2(\widetilde{r}/\lambda)\, ds_{n-1}^2 + \lambda^2 f^2(\widetilde{r}/\lambda)\, ds_m^2$$

for $\widetilde{r} \in [0, \lambda a]$. That is, by introducing a rescaled parameter, we can write the rescaled metric in the form $g(\widetilde{f}, \widetilde{h}_\infty)$ for $\widetilde{r} \in [0, \widetilde{a}]$, with $\widetilde{f} = \lambda f$, $\widetilde{h}_\infty = \lambda h_\infty$ and $\widetilde{a} = \lambda a$. In the interests of notational simplicity, we will assume from now on, by ignoring the

tilde, that the parameter r, the number a and the functions $f(r)$ and $h_\infty(r)$ all refer to the relevant quantities for the *rescaled* metric.

The adjustment of f to meet the boundary conditions required for smooth gluing (that is, the bending of f from having constant slope near the boundary to joining smoothly with $N \sin((R - a + r)/N)$ at $r = a$) will need our careful attention. In particular, we claim that this adjustment can be made within positive scalar (and in fact locally, nonnegative Ricci) curvature in such a way that the inequality

$$(*) \qquad\qquad f'(r) \le \sqrt{1 - (f(r)/N)^2}$$

holds for all r. This inequality will be important later on in the proof.

After the global rescale and prior to the adjustment, we can assume the function $f(r)$ takes the following form in a neighbourhood of $r = a$:

$$f(r) = \begin{cases} (r - a) \cos(R/N) + N \sin(R/N) & r < a, \\ N \sin\left(\frac{R - a + r}{N}\right) & r \ge a. \end{cases}$$

The join at $r = a$ is C^1 but not smooth.

Notice that $(*)$ holds with equality at all points on the sine curve. For $r < a$, we have by Theorem 3.2 (2) that f and f' are increasing. As equality in $(*)$ occurs in the limit as $r \to a^+$, we deduce that $(*)$ holds strictly for $r < a$.

Smooth $f(r)$ as follows. First we rechoose the joining point to be $r = a + \epsilon$ for some very small $\epsilon > 0$. Replace the function $f(r)$ above by the following:

$$f(r) = \begin{cases} (r - a) \cos(R/N) + N \sin(R/N) & r < a, \\ \psi(r) & r \in [a, a + \epsilon), \\ N \sin\left(\frac{R - (a + \epsilon) + r}{N}\right) & r \ge a + \epsilon, \end{cases}$$

where $\psi(r)$ is any concave down function which joins smoothly with the sine function at $r = a + \epsilon$ and satisfies

$$0 < \psi'(r) < \cos\left(\frac{R - (a + \epsilon) + r}{N}\right)$$

for all $r \in [a, a + \epsilon)$. Such a ψ clearly exists. For this redefined function it is clear that $(*)$ holds with strict inequality, and in particular at $r = a$, the single nonsmooth point. It is now clear that we can smooth at this point whilst maintaining $(*)$, as such a smoothing can have negligible effect on the values of the function, while the first derivatives interpolate between the values on either side of the join.

It is evident from the scalar curvature formula in Section 4 that the above adjustment to f also maintains positive scalar curvature.

Having made the adjustment, for notational simplicity we will rechoose the value of a to be $a + \epsilon$.

The rescaled, adjusted metric

$$g(f, h_\infty) = dr^2 + h_\infty(r)\, ds_{n-1}^2 + f(r)^2\, ds_m^2 \quad \text{on } [0, a] \times S^{n-1} \times S^m$$

can be viewed as consisting of a "cap"

$$([0, a] \times S^{n-1} \,;\, dr^2 + h_\infty^2(r)\, ds_{n-1}^2)$$

which is topologically a disc D^n, and a "tube"

$$([0, a] \times S^m \,;\, dr^2 + f^2(r)\, ds_m^2),$$

which share the parameter r. Let us consider the issue of smoothly gluing this object into $M \setminus \phi(D_R^{m+1}(N) \times S^{n-1}(\rho))$. By considering a slightly smaller disc D^{m+1} if necessary, we may assume that for some $\epsilon' > 0$, a neighbourhood of the boundary in $M \setminus \phi(D^{m+1} \times S^{n-1})$ takes the form

$$([R, R+\epsilon'] \times S^{n-1} \times S^m \,;\, dt^2 + \zeta^2(0)\, ds_{n-1}^2 + N^2 \sin^2(t/N)\, ds_m^2).$$

We can then split the gluing problem into two: the problem of gluing the "cap" onto its "continuation"

$$([-\epsilon, 0] \times S^{n-1} \,;\, dr^2 + \zeta^2(0)\, ds_{n-1}^2);$$

and the problem of gluing the "tube" onto its continuation

$$([R, R+\epsilon'] \times S^m \,;\, dt^2 + N^2 \sin^2(t/N))\, ds_m^2).$$

Clearly, what we need is the smooth joining of the functions $h_\infty(r)$ to the constant function with value $\zeta(0)$, and $f(r)$ to $N \sin(t/N)$, when the r and t parameters are suitably concatenated.

We want to go from this surgery picture to adding the handle $\bar{E}$ to $\bar{M}$ in such a way that the boundary metric agrees with that for the surgery, and of course whole metric must be smooth and have positive scalar curvature. Topologically, going from E to the handle $\bar{E}$ requires the "filling-in" of the fibre spheres S^m, that is, replacing the "tube" by a corresponding "solid tube". (See Figure 9.) Again, the smooth gluing problem can be divided into gluing the "cap" onto its continuation

$$([-\epsilon, 0] \times S^{n-1} \,;\, ds^2 + \zeta^2(s)\, ds_{n-1}^2)$$

and the "solid tube" onto its continuation

$$([-\epsilon, 0] \times D^{m+1} \,;\, ds^2 + dt^2 + N^2 \sin^2(t/N) \, ds_m^2).$$

Note that in this handle picture, the sphere radii in the cap are naturally parametrised by s, the parameter running along the central axis in the solid tube. Thus s will be a shared parameter in the handle metric in the same way that r was a shared parameter for the surgery metric.
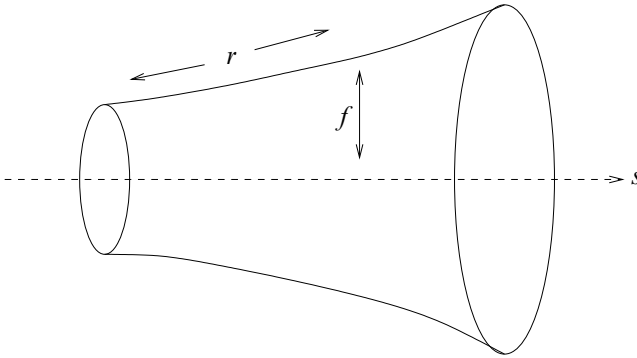

Figure 9: The "solid tube"

We focus first on the "solid tube". Our approach to defining a metric on this is to view it as a subset of a "big tube" $([0, b] \times D^{m+1} \,;\, ds^2 + dt^2 + N^2 \sin^2(t/N) \, ds_m^2)$, for some $b > 0$ to be chosen later. (See Figure 10.)
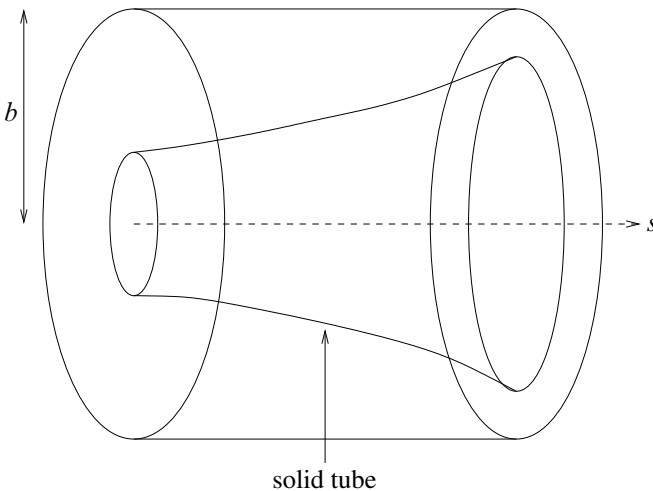

solid tube

Figure 10: The "big tube"

For simplicity, we are using the same symbols s and t for the parameters here as for the corresponding parameters in $\overline{M}$, on the understanding that a (metrically insignificant) adjustment will need to be made to the s parameter on one side of the join in order for concatenation to take place. This "big tube" will clearly glue onto $([-\epsilon, 0] \times D^{m+1}; ds^2 + dt^2 + N^2 \sin^2(t/N) ds_m^2)$ on concatenation, to give a smooth positive scalar curvature metric *on the interior*. We must therefore show how to cut the "solid tube" from the "big tube" in such a way that the boundary of the resulting object is smooth, and has a metric which agrees with that coming from the corresponding surgery.

We will define a curve $\gamma(r)$ in the $(s-t)$–plane (with metric $ds^2 + dt^2$) which will define the solid tube. The s–coordinate will parameterize the central axis of the solid tube, and the t–coordinate will determine the radius of the boundary spheres S^m. We will write $\gamma(r) = (\gamma_s(r), \gamma_t(r))$.

Recall that the (boundary) tube must have induced metric $dr^2 + f^2(r) ds_m^2$. For such a warped product, the parameter r measures distance *around* the tube (as opposed to along the central axis). For this reason we clearly need $\gamma(r)$ to be a unit speed curve, that is, $\gamma_s'(r) + \gamma_t'(r) \equiv 1$.

We need to choose $\gamma_t(r)$ so the distance sphere in $(D^{m+1}; dt^2 + N^2 \sin^2(t/N) ds_m^2)$ at a distance of $\gamma_t(r)$ from the centre is round of radius $f(r)$. In other words we need $N \sin(\gamma_t(r)/N) = f(r)$, and therefore

$$\gamma_t(r) = N \sin^{-1}(N^{-1} f(r)).$$

Using this together with $\gamma_s'(r) + \gamma_t'(r) \equiv 1$ allows us to solve for $\gamma_s(r)$: namely

$$\gamma_s(r) = \int_0^r \left(1 - \frac{f'^2(u)}{1 - N^{-2} f^2(u)} \right)^{1/2} du.$$

Note that the value of b defining the range of s above, is equal to $\gamma_s(a)$, where $r = a$ corresponds to the end of the tube to be glued onto $\overline{M}$. (See Figure 11.)

Note that the above expression for $\gamma_s(r)$ is well defined because the inequality $(*)$ is satisfied, as discussed above.

To simplify notation, denote the solid tube and its metric by $([0, b] \times D^{m+1}; ds^2 + g_s)$ where g_s is the metric $dt^2 + N^2 \sin^2(t/N) ds_m^2$ for $t \in [0, \gamma_t(\iota(s))]$, where $\iota: [0, b] \to \mathbb{R}$ is defined as follows. For $s < b$ the value of $\iota(s)$ is determined by the equation $\gamma_s(\iota(s)) = s$. Notice that $\iota(s)$ is smooth for these values of s since γ is a smooth curve. For $s = b$ set $\iota(b) = a$, which means that $\gamma_t(\iota(b)) = R$. Note that ι will in general not be continuous at $s = b$. The idea here is that the function $\iota(s)$ allows us
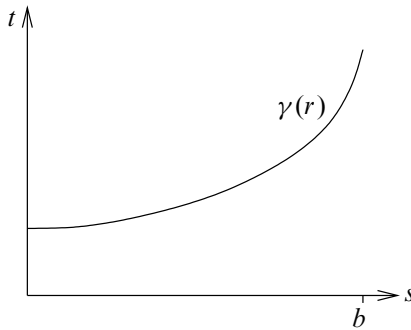
Figure 11

to replace the parameter r in our subsequent calculations by the parameter s running along the central axis of the solid tube.

We turn our attention now to the "cap" D^n. In the surgery picture, we require the distance sphere at radius r to have the metric $h_\infty^2(r) \, ds_{n-1}^2$. In the handle picture, these spheres will be parametrised by s, as discussed above. Clearly, the required metric for any given value of s is $h_\infty^2(\iota(s)) \, ds_{n-1}^2$. Notice that $h(\iota(s))$ is smooth on $[0, b]$, despite the fact that ι cannot be assumed continuous at b. This is because $h_\infty(r)$ is constant for $r \in [R', a]$.

A fundamental difference between the handle and surgery cases is that the function $h(\iota(s))$ needs to be joined to the function $\zeta(s)$, as opposed to a constant function. However, since $\zeta' > 0$ and $\zeta'' < 0$, this joining can easily be achieved by a small concave down deformation of $h(\iota(s))$ in a neighbourhood of the joining point. Such a bend can only have a positive effect on the scalar curvature. Note that the second derivative of $h_\infty(\iota(s))$ is nonpositive. To see this, recall that $f(r) \equiv 1$ for small r, and this means that for s in the same range we have $\iota(s) = s$. Also, where f is nonconstant we have $h_\infty(r) \equiv c$. The assertion now follows immediately since the desired second derivative is equivalent to $h_\infty''(\iota(s))(\iota'(s))^2 + h_\infty'(\iota(s))\iota''(s)$. As a result we deduce that the cap itself has positive scalar curvature (and nonnegative Ricci curvature).

Using the above notation, the whole handle and its metric can now be expressed as

$$(\ddagger) \qquad ([0, b] \times S^{n-1} \times D^{m+1}, ds^2 + h_\infty^2(\iota(s)) + g_s).$$

As the boundary metric by construction agrees with the surgery metric, it is clear that $\overline{M} \cup \overline{E}$ admits a smooth positive scalar curvature metric, as claimed.

Finally, we must consider the situation when the connection ∇ is not flat (so the metric $g(f, h_\infty, \nabla)$ is not a warped product). Clearly, the technique of cutting out the "solid tube" from the "big tube" explained above requires the metric to be a warped product,

and so will not work here. However, recall that the nonflat part of the connection is restricted to the region $r \in (\delta/3, 2\delta/3)$. Accordingly, we are free to assume that $g(f, h_\infty, \nabla)$ is a warped product outside this region.

For $r \in [0, \delta]$, recall that the fibres of E are all isometric to a round sphere of radius ρ'. There is clearly a submersion metric on $\bar{E}$ for $r \in [0, \delta]$, having fibres isometric to $dt^2 + N^2 \sin^2(t/N) \, ds_m^2$, for which the induced metric on the boundary agrees with that of E. We choose a value for ρ' for which guarantees the Ricci positivity of E, and also the Ricci positivity of $\bar{E}$ restricted to the nonflat region $r \in (\delta/3, 2\delta/3)$. That such a value of ρ' exists follows from Lemma 5.1.

For $r \geq \delta$ we use the metric constructed for the product case. This clearly glues smoothly with the above metric for $r \in [0, \delta]$ to create the desired metric on $\bar{E}$. $\qquad \square$

Let Σ now be a homotopy sphere representing any element of bP_{4k}, and let X_p be a plumbed manifold as discussed in Section 2 with $\Sigma = \partial X_p$.

Corollary 5.3 *The positive scalar curvature metric $g_{p,\infty}$ on Σ can be extended across X_p to give a positive scalar curvature metric $\bar{g}_{p,\infty}$ on X_p.*

Proof The existence of this positive scalar curvature metric on X_p follows immediately from repeated application of Proposition 5.2, starting with the Ricci positive metric on the "central" disc-bundle guaranteed by Lemma 5.1 and systematically extending the metric over each plumbing in turn. $\qquad \square$

Fixing this metric on X_p, consider a neighbourhood of the boundary with normal parameter $w \in [-\epsilon, 0]$. By the compactness of X_p there exists an $\epsilon > 0$ such that this neighbourhood is diffeomorphic via the exponential map to $[-\epsilon, 0] \times \partial X_p$. We will express the metric here as $dw^2 + g(w)$, with $g(w)$ a metric on the equidistant hypersurface at a distance $|w|$ from ∂X. By the openness of the positive scalar curvature condition, it follows that $\mathrm{scal}(g(w)) > 0$ for all w suitably close to 0, as well as $\mathrm{scal}(dw^2 + g(w)) > 0$.

Fix an arbitrary local coordinate system on ∂X_p, $\{x_1, \ldots, x_{4k-1}\}$, and extend to a local coordinate system on the collar neighbourhood $\{w, x_1, \ldots, x_{4k-1}\}$ via the identification of the collar with $[-\epsilon, 0] \times \partial X_p$. In the following Lemma, the subscript i refers to the coordinate system on ∂X_p.

Lemma 5.4 *For w suitably close to 0, the metrics $g(w)$ above satisfy*

$$\frac{\partial g_{ii}(w)}{\partial w} \geq 0.$$

Proof To begin with, we will choose the coordinate system $\{x_1, \ldots, x_{4k-1}\}$ to lie entirely on the boundary of one of the handles $\bar{E}$ used in the construction of X_p.

First assume that the metric on $\bar{E}$ takes the form (‡), and that the connection is chosen to be flat. We can identify four mutually orthogonal directions: the s–direction with metric ds^2; the t–direction with metric dt^2; the S^m direction with metric $N^2 \sin^2(t/N) \, ds_m^2$; and the S^{n-1} direction with metric $h_\infty^2(\iota(s)) \, ds_{n-1}^2$. The boundary of solid tube is determined by the function f, and in turn this determines the direction of the normal vector $\partial/\partial w$.

Assume the coordinate system lies in the region where $f(\iota(s)) \equiv 1$. Here we have $\partial/\partial w = \partial/\partial t$. Clearly g_{ii} is nondecreasing along t parameter lines, hence the result.

If the coordinate system lies in the region corresponding where $f(\iota(s))$ is increasing, we have that $\partial/\partial w$ can be decomposed into nonnegative components in the $-\partial/\partial s$ and $\partial/\partial t$ directions. As $h_\infty(\iota(s))$ is constant at these values of s, we see that the metric itself is constant along s–parameter lines. Again, we have that g_{ii} is nondecreasing along t parameter lines, hence the result in this case.

Now suppose that the connection on E is not everywhere flat. Recall that the nonflatness is constrained to lie within the region corresponding to $r \in (\delta/3, 2\delta/3)$, which is equivalent to $s \in (\delta/3, 2\delta/3)$. Note that changing the principal connection on the associated $SO(m+1)$–bundle only alters the metric on $\bar{E}$ in directions tangent to the distance sphere bundles. In particular, normal directions to the distance sphere bundles are unaffected, so in this region $\partial/\partial w = \partial/\partial t$ as before. The same arguments as for the flat case again yield the result that g_{ii} is nondecreasing in normal directions.

Next suppose that our initial coordinate system $\{x_1, \ldots, x_{4k-1}\}$ lies on the disc-bundle with which we begin our construction of X_p, where the metric takes the form described in [Lemma 5.1](). Arguments analogous to the above show that $\partial g_{ii}(w)/\partial w \geq 0$, since $\theta(s)$ is a strictly increasing function.

Finally, if we choose a coordinate system $\{x_1, \ldots, x_{4k-1}\}$ which crosses the boundary between a handle and the original disc-bundle, or between two handles, the nonnegativity of a derivative arbitrarily close to the boundary on either side is clearly sufficient to guarantee nonnegativity throughout the region. $\qquad\square$

6 Paths of metrics

In this section we will ultimately prove [Theorem A](). Firstly, however, we must study the problem of deforming a positive scalar curvature metric on a manifold with boundary to a product in some neighbourhood of the boundary, in such a way that the positive

scalar curvature is preserved. For the manifolds and metrics we are concerned about in this paper, Lemma 5.4 (above) is crucial to being able to perform such a deformation. The importance of this Lemma arises from the following result:

Lemma 6.1   *Consider a metric of the form $\eta^2(w)dw^2 + g(w)$ on $I \times M$ where I is an interval parametrised by w, M^n a manifold, $\eta(w)$ a smooth real-valued function and $g(w)$ a smooth path of metrics on M. Given a local coordinate system $\{x_1, \ldots, x_n\}$ for M, consider the local coordinate system $\{w, x_1, \ldots, x_n\}$ on $I \times M$. With respect to this coordinate system the scalar curvature of the above metric is given by*

$$\operatorname{scal}(\eta^2(w)dw^2 + g(w)) = \operatorname{scal}(g(w)) + \frac{H(w)}{\eta^2(w)} + \frac{\eta'(w)}{\eta^3(w)} \sum_{i=1}^{n}\left(\frac{\partial g_{ii}(w)}{\partial w} \Big/ g_{ii}(w)\right),$$

where $H(w)$ is a smooth one-parameter family of real-valued functions on the domain of the coordinate system $\{x_1, \ldots, x_n\}$, which is independent of $\eta(w)$ and given by

$$H(w) = \frac{1}{2}\sum_{i<j}\left(\frac{\partial g_{ij}}{\partial w}\sum_{k=1}^{n}\sum_{p=1}^{n}g^{kp}\frac{\partial g_{jp}}{\partial w}g_{ki} - \frac{\partial g_{jj}}{\partial w}\sum_{k=1}^{n}\sum_{p=1}^{n}g^{kp}\frac{\partial g_{ip}}{\partial w}g_{ki}\right)\Big/(g_{ii}g_{jj} - g_{ij}^2)$$

$$+ \sum_{i=1}^{n}\frac{1}{2g_{ii}}\left(\sum_{k=1}^{n}\sum_{p=1}^{n}g^{kp}\frac{\partial g_{ik}}{\partial w}\frac{\partial g_{ip}}{\partial w} - 2\frac{\partial^2 g_{ii}}{\partial w^2}\right).$$

Proof   This result follows from an elementary calculation (for example by computing Christoffel symbols for the coordinate system), and we omit the details.   □

Lemma 6.2   *Assume that the metric $dw^2 + g(w)$ on $[-\epsilon, 0] \times M$ has positive scalar curvature. Given a local coordinate system on M, suppose that*

$$\sum_{i=1}^{n}\frac{\partial g_{ii}(w)}{\partial w} \geq 0$$

at all points of $[-\epsilon', 0] \times M$ covered by the coordinate system, for some $\epsilon' \leq \epsilon$. Then there exists a smooth positive function $\beta(w)$ for $w \in [-\epsilon', 0]$ with

$$\beta(w) = 1 \quad \text{for } w \in [-\epsilon', -\epsilon'/2],$$
$$\beta(w) = \Lambda \quad \text{for } w \in [-\epsilon'/4, 0],$$

for any given large constant Λ, such that the metric $\beta^2(w)dw^2 + g(w)$ has positive scalar curvature.

Proof   By Lemma 6.1, $\mathrm{scal}(dw^2 + g(w)) = \mathrm{scal}(g(w)) + H(w)$ for all $w \in [-\epsilon, 0]$, and this is positive by hypothesis. Since $\mathrm{scal}(g(w)) > 0$ we have $\mathrm{scal}(g(w)) + H(w)/\beta > 0$ for any $\beta \geq 1$. It is then clear that for any *increasing* function $\beta(w)$ of the required form,

$$\mathrm{scal}(g(w)) + \frac{1}{\beta^2(w)} H(w) + \frac{\beta'(w)}{\beta^3(w)} \sum_{i=1}^{n} \left(\frac{\partial g_{ii}(w)}{\partial w} \Big/ g_{ii}(w) \right) > 0$$

since the last term is nonnegative by hypothesis.   □

Lemma 6.3   *If $g(w)$ is any smooth path through positive scalar curvature metrics on a compact manifold M, and w belongs to a compact interval I, then the metric $\Lambda^2 dw^2 + g(w)$ has positive scalar curvature on $I \times M$ for all Λ sufficiently large.*

Proof   By Lemma 6.1, the scalar curvature of $\Lambda^2 dw^2 + g(w)$ is $\mathrm{scal}(g(w)) + H(w)/\Lambda^2$. By the compactness of $I \times M$, there exists a constant $C \geq 0$ such that

$$\mathrm{scal}(\Lambda^2 dw^2 + g(w)) \geq \mathrm{scal}(g(w)) - C/\Lambda^2.$$

Since $\mathrm{scal}(g(w)) > 0$ for all $w \in I$, we can choose Λ sufficiently large so that $\mathrm{scal}(g(w)) - C/\Lambda^2 > 0$, and this guarantees the positivity of $\mathrm{scal}(\Lambda^2 dw^2 + g(w))$.   □

Corollary 6.4   *For a given homotopy sphere $\Sigma \in bP_{4k}$, suppose that X_p is a plumbed manifold as discussed in Section 2 with $\partial X_p = \Sigma$. Then X_p can be equipped with a positive scalar curvature metric which is a product $dw^2 + g$ in a neighbourhood of the boundary, and where the component of the moduli space of positive scalar curvature metrics on Σ containing g also contains a Ricci positive metric.*

Proof   We equip X_p with the positive scalar curvature metric $\bar{g}_{p,\infty}$ from Corollary 5.3. In a neighbourhood of the boundary of X_p, this takes the form $dw^2 + g(w)$ ($w \leq 0$), for some smooth path of metrics $g(w)$ on Σ, which for w sufficiently close to zero can be assumed to have positive scalar curvature. By Lemma 5.4 and Lemma 6.2, we can adjust this metric in a neighbourhood of the boundary to take the form $\Lambda^2 dw^2 + g(w)$ for any large constant Λ, whilst preserving positive scalar curvature. Next, adjoin a collar $\Sigma \times [0, 1]$ to ∂X_p and choose a smooth path of positive scalar curvature metrics $h(w)$ ($w \in [0, 1]$) on Σ which smoothly extends the path $g(w)$, and for which $h(w) = h(1)$ for all w sufficiently close to 1. By Lemma 6.3 it follows that provided Λ is chosen sufficiently large, $\Lambda^2 dw^2 + h(w)$ will have positive scalar curvature. Thus we have a metric on $X_p \cup (\Sigma \times [0, 1])$ which has positive scalar curvature and takes the form $\Lambda^2 dw^2 + h(1)$ in a neighbourhood of the boundary. Clearly, a similar statement can be made for X itself, after pulling back this metric via a suitable diffeomorphism

$X_p \cong X_p \cup (\Sigma \times [0, 1])$, and after a global rescale we can assume a positive scalar curvature metric on X_p taking the required form near the boundary.

By Corollary 4.8, it is clear that $h(1)$ belongs to the same component of positive scalar curvature metrics as the Ricci positive metric g_p on the boundary of X_p constructed in [18] (see Notation 3.6). It follows automatically that the same is true when we pass to the moduli space. $\square$

Proof of Theorem A Regard a homotopy sphere $\Sigma \in bP_{4k}$ as the boundary of a manifold X_p as described in Section 2. As noted at the end of Section 2, Σ is a spin manifold with vanishing real Pontrjagin classes and X_p is a parallelisable manifold. Equip X_p with the positive scalar curvature metric guaranteed by Corollary 6.4. Denote this metric $\overline{g}_p$, and let g_p be its restriction to $\Sigma = \partial X_p$. Lemma 2.1 applies to the Riemannian manifolds (Σ, g_p) and $(X_p, \overline{g}_p)$, and in this case asserts that

$$s(\Sigma, g_p) = \frac{1}{2^{2k+1}(2^{2k-1} - 1)} \sigma(X_p)$$
$$= \frac{8(p|bP_{4k}| + q)}{2^{2k+1}(2^{2k-1} - 1)}$$

for some integer q depending on Σ. Thus if $p \neq p'$, $|s(\Sigma, g_p)| \neq |s(\Sigma, g_{p'})|$, and therefore by [13, Proposition 2.14] g_p and $g_{p'}$ belong to different components of the moduli space of positive scalar curvature metrics on Σ. It follows immediately that the moduli space of Ricci positive metrics on Σ must have infinitely many components. $\square$

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