

Dominating surface group representations and deforming closed anti-de Sitter 3-manifolds

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Let S be a closed oriented surface of negative Euler characteristic and M a complete contractible Riemannian manifold. A Fuchsian representation $j: \pi_1(S) \rightarrow \text{Isom}^+(\mathbb{H}^2)$ *strictly dominates* a representation $\rho: \pi_1(S) \rightarrow \text{Isom}(M)$ if there exists a (j, ρ) -equivariant map from \mathbb{H}^2 to M that is λ -Lipschitz for some $\lambda < 1$. In a previous paper by Deroin and Tholozan, the authors construct a map Ψ_ρ from the Teichmüller space $\mathcal{T}(S)$ of the surface S to itself and prove that, when M has sectional curvature at most -1 , the image of Ψ_ρ lies (almost always) in the domain $\text{Dom}(\rho)$ of Fuchsian representations strictly dominating ρ . Here we prove that $\Psi_\rho: \mathcal{T}(S) \rightarrow \text{Dom}(\rho)$ is a homeomorphism. As a consequence, we are able to describe the topology of the space of pairs of representations (j, ρ) from $\pi_1(S)$ to $\text{Isom}^+(\mathbb{H}^2)$ with j Fuchsian strictly dominating ρ . In particular, we obtain that its connected components are classified by the Euler class of ρ . The link with anti-de Sitter geometry comes from a theorem of Kassel, stating that those pairs parametrize deformation spaces of anti-de Sitter structures on closed 3-manifolds.

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Introduction

0.1 Closed AdS 3-manifolds

An *anti-de Sitter* (AdS) manifold is a smooth manifold equipped with a Lorentz metric of constant negative sectional curvature. In dimension 3, these manifolds are locally modelled on $\text{PSL}(2, \mathbb{R})$ with its Killing metric, whose isometry group identifies to a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ extension of $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ acting by left and right translations, ie

$$(g_1, g_2) \cdot x = g_1 x g_2^{-1}$$

for all $(g_1, g_2) \in \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ and all $x \in \text{PSL}(2, \mathbb{R})$.

The connection between closed anti-de Sitter 3-manifolds and surface group representations was established by the following theorem of Kulkarni and Raymond:

Theorem [16] *Every closed complete anti-de Sitter 3–manifold has, up to finite cover, the form*

$$j \times \rho(\Gamma) \backslash \mathrm{PSL}(2, \mathbb{R}),$$

where Γ is the fundamental group of a closed surface of negative Euler characteristic and j and ρ are two representations of Γ into $\mathrm{PSL}(2, \mathbb{R})$, one of which is Fuchsian (ie discrete and faithful).

Remark 0.1 In contrast to the Riemannian setting, a Lorentz metric on a closed manifold may not be complete. However, Klingler [15] proved later, generalizing Carrière’s theorem [2], that closed Lorentz manifolds of constant curvature are complete. Hence the completeness assumption in Kulkarni and Raymond’s work can be removed.

A pair of representations (j, ρ) such that $j \times \rho(\Gamma)$ does act properly discontinuously and cocompactly on $\mathrm{PSL}(2, \mathbb{R})$ is called an *admissible pair*. The first examples of such pairs are when (up to switching the factors) j is Fuchsian and ρ takes values in $\mathrm{SO}(2)$. The resulting AdS 3–manifold is then called *standard*. Goldman [10] constructed the first examples of nonstandard AdS 3–manifolds by deforming standard ones. Later, Salein [18] found a sufficient properness criterion for the action of $j \times \rho(\Gamma)$, allowing him to construct admissible pairs with ρ of any nonextremal Euler class (and thus nonstandard AdS 3–manifolds that cannot be deformed into standard ones). Finally, Kassel proved in her thesis that Salein’s criterion is also necessary, leading to the following characterization of admissible pairs. Recall that $\mathrm{PSL}(2, \mathbb{R})$ is the group of orientation preserving isometries of the hyperbolic plane \mathbb{H}^2 .

Theorem (Salein [18], Kassel [14]) *Let S be a closed surface of negative Euler characteristic and j, ρ two representations of $\pi_1(S)$ into $\mathrm{PSL}(2, \mathbb{R})$, with j Fuchsian. Then the pair (j, ρ) is admissible if and only if there exists a (j, ρ) –equivariant map from \mathbb{H}^2 to \mathbb{H}^2 that is λ –Lipschitz for some $\lambda < 1$.*

The primary purpose of this paper is to describe the space of pairs (j, ρ) satisfying this admissibility criterion, leading to a description of the “deformation space” of anti-de Sitter 3–manifolds.

0.2 Dominated representations

Kassel’s criterion for admissibility raises many questions that may turn out to be interesting beyond the scope of 3–dimensional AdS geometry. Consider more generally a closed surface S of negative Euler characteristic and a contractible Riemannian manifold (M, g_M) . Let $j: \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{R})$ be a Fuchsian representation, and ρ

a representation of $\pi_1(S)$ into $\text{Isom}(M, g_M)$. Since M is contractible, there always exists a smooth (j, ρ) -equivariant map from \mathbb{H}^2 to M . Since S is compact, this map is Lipschitz. We can thus define the *minimal Lipschitz constant*

$$\text{Lip}(j, \rho) = \inf\{\lambda \in \mathbb{R}_+ \mid \text{there is a } (j, \rho)\text{-equivariant and } \lambda\text{-Lipschitz map } \mathbb{H}^2 \rightarrow M\}.$$

We will say that j *strictly dominates* ρ if $\text{Lip}(j, \rho) < 1$.

Let $\text{Rep}(S, G)$ denote the quotient of the space of representations of $\pi_1(S)$ into a group G by the conjugation action of G . This quotient is a (not necessarily Hausdorff) topological space. Let $\mathcal{T}(S)$ denote the *Teichmüller space* of S , which we view as the connected component of $\text{Rep}(S, \text{PSL}(2, \mathbb{R}))$ containing Fuchsian representations of maximal Euler class. The function Lip is clearly invariant by conjugation of j and ρ . It can thus be viewed as a map from $\mathcal{T}(S) \times \text{Rep}(S, \text{Isom}(M))$ to \mathbb{R}_+ . The following proposition is proved by Guéritaud and Kassel [11].¹

Proposition 0.2 (Guéritaud and Kassel) *The function*

$$\text{Lip}: \mathcal{T}(S) \times \text{Rep}(S, \text{Isom}(M)) \rightarrow \mathbb{R}_+$$

is continuous.

Given a representation $\rho: \pi_1(S) \rightarrow \text{Isom}(M)$, it is natural to ask whether ρ can be dominated by a Fuchsian representation and, if so, what the domain of Fuchsian representations dominating ρ looks like. The first question was answered by Deroín and the author [6] when (M, g_M) is a Riemannian CAT(−1) space. This applies for instance when M is the symmetric space of a simple Lie group of real rank 1 (with a suitable normalization of the metric), and in particular for representations in $\text{PSL}(2, \mathbb{R})$. In that case, it was obtained independently and with other methods by Guéritaud, Kassel and Wolff [12].

Definition 0.3 Let (M, g_M) be a Riemannian CAT(−1) space (ie a complete simply connected Riemannian manifold of sectional curvature bounded above by −1). Let S be a closed connected oriented surface of negative Euler characteristic and ρ a representation of the fundamental group of S into $\text{Isom}(M)$. We say that ρ is Fuchsian if it preserves a totally geodesic plane $\mathbb{H} \subset M$ of curvature −1 and the induced representation in $\text{Isom}(\mathbb{H})$ is Fuchsian.

Theorem (Deroín and Tholozan) *Let S be a closed connected oriented surface of negative Euler characteristic, (M, g_M) a Riemannian CAT(−1) space and ρ a representation of $\pi_1(S)$ into $\text{Isom}(M)$. Then either ρ is Fuchsian or there exists a Fuchsian representation in $\text{PSL}(2, \mathbb{R})$ that strictly dominates ρ .*

¹In their paper the context is slightly different. Their proof was adapted to our setting in the author's thesis [21, Section 3.1].

Remark 0.4 By a simple volume argument of Thurston [22, Proposition 2.1], a Fuchsian representation cannot be strictly dominated. Thus the theorem is optimal.

Let us denote by $\text{Dom}(\rho)$ the domain of $\mathcal{T}(S)$ consisting of representations of Euler class $-\chi(S)$ strictly dominating a given representation ρ , and by $\text{Dom}(S, \text{Isom}(M))$ the domain of pairs $(j, \rho) \in \mathcal{T}(S) \times \text{Rep}(S, \text{Isom}(M))$ such that j strictly dominates ρ . By continuity of the function Lip , both $\text{Dom}(\rho)$ and $\text{Dom}(S, \text{Isom}(M))$ are open domains. To prove their theorem, Deroin and the author [6] consider a certain map $\Psi_\rho: \mathcal{T}(S) \rightarrow \mathcal{T}(S)$ and show that its image lies in $\text{Dom}(\rho)$ (see Section 2.1). Here, we prove that this map is a homeomorphism from $\mathcal{T}(S)$ to $\text{Dom}(\rho)$ and that it varies continuously with ρ . We thus obtain the following:

Theorem 0.5 *Let S be a closed oriented surface of negative Euler characteristic and (M, g_M) a Riemannian $\text{CAT}(-1)$ space. Then the domain $\text{Dom}(S, \text{Isom}(M))$ is homeomorphic to*

$$\mathcal{T}(S) \times \text{Rep}_{nf}(S, \text{Isom}(M)),$$

where $\text{Rep}_{nf}(S, \text{Isom}(M))$ denotes the domain of $\text{Rep}(S, \text{Isom}(M))$ consisting of representations that are not Fuchsian.

What's more, this homeomorphism is fiberwise, meaning that it restricts to a homeomorphism from $\mathcal{T}(S) \times \{\rho\}$ to $\text{Dom}(\rho) \times \{\rho\}$ for any $\rho \in \text{Rep}_{nf}(S, \text{Isom}(M, g_M))$.

Specializing to the case where M is the hyperbolic plane \mathbb{H}^2 , we obtain a topological description of the space of admissible pairs and we classify in particular its connected components. Recall that, by work of Goldman [9], the space $\text{Rep}(S, \text{PSL}(2, \mathbb{R}))$ has $2|\chi(S)| + 1$ connected components, classified by the Euler class. Moreover, Fuchsian representations are exactly the representations of Euler class $\pm\chi(S)$. Let us denote by $\text{Rep}_k(S, \text{PSL}(2, \mathbb{R}))$ the connected component of representations of Euler class k . The precise topology of $\text{Rep}_k(S, \text{PSL}(2, \mathbb{R}))$ has been described by Hitchin [13].

Corollary 0.6 *The space $\text{Dom}(S, \text{PSL}(2, \mathbb{R}))$ of admissible pairs is homeomorphic to*

$$\mathcal{T}(S) \times \bigsqcup_{\chi(S) < k < -\chi(S)} \text{Rep}_k(S, \text{PSL}(2, \mathbb{R})).$$

In particular, it has $2|\chi(S)| - 1$ connected components, classified by the Euler class of the non-Fuchsian representation in each pair.

This answers Question 2.2 in a recent survey on anti-de Sitter geometry coauthored by Barbot, Bonsante, Danciger, Goldman, Guéritaud, Kassel, Krasnov, Schlenker and Zeghib [1].

0.3 Application to Thurston’s asymmetric distance

The map Ψ_ρ depends nontrivially on the choice of a normalization of the metric on M . Fix a metric g_0 on M of sectional curvature ≤ -1 and a constant $\alpha \geq 1$. Then the metric $(1/\alpha^2)g_0$ still has sectional curvature ≤ -1 . To mark the dependence of the function Lip on the metric on the target, we denote by $\text{Lip}_g(j, \rho)$ the minimal Lipschitz constant of a (j, ρ) -equivariant map from (\mathbb{H}^2, g_P) to (M, g) . Then we clearly have $\text{Lip}_{(1/\alpha^2)g_0} = (1/\alpha) \text{Lip}_{g_0}$. Hence, if we apply [Theorem 0.5](#) to $(M, (1/\alpha^2)g_0)$, we obtain a description of the space of pairs $(j, \rho) \in \mathcal{T}(S) \times \text{Rep}(\pi_1(S), G)$ such that $\text{Lip}_{g_0}(j, \rho) < \alpha$. For $\alpha > 1$, the curvature of $(M, (1/\alpha^2)g_0)$ is everywhere < -1 . Therefore, there is no totally geodesic plane of curvature -1 in $(M, (1/\alpha^2)g_0)$ and we obtain the following:

Corollary 0.7 *Let S be a closed oriented surface of negative Euler characteristic, (M, g_M) a Riemannian CAT(−1) space and α a constant bigger than 1. Then the domain*

$$\{(j, \rho) \in \mathcal{T}(S) \times \text{Rep}(S, \text{Isom}(M)) \mid \text{Lip}_{g_M}(j, \rho) < \alpha\}$$

is fiberwise homeomorphic to

$$\mathcal{T}(S) \times \text{Rep}(S, \text{Isom}(M)).$$

When applied to $M = (\mathbb{H}^2, g_P)$ — where g_P denotes the Poincaré metric — we obtain a description of left open balls for Thurston’s *asymmetric distance* on $\mathcal{T}(S)$ [\[22\]](#). Recall that this distance is defined by

$$d_{\text{Th}}(j, j') = \log(\text{Lip}_{g_P}(j, j'))$$

for j and j' any two representations of Euler class $-\chi(S)$. The function d_{Th} is continuous, positive whenever j and j' are distinct, and satisfies the triangular inequality. However, it is not symmetric.

Fix a point j_0 in $\mathcal{T}(S)$ and a constant $C > 0$. Then, using the convexity of length functions on $\mathcal{T}(S)$ (see for instance the work of Wolpert [\[26\]](#)) — together with an alternative definition of the asymmetric distance — one can see that the domain

$$\{j \in \mathcal{T}(S) \mid d_{\text{Th}}(j_0, j) < C\}$$

is an open convex domain of $\mathcal{T}(S)$ for the Weil–Petersson metric. In particular, it is homeomorphic to a ball of dimension $-3\chi(S)$. Since the distance is asymmetric, it is not clear whether the same holds for

$$\{j \in \mathcal{T}(S) \mid d_{\text{Th}}(j, j_0) < C\}.$$

However, since

$$d_{\text{Th}}(j, j_0) < C \iff \text{Lip}(j, j_0) < e^C,$$

we can apply [Corollary 0.7](#) and obtain:

Corollary 0.8 *Let S be a closed oriented surface of negative Euler characteristic and j_0 a point in $\mathcal{T}(S)$. Then, for any $C > 0$, the domain*

$$\{j \in \mathcal{T}(S) \mid d_{\text{Th}}(j, j_0) < C\}$$

is homeomorphic to $\mathcal{T}(S)$. In other words, left open balls for Thurston's asymmetric distance on $\mathcal{T}(S)$ are contractible.

0.4 Admissible pairs and anti-de Sitter 3-manifolds

We conclude this introduction by a few remarks on the relation between the space of admissible pairs and the deformation space of anti-de Sitter 3-manifolds. Some more details are given in the author's thesis [\[21, Section 4.4\]](#).

Topology of the quotients Let $\text{Adm}_k(S)$ denote the space of admissible pairs (j, ρ) with j Fuchsian of positive Euler class and ρ of Euler class k , modulo the action of $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ by conjugation. If (j, ρ) is in $\text{Adm}_k(S)$, then the quotient

$$j \times \rho(\pi_1(S)) \backslash \text{PSL}(2, \mathbb{R})$$

is a circle bundle over S of Euler class $|\chi(S)| - k$. In particular, different values of k lead to nonhomeomorphic anti-de Sitter 3-manifolds.

Deformation space Let E be a circle bundle over S of Euler class $|\chi(S)| - k$, with $\chi(S) < k < -\chi(S)$. The *deformation space* of anti-de Sitter structures on E is the space of anti-de Sitter metrics on E modulo isotopy. All these anti-de Sitter structures arise from identifying E with a quotient of a *finite cover* of $\text{PSL}(2, \mathbb{R})$. It follows from [Corollary 0.6](#) that the connected components of this deformation space are essentially classified by the degree of this cover. Among these connected components, the one corresponding to quotients of $\text{PSL}(2, \mathbb{R})$ is homeomorphic to an abelian covering of $\text{Adm}_k(S) \simeq \mathcal{T}(S) \times \text{Rep}_k(S)$. (This abelian covering comes from the fact that the group of isotopy classes of bundle isomorphisms of E is \mathbb{Z}^{2g} , where g is the genus of S .)

AdS circle bundles of higher Euler class A circle bundle over S of Euler class $c \geq 2|\chi(S)|$ also admits anti-de Sitter structures. However, it is known since the work of Kulkarni and Raymond [\[16\]](#) that these structures are all standard. To construct them, one can lift the action on the left of a Fuchsian representation to the covering of degree $|\chi(S)|$ of $\text{PSL}(2, \mathbb{R})$, and then quotient on the right by a cyclic subgroup of order c .

Some deformation spaces are not Hausdorff Note that $\text{Rep}_0(S)$ and thus $\text{Adm}_0(S)$ is not Hausdorff, because the orbit under conjugation of a nonabelian parabolic representation (ie fixing a point at infinity in \mathbb{H}^2) is not closed. This actually reflects an irregularity of some deformation space: if (E, g) is an anti-de Sitter 3-manifold isometric to

$$j \times \rho(\pi_1(S)) \backslash \text{PSL}(2, \mathbb{R})$$

with ρ parabolic and nonabelian, then there exists a sequence ϕ_n of isotopies of E such that $\phi_n^* g$ converges to an anti-de Sitter metric on E that is not isometric to g . A similar phenomenon occurs for instance in the deformation space of certain closed anti-de Sitter 3-manifolds with tachyons; see Danciger [4]. This could not happen if the metric g were Riemannian.

Moduli space The isomorphism between $\mathcal{T}(S) \times \text{Rep}_k(S, \text{PSL}(2, \mathbb{R}))$ and $\text{Adm}_k(S)$ that we will construct is naturally equivariant with respect to the diagonal action of the mapping class group of S . The quotient by this diagonal action parametrizes a connected component of the *moduli space* of anti-de Sitter 3-manifolds homeomorphic to a circle bundle over S of Euler class $|\chi(S)| - k$.

Reversing the orientation of ρ There is a curious duality between closed anti-de Sitter 3-manifolds. Let $\sigma: \text{PSL}(2, \mathbb{R}) \rightarrow \text{PSL}(2, \mathbb{R})$ denote the conjugation by an orientation-reversing isometry of \mathbb{H}^2 . If (j, ρ) is an admissible pair, then obviously $(j, \sigma \circ \rho)$ is also an admissible pair. However, since $\mathbf{eu}(\sigma \circ \rho) = -\mathbf{eu}(\rho)$, the anti-de Sitter 3-manifolds

$$j \times \rho(\pi_1(S)) \backslash \text{PSL}(2, \mathbb{R})$$

and

$$j \times (\sigma \circ \rho)(\pi_1(S)) \backslash \text{PSL}(2, \mathbb{R})$$

are not homeomorphic when $\mathbf{eu}(\rho) \neq 0$.

0.5 Structure of the article and strategy of the proof

The article is organized as follows. In the next section we recall some fundamental results about harmonic maps from a surface. In particular, we recall the Corlette–Labourie theorem of existence of equivariant harmonic maps and the Sampson–Hitchin–Wolf parametrization of the Teichmüller space by means of quadratic differentials. In the second section, we start by using those theorems to construct the map Ψ_ρ studied in Deroin and Tholozan [6], and we prove that Ψ_ρ is a homeomorphism from $\mathcal{T}(S)$ to $\text{Dom}(\rho)$. To do so, we make explicit the inverse of the map Ψ_ρ . It will appear that reverse images of a point j in $\mathcal{T}(S)$ by Ψ_ρ are exactly critical points of a certain functional $F_{j,\rho}$. We will prove that when j is in $\text{Dom}(\rho)$, the functional $F_{j,\rho}$ is

proper and admits a unique critical point, which is a global minimum. Hence, the map Ψ_ρ is bijective. What's more, the functionals $F_{j,\rho}$ vary continuously with (j, ρ) , and so does their unique minimum. This will prove the continuity of Ψ_ρ^{-1} and its continuous dependence in ρ .

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1 Representations of surface groups, harmonic maps and functionals on Teichmüller space

In this section, we introduce briefly the tools from the theory of harmonic maps that we will need later. We refer to [5] for a more thorough survey.

1.1 Existence theorems

Recall that the *energy density* of a nonnegative symmetric 2-form h on a Riemannian manifold (X, g) is the function on X defined as

$$e_g(h) = \frac{1}{2} \text{Tr}(A(h)),$$

where $A(h)$ is the unique field of endomorphisms of the tangent bundle such that, for all $x \in X$ and all $u, v \in T_x X$,

$$h_x(u, v) = g_x(u, A(h)_x v).$$

If f is a map between two Riemannian manifolds (X, g_X) and (Y, g_Y) , then its energy density is the energy density on X of the pull-back metric:

$$e_{g_X}(f) = e_{g_X}(f^* g_Y).$$

This energy density can be integrated against the volume form vol_{g_X} associated to g_X , giving the *total energy* of f :

$$E_{g_X}(f) = \int_X e_{g_X}(f) \text{vol}_{g_X}.$$

The map f is called *harmonic* if it is, in some sense, a critical point of the total energy. For instance, harmonic maps from \mathbb{R} to a Riemannian manifold are geodesics and harmonic maps from a Riemannian manifold to \mathbb{R} are harmonic functions. In general, a map f is harmonic if it satisfies a certain partial differential equation that can be expressed as the vanishing of the *tension field*. Here we will be satisfied with existence results and some fundamental properties of these maps.

The first existence result, due to Eells and Sampson [8], deals with harmonic maps between compact manifolds. Their paper contains a thorough study of the analytic aspects of harmonicity, which allows extensions of their theorem in several cases. The one we will be interested in is an equivariant version.

Consider (X, g_X) a closed Riemannian manifold, (\tilde{X}, \tilde{g}_X) its universal cover, another Riemannian manifold (M, g_M) and ρ a representation of $\pi_1(X)$ into $\text{Isom}(M)$. A map $f: \tilde{X} \rightarrow M$ is called ρ -equivariant if, for all $x \in \tilde{X}$ and all $\gamma \in \pi_1(X)$, we have

$$f(\gamma \cdot x) = \rho(\gamma) \cdot f(x).$$

Given such a map, the symmetric 2-form f^*g_M on \tilde{X} is invariant under the action of $\pi_1(X)$ and thus induces a symmetric 2-form on X . We define the energy density and the total energy of f as the energy density and the total energy of this symmetric 2-form.

The equivariant version of Eells and Sampson's theorem is due to Donaldson [7] and Corlette [3] in the specific case where M is a symmetric space of noncompact type, and to Labourie [17] in the more general case of a complete simply connected Riemannian manifold of nonpositive curvature. We state it in the particular case where (M, g_M) is a Riemannian CAT(−1) space (ie a complete simply connected Riemannian manifold of sectional curvature ≤ -1). Recall that a CAT(−1) space is in particular *Gromov hyperbolic*. It thus admits a *boundary at infinity* such that every isometry of M extends to a homeomorphism of the boundary.

Theorem (Donaldson, Corlette, Labourie) *Let (X, g_X) be a closed Riemannian manifold, \tilde{X} its universal cover, and (M, g_M) a Riemannian CAT(−1) space. Let ρ be a representation of $\pi_1(X)$ into $\text{Isom}(M)$. Assume that ρ does not fix a point in $\partial_\infty M$. Then there exists a unique harmonic map from $(\tilde{X}, g_{\tilde{X}})$ to (M, g_M) that is ρ -equivariant. This map minimizes the energy among all such equivariant maps.*

1.1.1 Dealing with representations fixing a point at infinity In the sequel, representations fixing a point in $\partial_\infty M$ will be called *parabolic representations*. When M is the symmetric space of some rank 1 Lie group G , those are exactly the representations taking values into some parabolic subgroup of G . For some of these representations,

the theorem of Corlette and Labourie does not apply, and we must say a few words in order to be able to include them in the proof later.

The main problem with a parabolic representation ρ is that a sequence of equi-Lipschitz ρ -equivariant maps from \tilde{X} to M may not converge. To deal with them, one can “replace” them by representations into the group of translations of the real line. More precisely, if p is a fixed point of ρ in $\partial_\infty M$, consider B_p , a Busemann function centred at p , and set

$$m_\rho: \pi_1(X) \rightarrow \mathbb{R},$$

$$\gamma \mapsto B_p(\gamma \cdot x) - B_p(x),$$

for some base point x . Then m_ρ is a homomorphism into the group of translations of the real line and, for any Fuchsian representation j , one has

$$\text{Lip}(j, \rho) = \text{Lip}(j, m_\rho).$$

Moreover, if f_n is a sequence of smooth ρ -equivariant maps from \tilde{X} to M whose energy converges to the infimum of the energies of all such equivariant maps, then the symmetric 2-form $f_n^* g_M$ converges in L^2 -norm to

$$f^* dx^2,$$

where f is an m_ρ -equivariant harmonic function from \tilde{X} to \mathbb{R} and dx^2 is the canonical metric on \mathbb{R} (see [21, Proposition 3.3.8]). The function f is obtained by integrating the unique harmonic 1-form on X whose periods are given by m_ρ , and is unique up to a translation.

In some sense, this m_ρ -equivariant function f is the natural extension of the notion of ρ -equivariant harmonic map to parabolic representations. In particular, when ρ fixes two points in $\partial_\infty M$, it stabilizes a unique geodesic $c: \mathbb{R} \rightarrow M$; in that case, ρ -equivariant harmonic maps exist and have the form

$$c \circ f.$$

1.2 Harmonic maps from a surface, Hopf differential, and Teichmüller space

From now on we will restrict to harmonic maps from a Riemann surface. In that case, the energy of a map only depends on the conformal class of the Riemannian metric on the base. For the same reason, harmonicity is invariant under a conformal change of the metric.

1.2.1 Hopf differential Let S be an oriented surface equipped with a Riemannian metric g . Let (M, g_M) be a Riemannian manifold, and $f: S \rightarrow M$ a smooth map. The conformal class of g induces a complex structure on S . The symmetric 2-form f^*g_M can thus be uniquely decomposed into a $(1, 1)$ -part, a $(2, 0)$ -part and a $(0, 2)$ -part. One can check that the $(1, 1)$ -part is $e_g(f)g$, and we thus have

$$f^*g_M = e_g(f)g + \Phi_f + \bar{\Phi}_f,$$

where Φ_f is a *quadratic differential*, ie a section of the square of the canonical bundle of $(S, [g])$, called the *Hopf differential* of f . The following proposition is classical.

Proposition 1.1 *If the map f is harmonic, then its Hopf differential is holomorphic. The converse is true if M is also a surface.*

1.2.2 The Teichmüller space Let S be a closed oriented surface of negative Euler characteristic. Recall that the Teichmüller space of S can be seen alternatively as the space of complex structures on S modulo isotopy, the space of hyperbolic structures on S modulo isotopy or the space of Fuchsian representations of $\pi_1(S)$ into $\mathrm{PSL}(2, \mathbb{R})$ of positive Euler class modulo conjugation. Throughout the paper, a point in $\mathcal{T}(S)$ is denoted alternately by the letter X when we think of it as the surface S equipped with a complex structure, or by the letter j when we think of it as a Fuchsian representation.

It is well known that the Teichmüller space is a manifold diffeomorphic $\mathbb{R}^{-3\chi(S)}$ and that it carries a complex structure. Consider two points X_1 and X_2 in $\mathcal{T}(S)$ corresponding to two hyperbolic metrics g_1 and g_2 on S . Then, by the Eells–Sampson theorem, there is a unique harmonic map $f_{g_1, g_2}: (S, g_1) \rightarrow (S, g_2)$ homotopic to the identity map. Schoen and Yau’s theorem [20] (also proved by Sampson [19] in that specific case) states that this map is a diffeomorphism.

Of course, the map f_{g_1, g_2} depends on the choice of g_1 and g_2 up to isotopy. Actually, fixing g_1 or g_2 , one can choose the other metric so that the identity map itself is harmonic (by replacing g_2 by $f_{g_1, g_2}^*g_2$ or g_1 by $f_{g_1, g_2}^*g_1$). On the other hand, the total energy of f_{g_1, g_2} is invariant under changing one of the metrics by isotopy, and thus gives a well-defined functional

$$E: \mathcal{T}(S) \times \mathcal{T}(S) \rightarrow \mathbb{R}_+.$$

Besides, the Hopf differential of f_{g_1, g_2} is invariant under isotopic changes of g_2 , and if h is a diffeomorphism of S , one has

$$\Phi_{f_{h^*g_1, g_2}} = h^*\Phi_{f_{g_1, g_2}}.$$

The Hopf differential thus induces a well defined map

$$\Phi: \mathcal{T}(S) \times \mathcal{T}(S) \rightarrow \mathrm{QDT}(S),$$

where $\mathrm{QDT}(S)$ denotes the complex bundle of holomorphic quadratic differentials on $\mathcal{T}(S)$, that is, $\mathrm{QD}_X \mathcal{T}(S)$ is the space of holomorphic quadratic differentials on X , which identifies to the cotangent bundle to $\mathcal{T}(S)$.

Sampson [19] proved that the map Φ is an injective immersion, and Wolf [25] proved its surjectivity. This was obtained independently by Hitchin [13], as the first construction of a section to the *Hitchin fibration*.

Theorem (Sampson, Hitchin, Wolf) *The map $\Phi: \mathcal{T}(S) \times \mathcal{T}(S) \rightarrow \mathrm{QDT}(S)$ is a homeomorphism.*

1.3 Equivariant harmonic maps and functionals on $\mathcal{T}(S)$

The theorem of Corlette and Labourie allows us to extend the maps E and Φ to $\mathcal{T}(S) \times \mathrm{Rep}(S, \mathrm{Isom}(M))$. Given a point in $X_0 \in \mathcal{T}(S)$, represented by a hyperbolic metric g_0 , and a point ρ in $\mathrm{Rep}(\pi_1(S), \mathrm{Isom}(M))$, one can consider $f_{g_0, \rho}$, the unique ρ -equivariant harmonic map from (\tilde{S}, \tilde{g}_0) to (M, g_M) . (See Remark 1.3 for the case where ρ is parabolic.)

The energy density and the Hopf differential of $f_{g_0, \rho}$ only depend on the conjugacy class of the representation ρ . We thus obtain two well-defined maps

$$E: \mathcal{T}(S) \times \mathrm{Rep}(S, \mathrm{Isom}(M)) \rightarrow \mathbb{R}_+,$$

$$(X_0, \rho) \mapsto \int_S e_{g_0}(f_{g_0, \rho}) \mathrm{vol}_{g_0},$$

and

$$\Phi: \mathcal{T}(S) \times \mathrm{Rep}(S, \mathrm{Isom}(M)) \rightarrow \mathrm{QDT}(S),$$

$$(X_0, \rho) \mapsto \Phi_{f_{g_0, \rho}}.$$

Remark 1.2 If $M = \mathbb{H}^2$, we have $\mathrm{Isom}^+(M) \simeq \mathrm{PSL}(2, \mathbb{R})$. Consider two points $X, X_1 \in \mathcal{T}(S)$, and let j_1 be a holonomy representation of a hyperbolic metric g_1 corresponding to X_1 . Then a harmonic map from X to X_1 isotopic to the identity lifts to a j_1 -equivariant harmonic map from \tilde{X} to \mathbb{H}^2 . We thus have

$$E(X, j_1) = E(X, X_1)$$

and

$$\Phi(X, j_1) = \Phi(X, X_1).$$

In other words, the new definition of E and Φ extends the one given in the previous paragraph, via the identification of $\mathcal{T}(S)$ with the component of maximal Euler class in $\text{Rep}(S, \text{PSL}(2, \mathbb{R}))$.

Remark 1.3 E and Φ are a priori defined for representations ρ that are not parabolic. When ρ is parabolic, we introduce the representation $m_\rho: \Gamma \rightarrow \mathbb{R}$ given in [Section 1.1.1](#). Let $f_{X,m_\rho}: \tilde{X} \rightarrow \mathbb{R}$ be a ρ -equivariant harmonic function. We set

$$E(X, \rho) = E(f_{X,m_\rho}) = \inf\{E(f) \mid f: \tilde{X} \rightarrow M \text{ } \rho\text{-equivariant}\}$$

and

$$\Phi(X, \rho) = \Phi_{f_{X,m_\rho}}.$$

Proposition 1.4 *The function $E(X, \rho)$ and the map $\Phi(X, \rho)$ are continuous with respect to X and ρ .*

Proof This is a classical consequence of the ellipticity of the equation defining harmonic maps. Let (j_n, ρ_n) converge to (j, ρ) . Then the derivatives of a (j_n, ρ_n) -equivariant harmonic map f_n can be uniformly controlled by its total energy. When ρ is not parabolic, one deduces that the sequence f_n converges in C^1 topology (up to taking a subsequence) to a (j, ρ) -equivariant harmonic map f . The proposition easily follows. When ρ fixes a point p in $\partial_\infty M$, one can use the fact that the map $B_p \circ f_n$ converges up to translation to a m_ρ -equivariant harmonic function. \square

We will make crucial use of the following results:

Proposition 1.5 (see [\[24\]](#)) *Let (M, g_M) be a Riemannian CAT(−1) space and ρ a representation of $\pi_1(S)$ into $\text{Isom}(M)$. Then the functional*

$$\begin{aligned} E(\cdot, \rho): \mathcal{T}(S) &\rightarrow \mathbb{R}_+, \\ X &\mapsto E(X, \rho), \end{aligned}$$

is C^1 and its differential at a point $X_0 \in \mathcal{T}(S)$ is given by

$$dE(\cdot, \rho)(X_0) = -4\Phi(X_0, \rho).$$

Theorem 1.6 (Tromba [\[23\]](#)) *Consider a point j in $\mathcal{T}(S)$. Then the function*

$$\begin{aligned} E(\cdot, j): \mathcal{T}(S) &\rightarrow \mathbb{R}_+, \\ X &\mapsto E(X, j), \end{aligned}$$

is proper.

2 The homeomorphism from $\mathcal{T}(S)$ to $\text{Dom}(\rho)$

We are now in possession of all the tools required to define the map Ψ_ρ introduced in [6] and to prove that it is a homeomorphism from $\mathcal{T}(S)$ to $\text{Dom}(\rho)$.

2.1 Construction of the map Ψ_ρ

Let S be a closed oriented surface of negative Euler characteristic and ρ a representation of $\pi_1(S)$ into the isometry group of a Riemannian $\text{CAT}(-1)$ space (M, g_M) . Let X_1 be a point in $\mathcal{T}(S)$. Then $\Phi(X_1, \rho)$ is a holomorphic quadratic differential on X_1 , and the theorem of Sampson, Hitchin and Wolf asserts that there is a unique point j_2 in $\mathcal{T}(S)$ such that $\Phi(X_1, j_2) = \Phi(X_1, \rho)$. Setting

$$\Psi_\rho(X_1) = j_2,$$

we obtain a well-defined map

$$\Psi_\rho: \mathcal{T}(S) \rightarrow \mathcal{T}(S).$$

This map only depends on the class of ρ under conjugation by $\text{Isom}(M)$. We can thus define a map

$$\begin{aligned} \Psi: \mathcal{T}(S) \times \text{Rep}(S, \text{Isom}(M)) &\rightarrow \mathcal{T}(S) \times \text{Rep}(S, \text{Isom}(M)), \\ (X, \rho) &\mapsto (\Psi_\rho(X), \rho). \end{aligned}$$

Deroin and the author proved the following:

Theorem [6] *If ρ is not Fuchsian, then the image of the map Ψ_ρ lies in $\text{Dom}(\rho)$. In particular, $\text{Dom}(\rho)$ is nonempty.*

2.2 Surjectivity of the map Ψ_ρ

We first prove the following:

Proposition 2.1 *The map $\Psi_\rho: \mathcal{T}(S) \rightarrow \text{Dom}(\rho)$ is surjective.*

Proof Fix $j_0 \in \mathcal{T}(S)$ and $\rho \in \text{Rep}(S, \text{Isom}(M))$. Let us introduce the functional

$$\begin{aligned} F_{j_0, \rho}: \mathcal{T}(S) &\rightarrow \mathbb{R}, \\ X &\mapsto E(X, j_0) - E(X, \rho). \end{aligned}$$

By Proposition 1.5, the map $F_{j_0, \rho}$ is C^1 and its differential is given by

$$dF_{j_0, \rho}(X) = -4\Phi(X, j_0) + 4\Phi(X, \rho).$$

Hence X_1 is a critical point of $F_{j_0, \rho}$ if and only if $\Phi(X_1, j_0) = \Phi(X_1, \rho)$, which means precisely that

$$\Psi_\rho(X_1) = j_0.$$

Proving that j_0 is in the image of Ψ_ρ is thus equivalent to proving that the map $F_{j_0, \rho}$ admits a critical point. This will be a consequence of the following lemma:

Lemma 2.2 *For $j_0 \in \mathcal{T}(S)$ and $\rho \in \text{Rep}(S, \text{Isom}(M))$, we have*

$$F_{j_0, \rho} \geq (1 - \text{Lip}(j_0, \rho)^2)E(\cdot, j_0).$$

Proof Let f be a (j_0, ρ) -equivariant Lipschitz map from \mathbb{H}^2 to M with Lipschitz constant $\text{Lip}(j_0, \rho) + \epsilon$. Let X be a point in $\mathcal{T}(S)$ represented by some hyperbolic metric g , and let h be a j_0 -equivariant map from \tilde{X} to \mathbb{H}^2 such that $E_g(h) \leq (1 + \epsilon)E(X, j_0)$. The map $f \circ h$ is a ρ -equivariant map from \tilde{X} to M , and we thus have

$$E(X, \rho) \leq E_g(f \circ h).$$

Since f is $(\text{Lip}(j_0, \rho) + \epsilon)$ -Lipschitz, we have

$$E(X, \rho) \leq E_g(f \circ h) \leq (\text{Lip}(j_0, \rho) + \epsilon)^2 E_g(h) = (1 + \epsilon)(\text{Lip}(j_0, \rho) + \epsilon)^2 E(X, j_0),$$

from which we get

$$F_{j_0, \rho}(X) = E(X, j_0) - E(X, \rho) \geq (1 - (1 + \epsilon)(\text{Lip}(j_0, \rho) + \epsilon)^2)E(X, j_0).$$

This being true for any $\epsilon > 0$, we obtain the required inequality. \square

Now, by [Theorem 1.6](#), the map $X \rightarrow E(X, j_0)$ is proper. Therefore, if j_0 is in $\text{Dom}(\rho)$, we have $1 - \text{Lip}(j_0, \rho)^2 > 0$ and the function $F_{j_0, \rho}$ is also proper. Hence $F_{j_0, \rho}$ admits a minimum. This minimum is a critical point, and thus a preimage of j_0 by the map Ψ_ρ . We obtain that $\Psi_\rho: \mathcal{T}(S) \rightarrow \text{Dom}(\rho)$ is surjective. \square

We proved that if j_0 is in $\text{Dom}(\rho)$, the functional $F_{j_0, \rho}$ is proper. Note that, conversely, if $F_{j_0, \rho}$ is proper, then it admits a critical point. Hence j_0 is in the image of Ψ_ρ , which implies that j_0 lies in $\text{Dom}(\rho)$ by [\[6\]](#). We thus obtain the following corollary, which might be interesting in its own right:

Corollary 2.3 *The following are equivalent:*

- (i) *The map $F_{j_0, \rho}$ is proper.*
- (ii) *The map $F_{j_0, \rho}$ admits a critical point.*
- (iii) *The representation j_0 strictly dominates ρ .*

2.3 Injectivity of the map Ψ_ρ

To prove injectivity, we need to prove that, when j_0 is in $\text{Dom}(\rho)$, the critical point of $F_{j_0, \rho}$ is unique. To do so, we prove that any critical point of $F_{j_0, \rho}$ is a strict minimum of $F_{j_0, \rho}$.

Let X_1 be a critical point of $F_{j_0, \rho}$, and X_2 another point in $\mathcal{T}(S)$. Choose a hyperbolic metric g_1 on S representing X_1 . Let g_0 be the hyperbolic metric of holonomy j_0 such that $\text{Id}: (S, g_1) \rightarrow (S, g_0)$ is harmonic, and g_2 the hyperbolic metric representing X_2 such that $\text{Id}: (S, g_2) \rightarrow (S, g_0)$ is harmonic. Let $f: (\tilde{S}, \tilde{g}_1) \rightarrow (M, g_M)$ be a ρ -equivariant harmonic map. We have the decompositions

$$\begin{aligned} g_0 &= e_{g_1}(g_0)g_1 + \Phi + \bar{\Phi}, \\ f^*g_M &= e_{g_1}(f)g_1 + \Phi + \bar{\Phi}, \\ g_2 &= e_{g_1}(g_2)g_1 + \Psi + \bar{\Psi}, \end{aligned}$$

where Φ and Ψ are quadratic differentials on S , with Φ holomorphic with respect to the complex structure induced by g_1 . Note that the same Φ appears in the decomposition of g_0 and f^*g_M because X_1 is a critical point of $F_{j_0, \rho}$, and thus $\Phi(X_1, j_0) = \Phi(X_1, \rho)$.

Remark 2.4 If ρ is parabolic, the proof is still valid, provided that we replace f^*g_M by f^*dx^2 , where f is an m_ρ -equivariant harmonic function.

Lemma 2.5 *We have the following identity:*

$$E_{g_2}(g_0) - E_{g_2}(f^*g_M) = \int_S \frac{1}{\sqrt{1 - 4\|\Psi\|_{g_1}^2/e_{g_1}(g_2)^2}} (e_{g_1}(g_0) - e_{g_1}(f^*g_M)) \text{vol}_{g_1}.$$

Proof of Lemma 2.5 This is a rather basic computation that we will carry out in local coordinates. Let $z = x + iy$ be a local complex coordinate with respect to which g_1 is conformal. We denote by $\langle \cdot, \cdot \rangle$ the standard scalar product on \mathbb{C}

Any symmetric 2-form on \mathbb{C} can be written under the form $\langle \cdot, G \cdot \rangle$, where G is a field of symmetric endomorphisms of \mathbb{R}^2 depending on the coordinates (x, y) . We will represent such an endomorphism by its matrix in the canonical frame $(\partial/\partial x, \partial/\partial y)$. In local coordinates, we can thus write

$$\begin{aligned} g_0 &= \langle \cdot, G_0 \cdot \rangle, \\ g_1 &= \langle \cdot, G_1 \cdot \rangle, \\ g_2 &= \langle \cdot, G_2 \cdot \rangle, \\ f^*g_M &= \langle \cdot, G_f \cdot \rangle. \end{aligned}$$

Now, since g_1 is conformal with respect to the coordinate z , we have $g_1 = \alpha \langle \cdot, \cdot \rangle$ for some positive function α , and we can write

$$\Phi = \phi \, dz^2,$$

$$\Psi = \psi \, dz^2,$$

for some complex valued functions ϕ and ψ . (Since Φ is holomorphic, ϕ must be holomorphic, but we won't need it for our computation.)

We can now express G_0 , G_1 , G_2 , G_f , vol_{g_1} and vol_{g_2} in terms of α , $e_{g_1}(g_0)$, $e_{g_1}(g_2)$, $e_{g_1}(f)$, ϕ and ψ . We easily check that

$$G_1 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix},$$

$$G_0 = \begin{pmatrix} \alpha e_{g_1}(g_0) + 2 \operatorname{Re}(\phi) & -2 \operatorname{Im}(\phi) \\ -2 \operatorname{Im}(\phi) & \alpha e_{g_1}(g_0) - 2 \operatorname{Re}(\phi) \end{pmatrix},$$

$$G_f = \begin{pmatrix} \alpha e_{g_1}(g_f) + 2 \operatorname{Re}(\phi) & -2 \operatorname{Im}(\phi) \\ -2 \operatorname{Im}(\phi) & \alpha e_{g_1}(g_f) - 2 \operatorname{Re}(\phi) \end{pmatrix},$$

$$G_2 = \begin{pmatrix} \alpha e_{g_1}(g_2) + 2 \operatorname{Re}(\psi) & -2 \operatorname{Im}(\psi) \\ -2 \operatorname{Im}(\psi) & \alpha e_{g_1}(g_2) - 2 \operatorname{Re}(\psi) \end{pmatrix},$$

$$\text{vol}_{g_1} = \alpha \, dz \, d\bar{z},$$

$$\text{vol}_{g_2} = \sqrt{\det G_2} \, dz \, d\bar{z} = \sqrt{\alpha^2 e_{g_1}(g_2)^2 - 4|\psi|^2} \, dz \, d\bar{z}.$$

We now want to express $e_{g_2}(g_0)$ and $e_{g_2}(f^* g_M)$. To do so, note that we can write

$$g_0(\cdot, \cdot) = \langle \cdot, G_0 \cdot \rangle = \langle \cdot, G_2(G_2^{-1} G_0) \cdot \rangle = g_2(\cdot, G_2^{-1} G_0 \cdot).$$

By definition of the energy density, we thus obtain that

$$\begin{aligned} e_{g_2}(g_0) &= \frac{1}{2} \operatorname{Tr}(G_2^{-1} G_0) \\ &= \frac{1}{2} \operatorname{Tr} \left[\frac{1}{\det G_2} \begin{pmatrix} \alpha e_{g_1}(g_2) - 2 \operatorname{Re}(\psi) & 2 \operatorname{Im}(\psi) \\ 2 \operatorname{Im}(\psi) & \alpha e_{g_1}(g_2) + 2 \operatorname{Re}(\psi) \end{pmatrix} \right. \\ &\quad \left. \times \begin{pmatrix} \alpha e_{g_1}(g_0) + 2 \operatorname{Re}(\phi) & -2 \operatorname{Im}(\phi) \\ -2 \operatorname{Im}(\phi) & \alpha e_{g_1}(g_0) - 2 \operatorname{Re}(\phi) \end{pmatrix} \right] \\ &= \frac{1}{\alpha^2 e_{g_1}(g_2)^2 - 4|\psi|^2} (\alpha^2 e_{g_1}(g_2) e_{g_1}(g_0) - (\phi \bar{\psi} + \bar{\phi} \psi)). \end{aligned}$$

Similarly, we get that

$$e_{g_2}(f^* g_M) = \frac{1}{\alpha^2 e_{g_1}(g_2)^2 - 4|\psi|^2} (\alpha^2 e_{g_1}(g_2) e_{g_1}(f) - (\phi \bar{\psi} + \bar{\phi} \psi)).$$

When computing the difference, the terms $\phi\bar{\psi} + \bar{\phi}\psi$ simplify. (Here we use the fact that f^*g_M and g_0 have the same $(2, 0)$ -part.) We eventually obtain

$$\begin{aligned}(e_{g_2}(g_0) - e_{g_2}(f^*g_M))\text{vol}_{g_2} &= \frac{\alpha^2 e_{g_1}(g_2)(e_{g_2}(g_0) - e_{g_2}(f^*g_M))}{\sqrt{\alpha^2 e_{g_1}(g_2)^2 - 4|\psi|^2}} dz d\bar{z} \\ &= \frac{e_{g_2}(g_0) - e_{g_2}(f^*g_M)}{\sqrt{1 - 4\|\Psi\|_{g_1}^2/e_{g_1}(g_2)^2}} \text{vol}_{g_1}.\end{aligned}$$

Now, the parameters of the last expression are well-defined functions on S and the identity is true in any local chart. It is thus true everywhere on S and, when integrating, we obtain [Lemma 2.5](#). \square

From [Lemma 2.5](#), we obtain that

$$\begin{aligned}E_{g_2}(g_0) - E_{g_2}(f^*g_M) &\geq \int_S (e_{g_1}(g_0) - e_{g_1}(f^*g_M)) \text{vol}_{g_1} \\ &= E_{g_1}(g_0) - E_{g_1}(f^*g_M) = F_{j_0, \rho}(X_1),\end{aligned}$$

with equality if and only if $\|\Psi\|_{g_1} \equiv 0$, that is, if g_1 is conformal to g_2 .

On the other side, we have $E_{g_2}(g_0) = E(X_2, j_0)$ (since, by hypothesis, the identity map from (S, g_2) to (S, g_0) is harmonic) and $E_{g_2}(f^*g_M) \geq E(X_2, \rho)$, from which we deduce that

$$E_{g_2}(g_0) - E_{g_2}(f^*g_M) \leq F_{j_0, \rho}(X_2).$$

Combining the two inequalities, we obtain that

$$F_{j_0, \rho}(X_2) \geq F_{j_0, \rho}(X_1),$$

with equality if and only if $X_1 = X_2$.

Now, if X_1 and X_2 are two critical points of $F_{j_0, \rho}$, then by symmetry we must have $F_{j_0, \rho}(X_2) = F_{j_0, \rho}(X_1)$, and therefore $X_2 = X_1$. The functional $F_{j_0, \rho}$ admits a unique critical point, and j_0 admits a unique preimage by Ψ_ρ . Thus Ψ_ρ is injective.

2.4 Bicontinuity of Ψ

Recall that one has, by definition,

$$(X, \Psi_\rho(X)) = \Phi^{-1}(X, \Phi(X, \rho)),$$

where Φ^{-1} denotes the inverse of the map $\Phi: \mathcal{T}(S) \times \mathcal{T}(S) \rightarrow \text{QDT}(S)$, which is a homeomorphism by the Sampson–Hitchin–Wolf theorem. Therefore, by [Proposition 1.4](#), the maps Ψ_ρ and Ψ are continuous.

Let us now prove that Ψ^{-1} is continuous. We saw that $\Psi^{-1}(j, \rho)$ is the unique critical point of a proper function $F_{j, \rho}$ on $\mathcal{T}(S)$ which depends continuously on j and ρ . The continuity of Ψ^{-1} will follow from the fact that the functions $F_{j, \rho}$ are in some sense *locally uniformly proper*.

Definition 2.6 Let X be a metric space, Y a topological space and $(F_y)_{y \in Y}$ a family of continuous functions from X to \mathbb{R} depending continuously on y for the compact-open topology. We say that the family $(F_y)_{y \in Y}$ is *uniformly proper* if, for any $C \in \mathbb{R}$, there exists a compact subset K of X such that, for all $y \in Y$ and all $x \in X \setminus K$, $F_y(x) > C$.

We say that the family $(F_y)_{y \in Y}$ is *locally uniformly proper* if for any $y_0 \in Y$ there is a neighbourhood U of y_0 such that the subfamily $(F_y)_{y \in U}$ is uniformly proper.

Proposition 2.7 Let X be a metric space, Y a topological space and $(F_y)_{y \in Y}$ a locally uniformly proper family of continuous functions from X to \mathbb{R} depending continuously on y (for the compact-open topology). Assume that each F_y achieves its minimum at a unique point $x_m(y) \in X$. Then the function

$$y \mapsto x_m(y)$$

is continuous.

Proof Let us denote by $m(y) = F_y(x_m(y))$ the minimum value of F_y . Fix $y_0 \in Y$. Let U be a neighbourhood of y_0 and K a compact subset of X such that for all $y \in U$ and all $x \in X \setminus K$, we have

$$F_y(x) > m(y_0) + 1.$$

For $\epsilon > 0$, define

$$V_\epsilon = \{x \in X \mid F_{y_0}(x) < m(y_0) + \epsilon\}.$$

Since F_{y_0} is proper and achieves its minimum at a single point $x_m(y_0)$, the family $(V_\epsilon)_{\epsilon > 0}$ forms a basis of neighbourhoods of $x_m(y_0)$. Let U_ϵ be a neighbourhood of y_0 included in U such that, for all $y \in U_\epsilon$ and all $x \in K$,

$$|F_y(x) - F_{y_0}(x)| < \frac{\epsilon}{2}.$$

(U_ϵ exists because the map $y \mapsto F_y$ is continuous for the compact-open topology.) Since $x_m(y_0)$ is obviously in K , we have, for all $y \in U_\epsilon$,

$$F_y(x_m(y_0)) < m(y_0) + \frac{\epsilon}{2},$$

hence the minimum value $m(y)$ of F_y is smaller than $m(y_0) + \frac{\epsilon}{2}$. In particular, for $\epsilon < 2$, this minimum is achieved in K (since outside K , we have $F_y \geq m(y_0) + 1$). We thus have $x_m(y) \in K$, from which we deduce

$$F_{y_0}(x_m(y)) < F_y(x_m(y)) + \frac{\epsilon}{2} = m(y) + \frac{\epsilon}{2} < m(y_0) + \epsilon.$$

We have thus proved that $x_m(y) \in V_\epsilon$ for all $y \in U_\epsilon$. Since $(V_\epsilon)_{\epsilon>0}$ is a basis of neighbourhoods of $x_m(y_0)$, this proves that $y \mapsto x_m(y)$ is continuous at y_0 . \square

To prove the continuity of Ψ^{-1} , we can apply [Proposition 2.7](#) to the family $F_{j,\rho}$ of functions on $\mathcal{T}(S)$ depending on the parameter (j, ρ) . The continuity of $(j, \rho) \mapsto F_{j,\rho}$ comes from [Proposition 1.4](#). The only thing we need to check is thus that the family

$$(F_{j,\rho})_{(j,\rho) \in \text{Dom}(S, \text{Isom}(M))}$$

is locally uniformly proper. But this follows easily from the continuity of the minimal Lipschitz constant. Indeed, let (j_0, ρ_0) be a point in $\text{Dom}(S, \text{Isom}(M))$. We thus have $\text{Lip}(j_0, \rho_0) < 1$. By continuity of the function Lip , there exists a neighbourhood U of (j_0, ρ_0) and a $\lambda < 1$ such that for all $(j, \rho) \in U$, we have $\text{Lip}(j, \rho) \leq \lambda$. By [Lemma 2.2](#), we thus have

$$F_{j,\rho}(Y) \geq (1 - \lambda^2)E(Y, j)$$

for all $(j, \rho) \in U$ and all $Y \in \mathcal{T}(S)$. Since the function $Y \mapsto E(Y, j)$ is proper by [Theorem 1.6](#), we obtain that the family $(F_{j,\rho})$ is uniformly proper on U . Hence it is locally uniformly proper.

By [Proposition 2.7](#), we deduce that the unique minimum of $F_{j,\rho}$ varies continuously with (j, ρ) . Since this minimum is precisely $\Psi^{-1}(j, \rho)$, we proved that Ψ^{-1} is continuous. This concludes the proof of [Theorem 0.5](#).

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