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ON SOME RESULTS OF M.A. MALIK CONCERNING POLYNOMIALS

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Abstract: If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k, k \leqslant 1$, Rather, Gulzar and Ahangar [8] proved that for every $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant k$ and $\gamma > 0$,

$$n(|\alpha| - k)\left\{ \int_0^{2\pi} \Big|\frac{P(e^{i\theta})}{D_\alpha P(e^{i\theta})}\Big|^\gamma d\theta \right\}^{\frac{1}{\gamma}} \leqslant \left\{ \int_0^{2\pi} \Big|1 + ke^{i\theta}\Big|^\gamma d\theta \right\}^{\frac{1}{\gamma}}.$$

In this paper, we shall obtain a result which generalizes and sharpens the above inequality by obtaining a bound that depends upon the location of all the zeros of $P(z)$ rather than just on the location of the zero of largest modulus.

Keywords: polar derivative, polynomials, integral mean estimate.

1. Introduction

Let $P(z)$ be a polynomial of degree n. It was shown by Turán [10] that if $P(z)$ has all its zeros in $|z| \leqslant 1$, then

$$n \max_{|z|=1} |P(z)| \leqslant 2 \max_{|z|=1} |P'(z)|. \tag{1.1}$$

Inequality (1.1) is best possible with equality holds for $P(z) = \alpha z^n + \beta$ where $|\alpha| = |\beta| \neq 0$.

As an extension of (1.1), Malik [5] proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$ where $k \leqslant 1$, then

$$n \max_{|z|=1} |P(z)| \leqslant (1 + k) \max_{|z|=1} |P'(z)|. \tag{1.2}$$

Equality in (1.2) holds for $P(z) = (z + k)^n$.

Malik [6] obtained a generalization of (1.1) in the sense that the left-hand side of (1.1) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$.

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In fact, he proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then for each $\gamma > 0$,

$$n\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^{\gamma} d\theta\right\}^{\frac{1}{\gamma}} \leqslant \left\{\int_0^{2\pi} \left|1+e^{i\theta}\right|^{\gamma} d\theta\right\}^{\frac{1}{\gamma}} \max_{|z|=1} |P'(z)|. \tag{1.3}$$

The corresponding extension of (1.2), which is also a generalization of (1.3), was obtained by Aziz [1] who proved that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$ where $k \leqslant 1$, then for each $\gamma > 0$,

$$n\left\{\int_0^{2\pi} \left|\frac{P(e^{i\theta})}{P'(e^{i\theta})}\right|^{\gamma} d\theta\right\}^{\frac{1}{\gamma}} \leqslant \left\{\int_0^{2\pi} \left|1+ke^{i\theta}\right|^{\gamma} d\theta\right\}^{\frac{1}{\gamma}}. \tag{1.4}$$

The estimate (1.4) is best possible and equality holds for $P(z) = (\alpha z+\beta k)^n$, where $|\alpha| = |\beta|$.

Since $|P'(e^{i\theta})| \leqslant \max_{|z|=1} |P'(z)|, 0 \leqslant \theta < 2\pi$, it follows from (1.4) that if $P(z)$ has all its zeros in $|z| \leqslant k$ where $k \leqslant 1$, then

$$n\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^{\gamma} d\theta\right\}^{\frac{1}{\gamma}} \leqslant \left\{\int_0^{2\pi} \left|1+ke^{i\theta}\right|^{\gamma} d\theta\right\}^{\frac{1}{\gamma}} \max_{|z|=1} |P'(z)|. \tag{1.5}$$

Inequality (1.5) reduces to the inequality (1.2) by letting $\gamma \to \infty$.

Let $D_\alpha P(z)$ denote the polar derivative of a polynomial $P(z)$ of degree n with respect to a point $\alpha \in \mathbb{C}$, then (see [7])

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$

Recently, Rather, Gulzar and Ahangar [8] extended (1.4) to the polar derivative of a polynomial by proving the following result.

Theorem 1.1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant k$ where $k \leqslant 1$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant k$, and for each $\gamma > 0$,*

$$n(|\alpha|-k)\left\{\int_0^{2\pi} \left|\frac{P(e^{i\theta})}{D_\alpha P(e^{i\theta})}\right|^{\gamma} d\theta\right\}^{\frac{1}{\gamma}} \leqslant \left\{\int_0^{2\pi} \left|1+ke^{i\theta}\right|^{\gamma} d\theta\right\}^{\frac{1}{\gamma}}. \tag{1.6}$$

The result is best possible and equality in (1.6) holds for $P(z) = (z-k)^n$.

2. Main results

It is easy to see that the bound obtained in (1.6) depends only on the zero of largest modulus and not on other zeros even if some of them are very close to the origin. It would therefore be interesting to obtain a bound in Theorem 1.1 which depends on the location of all zeros of the polynomial $P(z)$. In this connection, we prove the following:

Theorem 2.1. *If* $P(z) = a_n \prod_{j=1}^{n}(z - z_j) = \sum_{j=0}^{n} a_j z^j, a_0 a_n \neq 0$ *is a polynomial of degree* n *such that* $|z_j| \leqslant k_j \leqslant 1, 1 \leqslant j \leqslant n$, *then for every* $\alpha \in \mathbb{C}$ *with* $|\alpha| \geqslant S_0$ *and for each* $\gamma > 0$,

$$n(|\alpha| - S_0)\left\{\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{D_\alpha P(e^{i\theta})}\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}} \leqslant \left\{\int_0^{2\pi}\left|1 + S_0 e^{i\theta}\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}, \tag{2.1}$$

where

$$S_0 = 1 - \frac{n}{\sum_{j=1}^{n}\left(\frac{1}{1-k_j}\right)}. \tag{2.2}$$

The result is best possible in the case $k_j = k, 1 \leqslant j \leqslant n$ *and equality in* (2.1) *holds for* $P(z) = (z-k)^n$, *where* α *is real with* $\alpha \geqslant k$.

Remark 2.1. It can be easily seen that Theorem 2.1 includes as a special case Theorem 1.1 when $k_j = k$ for $1 \leqslant j \leqslant n$.

Remark 2.2. Since $S_0 \leqslant k$, therefore, Theorem 2.1 holds for every $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant k$ as well. Using the fact that

$$\int_0^{2\pi}\left|1 + ae^{i\theta}\right|^\gamma d\theta \leqslant \int_0^{2\pi}\left|1 + be^{i\theta}\right|^\gamma d\theta, \ for\ 0 < a \leqslant b,$$

we can easily see that inequality (2.1) is a refinement of (1.6).

By letting $|\alpha| \to \infty$ in (2.1), we obtain the following generalization and refinement of (1.4).

Corollary 2.1. *If* $P(z) = a_n \prod_{j=1}^{n}(z - z_j) = \sum_{j=0}^{n} a_j z^j, a_0 a_n \neq 0$ *is a polynomial of degree* n *such that* $|z_j| \leqslant k_j \leqslant 1, 1 \leqslant j \leqslant n$, *then for each* $\gamma > 0$,

$$n\left\{\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{P'(e^{i\theta})}\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}} \leqslant \left\{\int_0^{2\pi}\left|1 + S_0 e^{i\theta}\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}, \tag{2.3}$$

where S_0 *is given by* (2.2).

The result is best possible in the case $k_j = k, 1 \leqslant j \leqslant n$, *and equality in* (2.3) *holds for* $P(z) = (z+k)^n$.

Using the fact that $|P'(e^{i\theta})| \leqslant \max_{|z|=1} |P'(z)|, 0 \leqslant \theta < 2\pi$ and letting $\gamma \to \infty$ in (2.3), we get the following result.

Corollary 2.2. *If* $P(z) = a_n \prod_{j=1}^{n} (z - z_j) = \sum_{j=0}^{n} a_j z^j, a_0 a_n \neq 0$ *is a polynomial of degree* n *such that* $|z_j| \leqslant k_j \leqslant 1, 1 \leqslant j \leqslant n$, *then*

$$n \max_{|z|=1} |P(z)| \leqslant (1 + S_0) \max_{|z|=1} |P'(z)|, \tag{2.4}$$

where S_0 *is given by* (2.2).

The result is best possible in the case $k_j = k, 1 \leqslant j \leqslant n$, *and equality holds for* $P(z) = (z + k)^n$.

Remark 2.3. By putting the value of S_0 in (2.4) and after simplification, we get

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \left\{ 1 + \frac{1}{1 + \frac{2}{n} \sum_{j=1}^{n} \frac{k_j}{1-k_j}} \right\} \max_{|z|=1} |P(z)|. \tag{2.5}$$

The above inequality (2.5) is a refinement of (1.1),(1.2) and a result of Giroux, Rahman and Schmeisser [4].

Remark 2.4. The above inequality (2.5) was also proved by Aziz and Ahmad [2]. It is worth to mention that the proof of (2.5) given by Aziz and Ahmad is valid only when $P(0) \neq 0$.

Definition. Let f and F be two analytic functions in $|z| < 1$. We say that f is sub-ordinate to F if there exists a function ω, analytic in $|z| < 1$, satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$, such that

$$f(z) = F(\omega(z)) \qquad (|z| < 1).$$

3. Lemma

For the proof of the Theorem 2.1, we need the following lemmas. The following lemma is due to Gardner and Govil [3].

Lemma 3.1. *If* $P(z) = a_n \prod_{j=1}^{n} (z - z_j)$ *is a polynomial of degree* n *with* $|z_j| \geqslant k_j \geqslant 1, 1 \leqslant j \leqslant n$, *and* $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, *then*

$$\left| \frac{Q'(z)}{P'(z)} \right| \geqslant 1 + \left(\frac{n}{\sum_{j=1}^{n} \frac{1}{k_j - 1}} \right),$$

for $|z| = 1$.

The following lemma is due to Rahman and Schmeisser [9, Theorem 1.6.17].

Lemma 3.2. *Let f be sub-ordinate to F in $|z| < 1$. In addition, let F be univalent in $|z| < 1$. Then for all $\gamma > 0$, we have*

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^{\gamma} d\theta \leqslant \int_{-\pi}^{\pi} |F(re^{i\theta})|^{\gamma} d\theta \qquad (0 \leqslant r < 1).$$

4. Proof of the Theorem

Proof of Theorem 2.1. Let $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, then $P(z) = z^n \overline{Q(\frac{1}{\bar{z}})}$ and it can be easily verified that for $|z| = 1$,

$$|P'(z)| = |nQ(z) - zQ'(z)| \tag{4.1}$$

and

$$|Q'(z)| = |nP(z) - zP'(z)|. \tag{4.2}$$

Since $P(z) = a_n \prod_{j=1}^{n} (z - z_j)$ is a polynomial of degree n with $0 < |z_j| \leqslant k_j \leqslant 1, 1 \leqslant j \leqslant n$, therefore, $Q(z) = \bar{a}_n \prod_{j=1}^{n} (1 - z\bar{z}_j)$ is a polynomial of degree n with $|\frac{1}{z_j}| \geqslant \frac{1}{k_j} \geqslant 1, 1 \leqslant j \leqslant n$. On applying Lemma 3.1 to $Q(z)$, we get for $|z| = 1$,

$$\Big|\frac{P'(z)}{Q'(z)}\Big| \geqslant 1 + \left\{ \frac{n}{\sum_{j=1}^{n} (\frac{k_j}{1-k_j})} \right\} = \frac{\sum_{j=1}^{n} (\frac{k_j}{1-k_j}) + n}{\sum_{j=1}^{n} (\frac{k_j}{1-k_j})}.$$

This implies for $|z| = 1$,

$$\Big|\frac{Q'(z)}{P'(z)}\Big| \leqslant \frac{\sum_{j=1}^{n} (\frac{1}{1-k_j}) - n}{\sum_{j=1}^{n} (\frac{1}{1-k_j})} = 1 - \left\{ \frac{n}{\sum_{j=1}^{n} (\frac{1}{1-k_j})} \right\} = S_0.$$

This gives with the help of (4.1) that

$$\frac{|Q'(z)|}{S_0 |nQ(z) - zQ'(z)|} \leqslant 1 \qquad \text{for} \quad |z| = 1. \tag{4.3}$$

Since all the zeros of $P(z)$ lie in $|z| \leqslant 1$, by Gauss-Lucas Theorem all the zeros of $P'(z)$ also lie in $|z| \leqslant 1$. This implies that the polynomial

$$z^{n-1} \overline{P'\left(\frac{1}{\bar{z}}\right)} = nQ(z) - zQ'(z)$$

has all its zeros in $|z| \geqslant 1$. Therefore, it follows from (4.3) and the Maximum Modulus Theorem that the function

$$W(z) = \frac{zQ'(z)}{S_0(nQ(z) - zQ'(z))}$$

is analytic for $|z| \leqslant 1$, $|W(z)| \leqslant 1$ for $|z| \leqslant 1$. Furthermore, $W(0) = 0$ and so the function $f(z) = 1 + S_0 W(z)$ is subordinate to the function $F(z) = 1 + S_0 z$ for $|z| \leqslant 1$. Hence by Lemma 3.2, we have for each $\gamma > 0$,

$$\int_0^{2\pi} |1 + S_0 W(e^{i\theta})|^\gamma d\theta \leqslant \int_0^{2\pi} |1 + S_0 e^{i\theta}|^\gamma d\theta. \tag{4.4}$$

Now

$$1 + S_0 W(z) = \frac{nQ(z)}{nQ(z) - zQ'(z)},$$

which gives with the help of (4.1) that for $|z| = 1$,

$$n|Q(z)| = |1 + S_0 W(z)||P'(z)|. \tag{4.5}$$

Since for $|z| = 1$, we have $|P(z)| = |Q(z)|$, therefore, from (4.5), we get

$$n|P(z)| = |1 + S_0 W(z)||P'(z)| \qquad \text{for} \quad |z| = 1. \tag{4.6}$$

For every $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant S_0$, we have

$$|D_\alpha P(z)| = |nP(z) + (\alpha - z)P'(z)| \geqslant |\alpha||P'(z)| - |nP(z) - zP'(z)|,$$

which gives with the help of (4.1), (4.2) and (4.3), that for $|z| = 1$,

$$|D_\alpha P(z)| \geqslant (|\alpha| - S_0)|P'(z)|. \tag{4.7}$$

Combining (4.6) and (4.7), we get for $|\alpha| \geqslant S_0$ and $|z| = 1$,

$$n(|\alpha| - S_0)|P(z)| \leqslant |D_\alpha P(z)||1 + S_0 W(z)|. \tag{4.8}$$

From (4.4) and (4.8), we deduce for each $\gamma > 0$,

$$n^\gamma(|\alpha| - S_0)^\gamma \int_0^{2\pi} \left|\frac{P(e^{i\theta})}{D_\alpha P(e^{i\theta})}\right|^\gamma d\theta \leqslant \int_0^{2\pi} |1 + S_0 e^{i\theta}|^\gamma d\theta,$$

which gives

$$n(|\alpha| - S_0)\left\{\int_0^{2\pi} \left|\frac{P(e^{i\theta})}{D_\alpha P(e^{i\theta})}\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}} \leqslant \left\{\int_0^{2\pi} \left|1 + S_0 e^{i\theta}\right|^\gamma d\theta\right\}^{\frac{1}{\gamma}}.$$

This completes the proof of Theorem 2.1. ∎

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