Functiones et Approximatio
45.1 (2011), 139–153

UNITS IN REAL CYCLIC FIELDS

Roman Marszałek

Abstract: Let $N/\mathbb{Q}$ be a real cyclic and tame extension of prime degree l with $\Gamma = \mathscr{G}al(N/\mathbb{Q})$. We give the Hom description of the class of the torsion-free part of the group of units in N in the class group of the order $\mathbb{Z}\Gamma/(\sum_{\gamma\in\Gamma}\gamma)$. This representation depends only on the structure of the ideal class group of N and determines the Galois module structure of the torsion-free part of the group of units in N as an ideal of the lth cyclotomic field. Using this approach we derive necessary and sufficient conditions for all real and tame cyclic fields of prime degree to have Minkowski units. We extend also the class of known cyclic real fields with Minkowski units.

Keywords: Galois module structure, cyclic fields, Minkowski units.

1. Introduction

In this paper we consider the Galois module structure of the group of units of real cyclic and tame fields of prime degree. Let N be such a field with the prime $l = [N : \mathbb{Q}]$ and the Galois group $\Gamma = \mathscr{G}al(N/\mathbb{Q})$. Let E_N be the group of units of N and denote by $\overline{E}_N$ the torsion-free part of E_N, i.e. $\overline{E}_N = E_N/\{\pm 1\}$. Moreover h_N is the class-number of N. The group $\overline{E}_N$ in natural way admits the structure of $\mathscr{A}$-module where by $\mathscr{A}$ we denote the factor ring $\mathbb{Z}\Gamma/(\sum_{\gamma\in\Gamma}\gamma)$.

The main purpose of this paper is to describe the class of $\overline{E}_N$ in the locally free class group of $\mathscr{A}$ denoted by $\mathrm{Cl}(\mathscr{A})$. In [5] the so-called Hom description of the class of $\overline{E}_N$ in $\mathrm{Cl}(\mathscr{A})$ is given by means of a representing homomorphism on the group of characters of $\mathscr{G}al(N/\mathbb{Q})$. This representing homomorphism (see Theorem 5.6. in [5]) involves Gauss sums, Galois Gauss sums, values of Dirichlet and p-adic L-functions at 1. Using the approach introduced in [5], we provide another representing homomorphism of the class of $\overline{E}_N$ which depends only on the structure of the class group of N (see Theorem 2.2). Since a class equality in $\mathrm{Cl}(\mathscr{A})$ implies an isomorphism of $\mathscr{A}$- modules ($\mathscr{A}$ is commutative), this representation gives a simple and complete classification of $\overline{E}_N$ as $\mathscr{A}$-module for real tame and cyclic fields of prime degree over $\mathbb{Q}$.

The order $\mathscr{A}$ is isomorphic to the ring of integers of the lth cyclotomic field, so the locally free class group of $\mathscr{A}$ is the class group of that field and $\overline{E}_N$ is

2010 Mathematics Subject Classification: primary: 11R33; secondary: 11S40

isomorphic to an ideal of $\mathbb{Z}[\zeta_l]$. This observation and Theorem 2.2 enable us to derive an even more explicit and natural classification of $\mathscr{A}$-modules $\overline{E}_N$. In Theorem 2.3 we shall prove that $\overline{E}_N$ is isomorphic to a product of powers of prime ideals of $\mathbb{Z}[\zeta_l]$ lying above prime divisors of h_N. The exponents of these prime ideals in this product depend only on the $\mathbb{Q}_p\Gamma$-structure of the class group of N for any prime p dividing h_N.

Note that up till now there are no results giving a global Galois module classification of $\overline{E}_N$ for large classes of real Abelian fields as those obtained in Theorem 2.2 and 2.3.

In Section 3 (Theorem 3.1) we shall prove that conditions given in [5] (Theorem 6.1) are also necessary and sufficient conditions for fields to have Minkowski units in the whole class of real cyclic and tame fields of prime degree (the degree $[N:\mathbb{Q}]$ need not to be a regular prime as it is assumed in Theorem 6.1 in [5]). Note also that the above criterion is the first one where the sufficient and necessary conditions for the existence of Minkowski units are given for such large class of real Abelian fields.

Finally in Corollaries 3.2 and 3.3 , by applying Theorem 3.1 , we shall extend the class of the fields with Minkowski units given in [5](see Corollaries 6.2, 6.3).

We shall adopt most notations and definitions from [5] and [7].

2. The Hom description of the class $(\overline{E}_N)_{\mathscr{A}}$

In order to formulate the result we shall use in this section we need some notation.

First we put

$$s_p = \begin{cases} 0 & \text{if } f_p = g_p = 1,\ l \mid p-1, \\ 2 & \text{if } f_2 = 1 \text{ or } g_2 = l, \\ 1 & \text{otherwise}, \end{cases}$$

where f_p and g_p are the residue class degree of p in N and the number of prime ideals in $\mathscr{O}_N$ above p respectively.

We put

$$\begin{aligned} d_{p,\chi} &= \nu_p(L_p(1,\chi_*)) - \nu_p(h_p^\chi)/\Phi_{\chi,p}(1) \qquad \text{for odd primes } p\ , \\ d_{2,\chi} &= \nu_2(L_2(1,\chi_*)) - 1 - \nu_2(h_2^\chi)/\Phi_{\chi,2}(1), \end{aligned}$$

where $\Phi_{\chi,p}$ is a $\mathbb{Q}_p$-irreducible character of Γ with χ as its summand, h_p^χ is the order of the $\Phi_{\chi,p}$-component of the class group for N and χ_* is the primitive character corresponding to χ.

We also put

$$m_l = \nu_l\left(\frac{lR_l(N)}{\sqrt{d_N}\, n_l}\right) + 1, \qquad \text{where } n_l \text{ is the absolute norm of } \prod_{t\in T_l} t(\mathfrak{l}).$$

The following theorem is fundamental for our study of $\overline{E}_N$ as a module over $\mathbb{Z}\Gamma/(\widetilde{\Gamma})$ (For a proof see [5]).

Theorem 2.1. *Let N be a tame real cyclic field of prime degree $l > 2$ and let γ_0 generate Γ. For any prime $p \neq l$ put*

$$\omega_{p,\chi} = p^{d_{p,\chi}+s_p}, \qquad \omega_{l,\chi} = \mathrm{Det}_\chi(1-\widetilde{\gamma}_0)^{m_l} l \qquad \textit{and} \qquad \omega_{\infty,\chi} = 1.$$

Then the map

$$\chi \longmapsto \tau(N/\mathbb{Q},\chi)\left(\frac{[(\chi_*(p)/p-1)\tau(\chi_*)]^\delta}{L_p(1,\chi_*^\delta)}\omega_{p,\chi^\delta}\right)_{\delta,p} \qquad \textit{with} \quad \delta \in \check{T}_p S_p$$

is a representative of $(\overline{E}_N)_{\mathscr{A}}$ in

$$\mathrm{Hom}_\Omega(R'_\Gamma, \mathscr{J}(F)))/[\mathrm{Hom}_\Omega(R'_\Gamma, F^*)\mathrm{Det}(\mathfrak{U}(\mathscr{A}))],$$

where $\tau(N/\mathbb{Q},\chi)$ is the Galois Gauss sum and $\tau(\chi_*)$ is the Gauss sum.

Now we prove the main result of this paper. We shall construct the representing homomorphism of the isomorphism class of $\overline{E}_N$ which depends only on the $\mathbb{Q}_p\Gamma$-structure of the class group of N.

Theorem 2.2. *Let N be a tame real cyclic field of prime degree $l > 2$. For any prime $p \neq l, \infty$ and $\chi \in \hat{\Gamma} \setminus \{1_\Gamma\}$ put*

$$a_{p,\chi} = -\nu_p(h_p^\chi)/\Phi_{\chi,p}(1) \qquad \textit{and} \qquad a_{l,\chi} = a_{\infty,\chi} = 0.$$

Then the map $\mathscr{F}$ defined by

$$\chi \longmapsto (p^{a_{p,\chi^\delta}})_{\delta,p} \qquad \textit{with} \quad \delta \in T_{F,p}$$

is a representative of $(\overline{E}_N)_{\mathscr{A}}$ in

$$\mathrm{Hom}_\Omega(R'_\Gamma, \mathscr{J}(F))/[\mathrm{Hom}_\Omega(R'_\Gamma, F^*)\mathrm{Det}(\mathfrak{U}(\mathscr{A}))].$$

Proof. Note that, by Remark 3.2 of [5], we may always assume that the components at ∞ of representing maps of any class in $\mathscr{A}$ are equal to 1 e.i. $a_{\infty,\chi} = 0$.

Let q be the conductor of the field N, so q is the least integer such that $N \subseteq \mathbb{Q}(\zeta_q)$. As N is tame, q is also square-free.

From now on we choose F to be the cyclotomic field $\mathbb{Q}(\zeta_{lq})$ so F contains N and the values of characters from $\widehat{\Gamma}$. Thus $\Omega = \mathscr{G}al(F/\mathbb{Q}) = \mathscr{G}al(\mathbb{Q}(\zeta_{lq})/\mathbb{Q})$ and by putting $H = \mathscr{G}al(F/N)$, we have

$$\Omega/H \cong \mathscr{G}al(N/\mathbb{Q}) = \Gamma \tag{2.1}$$

so we can identify Γ with Ω/H.

By (2.1) any character of Γ can be treated as a suitable character of Ω which is trivial on H.

For any $\gamma \in \Gamma$ let $\bar{\gamma}$ denote an automorphism of $\mathbb{Q}(\zeta_{lq})$ such that $\bar{\gamma}H$ corresponds to γ via the isomorphism in (2.1). Thus for any $\chi \in \widehat{\Gamma}$, there is $\chi_0 \in \widehat{\Omega}$ such that $\chi(\gamma) = \chi(\bar{\gamma}H) = \chi_0(\bar{\gamma})$ for any $\gamma \in \Gamma$. Using the isomorphism $\Omega \cong (\mathbb{Z}/lq\mathbb{Z})^*$ we assign to each $\bar{\gamma} \in \Omega$ an integer $\bar{\gamma}_* \mod lq$ satisfying

$$\bar{\gamma}(\zeta_{lq}) = \zeta_{lq}^{\bar{\gamma}_*}$$

Since we identify the group Ω with $(\mathbb{Z}/lq\mathbb{Z})^*$, we can regard the character χ_0 as a Dirichlet character modulo lq. Subsequently we assign to this character the primitive character χ_* with the conductor q.

Thus $\chi \mapsto \chi_*$ is a correspondence from $\widehat{\Gamma}$ to the group of primitive Dirichlet characters associated with the field N.

We also have

$$\chi(\gamma) = \chi_0(\bar{\gamma}) = \chi_0(\bar{\gamma}_*) = \chi_*(\gamma_*),$$

where γ_* is defined by $\gamma_* = \bar{\gamma}^* \mod q$.

Assume that $\zeta_q = \zeta_{lq}^l$. Then $\bar{\gamma}(\zeta_q) = \bar{\gamma}(\zeta_{lq}^l) = \zeta_q^{\bar{\gamma}} = \zeta_q^{\gamma_*}$, whence

$$\bar{\gamma}(\zeta_q) = \zeta_q^{\gamma_*}.$$

For any $\bar{\gamma} \in \Omega$ (each element of Ω has the form $\bar{\gamma}$ for some $\gamma \in \Gamma$) and any $\chi \neq 1_\Gamma$ we compute

$$\bar{\gamma}(\tau(\chi_*)) = \sum_{a=1}^{q} \chi_*^{\bar{\gamma}}(a)\zeta_q^{a\gamma_*} = \sum_{b=1}^{q} \chi_*^{\bar{\gamma}}(\gamma_*^{-1}b)\zeta_q^b = \sum_{b=1}^{q} \chi_*^{\bar{\gamma}}(b)\zeta_q^b$$

where γ_*^{-1} is the inverse of $\gamma_* \mod q$. Thus we get

$$\bar{\gamma}(\tau(\chi_*)) = \chi_*^{\bar{\gamma}}(\gamma_*^{-1})\tau(\chi_*^{\bar{\gamma}}) = \chi^{\bar{\gamma}}(\gamma^{-1})\tau(\chi_*^{\bar{\gamma}}). \tag{2.2}$$

Now we apply Theorem 20B.(ii) of [4] (p.119) in the case $K = \mathbb{Q}$. Then the co-transfer $\mathrm{v}_{K/\mathbb{Q}}$ is trivial and considering $\bar{\gamma} \in \Omega$ as the restriction of an automorphism (which we also denote by $\bar{\gamma}$) of the algebraic closure of $\mathbb{Q}$ we obtain

$$\bar{\gamma}(\tau(\chi^{\bar{\gamma}^{-1}}, N/\mathbb{Q})) = \tau(\chi, N/\mathbb{Q})\mathrm{det}_\chi(\bar{\gamma}),$$

where det_χ is a character of the group $\mathscr{G}al(\overline{\mathbb{Q}}/\mathbb{Q})$ defined via the surjection $\mathscr{G}al(\overline{\mathbb{Q}}/\mathbb{Q}) \longmapsto \Gamma$. Hence $\mathrm{det}_\chi(\bar{\gamma}) = \chi(\gamma)$ and so by replacing χ by $\chi^{\bar{\gamma}}$ we get

$$\bar{\gamma}(\tau(\chi, N/\mathbb{Q})) = \tau(\chi^{\bar{\gamma}}, N/\mathbb{Q})\chi^{\bar{\gamma}}(\gamma). \tag{2.3}$$

Now it follows from (2.2) and (2.3) that

$$\omega(\tau(\chi, N/\mathbb{Q})\tau(\chi_*)) = \tau(\chi^\omega, N/\mathbb{Q})\tau(\chi_*^\omega))$$

for any $\omega \in \Omega$. Thus the map defined by

$$\chi \longmapsto (\tau(\chi^\delta, N/\mathbb{Q})\tau(\chi_*^\delta))_{\delta,p} \quad \text{with} \quad \delta \in \check{T}_p S_p$$

is an element of $\mathrm{Hom}_\Omega(R'_\Gamma, F^*)$.

Consequently if $\mathscr{R}_0$ denotes the representing map defined in Theorem 2.1 , then the map

$$\mathscr{R}(\chi) = \mathscr{R}_0(\chi)[(\tau(\chi^\delta, N/\mathbb{Q})\tau(\chi_*^\delta))_{\delta,p}]^{-1} = \left(\frac{(\chi_*^\delta(p)/p - 1)}{L_p(1,\chi_*^\delta)} \omega_{p,\chi^\delta} \right)_{\delta,p}$$

is a representative of $(\overline{E}_N)_{\mathscr{A}}$.

After putting $V_{p,\chi} = \frac{pL_p(1,\chi_*)}{\chi_*(p)-p}$ we can write $\mathscr{R}(\chi) = \left(\omega_{p,\chi^\delta}/V_{p,\chi}\right)_{\delta,p}$ for any $\chi \neq 1_\Gamma$.

Next we shall prove that for any $\sigma \in \mathscr{G}al(\mathbb{Q}_p(\zeta_{lq})/\mathbb{Q}_p)$

$$\sigma(V_{p,\chi}) = V_{p,\chi^\sigma}. \tag{2.4}$$

To this end we apply Theorem 3.5 of [5] and obtain

$$V_{p,\chi} = \frac{\tau(\chi_*)}{q} \sum_{a=1}^{q} \bar{\chi}_*(a) \log_p(1-\zeta_q^a).$$

Then using (2.2) for $\sigma \in \mathscr{G}al(\mathbb{Q}_p(\zeta_{lq})/\mathbb{Q}_p)$ (we treat such σ as an extension of a suitable element of Ω) we get

$$\begin{aligned}
\sigma(V_{p,\chi}) &= \frac{\chi_*^\sigma(\sigma_*^{-1})}{q}\tau(\chi_*^\sigma) \sum_{a=1}^{q} \bar{\chi}_*(a) \log_p(1-\zeta_q^{a\sigma_*}) \\
&= \frac{\chi_*^\sigma(\sigma_*^{-1})}{q}\tau(\chi_*^\sigma) \sum_{b=1}^{q} \bar{\chi}_*(\sigma_*^{-1}b) \log_p(1-\zeta_q^{b}) \\
&= \frac{\tau(\chi_*^\sigma)}{q} \sum_{b=1}^{q} \bar{\chi}_*(b) \log_p(1-\zeta_q^{b}) = V_{p,\chi^\sigma}
\end{aligned}$$

where σ_* is an integer defined by $\sigma(\zeta_q) = \zeta_q^{\sigma_*}$ and σ_*^{-1} denotes the inverse of σ_* $\bmod q$.

Since the values of the characters of Γ are lth roots of unity, then (2.4) , applied for $\sigma \in \mathscr{G}al(\mathbb{Q}_p(\zeta_{lq})/\mathbb{Q}_p(\zeta_l))$, gives

$$V_{p,\chi^\sigma} \in \mathbb{Q}_p(\zeta_l). \tag{2.5}$$

By the definition of $\omega_{p,\chi}$ for $p \neq l$ one has

$$\nu_2(\omega_{2,\chi}/V_{2,\chi}) = s_2 + \nu_2(\chi_*(2)-2) - \nu_2(h_2^\chi)/\Phi_{\chi,2}(1) - 1$$

and for $p \neq 2$

$$\nu_p(\omega_{p,\chi}/V_{p,\chi}) = s_p + \nu_p(\chi_*(p)-p) - \nu_p(h_p^\chi)/\Phi_{\chi,p}(1) - 1.$$

Observe that $\nu_p(\chi_*(p)-p) = 0$ for $p \nmid q$ and $\nu_p(\chi_*(p)-p) = 1$ for $p \mid q$. Hence, as $2 \nmid q$, we obtain

$$\nu_p(\omega_{p,\chi}/V_{p,\chi}) = \begin{cases} s_p - \nu_p(h_p^\chi)/\Phi_{\chi,p}(1) - 1 & \text{for } p \nmid q \\ s_p - \nu_p(h_p^\chi)/\Phi_{\chi,p}(1) & \text{for } p \mid q \text{ or } p = 2. \end{cases} \tag{2.6}$$

In order to evaluate $\nu_l(\omega_{l,\chi}/V_{l,\chi})$ we apply Leopoldt's class number formula (see Theorem 5.24 in [7])

$$\frac{2^{l-1}h_N R_l(N)}{\sqrt{d_N}} = \prod_{\chi_*\neq 1}\left(1-\frac{\chi_*(l)}{l}\right)^{-1} L_l(1,\chi_*)$$

and we obtain

$$2^{l-1}h_N R_l(N)/\sqrt{d_N} = \prod_{\chi\neq 1_\Gamma} V_{\chi,l}. \tag{2.7}$$

Since all nontrivial characters of Γ are conjugate over $\mathbb{Q}_l$, it follows from (2.4) that all $V_{p,\chi}$ are also conjugate over $\mathbb{Q}_l$. Hence for any nontrivial μ, χ we have $\nu_l(V_{l,\mu}) = \nu_l(V_{l,\chi})$ and so by (2.7)

$$\nu_l(V_{l,\chi}) = \frac{1}{l-1}\nu_l(2^{l-1}h_N R_l(N)/\sqrt{d_N}).$$

As $N/\mathbb{Q}$ is tame and $l\neq 2$ we get

$$\nu_l(V_{l,\chi}) = \frac{1}{l-1}\nu_l(h_N R_l(N)).$$

Now we calculate

$$\begin{aligned}\nu_l(\omega_{l,\chi}/V_{l,\chi}) &= \nu_l(\mathrm{Det}_\chi(1-\widetilde{\gamma}_0)^{m_l}l) - \nu_l(V_{l,\chi})\\ &= \nu_l((1-\zeta_l)^{m_l}l) - \nu_l(V_{l,\chi}) = (1-\nu_l(h_N))/(l-1)\end{aligned}$$

and we have

$$\nu_l(\omega_{l,\chi}/V_{l,\chi}) = \frac{1-\nu_l(h_N)}{l-1}. \tag{2.8}$$

Observe that by (2.5) we may write

$$\mathscr{R}(\chi) = (\mathscr{R}_p(\chi^\delta))_{p,\delta} \in \mathrm{Hom}_\Omega(R'_\Gamma, \mathscr{J}(F)) \qquad \text{with} \quad \delta\in \check{T}_p S_p$$

where $\mathscr{R}_p(\chi) = t_{p,\chi}p^{r_{p,\chi}}$, $t_{p,\chi}\in\mathbb{Z}_p[\zeta_l]^*$ and $r_{p,\chi}\in\mathbb{Z}$ for $p\neq l$, and

$$\mathscr{R}_l(\chi) = t_{l,\chi}(1-\chi(\gamma_0))^{r_{l,\chi}}$$

where $t_{l,\chi}\in\mathbb{Z}_l[\zeta_l]^*$, $r_{l,\chi}\in\mathbb{Z}$.

Since $\mathscr{R}(\chi) = \left(\omega_{p,\chi^\delta}/V_{p,\chi^\delta}\right)_{\delta,p}$, by virtue of (2.6) and (2.8) we have

$$r_{p,\chi} = \begin{cases} s_p - \nu_p(h_p^\chi)/\Phi_{\chi,p}(1) - 1 & \text{for } p\nmid q\\ s_p - \nu_p(h_p^\chi)/\Phi_{\chi,p}(1) & \text{for } p\mid q\\ s_2 - \nu_p(h_2^\chi)/\Phi_{\chi,2}(1) - 2 & \text{for } p=2\\ 1-\nu_l(h_N) & \text{for } p=l.\end{cases} \tag{2.9}$$

Now we put

$$\mathscr{F}_p(\chi) = p^{a_{p,\chi}} \qquad \text{where } a_{p,\chi} = -\nu_p(h_p^\chi)/\Phi_{\chi,p}(1) \text{ for } p \neq l \text{ and } a_{l,\chi} = 0,$$

$$\mathscr{S}_p(\chi) = \begin{cases} p^{s_p - 1} & \text{for } p \nmid q \\ p^{s_p} & \text{for } p \mid q \\ 2^{s_2 - 2} & \text{for } p = 2 \\ (1 - \chi(\gamma_0))^{1 - \nu_l(h_N)} & \text{for } p = l \end{cases},$$

$$\mathscr{T}_p(\chi) = t_{p,\chi} \in \mathbb{Z}_p[\zeta_l]^*$$

and

$$\overline{\mathscr{F}}_p(\chi) = (\mathscr{F}_p(\chi^\delta))_\delta, \qquad \overline{\mathscr{S}}_p(\chi) = (\mathscr{S}_p(\chi^\delta))_\delta, \qquad \overline{\mathscr{T}}_p(\chi) = (\mathscr{T}_p(\chi^\delta))_\delta$$

with δ running over $\check{T}_p S_p$. Note that $\overline{\mathscr{F}}_p(\chi), \overline{\mathscr{S}}_p(\chi), \overline{\mathscr{T}}_p(\chi) \in F_p^*$.

Finally we define

$$\mathscr{F}(\chi) = (\overline{\mathscr{F}}_p(\chi))_p, \qquad \mathscr{S}(\chi) = (\overline{\mathscr{S}}_p(\chi))_p, \qquad \mathscr{T}(\chi) = (\overline{\mathscr{T}}_p(\chi))_p$$

and by (2.9) we can write

$$\mathscr{R} = \mathscr{F}\mathscr{S}\mathscr{T}.$$

The proof of our theorem will be completed if we show that

$$\mathscr{S},\ \mathscr{T} \in \mathrm{Hom}_\Omega(R'_\Gamma, F^*)\mathrm{Det}(\mathfrak{U}(\mathscr{A}))$$

i.e.

$$\bar{\mathscr{S}}_p,\ \bar{\mathscr{T}}_p \in \mathrm{Hom}_\Omega(R'_\Gamma, F^*)\mathrm{Det}(\mathscr{A}_p^*) \qquad \text{for any prime } p.$$

We recall that we treat F as embedded into F_p by i_p and then into $\mathscr{J}(F)$.

Since for almost all primes p we have $s_p = 1$ and $\nu_p(h_p^\chi) = 0$, then $\mathscr{F}, \mathscr{S} \in \mathrm{Hom}(R'_\Gamma, \mathscr{J}(F))$. Note that for any $\sigma \in \Delta_{F,p}$

$$\sigma(\mathscr{F}_p(\chi)) = \mathscr{F}_p(\chi^\sigma) \qquad \text{and} \qquad \sigma(\mathscr{S}_p(\chi)) = \mathscr{S}_p(\chi^\sigma), \qquad \chi \in \widetilde{\Gamma} \setminus \{1_\Gamma\}.$$

Consequently by Remark 2.1 in [5] we infer that $\mathscr{F}_p, \mathscr{S}_p \in \mathrm{Hom}_\Omega(R'_\Gamma, F_p^*)$. Hence $\mathscr{F}$ and $\mathscr{S} \in \mathrm{Hom}_\Omega(R'_\Gamma, \mathscr{J}(F))$ and so by $\mathscr{T} = \mathscr{F}^{-1}\mathscr{S}^{-1}\mathscr{R}$ we get $\mathscr{T} \in \mathrm{Hom}_\Omega(R'_\Gamma, \mathscr{J}(F))$. Since, for any p any $\chi \neq 1_\Gamma$, $\mathscr{T}_p(\chi) = t_{p,\chi} \in \mathbb{Z}_p[\zeta_l]^*$, one has

$$\overline{\mathscr{T}}_p \in \mathrm{Hom}_\Omega(R'_\Gamma, \prod_t \mathbb{Z}_p[\zeta_l]^*) \subseteq \mathrm{Hom}_\Omega(R'_\Gamma, F_p^*)$$

where t runs over the set $T_{p,\mathbb{Q}(\zeta_l)}$ of representatives for the cosets of the decomposition group of p in $\mathbb{Q}(\zeta_l)$.

We shall prove that $\mathscr{T} \in \mathrm{Det}(\mathfrak{U}(\mathscr{A}))$ i.e. $\overline{\mathscr{T}}_p \in \mathrm{Det}(\mathscr{A}_p^*)$ for any prime p.

For any prime $p \neq l$ define $\widetilde{\mathscr{T}_p} \in \mathrm{Hom}_\Omega(R_\Gamma, \prod_t \mathbb{Z}_p[\zeta_l]^*)$ to be $\widetilde{\mathscr{T}_p}(\chi) = \bar{\mathscr{T}}_p(\chi)$ for $\chi \neq 1_\Gamma$ and $\widetilde{\mathscr{T}_p}(1_\Gamma) = 1$. Since $p \nmid l = |\Gamma|$, then $\mathbb{Z}_p\Gamma$ is a maximal order in $\mathbb{Q}_p\Gamma$ and by Proposition 2.2 in [4] we get $\mathrm{Hom}_\Omega(R_\Gamma, \prod_t \mathbb{Z}_p[\zeta_l]^*) = \mathrm{Det}((\mathbb{Z}_p\Gamma)^*)$.

Consequently $\widetilde{\mathscr{T}_p} \in \mathrm{Det}((\mathbb{Z}_p\Gamma)^*)$ and so $\overline{\mathscr{T}}_p \in \mathrm{Det}(\mathscr{A}_p^*)$ because the image (under $\gamma \mapsto \widetilde{\gamma}$) of any unit of $\mathbb{Z}_p\Gamma$ is a unit of $\mathscr{A}_p$ and they have the same Det_χ.

For $p = l$ let $\chi \neq 1_\Gamma$ and let γ_0 be a generator of Γ such that $\chi(\gamma_0) = \zeta_l$. Put $\mathscr{T}_l(\chi) = \sum_{j=0}^{l-2} a_j\zeta_l^j \in \mathbb{Z}_l[\zeta_l]^*$. Since the mapping $\sum_{j=0}^{l-1} x_j\widetilde{\gamma_0}^j \mapsto \sum_{j=0}^{l-1} x_j\zeta_l^j$ is a ring isomorphism $\mathscr{A}_l \cong \mathbb{Z}_l[\zeta_l]$, then $\sum_{j=0}^{l-1} a_j\widetilde{\gamma_0}^j \in \mathscr{A}_l^*$ and note that $\mathrm{Det}_\chi(\sum_{j=0}^{l-1} a_j\widetilde{\gamma_0}^j) = (\mathscr{T}_l(\chi^\delta))_\delta$, $\delta \in \check{T}_pS_p$. Let μ be any nontrivial character of Γ. Since, in this case, all nontrivial characters of Γ are conjugate over $\mathbb{Q}_l$, then $\mu = \chi^\rho$ for some $\rho \in \mathcal{G}al(\mathbb{Q}_l(\zeta_l)/\mathbb{Q}_l)$. Observe that any such ρ is a restriction of an element of the decomposition group $\Delta_{F,l}$ so, by Remark 2.1 in [5], one has $\rho(\mathscr{T}_l(\chi^\delta) = \mathscr{T}_l(\chi^{\rho\delta})$.

Then we compute

$$\begin{aligned}\mathrm{Det}_\mu(\sum_{j=0}^{l-1} a_j\widetilde{\gamma_0}^j) = \left(\sum_{j=0}^{l-1} a_j\chi^{\rho\delta}(\gamma_0^j)\right)_\delta &= (\rho(\mathscr{T}_l(\chi^\delta))_\delta \\ = (\mathscr{T}_l(\chi^{\rho\delta}))_\delta = (\mathscr{T}_l(\mu^\delta))_\delta &= \overline{\mathscr{T}}_l(\mu)\end{aligned}$$

which shows that $\overline{\mathscr{T}}_l \in \mathrm{Det}(\mathscr{A}_l^*)$. Thus we proved that $\mathscr{T} \in \mathrm{Det}(\mathfrak{U}(\mathscr{A}))$.

Observe that $\mathscr{S}_p(\chi) = 1$ for almost all primes p so we can put $S'(\chi) = \prod_p \mathscr{S}_p(\chi) \in F^*$ for any $\chi \neq 1_\Gamma$ and define the map $\widetilde{\mathscr{S}} : \chi \mapsto (S'(\chi^\sigma))_{p.\sigma}$. It is clear that $\widetilde{\mathscr{S}} \in \mathrm{Hom}_\Omega(R'_\Gamma, F^*)$. As $\mathscr{S}_p(\chi)$ is a power of p for $p \neq l$ and $\mathscr{S}_l(\chi)$ is a power of $1-\zeta_l$ times a unit of $\mathbb{Z}[\zeta_l]$ we deduce that $\mathscr{S}/\widetilde{\mathscr{S}} \in \mathrm{Hom}_\Omega\left(R'_\Gamma, \prod_p\prod_t \mathbb{Z}_p[\zeta_l]^*\right)$ where t runs over $T_{\mathbb{Q}(\zeta_l),p}$. Then after proceeding as for $\mathscr{T}$ we obtain $\mathscr{S}/\widetilde{\mathscr{S}} \in \mathrm{Det}(\mathfrak{U}(\mathscr{A}))$, whence

$$\mathscr{S} = \widetilde{\mathscr{S}}(\mathscr{S}/\widetilde{\mathscr{S}}) \in \mathrm{Hom}_\Omega(R'_\Gamma, F^*)\mathrm{Det}(\mathfrak{U}(\mathscr{A})).$$

Thus the homomorphism $\mathscr{F}$ is a representative of the class $(\overline{E}_N)_\mathscr{A}$. Since isomorphism classes in $\mathrm{Cl}(\mathscr{A})$ do not depend on the choice of the sets $T_{F,p}$ used in the construction of the representing homomomorphism, our theorem follows. ■

Now we present the first application of the Hom description of the class of $\overline{E}_N$ given in Theorem 2.2. Since $\mathscr{A} \cong \mathbb{Z}[\zeta_l]$ and $\mathrm{rank}_\mathbb{Z}(\mathscr{A}) = l-1$, then $\overline{E}_N$ is isomorphic to an ideal of $\mathbb{Z}[\zeta_l]$. In the following theorem we give such an ideal explicitly as a product of powers of prime ideals. These prime ideals and their exponents depend only on the structure of the ideal class group of N.

Theorem 2.3. *Let N be a real cyclic and tame field of odd prime degree l and μ a nontrivial character of its Galois group. For any prime p dividing h_N, let $\mathscr{P}_p$ be a prime ideal above p in $\mathbb{Z}[\zeta_l]$ and R_p be a set of representatives of cosets with respect to the the decomposition group of $\mathscr{P}_p$. Then*

$$\overline{E}_N \cong \prod_{p|h_N}\prod_{\rho\in R_p} \rho(\mathscr{P}_p)^{a_{p,\mu^\rho}}$$

as $\mathscr{A}$-modules.

Proof. For any prime p let $\mathbb{P}_p$ be the maximal ideal in $\mathbb{C}_p$ and ξ_p be a nontrivial lth root of unity in $\mathbb{C}_p$. Let $\mathfrak{P}_p = \mathbb{P}_p \cap \mathbb{Q}(\xi_p)$ and let $\zeta = \zeta_l \in \mathbb{C}$ denote a nontrivial lth root of unity.

Without loss of generality we may assume that $\mu(\gamma_0) = \zeta$, where γ_0 is a generator of Γ.

Let $R_p = \{\rho_{p,1} = id, \rho_{p,2}, \ldots, \rho_{p,k_p}\}$ be a set of representatives of cosets with respect to the the decomposition group D_p of p in $\mathbb{Z}[\zeta]$. For any $j = 1, \ldots, k_p$ we define an integer $s_{p,j} \mod (l-1)$ by

$$\rho_{p,j}(\zeta) = \zeta^{s_{p,j}}.$$

Note that by definition $s_{p,1} = 1$. We also define field isomorphisms

$$\beta_{p,j} : \mathbb{Q}(\zeta) \longrightarrow \mathbb{Q}(\xi_p) \qquad \text{by } \ \beta_{p,j}(\zeta) = \xi_p^{u_{p,j}}$$

where $u_{p,j} = s_{p,j}^{-1} \mod (l-1)$ for $j = 1, \ldots, k_p$. Observe that by replacing ξ_p by its power, if necessary, we may assume that $\beta_{p,1}(\mathscr{P}_p) = \mathfrak{P}_p$.

For any $j = 1, \ldots, k_p$ put $\mathscr{P}_{p,j} = \beta_{p,j}^{-1}(\mathfrak{P}_p)$ and note that $\mathscr{P}_{p,1} = \mathscr{P}_p$. By $\beta_{p,j}^{-1} = \rho_{p,j} \circ \beta_{p,1}^{-1}$ we have $\mathscr{P}_{p,j} = \rho_{p,j}(\mathscr{P}_p)$ for $j = 1, \ldots, k_p$ and so $\mathscr{P}_{p,1}, \mathscr{P}_{p,2}, \ldots, \mathscr{P}_{p,k_p}$ are all distinct prime ideals of $\mathbb{Z}[\zeta]$ extending the ideal $p\mathbb{Z}$.

To any complex character χ of Γ we assign the $p-$adic character $\chi_p = \beta_{p,1} \circ \chi$ and we also write $\widetilde{\Gamma}_p = \{\chi_p : \chi \in \widetilde{\Gamma}\}$.

Let $\mathscr{F}$ be a homomorphism representing the class $(\overline{E}_N)_{\mathscr{A}}$ defined in Theorem 2.2. Observe that $a_{p,\chi} \leqslant 0$ for any p and $\chi \neq 1_\Gamma$, so the ideal in the assertion of the theorem may not be integral . Therefore it will be convenient to consider the representing map $\mathscr{F}^{-1}$ instead of $\mathscr{F}$. Then we have $\mathscr{F}_p^{-1}(\chi) = p^{-a_{p,\chi}}$.

For any prime $p \neq l$ put

$$\alpha_p = \sum_{\chi \neq 1_\Gamma} p^{-a_{p,\chi}} \tilde{e}_{\chi_p} \in \mathbb{Q}_p(\xi_p)\Gamma/(\widetilde{\Gamma}) \qquad \text{and} \qquad \alpha_l = 1, \tag{2.10}$$

where $\tilde{e}_{\chi_p} = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \chi_p(\gamma^{-1})\tilde{\gamma}$ is an idempotent in $\mathbb{Q}_p(\xi_p)\Gamma/(\widetilde{\Gamma})$. Now we shall prove that $\alpha_p \in A_p^*$. Note that if we write $\alpha_p = \sum_{\gamma\in\Gamma} b_\gamma \tilde{\gamma}$, then there exists $w \in \mathbb{Q}_p(\xi_p)$ such that $b_\gamma = \frac{1}{l}\sum_{\chi\neq 1_\Gamma} p^{-a_{p,\chi}}\chi_p(\gamma^{-1}) + w$ for each $\gamma \in \Gamma$.

For $p \neq l$ we identify D_p with $\mathscr{G}al(\mathbb{Q}_p(\xi_p)/\mathbb{Q}_p)$. As $\mathscr{F}$ is an Ω-homomorphism it follows from Remark 2.1 in [5] that $a_{p,\chi^\sigma} = a_{p,\chi}$ for any prime p, $\chi \neq 1_\Gamma$ and $\sigma \in \Delta_{F,p}$. Since each element of D_p is the restriction of a suitable automorphism from $\Delta_{F,p}$ we also have $a_{p,\chi^\delta} = a_{p,\chi}$ for $\delta \in D_p$. Note also that for any nontrivial character χ we have $\{\chi_p^{\rho\delta} : \rho \in R_p, \delta \in D_p\} = \widehat{\Gamma}_p \setminus \{1_\Gamma\}$. Therefore we obtain

$$b_\gamma - w = \frac{1}{l}\sum_{\rho\in R_p}\sum_{\delta\in D_p} p^{-a_{p,\chi^{\rho\delta}}}\chi_p^{\rho\delta}(\gamma^{-1}) = \frac{1}{l}\sum_{\rho\in R_p} p^{-a_{p,\chi^\rho}} \sum_{\delta\in D_p}\chi_p^{\rho\delta}(\gamma^{-1}).$$

Since clearly $\sum_{\delta\in D_p}\chi_p^{\rho\delta}(\gamma^{-1}) \in \mathbb{Q}_p$, we get $b_\gamma - w \in \mathbb{Q}_p$ for each γ which shows that $\alpha_p \in A_p$. Consequently $\alpha_p \in A_p^*$ because $p^{-a_{p,\chi}} \neq 0$ in the formula (2.10).

Observe that

$$\mathrm{Det}_\chi(\tilde{e}_{\chi_p}) = (1)_\delta, \qquad \text{and} \qquad \mathrm{Det}_\chi(\tilde{e}_{\nu_p}) = (0)_\delta \quad \text{for } \nu \neq \chi.$$

This with (2.10) gives

$$\mathrm{Det}_\chi(\alpha_p) = (p^{-a_{p,\chi^\delta}})_\delta, \qquad \delta \in \check{T}_p S_p \qquad \text{for any prime } p \text{ and } \chi \neq 1_\Gamma.$$

Thus by the definition of $\mathscr{F}$ we obtain

$$\mathscr{F}(\chi)^{-1} = (\overline{\mathscr{F}}_p(\chi))_p^{-1} = (\mathrm{Det}_\chi(\alpha_p))_p. \tag{2.11}$$

Note that $a_{p,\chi} = 0$ for $p \nmid h_N$ and for each $\chi \neq 1_\Gamma$, whence, by (2.10), $\alpha_p = 1$ for almost all p. Consequently $\alpha_p \in \mathscr{A}_p^*$ for almost all p i.e. $(\alpha_p)_p \in \mathscr{J}(A)$, where $\mathscr{J}(A) = \{(x_p)_p \in \prod_p A_p^* : x_p \in \mathscr{A}_p^* \text{ for almost all } p\}$ is the idele group of the algebra A. Put $\alpha = (\alpha_p)_p$.

It turns out that the idele α determines the locally free $\mathscr{A}$- module M_α by the formula

$$M_\alpha = A \cap \{\bigcap_p \mathscr{A}_p \alpha_p\} \qquad \text{(see (49.5) p. 219 in [2]).} \tag{2.12}$$

Since α satisfies (2.11) it follows from Corollary (52.12) in [2] that M_α is $\mathscr{A}$-isomorphic to any locally free module in the class represented by $\mathscr{F}^{-1}$. Therefore we have the equality

$$(\overline{E}_N)_{\mathscr{A}} = (M_\alpha)_{\mathscr{A}}^{-1} \tag{2.13}$$

in the classgroup $\mathrm{Cl}(\mathscr{A})$.

Now we define a $\mathbb{Q}_p$-algebra homomorphism

$$\Theta_p : A_p \longrightarrow \prod_{j=1}^{k_p} \mathbb{Q}_p(\xi_p)$$

by putting

$$\Theta_p\left(\sum_{i=1}^{l-1} a_i \tilde{\gamma_0}^i\right) = \left(\sum_{i=1}^{l-1} a_i \xi_p^{iu_{p,j}}\right)_{1 \leqslant j \leqslant k_p}$$

where $a_i \in \mathbb{Q}_p$ and γ_0 generates Γ. Observe that Θ_p is an isomorphism as the superposition of the following isomorphisms

$$A_p \cong \mathbb{Q}_p[x] / \left(\sum_{i=0}^{l-1} x^i\right) \cong \prod_{j=1}^{k_p} \mathbb{Q}_p[x]/(f_j) \cong \prod_{j=1}^{k_p} \mathbb{Q}_p(\xi_p^{u_{p,j}})$$

where f_j are $\mathbb{Q}_p-$irreducible factors of the polynomial $\sum_{i=0}^{l-1} x^i$ such that $f_j(\xi_p^{u_{p,j}}) = 0$.

It turns out that the restriction $\overline{\Theta}_p$ of Θ_p to $\mathscr{A}_p$ establishes an isomorphism

$$\overline{\Theta}_p : \mathscr{A}_p \longrightarrow \prod_{j=1}^{k_p} \mathbb{Z}_p[\xi_p].$$

To prove this it suffices to show that $\overline{\Theta}_p$ is surjective.

In the case $p = l$ one has $k_l = 1$ and $\overline{\Theta}_l$ is a ring isomorphism $\mathscr{A}_l \cong \mathbb{Z}_l[\xi_l]$.

Assume that $p \neq l$ and let $(y_j)_j \in \prod_{j=1}^{k_p} \mathbb{Z}_p[\xi_p]$. Since Θ_p is surjective, there exists $x = \sum_{i=1}^{l-1} a_i \tilde{\gamma_0}^i$ with $a_i \in \mathbb{Q}_p$ such that $\Theta_p(x) = (y_j)_j$ i.e. $y_j = \sum_{i=1}^{l-1} a_i \xi_p^{iu_{p,j}}$ for $1 \leqslant j \leqslant g_p$. Put $x = \sum_{\chi \neq 1_\Gamma} A_\chi \tilde{e}_{\chi_p}$, where $A_\chi \in \mathbb{Q}_p(\xi_p)$. Since $x \in A_p$, then

$$A_\chi = \sum_{i=0}^{l-1} a_i \chi_p(\gamma_0^i) \qquad \text{and} \qquad a_i = \frac{1}{l} \sum_{\chi \neq 1_\Gamma} A_\chi \chi_p(\gamma_0^i) + v, \tag{2.14}$$

for any $\chi \neq 1_\Gamma$, $j = 1, \dots, k_p$, and where $v \in \mathbb{Q}_p$. As $\mu_p(\gamma_0) = \xi_p$ we may write

$$y_j = \sum_{i=0}^{l-1} a_i \mu_p^{u_{p,j}}(\gamma_0^i) \qquad \text{and so} \qquad \delta(y_j) = \sum_{i=0}^{l-1} a_i \mu_p^{\delta u_{p,j}}(\gamma_0^i)$$

for any $j = 1, \dots, k_p$ and $\delta \in D_p$. Since $\{\mu_p^{\delta u_{p,j}} : 1 \leqslant j \leqslant k_p,\ \delta \in D_p\} = \widehat{\Gamma}_p \setminus \{1_\Gamma\}$ and $y_j \in \mathbb{Z}_p[\xi_p]$ we get $\sum_{i=0}^{l-1} a_i \chi_p(\gamma_0^i) \in \mathbb{Z}_p[\xi_p]$ for any $\chi \neq 1_\Gamma$, whence by the first part of (2.14) we have $A_\chi \in \mathbb{Z}_p[\xi_p]$. Then the second part of (2.14) gives $a_i - v \in \mathbb{Z}_p[\xi_p]$ $(p \neq l)$ and because of a_i, $v \in \mathbb{Q}_p$ we obtain $a_i - v \in \mathbb{Z}_p$ for any $1 \leqslant j \leqslant g_p$. This proves that $x \in \mathscr{A}_p$ and the surjectivity of $\overline{\Theta}_p$ follows.

For any prime p let z_p be a $\mathbb{Q}$-algebra monomorphism

$$z_p : \mathbb{Q}(\zeta_l) \longrightarrow \prod_{j=1}^{k_p} \mathbb{Q}(\xi_p) \qquad \text{defined by} \quad z_p(x) = (\beta_{p,j}(x))_j,\ x \in \mathbb{Q}(\zeta_l).$$

By Θ we denote the isomorphism $A \cong \mathbb{Q}(\zeta_l)$ where $\Theta\left(\sum_{i=0}^{l-1} a_i \tilde{\gamma}_0^i\right) = \sum_{i=0}^{l-1} a_i \xi_p^i$, $a_i \in \mathbb{Q}$. Then we may write

$$\Theta_p = z_p \circ \Theta \qquad \text{on} \ \ A \subset A_p. \tag{2.15}$$

Now we shall prove that Θ restricted to M_α establishes an $\mathscr{A}$-isomorphism

$$M_\alpha \cong \mathbb{Q}(\zeta) \cap \left\{ \bigcap_p z_p^{-1} \left(\prod_{j=1}^{k_p} \mathbb{Z}[\xi_p] \Theta_p(\alpha_p) \right) \right\}. \tag{2.16}$$

Since Θ is injective it suffices to show that the second module in (2.16) is the image of M_α under Θ. First observe that

$$\begin{aligned}
\Theta_p(\alpha_p) &= \Theta_p(\sum_{\gamma \in \Gamma} b_\gamma \tilde{\gamma}) = \Theta_p \left(\sum_{\gamma \in \Gamma} \frac{1}{l} \sum_{\chi \neq 1_\Gamma} p^{-a_{p,\chi}} \chi_p(\gamma^{-1}) \tilde{\gamma} \right) \\
&= \Theta_p \left(\frac{1}{l} \sum_{i=0}^{l-1} \sum_{\chi \neq 1_\Gamma} p^{-a_{p,\chi}} \chi_p(\gamma_0^{-i}) \tilde{\gamma}_0^i \right) = \left(\frac{1}{l} \sum_{i=0}^{l-1} \sum_{\chi \neq 1_\Gamma} p^{-a_{p,\chi}} \chi_p(\gamma_0^{-i}) \xi_p^{iu_{p,j}} \right)_j \\
&= \left(\frac{1}{l} \sum_{i=0}^{l-1} \sum_{\chi \neq 1_\Gamma} p^{-a_{p,\chi}} \chi_p(\gamma_0^{-i}) \mu_p^{u_{p,j}}(\gamma_0^i) \right)_j
\end{aligned}$$

and then using the orthogonality relation for characters we get

$$\Theta_p(\alpha_p) = (p^{a_{p,j}})_j, \qquad \text{where } \ a_{p,j} = -a_{p,\mu^{u_{p,j}}}\,,\ j = 1, \ldots, k_p. \tag{2.17}$$

Thus

$$\left(\prod_{j=1}^{k_p} \mathbb{Z}_p[\xi_p]\right) \Theta_p(\alpha_p) = \prod_{j=1}^{k_p} (\mathbb{Z}_p[\xi_p] p^{a_{p,j}})\,.$$

Let $x \in M_\alpha$ and for any prime p put $x = x_p \alpha_p$ for some $x_p \in \mathscr{A}_p$. This gives $\Theta_p(x) = \Theta_p(x_p)\Theta_p(\alpha_p)$ and so by (2.15) we obtain $z_p(\Theta(x)) = \Theta_p(x_p)\Theta_p(\alpha_p) \in \prod_{j=1}^{k_p} (\mathbb{Z}_p[\xi_p]p^{a_{p,j}})$. Hence $\Theta(x) \in z_p^{-1}\left(\prod_{j=1}^{k_p} (\mathbb{Z}_p[\xi_p]p^{a_{p,j}})\right)$ proving the inclusion

$$\Theta(M_\alpha) \subseteq \mathbb{Q}(\zeta) \cap \left\{\bigcap_p z_p^{-1}\left(\prod_{j=1}^{k_p} \mathbb{Z}_p[\xi_p]\Theta_p(\alpha_p)\right)\right\}.$$

Conversely suppose that $y \in \mathbb{Q}(\zeta) \cap \left\{\bigcap_p z_p^{-1}\left(\prod_{j=1}^{k_p} \mathbb{Z}_p[\xi_p]\Theta_p(\alpha_p)\right)\right\}$ and put $y = \sum_{i=0}^{l-2} c_i \zeta^i$, $c_i \in \mathbb{Q}$. Note that for any prime p

$$z_p(y) = \left(\sum_{i=0}^{l-2} c_i \xi_p^{iu_{p,j}}\right)_j \in \prod_{j=1}^{k_p} \mathbb{Z}_p[\xi_p]\Theta_p(\alpha_p).$$

Thus by (2.17) we can write

$$\sum_{i=0}^{l-2} c_i \xi_p^{iu_{p,j}} = x_{p,j} p^{a_{p,j}}, \qquad x_{p,j} \in \mathbb{Z}_p[\xi_p] \qquad \text{and} \qquad j = 1, \ldots, k_p.$$

Let $w = \sum_{i=0}^{l-2} c_i \tilde{\gamma}_0^i$ and observe that $\Theta(w) = y$. Indeed, since $z_p(\Theta(w)) = z_p(y)$, then by (2.15) $\Theta_p(w) = z_p(y) = (x_{p,j}p^{a_{p,j}})_j$. Consequently by (2.17) we have $w = \Theta_p^{-1}((x_{p,j})_j \Theta_p^{-1}(p^{a_{p,j}}) \in \mathscr{A}_p \alpha_p$ because $\Theta_p^{-1}((x_{p,j})_j \in \mathscr{A}_p$. This completes the proof of (2.16)

Our next step is to prove the following equality

$$\mathbb{Q}(\zeta) \cap \left\{\bigcap_p z_p^{-1}\left(\prod_{j=1}^{k_p} \mathbb{Z}_p[\xi_p]\Theta_p(\alpha_p)\right)\right\} = \bigcap_{p|h_N} \bigcap_{j=1}^{k_p} \mathscr{P}_{p,j}^{a_{p,j}}. \tag{2.18}$$

First we put

$$J_p = \mathbb{Q}(\zeta) \cap z_p^{-1}\left(\prod_{j=1}^{k_p} \mathbb{Z}_p[\xi_p]\Theta_p(\alpha_p)\right)$$

and suppose that $x \in J_p$. Then $x \in \mathbb{Q}(\zeta)$ and by (2.17) $z_p(x) = (\beta_{p,j}(x))_j \in \prod_{j=1}^{k_p} (\mathbb{Z}_p[\xi_p]p^{a_{p,j}})$, whence $\beta_{p,j}(x) \in \mathbb{Q}(\xi_p) \cap \mathbb{Z}_p[\xi_p]p^{a_{p,j}}$ for $j = 1, \ldots, k_p$. Thus there exist $c,\ d \in \mathbb{Z}[\xi_p]$ such that $\beta_{p,j}(x) = c/d$ and $c \in \mathfrak{P}_p^{a_{p,j}} \setminus \mathfrak{P}_p^{a_{p,j}+1}$, $d \notin \mathfrak{P}_p$. Subsequently let $c' = \beta_{p,j}^{-1}(c)$, $d' = \beta_{p,j}^{-1}(d) \in \mathbb{Z}[\zeta]$ and then $c' \in \mathscr{P}_{p,j}^{a_{p,j}} \setminus \mathscr{P}_{p,j}^{a_{p,j}+1}$, $d' \notin \mathscr{P}_{p,j}$ for $j = 1, \dots, k_p$, where $\mathscr{P}_{p,j}$ were defined as $\beta_{p,j}^{-1}(\mathfrak{P}_p)$.

Let $\mathbb{Z}[\zeta]_{\mathscr{P}_{p,j}}$ denote the localization of $\mathbb{Z}[\zeta]$ at the prime ideal $\mathscr{P}_{p,j}$ and let $\widetilde{\mathscr{P}}_{p,j}$ be its prime ideal. Thus $x = c'/d' \in \widetilde{\mathscr{P}}_{p,j}^{a_{p,j}}$ for $j = 1, \dots, k_p$, whence

$$J_p \subseteq \bigcap_{j=1}^{k_p} \widetilde{\mathscr{P}}_{p,j}^{a_{p,j}}. \tag{2.19}$$

Conversely let $x \in \bigcap_{j=1}^{k_p} \widetilde{\mathscr{P}}_{p,j}^{a_{p,j}}$. Then there exist $r,\ s \in \mathbb{Z}[\zeta]$ such that $x = r/s$, where $r \in \mathscr{P}_{p,j}^{a_{p,j}} \setminus \mathscr{P}_{p,j}^{a_{p,j}+1}$, $s \in \mathbb{Z}[\zeta] \setminus \mathscr{P}_{p,j}$ and $j = 1, \dots, k_p$. This is equivalent to

$$\beta_{p,j}(r) \in \beta_{p,j}(\mathscr{P}_{p,j}^{a_{p,j}}) \setminus \beta_{p,j}(\mathscr{P}_{p,j}^{a_{p,j}+1}) \qquad \text{and} \qquad \beta_{p,j}(s) \in \mathbb{Z}[\zeta] \setminus \beta_{p,j}(\mathscr{P}_{p,j})$$

and since $\beta_{p,j}(\mathscr{P}_j) = \mathfrak{P}_p$ we obtain $\beta_{p,j}(x) \in \mathbb{Z}_p[\xi_p]p^{a_{p,j}}$ for $j = 1, \dots, k_p$. Hence $z_p(x) = \prod_{i=1}^{g_p} \mathbb{Z}_p[\xi_p]p^{a_{p,j}}$ and so $x \in J_p$ which shows the inverse inclusion to that of (2.19). Thus we get for any prime p

$$J_p = \bigcap_{j=1}^{k_p} \widetilde{\mathscr{P}}_{p,j}^{a_{p,j}}$$

whence by the definition of J_p

$$\mathbb{Q}(\zeta) \cap \left\{ \bigcap_p z_p^{-1} \left(\prod_{j=1}^{k_p} \mathbb{Z}_p[\xi_p]\Theta_p(\alpha_p) \right) \right\} = \bigcap_p \bigcap_{j=1}^{k_p} \widetilde{\mathscr{P}}_{p,j}^{a_{p,j}}.$$

Now observe that (2.18) is a consequence of the above and the following sequence of equalities

$$\begin{aligned} \bigcap_p \bigcap_{j=1}^{k_p} \widetilde{\mathscr{P}}_{p,j}^{a_{p,j}} &= \left\{ \bigcap_{p \mid h_N} \bigcap_{j=1}^{k_p} \widetilde{\mathscr{P}}_{p,j}^{a_{p,j}} \right\} \cap \left\{ \bigcap_{p \nmid h_N} \bigcap_{j=1}^{k_p} \mathbb{Z}[\zeta]_{\mathscr{P}_{p,j}} \right\} \\ &= \left\{ \bigcap_{p \mid h_N} \bigcap_{j=1}^{k_p} \widetilde{\mathscr{P}}_{p,j}^{a_{p,j}} \right\} \cap \mathbb{Z}[\zeta] = \bigcap_{p \mid h_N} \bigcap_{j=1}^{k_p} (\widetilde{\mathscr{P}}_{p,j}^{a_{p,j}} \cap \mathbb{Z}[\zeta]) \\ &= \bigcap_{p \mid h_N} \bigcap_{j=1}^{k_p} \mathscr{P}_{p,j}^{a_{p,j}}. \end{aligned}$$

Since $\mathscr{P}_{p,j} = \rho_{p,j}(\mathscr{P}_p)$, $\mu^{u_{p,j}} = \mu^{\rho_{p,j}}$ and $a_{p,j} = -a_{p,\mu^{u_{p,j}}}$ it follows from (2.16) and (2.18) that

$$M_\alpha \cong \prod_{p \mid h_N} \prod_{\rho \in R_p} \rho(\mathscr{P}_p)^{-a_{p,\mu^\rho}}.$$

Hence by (2.13)

$$(\overline{E}_N)_{\mathscr{A}} = \left(\prod_{p|h_N} \prod_{\rho \in R_p} \rho(\mathscr{P}_p)^{a_{p,\mu\rho}} \right)_{\mathscr{A}} .$$

Now the theorem follows because a class equality in $\mathrm{Cl}(\mathscr{A})$ implies an $\mathscr{A}$-isomorphism of modules (see the arguments used in the proof of Theorem 6.1 in [5]). ■

3. Minkowski units

Now we shall give sufficient and necessary conditions for a real tame and cyclic field N of prime degree over $\mathbb{Q}$ to have a Minkowski unit i.e. $\overline{E}_N$ is $\mathscr{A}$-free. The following criterion is a generalization of Theorem 6.1 in [5] to all primes.

Theorem 3.1. *Let $N/\mathbb{Q}$ be a real tame and cyclic extension of odd prime degree l. Then N has a Minkowski unit if and only if*

$$\nu_p(h_p^{\chi})\Phi_{\mu,p}(1) = \nu_p(h_p^{\mu})\Phi_{\chi,p}(1) \tag{3.1}$$

for any $\chi,\ \mu \in \widehat{\Gamma} \setminus \{1_\Gamma\}$, and for any prime $p \neq l$ dividing h_N.

Proof. Similarly, as in the proof of Theorem 6.1 in [5], it is sufficient to examine when the class of $\overline{E}_N$ is trivial in the group $\mathrm{Cl}(\mathscr{A})$.

Assume first that the conditions of the theorem are satisfied. Then, by Theorem 2.2 , for any prime p and any $\sigma \in \Omega$, $\mathscr{F}_p(\chi^\sigma) = \sigma(\mathscr{F}_p(\chi))$ holds where $\mathscr{F}$ is the representative homomorphism defined in Theorem 2.2. Since $\mathscr{F}_p(\chi)$ is always a power of p and equals 1 for almost all p, we can apply the same arguments as used for the map $\mathscr{S}$ in the proof of Theorem 2.2 and get $\mathscr{F} \in \mathrm{Hom}_\Omega(R'_\Gamma, F^*)\mathrm{Det}(\mathfrak{U}(\mathscr{A}))$ which proves that $\overline{E}_N$ is $\mathscr{A}$-free.

Conversely suppose that $\overline{E}_N$ is $\mathscr{A}$-free. Then the representing function $\mathscr{F}$ of the class $(\overline{E}_N)_{\mathscr{A}}$ satisfies $\mathscr{F} \in \mathrm{Hom}_\Omega(R'_\Gamma, F^*)\mathrm{Det}(\mathfrak{U}(\mathscr{A}))$. Subsequently proceeding as in the final part of the proof of Theorem 6.1 in [5] we come to $a_{p,\chi} = a_{p,\mu}$ for any nontrivial characters of Γ and any prime $p \neq l$ dividing h_N. Hence the conditions (3.1) hold. ■

Now, as in [5], we can derive from the above theorem simple sufficient conditions for the existence of Minkowski units.

Corollary 3.2. *Let $N/\mathbb{Q}$ be a real tame and cyclic extension of prime degree $l > 2$. Then N has a Minkowski unit in the following two cases:*

(i) $h_N = 1$,
(ii) *any prime p dividing h_N is a primitive root* mod l.

Proof. (i) is an immediate consequence of Theorem 2.3. To prove the sufficiency of the condition (ii) we proceed as in the proof of Corollary 6.2 in [5] using Theorem 3.1 instead of Theorem 6.1 in [5]. ■

Note that the sufficiency of the condition (i) was proved in [3] (3.3.Lemme) and it is also a consequence of Theorem in [1].

Let $\mathbb{Q}(\zeta_q)^+$ denote the maximal real subfield of $\mathbb{Q}(\zeta_q)$ and h_q^+ its class number. Finally using the same arguments as in the proof of Corollary 6.3 in [5] we can extend the list of known cyclic fields having Minkowski units.

Corollary 3.3. *Let l and q be odd prime numbers such that $l \geqslant 23$ and $q \equiv 1 \pmod{l}$. Let N be the unique real subfield of $\mathbb{Q}(\zeta_q)$ such that $(N : \mathbb{Q}) = l$. Then N has a Minkowski unit for the following pairs (l, q) :*

(i) $(41, 83)$, $(53, 107)$ (*i.e.* $N = \mathbb{Q}(\zeta_{83})^+$, $\mathbb{Q}(\zeta_{107})^+$), $(23, 139)$, $(37, 149)$ (*tables in [7]*) *on the assumption that the generalized Riemann hypothesis holds.*

(ii) $l \mid q - 1$ *and q is a prime less than* 10 000 *from Schoof's table* (*[6]*) *such that $h_q^+ = 1$. There are* 800 *such pairs.*

(iii) $l \mid q - 1$ *and q is a prime less than* 10 000 *from of Schoof's table* (*[6]*) *such that $h_q^+ > 1$ and all prime factors of h_q^+ (possible factors of h_N) are primitive roots* mod l. *There are* 80 *such pairs.*

Similarly, as in Corollary 6.3 in [5], the correctness of the examples in (ii) and (iii) of the above corollary depends on whether the entries in Schoof's table are equal to h_q^+ for suitable q's.

References

[1] A. Brumer, *On the group of units of an absolutely cyclic number field of prime degree*, J. Math. Soc. Japan **21** (1969), 357–358.

[2] C. W. Curtis and I. Reiner, *Methods of Representation Theory II*, John Wiley & Sons, 1987.

[3] I. Dubois, *S-unités et S-groupe de classes d'un corps de nombres cyclique de degré premiere*, J. Number Theory **85** (2000), 35–58.

[4] A. Fröhlich, *Galois Module Structure of Algebraic Integers*, Springer, 1983.

[5] R. Marszałek, *Units in real Abelian fields*, Acta Arithmetica, **146.2** (2011), 115–151.

[6] R. Schoof, *Class numbers of real cyclotomic fields of prime conductor*, Math. Comp. **72** (2003), 913–937.

[7] L. Washington, *Introduction to Cyclotomic Fields*, Springer, 2000.

Address: Roman Marszałek: Institute of Mathematics and Computer Science, University of Opole, ul. Oleska 48, 45-052 Opole, Poland.

E-mail: Roman.Marszalek@math.uni.opole.pl

Received: 7 September 2010; **revised:** 29 September 2010