Chebyshev's Bias for Products of Two Primes

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CONTENTS

Under two assumptions, we determine the distribution of the difference between two functions each counting the numbers less than or equal to x that are in a given arithmetic progression modulo q and the product of two primes. The two assumptions are (i) the extended Riemann hypothesis for Dirichlet L-functions modulo q, and (ii) that the imaginary parts of the nontrivial zeros of these L-functions are linearly independent over the rationals. Our results are analogues of similar results proved for primes in arithmetic progressions by Rubinstein and Sarnak.

1. INTRODUCTION

1.1 Prime Number Races

Let $\pi(x;q,a)$ denote the number of primes in the progression a mod q. For fixed q, the functions $\pi(x;q,a)$ (for $a \in A_q$, the set of residues coprime to q) all satisfy

$$\pi(x,q,a) \sim \frac{x}{\varphi(q)\log x}, \tag{1–1}$$

where φ is Euler's totient function [Davenport 00]. There are, however, curious inequities. For example, $\pi(x;4,3) \geq \pi(x;4,1)$ seems to hold for most x, an observation of Chebyshev's from 1853 [Chebyshev 53]. In fact, $\pi(x;4,3) < \pi(x;4,1)$ for the first time at $x = 26{,}861$ [Leech 57]. More generally, one can ask various questions about the behavior of

$$\Delta(x;q,a,b) := \pi(x;q,a) - \pi(x;q,b) \tag{1–2}$$

for distinct $a,b \in A_q$. Does $\Delta(x;q,a,b)$ change sign infinitely often? Where is the first sign change? How many sign changes are there with $x \leq X$? What are the extreme values of $\Delta(x;q,a,b)$? Such questions are colloquially known as *prime race problems*, and were studied extensively by Knapowski and Turán in a series of papers beginning with [Knapowski and Turán 62]. See the survey articles [Ford and Konyagin 02] and [Granville and Martin 06] and references therein for an introduction to the subject and summary of major findings. Properties

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of Dirichlet L-functions lie at the heart of such investigations.

Despite the tendency of the function $\Delta(x; 4, 3, 1)$ to be negative, Littlewood showed that it changes sign infinitely often [Littlewood 14]. Similar results have been proved for other q, a, b (see [Sneed 10] and references therein). Still, in light of Chebyshev's observation, we can ask how frequently $\Delta(x; q, a, b)$ is positive and how often it is negative. These questions are best addressed in the context of *logarithmic density*. A set S of positive integers has logarithmic density

$$\delta(S) = \lim_{x\to\infty} \frac{1}{\log x} \sum_{\substack{n\le x\\ n\in S}} \frac{1}{n},$$

provided the limit exists. Let $\delta(q, a, b) = \delta(P(q, a, b))$, where $P(q, a, b)$ is the set of integers n with $\Delta(n; q, a, b) > 0$. It was shown in [Rubinstein and Sarnak 94] that $\delta(q; a, b)$ exists, assuming two hypotheses: (i) the extended Riemann hypothesis for Dirichlet L-functions modulo q (ERH_q), and (ii) that the imaginary parts of zeros of each Dirichlet L-function are linearly independent over the rationals (GSH_q, the grand simplicity hypothesis). The authors also gave methods to accurately estimate the "bias," for example showing that $\delta(4; 3, 1) \approx 0.996$ in Chebyshev's case. More generally, $\delta(q; a, b) = \frac{1}{2}$ when a and b are either both quadratic residues modulo q or both quadratic nonresidues (unbiased prime races), but $\delta(q; a, b) > \frac{1}{2}$ whenever a is a quadratic nonresidue and b is a quadratic residue. A bit later we will discuss the reasons behind these phenomena. Sharp asymptotics for $\delta(q; a, b)$ have recently been given in [Fiorilli and Martin 09], which explain other properties of these densities.

1.2 Quasiprime Races

In this paper we develop a parallel theory for comparison of functions $\pi_2(x; q, a)$, the number of integers $\le x$ that are in the progression a mod q and that are the product of two primes p_1p_2 ($p_1 = p_2$ allowed). Put

$$\Delta_2(x; q, a, b) := \pi_2(x; q, a) - \pi_2(x; q, b),$$

let $P_2(q, a, b)$ be the set of integers n with $\Delta_2(n; q, a, b) > 0$, and set $\delta_2(q, a, b) = \delta(P_2(q, a, b))$. Table 1 shows all such quasiprimes up to 100 grouped in residue classes modulo 4.

Observe that $\Delta_2(x; 4, 3, 1) \le 0$ for $x \le 100$, and in fact, the smallest x with $\Delta_2(x; 4, 3, 1) > 0$ is $x = 26{,}747$ (amazingly close to the first sign change of $\Delta(x; 4, 3, 1)$). Some years ago, Richard Hudson conjectured that the bias for products of two primes is always reversed from that of primes; i.e., $\delta_2(q; a, b) < \frac{1}{2}$ when a is a quadratic nonresidue modulo q and b is a quadratic residue. Under the same assumptions as [Rubinstein and Sarnak 94], namely ERH_q and GSH_q, we confirm Hudson's conjecture and also show that the bias is less pronounced than the bias for $\Delta(x; q, a, b)$.

$pq \equiv 1 \pmod 4$	$pq \equiv 3 \pmod 4$
9	15
21	35
25	39
33	51
49	55
57	87
65	91
69	95
77	
85	
93	

TABLE 1. All quasiprimes up to 100 grouped in residue classes modulo 4.

Theorem 1.1. *Let a, b be distinct elements of A_q. Assuming ERH_q and GSH_q, $\delta_2(q; a, b)$ exists. Moreover, if a and b are both quadratic residues modulo q or both quadratic nonresidues, then $\delta_2(q; a, b) = \frac{1}{2}$. Otherwise, if a is a quadratic nonresidue and b is a quadratic residue, then*

$$1 - \delta(q; a, b) < \delta_2(q; a, b) < \frac{1}{2}.$$

We can accurately estimate $\delta_2(q; a, b)$ borrowing methods from [Rubinstein and Sarnak 94, Section 4]. In particular, we have

$$\delta_2(4; 3, 1) \approx 0.10572.$$

We deduce Theorem 1.1 by connecting the distribution of $\Delta_2(x; q, a, b)$ with the distribution of $\Delta(x; q, a, b)$. Although the relationship is "simple," there is no elementary way to derive it, say by writing

$$\pi_2(x; q, a) = \frac{1}{2} \sum_{p\le x} \pi\left(\frac{x}{p}; q, ap^{-1} \bmod q\right) + \frac{1}{2} \sum_{\substack{p\le\sqrt{x}\\ p^2\equiv a \pmod q}} 1.$$

In particular, our result depends strongly on the assumption that the zeros of the L-functions modulo q have only simple zeros. Let $N(q, a)$ be the number of $x \in A_q$ with $x^2 \equiv a \pmod q$, and let $C(q)$ be the set of nonprincipal Dirichlet characters modulo q.

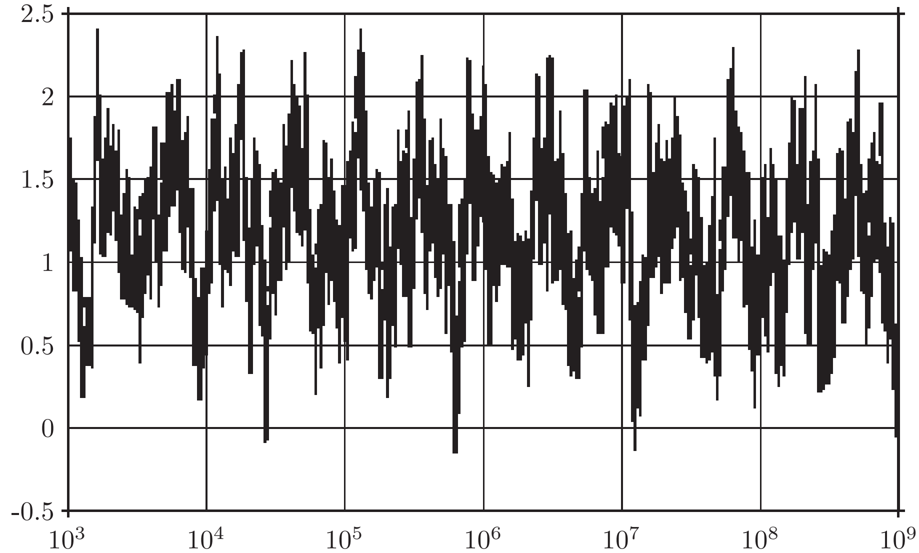

FIGURE 1. $\frac{\log x}{\sqrt{x}}\Delta(x;4,3,1)$.

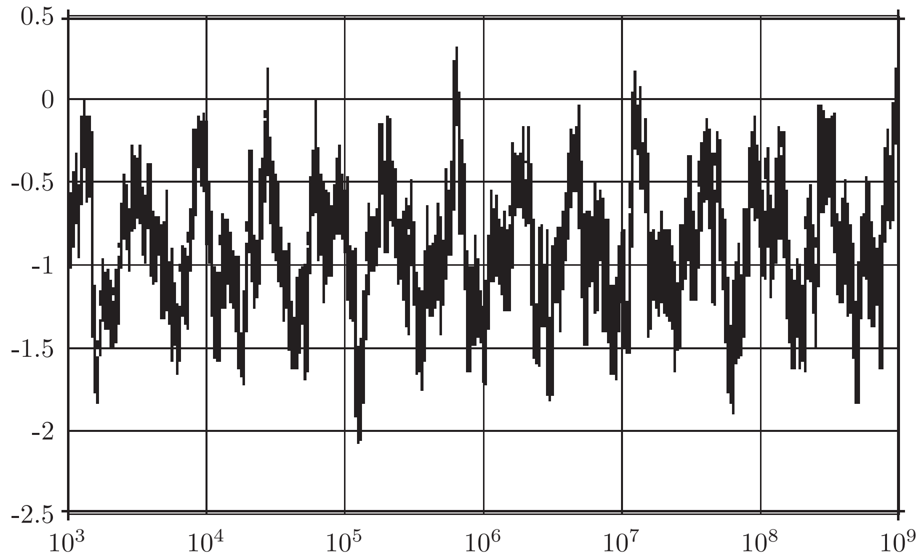

FIGURE 2. $\frac{\log x}{\sqrt{x}\log\log x}\Delta_2(x;4,3,1)$.

Theorem 1.2. *Assume* ERH_q *and for each* $\chi \in C(q)$, $L(\frac{1}{2},\chi) \neq 0$ *and the zeros of* $L(s,\chi)$ *are simple. Then*

$$\frac{\Delta_2(x;q,a,b)\log x}{\sqrt{x}\log\log x} = \frac{N(q,b)-N(q,a)}{2\phi(q)} - \frac{\log x}{\sqrt{x}}\Delta(x;q,a,b) + \Sigma(x;q,a,b),$$

where $\frac{1}{Y}\int_1^Y |\Sigma(e^y;q,a,b)|^2\,dy = o(1)$ *as* $Y \to \infty$.

The expression for Δ_2 given in Theorem 1.2 must be modified if some $L(s,\chi)$ has multiple zeros; see Section 3 for

Figures 1, 2, and 3 show graphs corresponding to $(q,a,b) = (4,3,1)$, plotted on a logarithmic scale from

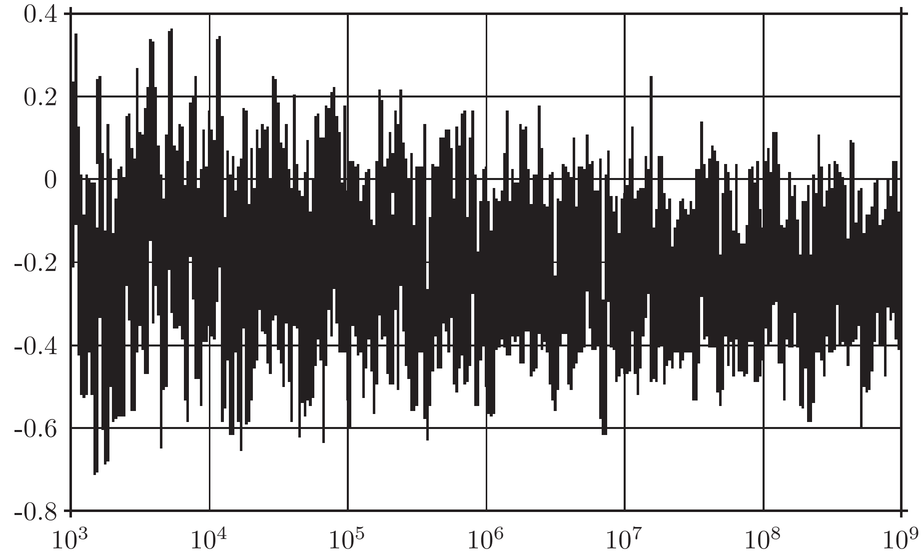

FIGURE 3. $\Sigma(x;4,3,1)$.

$x = 10^3$ to $x = 10^9$. While $\Sigma(x;4,3,1)$ appears to be oscillating around -0.2, this is caused by some terms in $\Sigma(x;4,3,1)$ of order $1/\log\log x$, and $\log\log 10^9 \approx 3.03$. By Theorem 1.2, $\Sigma(x;4,3,1)$ will (assuming ERH_4 and GSH_4) eventually settle down to oscillating about 0.

It is not immediate that Theorem 1.1 follows from Theorem 1.2. One first needs more precise information about the distribution of $\Delta(x;q,a,b)$ from [Rubinstein and Sarnak 94].

Theorem 1.3. [Rubinstein and Sarnak 94, Section 1] *Assume* ERH_q *and* GSH_q. *For any distinct* $a, b \in A_q$, *the function*

$$\frac{u\Delta(e^u;q,a,b)}{e^{u/2}} \tag{1–3}$$

has a probabilistic distribution. This distribution (i) has mean $(N(q,b)-N(q,a))/\phi(q)$, *(ii) is symmetric with respect to its mean, and (iii) has a continuous, positive density function.*

Assume that a is a quadratic nonresidue modulo q and that b is a quadratic residue. Then $N(q,b)-N(q,a) > 0$. Let f be the density function for the distribution of (1–3), that is,

$$f(t) = \frac{d}{dt}\lim_{U\to\infty}\frac{\mathrm{meas}\left\{0 \le u \le U : \frac{u\Delta(e^u;q,a,b)}{e^{u/2}} \le t\right\}}{U}.$$

We see from Theorem 1.3 that

$$\delta(q,a,b) = \int_0^\infty f(t)\,dt > \frac{1}{2}$$

and from Theorem 1.2 that

$$\delta_2(q,a,b) = \int_{-\infty}^{\frac{N(q,b)-N(q,a)}{2\phi(q)}} f(t)\,dt,$$

from which Theorem 1.1 follows.

Theorem 1.2 also determines the joint distribution of any vector function

$$\frac{u}{e^{u/2}\log u}\big(\Delta_2(e^u;q,a_1,b_1),\ldots,\Delta_2(e^u;q,a_r,b_r)\big). \tag{1–4}$$

Theorem 1.4. *If* $f(x_1,\ldots,x_r)$ *is the density function of*

$$\frac{u}{e^{u/2}}\big(\Delta(e^u;q,a_1,b_1),\ldots,\Delta(e^u;q,a_r,b_r)\big),$$

then the joint density function of (1–4) *is*

$$f\!\left(\frac{N(q,b_1)-N(q,a_1)}{2\phi(q)}-x_1,\ldots,\frac{N(q,b_r)-N(q,a_r)}{2\phi(q)}-x_r\right).$$

1.3 Origin of Chebyshev's Bias

From an analytic point of view (L-functions), the weighted sum

$$\Delta^*(x;q,a,b) = \sum_{\substack{n\le x\\ n\equiv a \bmod q}}\Lambda(n) - \sum_{\substack{n\le x\\ n\equiv b \bmod q}}\Lambda(n), \tag{1–5}$$

where Λ is the von Mangoldt function, is more natural than (1–2). Expressing $\Delta^*(x;q,a,b)$ in terms of sums over zeros of L-functions in the standard way [Davenport 00, Section 19], we obtain, on ERH_q,

$$e^{-u/2}\phi(q)\Delta^*(e^u;q,a,b) = -\sum_{\chi\in C(q)}(\overline{\chi}(a)-\overline{\chi}(b))\sum_{\gamma}\frac{e^{i\gamma u}}{1/2+i\gamma}+O(u^2e^{-u/2}),$$

where γ runs over the imaginary parts of the nontrivial zeros of $L(s,\chi)$ (counted with multiplicity). Hypothesis GSH_q implies, in particular, that $L(1/2,\chi)\neq 0$. Each summand $e^{i\gamma u}/(1/2+i\gamma)$ is thus a harmonic with mean zero as $u\to\infty$, and GSH_q implies that the harmonics behave independently. Hence, we expect that $e^{-u/2}\phi(q)\Delta^*(e^u;q,a,b)$ will behave like a mean-zero random variable. On the other hand, the right side of (1–5) contains not only terms corresponding to prime n but terms corresponding to powers of primes. Applying the prime number theorem for arithmetic progressions (1–1) to the terms $n=p^2$ in (1–5) gives

$$\begin{aligned}\Delta^*(x;q,a,b) &= \sum_{\substack{p\le x\\ p\equiv a \bmod q}}\log p-\sum_{\substack{p\le x\\ p\equiv b \bmod q}}\log p\\ &\quad+\frac{x^{1/2}}{\phi(q)}(N(q,a)-N(q,b))+O(x^{1/3}).\end{aligned}$$

Hence, on ERH_q and GSH_q, we expect the expression

$$\frac{1}{\sqrt{x}}\left(\sum_{\substack{p\le x\\ p\equiv a \bmod q}}\log p-\sum_{\substack{p\le x\\ p\equiv b \bmod q}}\log p\right) \tag{1–6}$$

to behave like a random variable with mean $(N(q,b)-N(q,a))/\phi(q)$. Finally, the distribution of $\Delta(x;q,a,b)$ is obtained from the distribution of (1–6) and partial summation.

1.4 Analyzing $\Delta_2(x;q,a,b)$

A natural analogue of $\Delta^*(x;q,a,b)$ is

$$\sum_{\substack{mn\le x\\ mn\equiv a \bmod q}}\Lambda(m)\Lambda(n)-\sum_{\substack{mn\le x\\ mn\equiv b \bmod q}}\Lambda(m)\Lambda(n). \tag{1–7}$$

As with $\Delta^*(x;q,a,b)$, the expression in (1–7) can be easily written as a sum over zeros of L-functions plus a small error. The main problem now is that the principal summands, namely $\log p_1\log p_2$ for primes p_1,p_2, are very irregular as a function of p_1p_2, and thus estimates for $\Delta_2(x;q,a,b)$ cannot be recovered by partial summation. We get around this problem using a double integration, a method that goes back to [Landau 74, Section 88]. We have

$$\begin{aligned}\Delta_2(x;q,a,b) &= \frac{1}{\phi(q)}\sum_{\chi\in C(q)}(\overline{\chi}(a)-\overline{\chi}(b))\sum_{\substack{n=p_1p_2\le x\\ p_1\le p_2}}\chi(n)\\ &= \frac{1}{2\phi(q)}\sum_{\chi\in C(q)}(\overline{\chi}(a)-\overline{\chi}(b))\int_0^\infty\!\!\int_0^\infty G(x,u,v;\chi)\,du\,dv\\ &\quad+O\left(\frac{\sqrt{x}}{\log x}\right),\end{aligned} \tag{1–8}$$

where

$$G(x,u,v;\chi)=\sum_{p_1p_2\le x}\frac{\chi(p_1p_2)\log p_1\log p_2}{p_1^up_2^v}. \tag{1–9}$$

The related functions

$$G^*(x,u,v;\chi)=\sum_{mn\le x}\frac{\chi(mn)\Lambda(m)\Lambda(n)}{m^un^v}$$

are more "natural" from an analytic point of view, being easily expressed in terms of zeros of Dirichlet L-functions. By the reasoning of the previous subsection, each $G^*(x,u,v;\chi)$ is expected to be unbiased, the bias in $\Delta_2(x;q,a,b)$ originating from the summands in $G^*(x,u,v;\chi)$ where m is not prime or n is not prime.

1.5 A Heuristic Argument for the Bias in $\Delta_2(x;q,a,b)$

We conclude this introduction with a heuristic evaluation of the bias in $\Delta_2(x;q,a,b)$, which originates from the difference between functions $G(x;u,v;\chi)$ and $G^*(x,u,v;\chi)$. For simplicity of exposition, we shall concentrate on the special case $(q,a,b)=(4,3,1)$. In this case, the bias arises from terms $p_1p_2^2$ and $p_1^2p_2^2$ that appear in $G^*(x;u,v;\chi)$ but not in $G(x,u,v;\chi)$. Let χ be the nonprincipal character modulo 4, so that

$$\begin{aligned}\frac12\int_0^\infty\!\!\int_0^\infty\big(G^*(x,u,v;\chi)-G(x,u,v;\chi)\big)\,du\,dv\\ =\frac12\sum_{\substack{p_1^ap_2^b\le x\\ \max(a,b)\ge 2}}\frac{\chi(p_1^ap_2^b)}{ab}.\end{aligned}$$

There are $O(x^{1/2}/\log x)$ terms with $\min(a,b)\ge 2$ and $\max(a,b)\ge 3$. By the prime number theorem and partial summation,

$$\frac12\sum_{p_1^2p_2^2\le x}\frac14=\frac18\sum_{p\le\sqrt{x}}\pi\left(\sqrt{x/p^2}\right)\sim\frac{x^{1/2}\log\log x}{2\log x}.$$

Thus,

$$\Delta_2(x;4,3,1) = -\frac{1}{2}\sum_{mn\le x}\frac{\chi(mn)\Lambda(m)\Lambda(n)}{\log m\log n} - \sum_{k=2}^{\infty}\frac{1}{k}\sum_{p_1^k\le x}\chi(p_1^k)\Delta(x/p_1^k;4,3,1) + \left(\frac{1}{2}+o(1)\right)\frac{x^{1/2}\log\log x}{\log x}.$$

By Theorem 1.3, $\Delta(y;4,3,1) = y^{1/2}/\log y + E(y)$, where $E(y)$ oscillates with mean 0. Thus,

$$\sum_{k=2}^{\infty}\frac{1}{k}\sum_{p_1^k\le x}\chi(p_1^k)\Delta(x/p_1^k;4,3,1) = \sum_{k=2}^{\infty}\frac{2}{k}\sum_{p_1^k\le x}\chi(p_1^k)\frac{\sqrt{x/p_1^k}}{\log(x/p_1^k)} + E'(x),$$

where $E'(x)$ is expected to oscillate with mean zero. The $k=2$ terms are

$$\sum_{p_1^2\le x}\frac{\sqrt{x/p_1^2}}{\log(x/p_1^2)} \sim \frac{\sqrt{x}\log\log x}{\log x},$$

while the terms corresponding to $k\ge 3$ contribute

$$\ll \sum_{k=3}^{\infty}\frac{1}{k}\sum_{p_1^k\le x}\frac{\sqrt{x/p_1^k}}{\log(x/p_1^k)} \ll \frac{\sqrt{x}}{\log x}.$$

Thus, we find that

$$\Delta_2(x;4,3,1) = -\frac{1}{2}\sum_{mn\le x}\frac{\chi(mn)\Lambda(m)\Lambda(n)}{\log m\log n} - \left(\frac{1}{2}+o(1)\right)\frac{x^{1/2}\log\log x}{\log x} + E'(x).$$

1.6 Further Problems

It is natural to consider the distribution, in arithmetic progressions, of numbers composed of exactly k prime factors, where $k\ge 3$ is fixed. As with the cases $k=1$ and $k=2$, we expect there to be no bias if we count all numbers $p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ with weight $(a_1\cdots a_k)^{-1}$. If, however, we count terms that are the product of precisely k primes (that is, numbers $p_1^{a_1}\cdots p_j^{a_j}$ with $a_1+\cdots+a_j=k$), then there will be a bias. Hudson has conjectured that the bias will be in the same direction as for primes when k is odd, and in the opposite direction for even k. We conjecture that in addition, the bias becomes less pronounced as k increases.

2. PRELIMINARIES

With χ fixed, the letter γ, with or without subscripts, denotes the imaginary part of a zero of $L(s,\chi)$ inside the critical strip. In sums over γ, each term appears with its multiplicity $m(\gamma)$ unless we specify that we sum over distinct γ. Constants implied by O and $\ll$ symbols depend only on χ (and hence on q) unless additional dependence is indicated with a subscript. Let

$$A(\chi) = \begin{cases}1 & \text{if } \chi^2=\chi_0,\\ 0 & \text{otherwise,}\end{cases}$$

where χ_0 is the principal character modulo q. That is, $A(\chi)=1$ if and only if χ is a real character. For $\chi\in C(q)$, define

$$F(s,\chi) = \sum_p \frac{\chi(p)\log p}{p^s}.$$

The following estimates are standard; see, for example, [Davenport 00, Sections 15, 16].

Lemma 2.1. *Let $\chi\in C(q)$, assume ERH_q, and fix $c>\frac{1}{3}$. Then $F(s,\chi) = -\frac{L'}{L}(s,\chi) + A(\chi)\frac{\zeta'}{\zeta}(2s) + H(s,\chi)$, where $H(s,\chi)$ is analytic and uniformly bounded in the half-plane $\Re s\ge c$.*

Lemma 2.2. *Let χ be a Dirichlet character modulo q. Let $N(T,\chi)$ denote the number of zeros of $L(s,\chi)$ with $0<\Re s<1$ and $|\Im s|<T$. Then*

(1) $N(T,\chi) = O(T\log(qT))$ *for* $T\ge 1$.

(2) $N(T,\chi) - N(T-1,\chi) = O(\log(qT))$ *for* $T\ge 1$.

(3) *Uniformly for $s=\sigma+it$ and $\sigma\ge -1$,*

$$\frac{L'(s,\chi)}{L(s,\chi)} = \sum_{|\gamma-t|<1}\frac{1}{s-\rho} + O(\log q(|t|+2)).$$

(4) $-\frac{\zeta'}{\zeta}(\sigma) = \frac{1}{\sigma-1} + O(1)$ *uniformly for* $\sigma\ge\frac{1}{2}$, $\sigma\ne 1$.

(5) $\left|\frac{\zeta'}{\zeta}(\sigma+iT)\right| \le -\frac{\zeta'}{\zeta}(\sigma)$ *for* $\sigma>1$.

For a suitably small fixed $\delta>0$, we say that a number $T\ge 2$ is *admissible* if for all $\chi\in C(q)\cup\{\chi_0\}$ and all zeros $\frac{1}{2}+i\gamma$ of $L(s,\chi)$, $|\gamma-T|\ge\delta(\log T)^{-1}$. By Lemma 2.2, we can choose δ small enough, depending on q, that there is an admissible T in $[U,U+1]$ for all $U\ge 2$. From Lemma 2.2 we obtain the following result.

Lemma 2.3. *Uniformly for $\sigma \ge \frac{2}{5}$ and admissible $T \ge 2$,*

$$|F(\sigma+iT,\chi)| = O(\log^2 T).$$

Lemma 2.4. *Fix $\chi \in C(q)$ and assume $L(\frac12,\chi) \ne 0$. For $A \ge 0$ and real $k \ge 0$,*

$$\sum_{\substack{|\gamma_1|,|\gamma_2|\ge A\\ |\gamma_1-\gamma_2|\ge 1}} \frac{\log^k(|\gamma_1|+3)\log^k(|\gamma_2|+3)}{|\gamma_1||\gamma_2||\gamma_1-\gamma_2|} \ll_k \frac{\log^{2k+3}(A+3)}{A+1}.$$

Proof: The sum in question is at most twice the sum of terms with $|\gamma_2| \ge |\gamma_1|$, which is

$$\ll \sum_{|\gamma_2|\ge A} \frac{\log^{2k}(|\gamma_2|+3)}{|\gamma_2|} \times \left(\frac{1}{|\gamma_2|} \sum_{|\gamma_1|<\frac{|\gamma_2|}{2}} \frac{1}{|\gamma_1|} + \frac{1}{|\gamma_2|} \sum_{\substack{\frac{|\gamma_2|}{2}\le|\gamma_1|\le|\gamma_2|\\ |\gamma_2-\gamma_1|\ge 1}} \frac{1}{|\gamma_2-\gamma_1|} \right).$$

By Lemma 2.2(1), the two sums over γ_1 are $O(\log^2(|\gamma_2|+3))$. A further application of Lemma 2.2(1) completes the proof. □

We conclude this section with a truncated version of the Perron formula for $G(x,u,v;\chi)$.

Lemma 2.5. *Uniformly for $x \le T \le 2x^2$, $x \ge 2$, $u \ge 0$, and $v \ge 0$, we have*

$$G(x,u,v;\chi) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} F(s+u,\chi)F(s+v,\chi)\frac{x^s}{s}ds + O(\log^3 x), \tag{2–1}$$

where $c = 1 + \frac{1}{\log x}$.

Proof: For $\Re s > 1$, we have

$$F(s+u,\chi)F(s+v,\chi) = \sum_{n=1}^{\infty} f(n)n^{-s},$$

$$f(n) = \sum_{p_1p_2=n} \frac{\chi(p_1p_2)\log p_1 \log p_2}{p_1^u p_2^v}.$$

Using the trivial estimate $|f(n)| \le \log^2 n$ and a standard argument [Davenport 00, Section 17, (3) and (5)], we obtain the desired bounds. □

3. OUTLINE OF THE PROOF OF THEOREM 1.2

Throughout the remainder of this paper, fix q, and assume ERH_q and that $L(\frac12,\chi) \ne 0$ for each $\chi \in C(q)$. Let

$$\varepsilon = \frac{1}{100}.$$

We next define a function $T(x)$ as follows. For each positive integer n, let T_n be an admissible value of T satisfying $\exp(2^{n+1}) \le T_n \le \exp(2^{n+1})+1$ and set $T(x) = T_n$ for $\exp(2^n) < x \le \exp(2^{n+1})$. In particular, we have

$$x \le T(x) \le 2x^2 \qquad (x \ge e^2).$$

Our first task is to express the double integrals in (1–8) in terms of sums over zeros of $L(s,\chi)$. This is proved in Section 4.

Lemma 3.1. *Let $\chi \in C(q)$ and let $T = T(x)$. Then*

$$\begin{aligned} x^{-1/2}&\int_0^\infty\int_0^\infty G(x,u,v;\chi)\,du\,dv \\ &= 2\int_0^{2\varepsilon}\int_0^{2\varepsilon} \sum_{|\gamma|\le T} \frac{F(\frac12+u-v+i\gamma,\chi)x^{-v+i\gamma}}{\frac12-v+i\gamma}du\,dv \\ &\quad + \frac{A(\chi)\log\log x + \Sigma_1(x;\chi) + O(1)}{\log x}, \end{aligned}$$

where $\int_1^Y |\Sigma_1(e^y;\chi)|^2 dy = O(Y)$.

The aggregate of terms $A(\chi)\log\log x/\log x$ accounts for the bias for products of two primes. As with the Chebyshev bias for primes, these terms arise from poles of $F(s)$ at $s = \frac12$ when $A(\chi) = 1$ (see Lemma 2.1) and correspond to the contribution to $F(s)$ from squares of primes. The double integral on the right side in Lemma 3.1 is complicated to analyze. In Section 5 we prove the following.

Lemma 3.2. *Let $\chi \in C(q)$. Let n be a positive integer, $2^n < \log x \le 2^{n+1}$, and $T = T(x)$. Then*

$$\begin{aligned} 2&\int_0^{2\varepsilon}\int_0^{2\varepsilon} \sum_{|\gamma|\le T} \frac{F(\frac12+u-v+i\gamma,\chi)x^{-v+i\gamma}}{\frac12-v+i\gamma}du\ dv \\ &= \frac{\Sigma_2(x;\chi)}{\log x} + 2\sum_{\substack{|\gamma|\le T\\ \gamma \text{ distinct}}} m^2(\gamma)x^{i\gamma}\left(\frac12+i\gamma\right) \\ &\quad \times \int_0^{2\varepsilon-2^{-n}} \frac{x^{-v}}{\frac12-v+i\gamma}\int_{v+2^{-n}}^{2\varepsilon} \frac{du\ dv}{(u-v)(\frac12-u+i\gamma)}, \end{aligned}$$

where $\int_1^Y |\Sigma_2(e^y;\chi)|^2 dy = o(Y\log^2 Y)$ as $Y \to \infty$.

The terms on the right in Lemma 3.2 with small $|\gamma|$ will give the main term, and terms with larger $|\gamma|$ are considered as error terms. The next lemma is proved in Section 6.

Lemma 3.3. *Let $\chi \in C(q)$. Let n be a positive integer, $2^n < \log x \le 2^{n+1}$, $T = T(x)$, and $2 \le T_0 \le T$. Then*

$$2 \sum_{\substack{|\gamma| \le T \\ \gamma \text{ distinct}}} m^2(\gamma) x^{i\gamma} \left(\frac{1}{2} + i\gamma\right)$$

$$\times \int_0^{2\varepsilon - 2^{-n}} \frac{x^{-v}}{\frac{1}{2} - v + i\gamma} \int_{v+2^{-n}}^{2\varepsilon} \frac{du}{(u-v)(\frac{1}{2} - u + i\gamma)} dv$$

$$= \frac{2\log\log x}{\log x} \sum_{\substack{|\gamma| \le T_0 \\ \gamma \text{ distinct}}} \frac{m^2(\gamma) x^{i\gamma}}{1/2 + i\gamma} + O\left(\frac{\log^3 T_0}{\log x}\right)$$

$$+ \frac{\Sigma_3(x, T_0; \chi)}{\log x},$$

where

$$\frac{1}{Y} \int_1^Y |\Sigma_3(e^y, T_0; \chi)|^2 \, dy \ll \frac{\log^5 T_0}{T_0} \log^2 Y.$$

Combining Lemmas 3.1, 3.2, and 3.3 with (1–8) yields (for fixed large T_0)

$$\Delta_2(x; q, a, b) = \frac{\sqrt{x}}{2\phi(q)} \sum_{\chi \in C(q)} (\overline{\chi}(a) - \overline{\chi}(b))$$

$$\times \left[\frac{\log\log x}{\log x}\left(A(\chi) + 2 \sum_{|\gamma| \le T_0, \gamma \text{ distinct}} \frac{m^2(\gamma) x^{i\gamma}}{1/2 + i\gamma}\right)\right.$$

$$\left. + \frac{\Sigma_1(x;\chi) + \Sigma_2(x;\chi) + \Sigma_3(x, T_0;\chi) + O(\log^3 T_0)}{\log x}\right],$$

where

$$\lim_{T_0 \to \infty}\left(\limsup_{Y \to \infty} \frac{1}{Y \log^2 Y} \sum_{\chi \in C(q)} \int_1^Y |\Sigma_1(e^y;\chi) + \Sigma_2(e^y;\chi)\right.$$

$$\left. + \Sigma_3(e^y; T_0; \chi)|^2 \, dy\right) = 0.$$

On the other hand, (cf. [Rubinstein and Sarnak 94]),

$$\Delta(x; q, a, b) = \frac{\sqrt{x}}{\log x}\left(\frac{N(q,b) - N(q,a)}{\phi(q)}\right.$$

$$\left. - \sum_{\chi \in C(q)} (\overline{\chi}(a) - \overline{\chi}(b)) \sum_{|\gamma| \le T_0} \frac{x^{i\gamma}}{1/2 + i\gamma} + \Sigma_4(x; T_0)\right),$$

where

$$\lim_{T_0 \to \infty}\left(\limsup_{Y \to \infty} Y^{-1} \int_1^Y |\Sigma_4(e^y; T_0)|^2 \, dy\right) = 0.$$

Now assume that $m(\gamma) = 1$ for all γ, and note that

$$\sum_{\chi \in C(q)} (\overline{\chi}(a) - \overline{\chi}(b)) \, A(\chi) = N(q,a) - N(q,b).$$

Letting $T_0 \to \infty$ finishes the proof of Theorem 1.2.

4. PROOF OF LEMMA 3.1

Assume ERH_q throughout. We first estimate $G(x, u, v; \chi)$ for different ranges of u, v.

Lemma 4.1. *Let $\chi \in C(q)$, $\chi \ne \chi_0$. For $x \ge 4$, the following hold:*

(1) *For $u \ge \varepsilon$ and $v \ge \varepsilon$, $G(x, u, v; \chi) \ll x^{\frac{1}{2} - \frac{\varepsilon}{2}} \log^5 x$.*

(2) *For $u \ge 2\varepsilon$, $v \le \varepsilon$ and $T = T(x)$,*

$$\frac{G(x,u,v;\chi)}{\sqrt{x}} = \sum_{|\gamma| \le T} \frac{F(\frac{1}{2} + u - v + i\gamma, \chi) x^{-v+i\gamma}}{\frac{1}{2} - v + i\gamma}$$

$$- A(\chi) \frac{F(\frac{1}{2} + u - v, \chi) x^{-v}}{1 - 2v} + O(x^{-\varepsilon} \log^5 x).$$

(3) *For $u \le 2\varepsilon$, $v \le 2\varepsilon$, $u \ne v$ and $T = T(x)$,*

$$\frac{G(x,u,v;\chi)}{\sqrt{x}} = \sum_{|\gamma| \le T} \frac{F(\frac{1}{2} + u - v + i\gamma, \chi) x^{-v+i\gamma}}{\frac{1}{2} - v + i\gamma}$$

$$+ \frac{F(\frac{1}{2} - u + v + i\gamma, \chi) x^{-u+i\gamma}}{\frac{1}{2} - u + i\gamma}$$

$$- A(\chi)\left(\frac{F(\frac{1}{2} + u - v, \chi) x^{-v}}{1 - 2v}\right.$$

$$\left. + \frac{F(\frac{1}{2} - u + v, \chi) x^{-u}}{1 - 2u}\right) + O(x^{-3\varepsilon} \log^5 x).$$

Proof: Assume $u \ge \varepsilon$ and $v \ge \varepsilon$. Start with the approximation of $G(x, u, v; \chi)$ given by Lemma 2.5, and then deform the segment of integration to the contour consisting of three straight segments connecting $c - iT$, $b - iT$, $b + iT$, and $c + iT$, where $b = \frac{1}{2} - \frac{\varepsilon}{2}$ and $T = T(x)$. The rectangle formed by the new and old contours does not contain any poles of $F(s+u, \chi) F(s+v, \chi) s^{-1}$. On the three new segments, by Lemmas 2.1, 2.2, and 2.3, we have $|F(s+u, \chi) F(s+v, \chi)| \ll \log^4 T$. Hence the integral of $F(s+u, \chi) F(s+v, \chi) x^s s^{-1}$ over the three segments is

$$\ll (\log^4 x)\left(\int_b^c \frac{x^\sigma}{|\sigma + iT|} \, d\sigma + \int_{-T}^T \frac{x^b}{|b + it|} \, dt\right) \ll x^b \log^5 x.$$

This proves (1).

We now consider the case $v \le \varepsilon$ and $u \ge 2\varepsilon$. We set $b = \frac{1}{2} - \frac{3\varepsilon}{2}$ and deform the contour of integration as in the previous case. Since $u + b \ge \frac{1}{2} + \frac{\varepsilon}{2}$ and $v + b \le \frac{1}{2} - \frac{\varepsilon}{2}$, we have by Lemma 2.3 that $|F(s+u, \chi) F(s+v, \chi)| \ll \log^4 T \ll \log^4 x$ on all three new segments. As in the proof of (1), the integral over the new contour is $\ll x^b \log^5 x$. We pick up residue terms from poles of

$F(s+v,\chi)$ inside the rectangle coming from the nontrivial zeros of $L(s,\chi)$, plus a pole at $s=\frac{1}{2}-v$ from the term $\frac{\zeta'}{\zeta}(2s+2v)$ if $\chi^2=\chi_0$. The sum of the residues is

$$\sum_{|\gamma|\le T}\frac{F(\frac{1}{2}+u-v+i\gamma,\chi)x^{\frac{1}{2}-v+i\gamma}}{\frac{1}{2}-v+i\gamma} - A(\chi)\frac{F(\frac{1}{2}+u-v,\chi)x^{\frac{1}{2}-v}}{1-2v},$$

and (2) follows.

Finally, consider the case $0\le u,v\le 2\varepsilon$. Let $b=\frac{1}{2}-3\varepsilon$ and deform the contour as in the previous cases. As before, the integral over the new contour is $O(x^b\log^5 x)$. This time, we pick up residues from poles of both $F(s+u,\chi)$ and $F(s+v,\chi)$ and (3) follows. □

Proof of Lemma 3.1: Begin with

$$\int_0^\infty\int_0^\infty G(x,u,v;\chi)\,du\,dv = I_1+I_2+2I_3+I_4,$$

where I_1 is the integral over $\max(u,v)\ge\log x$; I_2 is the integral over $2\varepsilon\le\max(u,v)\le\log x$ and $\min(u,v)\ge\varepsilon$; I_3 is the integral over $0\le v\le\varepsilon$, $2\varepsilon\le u\le\log x$; and I_4 is the integral over $0\le u,v\le 2\varepsilon$. For $\max(u,v)\ge\log x$,

$$|G(x,u,v;\chi)|\le\sum_{p\le x}\frac{\log p}{p^u}\sum_{p\le x}\frac{\log q}{q^v}\ll\frac{x}{2^{\max(u,v)}},$$

whence $I_1\ll x^{1-\log 2}$. By Lemma 4.1(1), $I_2\ll x^{1/2-\varepsilon/2}\log^7 x$.

By Lemma 4.1(2),

$$I_3 = x^{1/2}\int_0^\varepsilon\int_{2\varepsilon}^{\log x}\sum_{|\gamma|\le T}\frac{F(\frac{1}{2}+u-v+i\gamma,\chi)x^{-v+i\gamma}}{\frac{1}{2}-v+i\gamma} - A(\chi)\frac{F(\frac{1}{2}+u-v,\chi)x^{-v}}{1-2v}\,du\,dv + O(x^{1/2-\frac{3\varepsilon}{2}}\log^6 x). \tag{4–1}$$

By Lemmas 2.2 and 2.3,

$$\int_0^\varepsilon\int_{2\varepsilon}^{\log x}\frac{F(\frac{1}{2}+u-v,\chi)x^{-v}}{1-2v}du\ dv\ll\frac{1}{\log x}. \tag{4–2}$$

Let

$$\Sigma_1(x)=(\log x)\int_0^\varepsilon\int_{2\varepsilon}^{\log x}\sum_{|\gamma|\le T}\frac{F(\frac{1}{2}+u-v+i\gamma,\chi)x^{-v+i\gamma}}{\frac{1}{2}-v+i\gamma}\,du\,dv.$$

Since $\sigma:=\frac{1}{2}+u-v\ge\frac{1}{2}+\varepsilon$ for $0\le v\le\varepsilon$ and $2\varepsilon\le u\le\log x$, by Lemmas 2.1, 2.2, and 2.3,

$$F(\sigma+i\gamma,\chi)=-\frac{L'}{L}(\sigma+i\gamma,\chi)+O(1)\ll\log(|\gamma|+3).$$

We also have $F(1/2+u-v+i\gamma,\chi)\ll 2^{-u}$ for $u\ge 2$. Thus for positive integers n,

$$\int_{2^n}^{2^{n+1}}|\Sigma_1(e^y)|^2dy \ll 2^{2n}\sum_{|\gamma_1|,|\gamma_2|\le T}\frac{\log(|\gamma_1|+3)\log(|\gamma_2|+3)}{|\gamma_1\gamma_2|}\times\int_0^\varepsilon\int_0^\varepsilon\left|\int_{2^n}^{2^{n+1}}e^{y(-v_1+i\gamma_1-v_2-i\gamma_2)}dy\right|dv_1dv_2.$$

The triple integral is $\le\int_{2^n}^{2^{n+1}}(\int_0^\varepsilon e^{-vy}dv)^2dy\ll 2^{-n}$. Hence, the summands with $|\gamma_1-\gamma_2|<1$ contribute, by Lemma 2.2,

$$\ll 2^n\sum_{|\gamma|\le T}\frac{\log^3(|\gamma|+3)}{|\gamma|^2}\ll 2^n.$$

The summands with $|\gamma_1-\gamma_2|\ge 1$ contribute, by Lemma 2.4,

$$\ll\sum_{\substack{|\gamma_1|,|\gamma_2|<T\\|\gamma_1-\gamma_2|\ge1}}\frac{2^{2n}\log(|\gamma_1|+3)\log(|\gamma_2|+3)}{|\gamma_1||\gamma_2||\gamma_1-\gamma_2|}\Big(\int_0^\varepsilon e^{-v2^n}dv\Big)^2 \ll 1.$$

Thus $\int_{2^n}^{2^{n+1}}|\Sigma_1(e^y)|^2dy=O(2^n)$. Summing over

$$n\le\frac{\log Y}{\log 2}+1$$

yields

$$\int_1^Y|\Sigma_1(e^y)|^2\,dy=O(Y).$$

Finally, use Lemma 4.1(3) for I_4. It suffices to show, for $\chi^2=\chi_0$, that

$$\int_0^{2\varepsilon}\int_0^{2\varepsilon}\frac{F(\frac{1}{2}+u-v,\chi)x^{-v}}{1-2v}+\frac{F(\frac{1}{2}-u+v,\chi)x^{-u}}{1-2u}\,du\,dv = -\frac{\log\log x+O(1)}{\log x}. \tag{4–3}$$

Together with (4–1) and (4–2), this completes the proof of Lemma 3.1.

Note that $-F(\frac{1}{2}+w)=\frac{1}{2w}+O(1)$ by Lemmas 2.1 and 2.3. Replacing x with e^y, the integrand is equal to

$$-\frac{1}{2}\left(\frac{e^{-yv}}{(u-v)(1-2v)}+\frac{e^{-yu}}{(v-u)(1-2u)}\right)+O\left(e^{-yv}\right).$$

The integral of the error term above is $O(1/y)$. In the main term, when $|u-v| < 1/y$, the integrand is $O(ye^{-vy})$ and the corresponding part of the double integral is $O(1/y)$. When $u \ge v + 1/y$, the main part of the integrand is

$$-\frac{e^{-vy}}{2(u-v)} + O\left(\frac{ve^{-vy}+e^{-uy}}{u-v}\right)$$

and the corresponding part of the double integral is

$$-\frac{1}{2}\int_0^{2\varepsilon} e^{-vy}\log\left(\frac{y}{2\varepsilon - v}\right)\, dv + O\left(\frac{1}{y}\right) = \frac{-\log y + O(1)}{2y}.$$

The contribution from $u \le v - 1/y$ is, by symmetry, also $\frac{-\log y+O(1)}{2y}$. The asymptotic (4–3) follows. □

5. PROOF OF LEMMA 3.2

We begin with a lemma.

Lemma 5.1. *Uniformly for $y \ge 1$, $0 < |\xi| \le 1$, $|w| \ge \frac{1}{2}$, and $a \ge 0$, we have*

$$\left|\int_0^{2\varepsilon}\int_0^{2\varepsilon} \frac{v^a e^{-vy}\, du\, dv}{(u-v+i\xi)(w-v)}\right| \ll \frac{(4\varepsilon)^a \log\min(2y, \frac{2}{|\xi|})}{y|w|}.$$

Proof: Let I denote the double integral in the lemma. If $|\xi| \ge \frac{1}{y}$, then

$$\begin{aligned} I &\ll \frac{1}{|w|}\int_0^{2\varepsilon} v^a e^{-vy}\int_0^{2\varepsilon}\min\left(\frac{1}{|u-v|}, \frac{1}{|\xi|}\right) du\, dv \\ &\ll \frac{(2\varepsilon)^a}{|w|}\left(1+\log\frac{2}{|\xi|}\right)\int_0^{2\varepsilon} e^{-vy}\, dv \ll \frac{(2\varepsilon)^a\log(\frac{2}{|\xi|})}{y|w|}. \end{aligned}$$

If $|\xi| < \frac{1}{y}$, let $I = I_1 + I_2 + I_3$, where I_1 is the part of I coming from $|u-v| \le |\xi|$, I_2 is the part of I coming from $|\xi| < |u-v| \le \frac{1}{y}$, and I_3 is the part of I coming from $|u-v| > \frac{1}{y}$. We have

$$I_1 \ll \frac{1}{|w\xi|}\iint_{\substack{0\le u,v\le 2\varepsilon \\ |u-v|\le|\xi|}} v^a e^{-vy} du\, dv \ll \frac{(2\varepsilon)^a}{y|w|}$$

and

$$\begin{aligned} I_3 &\ll \frac{(2\varepsilon)^a}{|w|}\iint_{\substack{0\le u,v\le 2\varepsilon \\ |u-v|\ge\frac{1}{y}}} \frac{e^{-vy}}{|u-v|} du\, dv \\ &\ll \frac{(2\varepsilon)^a}{|w|}\int_0^{2\varepsilon} e^{-vy}(\log y + 1) dv \\ &\ll \frac{(2\varepsilon)^a\log(2y)}{y|w|}. \end{aligned}$$

By symmetry,

$$I_2 = \frac{1}{2}\iint_{|\xi|<|u-v|\le 1/y} \frac{v^a e^{-vy}}{(u-v+i\xi)(w-v)} + \frac{u^a e^{-uy}}{(v-u+i\xi)(w-u)}\, du\, dv.$$

Since, $|u^a - v^a| \le a|u-v|(2\varepsilon)^{a-1}$, we have

$$\begin{aligned} &u^a e^{-uy} - v^a e^{-vy} \\ &\quad = e^{-vy}v^a\left(e^{(v-u)y}-1\right) + e^{-vy}(u^a - v^a)e^{(v-u)y} \\ &\quad \ll e^{-vy}y|u-v|(4\varepsilon)^a. \end{aligned} \tag{5–1}$$

Writing $X = u^a e^{-uy} - v^a e^{-vy}$ and $Y = u^a e^{-uy}(u-v)^2$, we deduce that

$$\begin{aligned} I_2 &= \iint_{\substack{0\le u,v\le 2\varepsilon \\ |\xi|<|u-v|\le 1/y}} \frac{(w-u)(u-v)X + Y + O(|\xi w|(2\varepsilon)^a e^{-vy}) du\, dv}{2(u-v+i\xi)(v-u+i\xi)(w-u)(w-v)} \\ &\ll \frac{(4\varepsilon)^a}{|w|}\iint_{\substack{0\le u,v\le 2\varepsilon \\ |\xi|<|u-v|\le 1/y}} ye^{-vy} + \frac{|\xi|e^{-vy}}{|u-v|^2}\, du\, dv \ll \frac{(4\varepsilon)^a}{y|w|}. \end{aligned}$$

□

Proof of Lemma 3.2: Let $y = \log x$. By Lemmas 2.1 and 2.2

$$F\left(\frac{1}{2}+u-v+i\gamma, \chi\right) = \frac{m(\gamma)}{u-v} + R(\gamma, u-v) + R'(\gamma, u-v),$$

where

$$\begin{aligned} R(\gamma, w) &= \sum_{0<|\gamma'-\gamma|\le 1} \frac{1}{w + i(\gamma-\gamma')}, \\ R'(\gamma, w) &= O(\log(|\gamma|+3)). \end{aligned}$$

Then the double integral in Lemma 3.2 is equal to

$$\sum_{i=1}^{4}\Sigma_{2,i}(y) + 2\sum_{\substack{|\gamma|\le T \\ \gamma \text{ distinct}}} m^2(\gamma)e^{iy\gamma}\left(\frac{1}{2}+i\gamma\right)$$

$$\int_0^{2\varepsilon-2^{-n}} \frac{e^{-yv}}{\frac{1}{2}-v+i\gamma}\int_{v+2^{-n}}^{2\varepsilon} \frac{du\, dv}{(u-v)(\frac{1}{2}-u+i\gamma)},$$

where

$$\Sigma_{2,1}(y) = 2\int_0^{2\varepsilon}\int_0^{2\varepsilon}\sum_{|\gamma|\le T}\frac{R(\gamma,u-v)e^{y(-v+i\gamma)}}{\frac{1}{2}-v+i\gamma}du\,dv,$$

$$\Sigma_{2,2}(y) = 2\int_0^{2\varepsilon}\int_0^{2\varepsilon}\frac{R'(\gamma,u-v)e^{y(-v+i\gamma)}}{\frac{1}{2}-v+i\gamma}\,du\,dv,$$

$$\Sigma_{2,3}(y) = \sum_{\substack{|\gamma|\le T\\ \gamma \text{ distinct}}} m^2(\gamma)e^{iy\gamma}\left(\frac{1}{2}+i\gamma\right) \iint\limits_{\substack{0\le u,v\le 2\varepsilon\\ |u-v|\le 2^{-n}}}\frac{e^{-yv}-e^{-uy}}{(u-v)(\frac{1}{2}-v+i\gamma)(\frac{1}{2}-u+i\gamma)}dv\,du,$$

$$\Sigma_{2,4}(y) = 2\sum_{\substack{|\gamma|\le T\\ \gamma \text{ distinct}}} m^2(\gamma)e^{iy\gamma}\left(\frac{1}{2}+i\gamma\right) \int_{2^{-n}}^{2\varepsilon}\int_0^{v-2^{-n}}\frac{e^{-yv}}{(u-v)(\frac{1}{2}-v+i\gamma)(\frac{1}{2}-u+i\gamma)}du\,dv.$$

We show that $\sum_{j=1}^4 \Sigma_{2,j}(y)$ is small in mean square. Note that for $2^n < y \le 2^{n+1}$, $T = T(e^y)$ is constant. Also, by Lemma 2.2, we have

$$m(\gamma) \ll \log(|\gamma|+3). \tag{5–2}$$

First, by Lemmas 2.2 and 2.4,

$$\begin{aligned}\int_{2^n}^{2^{n+1}} |\Sigma_{2,2}(y)|^2\,dy & \qquad\qquad (5–3)\\ = 4\iiiint\limits_{[0,2\varepsilon]^4}\sum_{\substack{|\gamma_1|\le T\\ |\gamma_2|\le T}} &\frac{R'(\gamma_1,u_1-v_1)\overline{R'(\gamma_2,u_2-v_2)}}{(\frac{1}{2}-v_1+i\gamma_1)(\frac{1}{2}-v_2-i\gamma_2)} \times\int_{2^n}^{2^{n+1}} e^{y(-v_1-v_2+i\gamma_1-i\gamma_2)}\,dy\,du_j dv_j\\ \ll \sum_{|\gamma_1-\gamma_2|>1} &\frac{\log(|\gamma_1|+3)\log(|\gamma_2|+3)}{|\gamma_1\gamma_2|\cdot|\gamma_1-\gamma_2|} \iiiint\limits_{[0,2\varepsilon]^4} e^{-2^n(v_1+v_2)}\,du_j dv_j\\ &+\sum_{|\gamma_1-\gamma_2|\le 1}\frac{\log(|\gamma_1|+3)\log(|\gamma_2|+3)}{|\gamma_1\gamma_2|}\int_{2^n}^{2^{n+1}} \iiiint\limits_{[0,2\varepsilon]^4} e^{-y(v_1+v_2)}\,du_j dv_j dy\\ \ll 2^{-n}.&\end{aligned}$$

For the remaining sums, for brevity we define

$$\rho_1 = \frac{1}{2}+i\gamma_1, \qquad \rho_2 = \frac{1}{2}-i\gamma_2.$$

Next,

$$\begin{aligned}\int_{2^n}^{2^{n+1}} &|\Sigma_{2,3}(y)|^2 dy\\ &= \int_{2^n}^{2^{n+1}}\sum_{|\gamma_1|,|\gamma_2|\le T} m(\gamma_1)m(\gamma_2)e^{iy(\gamma_1-\gamma_2)}\rho_1\rho_2\\ &\times \iiiint\limits_{\substack{[0,2\varepsilon]^4\\ |u_j-v_j|\le 2^{-n}}}\frac{(e^{-v_1y}-e^{-u_1y})(e^{-v_2y}-e^{-u_2y})}{\prod_{j=1}^2(u_j-v_j)(\rho_j-v_j)(\rho_j-u_j)}dv_j dv_j\,dy.\end{aligned}$$

By (5–1), the integrand in the quadruple integral is $\ll y^2e^{-uy-u_1y}|\rho_1\rho_2|^{-2}$. By Lemma 2.2, for a given γ_1, there are $\ll \log(|\gamma_1|+3)$ zeros γ_2 with $|\gamma_1-\gamma_2|<1$. Hence the contribution from terms with $|\gamma_1-\gamma_2|<1$ is

$$\ll 2^{-n}\sum_{|\gamma_1-\gamma_2|<1}\frac{m(\gamma_1)m(\gamma_2)}{|\rho_1\rho_2|} \ll 2^{-n}\sum_{\gamma_1}\frac{\log^3(|\gamma_1|+3)}{|\gamma_1|^2} \ll 2^{-n}.$$

Using integration by parts, we have

$$\int_{2^n}^{2^{n+1}} e^{iy(\gamma_1-\gamma_2)}(e^{-v_1y}-e^{-u_2y})(e^{-v_1y}-e^{-u_2y})\,dy \ll \frac{2^{3n}|u_1-v_1|\,|u_2-v_2|e^{-2^n(u_1+u_2)}}{|\gamma_1-\gamma_2|}$$

uniformly in u_1, v_1, u_2, v_2. Thus, by (5–2) and Lemma 2.4, the contribution from terms with $|\gamma_1-\gamma_2|\ge 1$ is

$$\ll 2^{-n}\sum_{|\gamma_1-\gamma_2|\ge 1}\frac{m(\gamma_1)m(\gamma_2)}{|\rho_1\rho_2|\cdot|\gamma_1-\gamma_2|} \ll 2^{-n}.$$

Combining these estimates, we have

$$\int_{2^n}^{2^{n+1}} |\Sigma_{2,3}(y)|^2 dy \ll 2^{-n}. \tag{5–4}$$

In the same manner, we have

$$\begin{aligned}\int_{2^n}^{2^{n+1}} |\Sigma_{2,4}(y)|^2 dy &= \sum_{\substack{|\gamma_1|\le T\\ |\gamma_2|\le T}} m(\gamma_1)m(\gamma_2)\rho_1\rho_2\\ &\times\int_{2^n}^{2^{n+1}}\iiiint\limits_{\substack{[0,2\varepsilon]^4\\ u_j\le v_j-2^{-n}}}\frac{e^{y(-v_1-v_2+i(\gamma_1-\gamma_2))}du_j dv_j}{\prod_{j=1}^2(u_j-v_j)(\rho_j-v_j)(\rho_j-u_j)}dy.\end{aligned}$$

The contribution to the right side from terms with $|\gamma_1-\gamma_2|<1$ is

$$\ll \sum_{|\gamma_1-\gamma_2|<1} \frac{m(\gamma_1)m(\gamma_2)}{|\gamma_1\gamma_2|} \times \int_{2^n}^{2^{n+1}} \left(\int_{2^{-n}}^{2\varepsilon} \int_0^{v-2^{-n}} \frac{e^{-yv}}{(v-u)}\, du\, dv \right)^2$$

$$\ll \sum_{\gamma_1} \frac{\log^3(|\gamma_1|+3)}{|\gamma_1|^2} \int_{2^n}^{2^{n+1}} \left(\int_{1/y}^{\infty} e^{-yv} \log(yv)\, dv \right)^2 \ll 2^{-n}.$$

The terms with $|\gamma_1-\gamma_2|>1$ contribute

$$\ll \sum_{\substack{|\gamma_1|,|\gamma_2|<T \\ |\gamma_1-\gamma_2|>1}} \frac{m(\gamma_1)m(\gamma_2)}{|\gamma_1\gamma_2|\cdot|\gamma_1-\gamma_2|} \left(\int_{2^{-n}}^{2\varepsilon} \int_0^{v-2^{-n}} \frac{e^{-2^n v}}{v-u}\, du\ dv \right)^2$$

$$\ll \sum_{|\gamma_1-\gamma_2|>1} \frac{\log(|\gamma_1|+3)\log(|\gamma_2|+3)}{|\gamma_1\gamma_2|\cdot|\gamma_1-\gamma_2|} \left(\frac{1}{2^n}\right)^2 \ll \frac{1}{2^{2n}}.$$

Therefore,

$$\int_{2^n}^{2^{n+1}} |\Sigma_{2,4}(y)|^2 dy \ll 2^{-n}. \tag{5–5}$$

Estimating an average of $\Sigma_{2,1}(y)$ is more complicated, since $R(\gamma,w)$ could be very large if $|w|$ is small and there is another γ' very close to γ. We get around the problem by noticing that $R(\gamma,w)+R(\gamma,-w)$ is always small. We first have, by (5–1) and Lemma 2.2,

$$\int_{2^n}^{2^{n+1}} |\Sigma_{2,1}(y)|^2 dy \ll \sum_{\gamma_1,\gamma_2} \log^2(|\gamma_1|+3)\log^2(|\gamma_2|+3) \times \max_{\substack{0<|\gamma_1-\gamma_1'|\le 1 \\ 0<|\gamma_2-\gamma_2'|\le 1}} \int_{2^n}^{2^{n+1}} e^{iy(\gamma_1-\gamma_2)} \times \iiiint_{[0,2\varepsilon]^4} \frac{e^{-y(v_1+v_2)} du_j\, dv_j\, dy}{\prod_{j=1}^{2}(u_j-v_j+i\xi_j)(\rho_j-v_j)}, \tag{5–6}$$

where $\xi_1=\gamma_1-\gamma_1'$ and $\xi_2=-(\gamma_2-\gamma_2')$. Let

$$M(\gamma) = \max_{\substack{|\gamma-\gamma_1|\le 1 \\ 0<|\gamma_1-\gamma_1'|<1}} \frac{2}{|\gamma_1-\gamma_1'|}.$$

By Lemmas 2.3 and 5.1, the terms with $|\gamma_1-\gamma_2|<1$ contribute

$$\ll \sum_{|\gamma_1-\gamma_2|<1} \frac{\log^2(|\gamma_1|+3)\log^2(|\gamma_2|+3)}{|\gamma_1\gamma_2|} \times \int_{2^n}^{2^{n+1}} \frac{1}{y^2} \prod_{j=1}^{2} \log\left(\min\left(2y, \frac{2}{|\gamma_j-\gamma_j'|}\right)\right) dy$$

$$\ll \frac{1}{2^n}\sum_{\gamma_1} \frac{\log^5(|\gamma_1|+3)}{|\gamma_1|^2} \log^2\left(\min(2^{n+2}, M(\gamma))\right) = o\left(\frac{n^2}{2^n}\right) \quad (n\to\infty).$$

Now suppose $|\gamma_1-\gamma_2|>1$. With $\gamma_1,\gamma_2,\gamma_1',\gamma_2'$ all fixed, let $\Delta=\gamma_1-\gamma_2$. Fixing u_1,v_1,u_2,v_2, we integrate over y first. The quintuple integral in (5–6) is $J(2^{n+1})-J(2^n)$, where

$$J(y) = e^{iy\Delta} \times \iiiint_{[0,2\varepsilon]^4} \frac{e^{-y(v_1+v_2)} du_j\, dv_j}{(i\Delta-v_1-v_2)\prod_{j=1}^{2}(u_j-v_j+i\xi_j)(\rho_j-v_j)}.$$

Using

$$\frac{1}{i\Delta-v_1-v_2} = \frac{1}{i\Delta}\sum_{k=0}^{\infty}\left(\frac{v_1+v_2}{i\Delta}\right)^k = \sum_{a,b\ge 0}\binom{a+b}{a}\frac{v_1^a v_2^b}{(i\Delta)^{a+b}}$$

together with Lemma 5.1 yields

$$|J(y)| \ll \frac{\log^2 y}{|\rho_1\rho_2\Delta| y^2} \sum_{a,b\ge 0}\binom{a+b}{a}\left(\frac{4\varepsilon}{|\Delta|}\right)^{a+b} \ll \frac{\log^2 y}{|\rho_1\rho_2\Delta| y^2}.$$

Therefore, by Lemma 2.4,

$$\sum_{\gamma_1,\gamma_2} \log^2(|\gamma_1|+3)\log^2(|\gamma_2|+3) \times \max_{\substack{0<|\gamma_1-\gamma_1'|\le 1 \\ 0<|\gamma_2-\gamma_2'|\le 1}} |J(2^{n+1})-J(2^n)| \ll \frac{n^2}{2^{2n}},$$

and hence

$$\int_{2^n}^{2^{n+1}} |\Sigma_{2,1}(y)|^2 = o(n^2 2^{-n}). \tag{5–7}$$

Define

$$\Sigma_2(x;\chi) = (\log x)\sum_{j=1}^{4}\Sigma_{2,j}(\log x).$$

By (5–3), (5–4), (5–5), and (5–7),

$$\int_2^Y |\Sigma_2(e^y;\chi)|^2\,dy \ll \sum_{j=1}^{4}\sum_{n\le \frac{\log Y}{\log 2}+1} 2^{2n}\int_{2^n}^{2^{n+1}} |\Sigma_{2,j}(y)|^2\,dy$$
$$= o(Y\log^2 Y) \qquad (Y\to\infty).$$

This completes the proof of Theorem 3.2. □

6. PROOF OF LEMMA 3.3

Proof: Put $y=\log x$. For any γ we have

$$\int_0^{2\varepsilon-2^{-n}} \frac{e^{-yv}}{\frac12 - v + i\gamma}\int_{v+2^{-n}}^{2\varepsilon}\frac{du}{(u-v)(\frac12-u+i\gamma)}dv$$
$$= \int_0^{2\varepsilon-2^{-n}} e^{-yv}\left(\frac{1}{\frac12+i\gamma}+O\left(\frac{v}{\frac14+\gamma^2}\right)\right)$$
$$\times\int_{v+2^{-n}}^{2\varepsilon}\left(\frac{1}{\frac12+i\gamma}+O\left(\frac{u}{\frac14+\gamma^2}\right)\right)\frac{du}{u-v}dv$$
$$= \frac{M+E}{(1/2+i\gamma)^2},$$

where

$$M = \int_0^{2\varepsilon-2^{-n}} e^{-yv}\left(\log(2\varepsilon-v)+\log 2^n\right)\,dv$$
$$= \frac{\log y + O(1)}{y}$$

and

$$E \ll \int_0^{2\varepsilon-2^{-n}} e^{-yv}\int_{v+2^{-n}}^{2\varepsilon}\frac{u}{u-v}\,du\,dv$$
$$\ll \int_0^{2\varepsilon-2^{-n}} e^{-yv}\left(1+v\log 2^n + v\log(2\varepsilon-v)\right)\,dv$$
$$\ll \frac{1}{y}.$$

Hence, the zeros with $|\gamma|\le T_0$ contribute

$$\frac{2\log\log x}{\log x}\sum_{\substack{|\gamma|\le T_0\\ \gamma\ \text{distinct}}}\frac{m^2(\gamma)x^{i\gamma}}{1/2+i\gamma} + O\left(\frac{\log^3 T_0}{\log x}\right).$$

Next, let $\Sigma_3(x;T_0)$ be the sum over zeros with $T_0 < |\gamma| \le T$. We have

$$\int_{2^n}^{2^{n+1}} |\Sigma_3(e^y,T_0)|^2 dy$$
$$\le \sum_{T_0\le|\gamma_1|,|\gamma_2|\le T} 2^{2n+2}m(\gamma_1)m(\gamma_2)\left(\frac12+i\gamma_1\right)$$
$$\times\left(\frac12 - i\gamma_2\right)\int_{2^n}^{2^{n+1}} e^{yi(\gamma_1-\gamma_2)} \tag{6–1}$$
$$\times\iiiint_{u_j\ge v_j+2^{-n}}\frac{e^{-yv_1-yv_2}du_jdv_j\ dy}{\prod_{j=1}^{2}(u_j-v_j)(\frac12-v_j+i\gamma_j)(\frac12-u_j+i\gamma_j)}.$$

The sum over $|\gamma_1-\gamma_2|<1$ on the right side of (6–1) is

$$\ll \sum_{\substack{T_0\le|\gamma_1|,|\gamma_2|\le T\\ |\gamma_1-\gamma_2|<1}}\frac{2^{2n}m(\gamma_1)m(\gamma_2)}{|\gamma_1||\gamma_2|}$$
$$\times\int_{2^n}^{2^{n+1}}\iiiint_{u_j\ge v_j+2^{-n}}\frac{e^{-yv_1-yv_2}}{(u_1-v_1)(u_2-v_2)}du_jdv_j\,dy$$
$$\ll \sum_{\substack{T_0\le|\gamma_1|,|\gamma_2|\le T\\ |\gamma_1-\gamma_2|<1}}\frac{n^2 2^n m(\gamma_1)m(\gamma_2)}{|\gamma_1||\gamma_2|}$$
$$\ll n^2 2^n\sum_{|\gamma|\ge T_0}\frac{\log^3(|\gamma|+3)}{|\gamma|} \ll \frac{n^2 2^n\log^5 T_0}{T_0},$$

applying Lemma 2.2.

The terms where $|\gamma_1-\gamma_2|\ge 1$ on the right-hand side of (6–1) total

$$\ll \sum_{\substack{T_0\le|\gamma_1|,|\gamma_2|\le T\\ |\gamma_1-\gamma_2|>1}}\frac{2^{2n}m(\gamma_1)m(\gamma_2)}{|\gamma_1||\gamma_2||\gamma_1-\gamma_2|}$$
$$\times\iiiint_{u_j\ge v_j+2^{-n}}\frac{e^{-2^nv_1-2^nv_2}}{(u_1-v_1)(u_2-v_2)}du_jdv_j$$
$$\ll \sum_{\substack{T_0\le|\gamma_1|,|\gamma_2|\\ |\gamma_1-\gamma_2|>1}}\frac{n^2\log(|\gamma_1|+3)\log(|\gamma_2|+3)}{|\gamma_1||\gamma_2||\gamma_1-\gamma_2|}\ll n^2\frac{\log^5 T_0}{T_0}$$

by Lemma 2.4. Summing over n proves the lemma. □

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REFERENCES

[Chebyshev 53] P. L. Chebyshev. "Lettre de M. le professeur Tchébyshev à M. Fuss, sur un nouveau théorème rélatif aux nombres premiers contenus dans les formes $4n+1$ et $4n+3$." *Bull. de la Classe phys.-math. de l'Acad. Imp. des Sciences St. Petersburg* 11 (1853), 208.

[Davenport 00] H. Davenport. *Multiplicative Number Theory*, 3rd edition, Graduate Texts in Mathematics 74. New York: Springer, 2000.

[Fiorilli and Martin 09] D. Fiorilli and G. Martin. "Inequities in the Shanks–Rényi Prime Number Race: An Asymptotic Formula for the Densities." arXiv:0912.4908, 2009.

[Ford and Konyagin 02] K. Ford and S. Konyagin. "Chebyshev's Conjecture and the Prime Number Race." In *Fourth International Conference "Modern Problems of Number Theory and Its Applications": Current Problems, Part II (Tula, 2001)* (Russian), pp. 67–91. Moscow: Mosk. Gos. Univ. im. Lomonosova, Mekh.-Mat. Fak., 2002.

[Granville and Martin 06] A. Granville and G. Martin. "Prime Number Races." *Amer. Math. Monthly* 113:1 (2006), 1–33.

[Knapowski and Turán 62] S. Knapowski and P. Turán. "Comparative Prime Number Theory I." *Acta Math. Sci. Hungar.* 13 (1962), 315–342.

[Landau 74] E. Landau. *Handbuch der Lehre von der Verteilung der Primzahlen*, 3rd edition. New York: Chelsea, 1974.

[Leech 57] J. Leech. "Note on the Distribution of Prime Numbers." *J. London Math. Soc.* 32 (1957), 56–58.

[Littlewood 14] J. E. Littlewood. "Sur la distribution des nombres premiers." *C. R. Acad. des Sciences Paris* 158 (1914), 1869–1872.

[Rubinstein and Sarnak 94] M. Rubinstein and P. Sarnak. "Chebyshev's Bias." *Exper. Math.* 3 (1994), 173–197.

[Sneed 10] J. Sneed. "Lead Changes in the Prime Number Race." To appear in *Math. Comp.*, 2010.

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