Classification of Solvable Lie Algebras

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CONTENTS

In this paper we describe a simple method for obtaining a classification of small-dimensional solvable Lie algebras. Using this method, we obtain the classification of three- and four-dimensional solvable Lie algebras (over fields of any characteristic). Precise conditions for isomorphism are given.

1. INTRODUCTION

Several classifications of solvable Lie algebras of small dimension are known. Up to dimension 6 over a real field they were classified by G. M. Mubarakzjanov [Mubarakzjanov 63a, Mubarakzjanov 63b], and up to dimension 4 over any perfect field by J. Patera and H. Zassenhaus [Patera and Zassenhaus 90]. In this paper we explore the possibility of using the computer to obtain a classification of solvable Lie algebras. The possible advantages of this are clear. The problem of classifying Lie algebras needs a systematic approach, and the more the computer is involved, the more systematic the methods have to be. However, the drawback is that the computer can only handle finite data. For example, we will consider orbits of the action of the automorphism group of a Lie algebra on the algebra of its derivations. Now, if the ground field is infinite, then we know of no algorithm for obtaining these orbits. In our approach we use the computer (specifically the technique of Gröbner bases) to decide isomorphism of Lie algebras, and to obtain explicit isomorphisms if they exist.

The procedure that we use to classify solvable Lie algebras is based on some simple ideas, which are described in Section 2 (and for which we do not claim any originality). Then in Section 3 we describe the use of Gröbner bases for obtaining isomorphisms. In Section 4 solvable Lie algebras of dimension 3 over any field are classified. In Section 5 the same is done for dimension 4. We show that our classification in dimension 4 differs slightly from the one found in [Patera and Zassenhaus 90] (i.e., we find a few more Lie algebras).

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For the explicit calculations reported here we have used the computer algebra system Magma [Bosma et al. 97].

2. GENERAL LEMMAS

In the following we denote the field we work with by F. The strategy that we use for constructing solvable Lie algebras of dimension n is to extend a solvable Lie algebra of dimension $n-1$ by a derivation. More precisely, let K be a solvable Lie algebra of dimension $n-1$, and $d : K \to K$ a derivation. Then we construct a solvable Lie algebra of dimension n by setting $L = Fx_d \oplus K$. The Lie bracket on L is defined by $[\alpha x_d + y_1, \beta x_d + y_2] = \alpha d(y_2) - \beta d(y_1) + [y_1, y_2]$, where $\alpha, \beta \in F$ and $y_1, y_2 \in K$. We recall that the derivations of a Lie algebra L that are of the form $\mathrm{ad}x$ for $x \in L$ are called inner. Derivations not of that form are called outer.

Lemma 2.1. *Let L be a solvable Lie algebra. Then there is a subalgebra $K \subset L$ of codimension 1, and a derivation d of K such that $L = Fx_d \oplus K$. Moreover, if L is not Abelian, then d and K can be chosen such that d is an outer derivation of K.*

Proof: Let K be any subspace of codimension 1 containing $[L, L]$, and let $x \in L$ span a complementary subspace. Then K is an ideal of L and we get the result with $d = \mathrm{ad}x$ and $x_d = x$.

The proof of the second statement is by induction on $\dim L$. If $\dim L = 2$ then the statement is clear. Now suppose $\dim L = n > 2$ and write $L = Fy \oplus K$. Suppose that $\mathrm{ad}y$ is an inner derivation of K, i.e., $\mathrm{ad}y = \mathrm{ad}u$ for some $u \in K$. Set $z = y - u$; then $L = K \oplus Fz$ and $[z, K] = 0$. So K is non-Abelian and by induction we have $K = Fx \oplus K_1$, where $\mathrm{ad}x|_{K_1}$ is an outer derivation. Set $K_2 = K_1 \oplus Fz$, then also $\mathrm{ad}x|_{K_2}$ is an outer derivation, and $L = Fx \oplus K_2$. □

Lemma 2.2. *Let K be a solvable Lie algebra and d_1, d_2 derivations of K. Set $L_i = Fx_{d_i} \oplus K$, $i = 1, 2$. Suppose that there is an automorphism σ of K such that $\sigma d_1 \sigma^{-1} = \lambda d_2$, for some scalar $\lambda \neq 0$. Then L_1 and L_2 are isomorphic.*

Proof: Define a linear map $\tilde{\sigma} : L_1 \to L_2$ by $\tilde{\sigma}(y) = \sigma(y)$ for $y \in K$ and $\tilde{\sigma}(x_{d_1}) = \lambda x_{d_2}$. Then $\tilde{\sigma}$ is a bijective linear map. The fact that it is an isomorphism can be established by direct verification. □

The classification procedure based on these lemmas is as follows. Let K be a solvable Lie algebra of dimension n. We compute the automorphism group $\mathrm{Aut}(K)$ of K and the derivation algebra $\mathrm{Der}(K)$ of K. We denote the subalgebra of inner derivations by $\mathrm{Inn}(K)$. It is straightforward to see that Lie algebras defined by derivations in the same coset of $\mathrm{Inn}(K)$ in $\mathrm{Der}(K)$ are isomorphic. Now the group $G(K) = F^* \times \mathrm{Aut}(K)$ acts on the cosets $d + \mathrm{Inn}(K)$ for $d \in \mathrm{Der}(K)$ by $(\lambda, \sigma) \cdot d + \mathrm{Inn}(K) = \lambda \sigma d \sigma^{-1} + \mathrm{Inn}(K)$. We compute orbit representatives of the action of $G(K)$ on $\mathrm{Der}(K)/\mathrm{Inn}(K)$. For every such representative we get a solvable Lie algebra of dimension $n+1$. Subsequently we weed out the isomorphic ones.

When doing this we often deal with Lie algebras given by a multiplication table containing parameters. An easy trick that often works to reduce the number of parameters is to consider a diagonal base change. Let $\{x_1, \ldots, x_n\}$ be a basis of L, and set $y_i = \alpha_i x_i$. Then write down the multiplication table of L with respect to the y_i. Often it is possible to choose the α_i in such a way that we can get rid of one or more parameters.

When K is Abelian of dimension n we have that $\mathrm{Der}(K) = M_n(F)$ and $\mathrm{Aut}(K) = \mathrm{GL}(n, F)$. In this case representatives of the orbits of $\mathrm{Aut}(K)$ are known, by the following well-known theorem (for a proof we refer to [Hartley and Hawkes 70]).

Theorem 2.3. *Let A be an $n \times n$-matrix over a field F. Then A is similar over F to a unique block-diagonal matrix, containing the blocks $C(f_1), \ldots, C(f_s)$ where $C(f_k)$ is the companion matrix of the nonconstant monic polynomial f_k, and $f_k | f_{k+1}$ for $1 \leq k \leq s-1$.*

The unique block-diagonal matrix is called the *rational canonical form* of A.

In this paper we usually describe an n-dimensional Lie algebra by giving its multiplication table with respect to a basis, which on most occasions is denoted $x_1, \ldots, x_n$. In these multiplication tables we use the convention that products which are not listed are zero. Also when representing a linear map by a matrix we always use the column convention.

3. CONSTRUCTING ISOMORPHISMS

One of the main problems when classifying Lie algebras is to decide whether two of them are isomorphic. A very convenient tool for doing that is Gröbner bases (cf. [Gerdt and Lassner 93]). (For an introduction into

Gröbner bases we refer to [Cox et al. 92].) By way of example we describe how this works.

Consider the three-dimensional Lie algebra L_1 with basis x_1, x_2, x_3 and multiplication table

$$[x_1, x_2] = x_2, \quad [x_1, x_3] = ax_3,$$

and the three-dimensional Lie algebra L_2 with basis y_1, y_2, y_3 and multiplication table

$$[y_3, y_1] = y_2, \quad [y_3, y_2] = by_1 + y_2.$$

The question is whether L_1 and L_2 are isomorphic, and if so for which values of a, b. In that case we would also like to have an explicit isomorphism. An isomorphism will map the nilradical of L_1 onto the nilradical of L_2. So an isomorphism $\phi : L_1 \to L_2$ has the form $\phi(x_1) = a_1y_1 + a_2y_2 + a_3y_3$, $\phi(x_2) = b_1y_1 + b_2y_2$, $\phi(x_3) = c_1y_1 + c_2y_2$. Now this is an isomorphism if and only if the following polynomial equations are satisfied

$$ba_3b_2 - b_1 = 0, \quad a_3b_1 + a_3b_2 - b_2 = 0,$$
$$ba_3c_2 - ac_1 = 0, \quad a_3c_1 + a_3c_2 - ac_2 = 0,$$

and

$$D_1a_3 - 1 = 0, \quad D_2(b_1c_2 - b_2c_1) - 1 = 0.$$

The last two equations are added to ensure that the determinant is nonzero. Now in Magma we compute a Gröbner basis of the ideal of

$$\mathbb{Q}[D_1, D_2, a_1, a_2, a_3, b_1, b_2, c_1, c_2, a, b]$$

generated by the left hand sides of these equations. We use the lexicographical ordering, with $D_1 > D_2 > \dots > c_2 > a > b$. This leads to a Gröbner basis with a triangular structure, which on many occasions makes it possible to find an explicit solution. Also, we let a, b be the smallest variables in the ordering; this makes it likely that the Gröbner basis contains polynomials in only a and b (cf. [Cox et al. 92, Chapter 3, Theorem 2]). From these we can derive necessary conditions for isomorphism. Using Magma we find that the Gröbner basis contains 7 elements, including $a_3 - a - 1$, $b_1 - b_2ab - b_2b$, $c_1 + c_2ab + c_2b + c_2$, $a^2b + 2ab + a + b$. From the last expression we get

$$b = -\frac{a}{(a+1)^2}. \tag{3–1}$$

From this we also see that the algebras are not isomorphic if $a = -1$. Solutions to the other equations are easily found, e.g., $a_1 = a_2 = 0$, $a_3 = a + 1$, $b_1 = ab + b$, $b_2 = 1$, $c_1 = ab+b+1$, $c_2 = -1$. By direct verification we get that this indeed defines an isomorphism, if (3–1) holds, and $a \neq 1$ (otherwise the determinant is zero). If $a = 1$ then by a separate calculation we get that the Gröbner basis is $\{1\}$. So in that case L_1 and L_2 are not isomorphic. The conclusion is that $L_1 \cong L_2$ if and only if (3–1) and $a \neq \pm 1$. Moreover, in that case we also have an explicit isomorphism.

In the above discussion we have taken the ground field to be $\mathbb{Q}$. However, the conclusion holds over any field of characteristic 0, since over any such field the Gröbner basis will be the same. We can also easily reach the same conclusion for any field of characteristic $p > 0$. For that we note that the input polynomials g_i are defined over any field. Now by using the Magma function `Coordinates` we can find polynomials p_i such that $\sum p_ig_i = a^2b+2ab+a+b$. The coefficients of the p_i are rational numbers. So from these coefficients we find a finite set of characteristics over which they are not defined. We then have to do the computation separately over fields of those characteristics. In our example we find that the coordinates of $a^2b + 2ab + a + b$ with respect to the input polynomials all have integral coefficients. So over all fields we have that the ideal generated by the input polynomials contains $a^2b + 2ab + a + b$. We conclude that the isomorphism of L_1 and L_2 implies (3–1), independently of the base field. Furthermore, since the explicit isomorphism is defined over any field, we have that (3–1) and $a \neq \pm 1$ imply that L_1 and L_2 are isomorphic.

4. THE THREE-DIMENSIONAL CASE

There are only two (isomorphism classes of) Lie algebras of dimension 2.

First we consider the Lie algebra K spanned by x_1, x_2 with $[x_1, x_2] = 0$. Then $\mathrm{Aut}(K) = \mathrm{GL}(2, F)$, and $\mathrm{Der}(K) = M_2(F)$ (i.e., the space of all 2×2-matrices). In this case the rational canonical form of an element in $\mathrm{Der}(K)$ is either

$$\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

or

$$\begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix}.$$

$\lambda = 0$ gives a Lie algebra that is the three-dimensional Abelian Lie algebra. If $\lambda \neq 0$ then by Lemma 2.2 we may divide by λ and get the Lie algebra L^2 spanned by x_1, x_2, x_3 and nontrivial brackets $[x_3, x_1] = x_1$, $[x_3, x_2] = x_2$. If the derivation is of the second type, we get the

Lie algebras $L_{a,b}$ spanned by x_1, x_2, x_3 and multiplication table

$$[x_3, x_1] = x_2, \quad [x_3, x_2] = ax_1 + bx_2.$$

From a Gröbner basis computation we get that $L_{a,b} \cong L_{c,d}$ implies $ad^2 - b^2c = 0$ and $\alpha^2 c - a = 0$, for some nonzero $\alpha \in F$. Furthermore, this holds over any field since the coordinates of these polynomials (with respect to the input basis) all have integer coefficients.

Also by applying a diagonal base change we see that $L_{a,b} \cong L_{\alpha^2 a, \alpha b}$ for $\alpha \neq 0$ (the base change is $y_1 = x_1$, $y_2 = \alpha x_2$ $y_3 = \alpha x_3$). Now we distinguish two cases:

1. $b \neq 0$. Then $L_{a,b} \cong L_{a,1}$. So we get the class of Lie algebras $L_a^3 = L_{a,1}$. The above discussion implies that $L_a^3 \cong L_b^3$ if and only if $a = b$.

2. $b = 0$. We get the class of Lie algebras $L_a^4 = L_{a,0}$. In this case $L_a \cong L_b$ if and only if $a = \alpha^2 b$ for some nonzero $\alpha \in F$.

The Lie algebras L_a^3 and L_c^4 are never isomorphic. This can be established by a Gröbner basis computation. It can also be shown in the following way. Suppose that $L_a^3 \cong L_c^4$. Then $ad^2 - b^2c = 0$ amounts to $c = 0$. However, L_0^4 is nilpotent, and L_a^3 is not. We have established the nonisomorphism of L^2 with L_a^3, L_a^4 by Gröbner basis calculations.

The other two-dimensional Lie algebra (with basis x_1, x_2 and $[x_1, x_2] = x_2$) does not have to be considered, as it has no outer derivations.

Summarising we get the following solvable Lie algebras of dimension 3:

L^1 The Abelian Lie algebra.

L^2 $[x_3, x_1] = x_1$, $[x_3, x_2] = x_2$.

L_a^3 $[x_3, x_1] = x_2$, $[x_3, x_2] = ax_1 + x_2$.

L_a^4 $[x_3, x_1] = x_2$, $[x_3, x_2] = ax_1$. Condition of isomorphism: $L_a^4 \cong L_b^4$ if and only if there is an $\alpha \in F^*$ with $a = \alpha^2 b$.

We count the number of nonisomorphic solvable Lie algebras over the finite field with q elements. The classes L^1, L^2, L^3 always give $q + 2$ Lie algebras. If the characteristic of the ground field is not 2, then L_a^4 gives 3 more Lie algebras. In that case the total number is $q + 5$. If the characteristic is 2, then all elements of $\mathbb{F}_q$ are squares, meaning that the L_a^4 give two isomorphism classes of Lie algebras (i.e., L_0^4 and L_1^4). In that case the total number is $q + 4$.

Remark 4.1. Our classification is the same as the one obtained in [Patera and Zassenhaus 90]. More precisely, we have $L_{3,1} \cong L^1$, $L_{3,2} \cong L_0^4$, $L_{3,3} \cong L_0^3$, $L_{3,4} \cong L^2$, $L_{3,5} \cong L_\alpha^4$ (where α is as in [Patera and Zassenhaus 90]), $L_{3,6} \cong L_{-\alpha}^3$, and $L_{3,7} \cong L_{-1/4}^3$ (if the characteristic is not 2) and $L_{3,7} \cong L_1^4$ (if the characteristic is 2).

So we have the same classification, but with a shorter description.

Remark 4.2. From the method used, we get a simple algorithm for recognising a given three-dimensional Lie algebra K as one of the L^i. First we find a two-dimensional Abelian ideal. Let x span a complement to this ideal. Then we find the rational canonical form of the adjoint action of x on the ideal. From this we immediately see to which algebra K is isomorphic.

5. THE FOUR-DIMENSIONAL CASE

Here we have to find derivation algebras and automorphism groups of three-dimensional Lie algebras K. For every such K we have a subsection. The algebras that will appear in the final classification will be denoted M^i.

5.1 $K = L^1$

This Lie algebra is Abelian, so the orbits of the derivations under the action of $\mathrm{Aut}(K)$ are given by the rational canonical form of matrices. If this form consists of three 1×1 blocks, then because of the divisibility condition in Theorem 2.3, they have to be the same. After division we get two algebras: the 4-dimensional commutative algebra (denoted by M^1), and

$$M^2 : [x_4, x_1] = x_1, \quad [x_4, x_2] = x_2, \quad [x_4, x_3] = x_3.$$

If there is a 1×1-block and a 2×2-block, then again because of divisibility we have

$$D = \begin{pmatrix} s & 0 & 0 \\ 0 & 0 & -st \\ 0 & 1 & s+t \end{pmatrix}.$$

Denote the corresponding Lie algebra by $K_{s,t}$. After multiplying x_4, x_3 by α (and x_1, x_2 by 1) we see that this Lie algebra is isomorphic to $K_{\alpha s, \alpha t}$, where $\alpha \neq 0$. We consider the following cases:

1. $s \neq 0$. We can take $\alpha = s^{-1}$, and we get the Lie algebras

$$\begin{aligned} M_a^3 : [x_4, x_1] &= x_1, \\ [x_4, x_2] &= x_3, \\ [x_4, x_3] &= -ax_2 + (a+1)x_3. \end{aligned}$$

Gröbner basis computations reveal that $M^3_a \cong M^3_b$ if and only if $a = b$.

2. $s = 0$, $t \neq 0$. We take $\alpha = t^{-1}$, and get

$$M^4 : [x_4, x_2] = x_3, \quad [x_4, x_3] = x_3.$$

3. $s = t = 0$. We get

$$M^5 : [x_4, x_2] = x_3.$$

If there is a 3×3-block in the rational normal form, then we get the Lie algebras $K_{s,t,u}$:

$$[x_4, x_1] = x_2, \quad [x_4, x_2] = x_3, \quad [x_4, x_3] = sx_1 + tx_2 + ux_3.$$

Multiplying x_2, x_3, x_4 by α, α^2, α respectively, we see that $K_{s,t,u} \cong K_{\alpha^3 s, \alpha^2 t, \alpha u}$. Here there are two cases to consider:

1. $u \neq 0$. We take $\alpha = u^{-1}$, and get the Lie algebras

$$\begin{aligned} M^6_{a,b} : [x_4, x_1] &= x_2, \\ [x_4, x_2] &= x_3, \\ [x_4, x_3] &= ax_1 + bx_2 + x_3. \end{aligned}$$

A Gröbner basis computation shows that $M^6_{a,b} \cong M^6_{c,d}$ if and only if $a = c$ and $b = d$.

2. $u = 0$. We get the Lie algebras

$$\begin{aligned} M^7_{a,b} : [x_4, x_1] &= x_2, \\ [x_4, x_2] &= x_3, \\ [x_4, x_3] &= ax_1 + bx_2. \end{aligned}$$

From the above discussion we see that $M^7_{a,b} \cong M^7_{c,d}$ if $a = \alpha^3 c$ and $b = \alpha^2 d$ (for some $\alpha \neq 0$). From a Gröbner basis computation we get that this is also a necessary condition. So, if both parameters are nonzero, than by a suitable choice for α we can make them equal. Hence this class splits into three subclasses: $M^7_{a,a}$, $M^7_{a,0}$, $M^7_{0,b}$. Among the first class there are no isomorphisms.

5.2 $K = L^2$

The coset representatives of the outer derivations of K (modulo inner derivations) are

$$D = \begin{pmatrix} s & t & 0 \\ u & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We consider the cases $s \neq 0$ and $s = 0$ separately.

1. $s \neq 0$. We can divide by it and get that D is conjugate to

$$\begin{pmatrix} 1 & w & 0 \\ v & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some v, w. This leads to the Lie algebras $K_{v,w}$ with basis x_1, x_2, x_3, x_4 and nonzero commutators

$$[x_4, x_1] = x_1 + vx_2, \quad [x_4, x_2] = wx_1,$$
$$[x_3, x_1] = x_1, \quad [x_3, x_2] = x_2.$$

Here again there is a subdivision in several cases.

(a) $w \neq 0$. By setting $y_1 = wx_1$, $y_i = x_i$ for $i = 2, 3, 4$, we see that $K_{v,w} \cong K_{v',1}$. Denote this Lie algebra simply by K_v. By some calculations it is seen that the centraliser $C(\mathrm{ad}K_v)$ in the full (associative) matrix algebra is spanned by the identity and

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ v & 0 & 0 & 0 \\ 0 & 0 & 0 & v \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

The minimal polynomial of this last matrix is $T^2 - T - v$.

Suppose that the characteristic of F is not 2. Then, if $v = -\frac{1}{4}$, this algebra has a nonzero radical. We get the Lie algebra

$$\begin{aligned} N : [x_4, x_1] &= x_1 - \frac{1}{4}x_2, \\ [x_4, x_2] &= x_1, \\ [x_3, x_1] &= x_1, \\ [x_3, x_2] &= x_2. \end{aligned}$$

(We denote this algebra by N and not by M^8, because it is isomorphic to a Lie algebra that we define later). On the other hand, if $v \neq -\frac{1}{4}$ then $C(\mathrm{ad}K_v)$ is semisimple. Also, if $T^2 - T - v$ has a root in the base field, then it splits. This implies that K_v is isomorphic to the direct sum of two, two-dimensional Lie algebras (namely the noncommutative ones) (cf. [de Graaf 00, Rand et al. 88]). We get the Lie algebra

$$M^8 : [x_1, x_2] = x_2, \quad [x_3, x_4] = x_4.$$

Now suppose that $T^2 - T - v$ does not have a root in F. Then K_v is indecomposable. Suppose that $K_v \cong K_w$, where also $T^2 - T - w$

has no root in F. Then from the Gröbner basis it follows that $v + \frac{1}{4} = \alpha^2(w + \frac{1}{4})$ for some nonzero $\alpha \in F$. (There is also another argument to prove this: as seen above K_v splits over $F(\sqrt{1+4v})$ so also K_w splits over this field. Hence $\sqrt{1+4w} \in F(\sqrt{1+4v})$. This implies the claim.) Conversely, suppose that $v + \frac{1}{4} = \alpha^2(w + \frac{1}{4})$ for some nonzero $\alpha \in F$. Let $\phi : K_v \to K_w$ be the linear map given by $\phi(x_1) = \alpha y_1 + \frac{1}{2}(1-\alpha)y_2$, $\phi(y_2) = y_2$, $\phi(y_3) = y_3$, $\phi(x_4) = \frac{1}{2}(1-\alpha)y_3 + \alpha y_4$. Then ϕ is an isomorphism. We conclude that $K_v \cong K_w$ if and only if $v + \frac{1}{4} = \alpha^2(w + \frac{1}{4})$ for some nonzero $\alpha \in F$.

Now we deal with the case where the characteristic of F is 2. Just as above, if T^2+T+v factors over F, then K_v is isomorphic to a direct sum. If the polynomial does not factor, then K_v is indecomposable. From the Gröbner basis computation it follows that $K_v \cong K_w$ implies that $X^2+X+v+w$ has roots in F. Conversely, suppose that this equation has a root $\alpha \in F$. Then there is an isomorphism $\phi : K_v \to K_w$ given by $\phi(x_1) = y_1 + \alpha y_2$, $\phi(x_2) = y_2$, $\phi(x_3) = y_3$, $\phi(x_4) = \alpha y_3 + y_4$. So $K_v \cong K_w$ if and only if $X^2+X+v+w$ has roots in F. The conclusion is that we get the Lie algebras

$$\begin{aligned} M_a^9 : [x_4, x_1] &= x_1 + a x_2, \\ [x_4, x_2] &= x_1, \\ [x_3, x_1] &= x_1, \\ [x_3, x_2] &= x_2, \end{aligned}$$

where $a \in F$ is such that $T^2 - T - a$ has no root in the base field.

(b) $w = 0$. $K_{v,0}$ is the direct sum of ideals with bases $x_1 + v x_2$, x_4, and $x_2, x_3 - x_4$. So $K_{v,0} \cong M^8$.

2. $s = 0$. D is equal to

$$\begin{pmatrix} 0 & t & 0 \\ u & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now we consider the following cases.

(a) $u \neq 0$. We divide by u and obtain the derivation

$$\begin{pmatrix} 0 & a & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This leads to the Lie algebras

$$\begin{aligned} M_a^{10} : [x_4, x_1] &= x_2, \\ [x_4, x_2] &= a x_1, \\ [x_3, x_1] &= x_1, \\ [x_3, x_2] &= x_2. \end{aligned}$$

If the characteristic of F is not 2, then $M_a^{10} \cong M^9_{a-\frac{1}{4}}$. The isomorphism is given by $\phi(x_1) = 2y_2$, $\phi(x_2) = 2y_1 - y_2$, $\phi(x_3) = y_3$, $\phi(x_4) = -\frac{1}{2}y_3 + y_4$ (where the x_i are the basis elements of M_a^{10}). Note that, if $a = 0$, this gives an isomorphism with N.

If the characteristic is 2, then from a Gröbner basis computation it follows that M_a^{10} is not isomorphic to M_b^9. So in this instance, we have a new series of Lie algebras. From a Gröbner basis computation we get that $M_a^{10} \cong M_b^{10}$ implies that $Y^2 + X^2 b + a = 0$ is solvable in F, with $X \neq 0$. On the other hand, if $\alpha \neq 0$ and β are such that $\beta^2 + \alpha^2 b + a = 0$, then $\phi(x_1) = y_1$, $\phi(x_2) = \beta y_1 + \alpha y_2$, $\phi(x_3) = y_3$, $\phi(x_4) = \beta y_3 + \alpha y_4$ is an isomorphism. So $M_a^{10} \cong M_b^{10}$ if and only if $Y^2 + X^2 b + a = 0$ has a solution in F, with $X \neq 0$. In particular, if the field is perfect (i.e., $F^2 = F$) then $M_a^{10} \cong M_0^{10}$.

(b) $u = 0$, $t \neq 0$. We divide by t. The corresponding Lie algebra has multiplication table

$$[x_4, x_2] = x_1, \quad [x_3, x_1] = x_1, \quad [x_3, x_2] = x_2.$$

If the characteristic is not 2, then it is isomorphic to N, the isomorphism being $\phi(x_1) = 2y_1 - y_2$, $\phi(x_2) = y_2$, $\phi(x_3) = y_3$, $\phi(x_4) = -y_3 + 2y_4$. If the characteristic is 2, then this algebra is isomorphic to M_0^{10}, within this case, $\phi(x_1) = y_2$, $\phi(x_2) = y_1$, $\phi(x_3) = y_3$, $\phi(x_4) = y_4$.

(c) $u = t = 0$. The derivation is inner, and we obtain nothing new.

5.3 $K = L_a^3$

Its derivations consist of

$$\begin{pmatrix} u & av & s \\ v & u+v & t \\ 0 & 0 & 0 \end{pmatrix}.$$

If $a \neq 0$, this means that, then modulo scalar factors, there is only one outer derivation, namely

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

leading to the Lie algebras

$$[x_4, x_1] = x_1, \quad [x_4, x_2] = x_2,$$
$$[x_3, x_1] = x_2, \quad [x_3, x_2] = ax_1 + x_2.$$

However, by interchanging x_3, x_4 and x_1, x_2 we get the Lie algebra K_a considered before (leading to the algebras N, M^8, M^9_a).

If $a = 0$, then apart from the derivation above, we get two more:

$$D_1 = \begin{pmatrix} 1 & 0 & v \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } D_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let σ be the automorphism with matrix

$$\begin{pmatrix} 1 & 0 & v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\sigma D_1 \sigma^{-1}$ is equal to D, so we get nothing new from D_1. However, we can not get rid of D_2 in this way. It leads to the Lie algebra

$$[x_4, x_3] = x_1, \quad [x_3, x_1] = x_2, \quad [x_3, x_2] = x_2.$$

But this is isomorphic to $M^6_{0,0}$, given by $\phi(x_1) = y_2$, $\phi(x_2) = y_3$, $\phi(x_3) = y_4$, $\phi(x_4) = -y_1$.

5.4 $K = L^4_a$

If $a \neq 0$ and the characteristic of F is not 2, then the derivations of L^4_a are given by

$$\begin{pmatrix} u & av & s \\ v & u & t \\ 0 & 0 & 0 \end{pmatrix}.$$

Here modulo inner derivations, and scalar factors, there remains only one derivation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

leading to the Lie algebras

$$[x_4, x_1] = x_1, \quad [x_4, x_2] = x_2,$$
$$[x_3, x_1] = x_2, \quad [x_3, x_2] = ax_1.$$

This Lie algebra is isomorphic to $M^9_{a-\frac{1}{4}}$. The isomorphism is given by $\phi(x_1) = 2y_2$, $\phi(x_2) = 2y_1 - y_2$, $\phi(x_3) = -\frac{1}{2}y_3 + y_4$, $\phi(x_4) = y_3$.

If $a \neq 0$ and the characteristic of F is 2, then the derivations are given by

$$\begin{pmatrix} u & av & s \\ v & u+w & t \\ 0 & 0 & w \end{pmatrix}.$$

So modulo inner derivations we get

$$\begin{pmatrix} u & 0 & 0 \\ 0 & u+w & 0 \\ 0 & 0 & w \end{pmatrix}.$$

We consider a few cases.

1. $w = 0$. This leads to the algebra that we have seen in the case where the characteristic is not 2. In this case it is isomorphic to M^{10}_a, given by $\phi(x_1) = y_1$, $\phi(x_2) = y_2$, $\phi(x_3) = y_4$, $\phi(x_4) = y_3$.
2. $u \neq 0 \neq w$. After dividing by u and setting $b = 1 + w/u$, we get the derivations

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1+b \end{pmatrix}.$$

Here we assume that $b \neq 1$—we have already listed the corresponding algebra (it is isomorphic to M^{10}_a). The Lie algebras we now get are:

$$\begin{aligned} M^{11}_{a,b} : [x_4, x_1] &= x_1, \\ [x_4, x_2] &= bx_2, \\ [x_4, x_3] &= (1+b)x_3, \\ [x_3, x_1] &= x_2, \\ [x_3, x_2] &= ax_1. \end{aligned}$$

(Recall that, here $a \neq 0$, $b \neq 1$.) Let $c \neq 0$ and $d \neq 1$. Set $\delta = (b+1)/(d+1)$. We claim that $M^{11}_{a,b} \cong M^{11}_{c,d}$ if and only if $(\delta^2 + (b+1)\delta + b)/c$ and a/c are squares in F. The only if part follows from inspection of the Gröbner basis; the if part, from explicit construction of an isomorphism. Let $\gamma, \epsilon \in F$ be such that

$$\gamma^2 = \frac{1}{c}(\delta^2 + (b+1)\delta + b), \text{ and } \epsilon^2 = \frac{a}{c}.$$

If $\delta = 1$ then $b = d$ and isomorphism follows already from the dimension-3 isomorphism. So we suppose that $\delta \neq 1$, and we set $\beta = \delta + 1$, $\alpha = c\gamma$. Then $\phi : K_{a,b} \to K_{c,d}$ given by $\phi(x_1) = \alpha y_1 + \beta y_2$, $\phi(x_2) = c\epsilon\beta y_1 + \alpha\epsilon y_2$, $\phi(x_3) = \epsilon y_3$, $\phi(x_4) = \gamma y_3 + \delta y_4$, is an isomorphism. In particular, if F is perfect, then $M^{11}_{a,b} \cong M^{11}_{1,0}$.

3. $w \neq 0$, $u = 0$. We divide by w and get the algebra

$$[x_4, x_2] = x_2, \quad [x_4, x_3] = x_3,$$
$$[x_3, x_1] = x_2, \quad [x_3, x_2] = ax_1.$$

If $a \neq 1$ then this is isomorphic to $M^{11}_{a,a}$, given by $\phi(x_1) = y_1 + a^{-1}y_2$, $\phi(x_2) = y_1 + y_2$, $\phi(x_3) = y_3$, $\phi(x_4) = \frac{1}{a+1}(y_3 + y_4)$. If $a = 1$, then it is isomorphic to $M^{11}_{1,0}$, given by $\phi(x_1) = y_2$, $\phi(x_2) = y_1$, $\phi(x_3) = y_3$, $\phi(x_4) = y_4$.

Now suppose that $a = 0$. Then the derivations (modulo inner derivations) are

$$D = \begin{pmatrix} u_1 & 0 & v_1 \\ 0 & u_1 + v_3 & 0 \\ u_3 & 0 & v_3 \end{pmatrix}.$$

A general automorphism of L^4_0 is given by

$$\phi = \begin{pmatrix} \alpha_1 & 0 & \gamma_1 \\ \alpha_2 & \alpha_1\gamma_3 - \alpha_3\gamma_1 & \gamma_2 \\ \alpha_3 & 0 & \gamma_3 \end{pmatrix},$$

where $\alpha_1\gamma_3 - \alpha_3\gamma_1 \neq 0$. The entry at position $(3,3)$ of $(\alpha_1\gamma_3 - \alpha_3\gamma_1)\phi D\phi^{-1}$ is $-u_1\alpha_3\gamma_1 - u_3\gamma_1\gamma_3 + v_1\alpha_1\alpha_3 + v_3\alpha_1\gamma_3$. It is straightforward to see that, except in the case where $u_1 = v_3 \neq 0$ and $u_3 = v_1 = 0$, we can choose the α_i, γ_i such that this becomes zero. If $u_1 = v_3 \neq 0$ and $u_3 = v_1 = 0$ then we divide by u_1 and get the Lie algebra

$$\begin{aligned} M^{12} : [x_4, x_1] &= x_1, \\ [x_4, x_2] &= 2x_2, \\ [x_4, x_3] &= x_3, \\ [x_3, x_1] &= x_2. \end{aligned}$$

Otherwise D is conjugate to

$$D' = \begin{pmatrix} u_1 & 0 & v_1 \\ 0 & u_1 & 0 \\ u_3 & 0 & 0 \end{pmatrix}.$$

Now let ϕ be the automorphism given by the matrix

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha\beta & 0 \\ 0 & 0 & \beta \end{pmatrix},$$

where both α, β are nonzero. Then

$$\phi D' \phi^{-1} = \begin{pmatrix} u_1 & 0 & \frac{\alpha}{\beta}v_1 \\ 0 & u_1 & 0 \\ \frac{\beta}{\alpha}u_3 & 0 & 0 \end{pmatrix}.$$

We have a subdivision into four cases.

1. $u_1 \neq 0 \neq v_1$. We divide by u_1, and choose $\alpha = u_1$ and $\beta = v_1$. This leads to the Lie algebras

$$\begin{aligned} M^{13}_b : [x_4, x_1] &= x_1 + bx_3, \\ [x_4, x_2] &= x_2, \\ [x_4, x_3] &= x_1, \\ [x_3, x_1] &= x_2. \end{aligned}$$

From the Gröbner basis it follows that two of those algebras, with parameters b and c, are isomorphic if and only if $b = c$.

2. $u_1 \neq 0$, $v_1 = 0$. If $u_3 \neq 0$, then set $\alpha = u_3$, $\beta = u_1$. We get the Lie algebra

$$[x_4, x_1] = x_1 + x_3, \quad [x_4, x_2] = x_2, \quad [x_3, x_1] = x_2.$$

If we set $\tilde{x}_1 = x_1 + x_3$, $\tilde{x}_2 = -x_2$, $\tilde{x}_3 = x_1$, $\tilde{x}_4 = x_4$, then we see that with respect to this new basis the Lie algebra has the same multiplication table as M^{13}_0.

On the other hand, if $u_3 = 0$ then we get the Lie algebra

$$[x_4, x_1] = x_1, \quad [x_4, x_2] = x_2, \quad [x_3, x_1] = x_2.$$

In this case we set $\tilde{x}_1 = x_1$, $\tilde{x}_2 = x_2$, $\tilde{x}_3 = x_1 + x_3$, $\tilde{x}_4 = x_4$. Again we get the multiplication table of M^{13}_0.

3. $u_1 = 0$, $v_1 \neq 0$. We divide by v_1 and get the derivations

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix},$$

leading to the Lie algebras

$$\begin{aligned} M^{14}_b : [x_4, x_1] &= bx_3, \\ [x_4, x_3] &= x_1, \\ [x_3, x_1] &= x_2. \end{aligned}$$

By setting $y_i = \alpha x_i$ for $i = 1, 2, 4$ and $y_3 = x_3$, we see that this Lie algebra is isomorphic to the same one with parameter $\alpha^2 b$. On the other hand, from a Gröbner basis computation we get that $M^{14}_b \cong M^{14}_c$ implies $b = \alpha^2 c$ for some α.

4. $u_1 = v_1 = 0$. We get two more algebras. The first is a direct sum isomorphic to M^5. The other is

$$[x_4, x_1] = x_3, \quad [x_3, x_1] = x_2.$$

Here we set $\tilde{x}_1 = x_3$, $\tilde{x}_2 = -x_2$, $\tilde{x}_3 = x_1$, $\tilde{x}_4 = x_4$. This gives us the multiplication table of M^{14}_0.

5.5 Putting It Together

There are some additional isomorphisms between the algebras that we have found. If the characteristic of F is not 2, then $M_0^{13} \cong N$, given by $\phi(x_1) = y_1 + y_2$, $\phi(x_2) = 3y_1 - \frac{3}{2}y_2$, $\phi(x_3) = y_1 + y_2 - y_3 + 2y_4$, $\phi(x_4) = y_3$. For this reason we do not list N separately (if the characteristic is 2, then it does not exist).

$M_{0,0}^7 \cong M_0^{14}$, by $\phi(x_1) = -y_4$, $\phi(x_2) = y_1$, $\phi(x_3) = y_2$, $\phi(x_4) = y_3$.

If the characteristic is 2, and $a = \alpha^2 \neq 0$, then $M_a^{10} \cong M_0^{13}$, given by $\phi(x_1) = y_1$, $\phi(x_2) = \alpha y_1 + \alpha y_2$, $\phi(x_3) = y_1 + y_2 + y_4$, $\phi(x_4) = \alpha y_3 + \alpha y_4$. Note that if $a = 0$ we also have isomorphism: $M_0^{10} \cong M_1^{10} \cong M_0^{13}$.

I have established the nonisomorphism of the remaining Lie algebras M^i by Gröbner basis computations.

Remark 5.1. The Lie algebras $M_{0,b}^{11}$ do exist, and one may wonder where they occur in the list. We have $M_{0,0}^{11} \cong M^{12}$, and $M_{0,b}^{11} \cong M_{(b+1)/b^2}^{13}$ if $b \neq 0, b \neq 1$.

Summarising, we have the following classes of solvable Lie algebras of dimension 4:

M^1 The Abelian Lie algebra.

M^2 $[x_4, x_1] = x_1$, $[x_4, x_2] = x_2$, $[x_4, x_3] = x_3$.

M_a^3 $[x_4, x_1] = x_1$, $[x_4, x_2] = x_3$, $[x_4, x_3] = -ax_2 + (a+1)x_3$.

M^4 $[x_4, x_2] = x_3$, $[x_4, x_3] = x_3$.

M^5 $[x_4, x_2] = x_3$.

$M_{a,b}^6$ $[x_4, x_1] = x_2$, $[x_4, x_2] = x_3$, $[x_4, x_3] = ax_1 + bx_2 + x_3$.

$M_{a,b}^7$ $[x_4, x_1] = x_2$, $[x_4, x_2] = x_3$, $[x_4, x_3] = ax_1 + bx_2$. Isomorphism condition: $M_{a,b}^7 \cong M_{c,d}^7$ if and only if there is an $\alpha \in F^*$ with $a = \alpha^3 c$ and $b = \alpha^2 d$.

M^8 $[x_1, x_2] = x_2$, $[x_3, x_4] = x_4$.

M_a^9 $[x_4, x_1] = x_1 + ax_2$, $[x_4, x_2] = x_1$, $[x_3, x_1] = x_1$, $[x_3, x_2] = x_2$. Condition on the parameter a: $T^2 - T - a$ has no roots in F. Isomorphism condition: $M_a^9 \cong M_b^9$ if and only if the characteristic of F is not 2 and there is an $\alpha \in F^*$ with $a + \frac{1}{4} = \alpha^2(b + \frac{1}{4})$, or the characteristic of F is 2 and $X^2 + X + a + b$ has roots in F.

M_a^{10} $[x_4, x_1] = x_2$, $[x_4, x_2] = ax_1$, $[x_3, x_1] = x_1$, $[x_3, x_2] = x_2$. Condition on F: the characteristic of F is 2. Condition on the parameter a: $a \notin F^2$. Isomorphism condition: $M_a^{10} \cong M_b^{10}$ if and only if $Y^2 + X^2 b + a$ has a solution $(X, Y) \in F \times F$ with $X \neq 0$.

$M_{a,b}^{11}$ $[x_4, x_1] = x_1$, $[x_4, x_2] = bx_2$, $[x_4, x_3] = (1+b)x_3$, $[x_3, x_1] = x_2$, $[x_3, x_2] = ax_1$. Condition on F: the characteristic of F is 2. Condition on the parameters a, b: $a \neq 0$, $b \neq 1$. Isomorphism condition: $M_{a,b}^{11} \cong M_{c,d}^{11}$ if and only if $\frac{a}{c}$ and $(\delta^2 + (b+1)\delta + b)/c$ are squares in F, where $\delta = (b+1)/(d+1)$.

M^{12} $[x_4, x_1] = x_1$, $[x_4, x_2] = 2x_2$, $[x_4, x_3] = x_3$, $[x_3, x_1] = x_2$.

M_a^{13} $[x_4, x_1] = x_1 + ax_3$, $[x_4, x_2] = x_2$, $[x_4, x_3] = x_1$, $[x_3, x_1] = x_2$.

M_a^{14} $[x_4, x_1] = ax_3$, $[x_4, x_3] = x_1$, $[x_3, x_1] = x_2$. Condition on parameter a: $a \neq 0$. Isomorphism condition: $M_a^{14} \cong M_b^{14}$ if and only if there is an $\alpha \in F^*$ with $a = \alpha^2 b$.

We count the number of solvable Lie algebras of dimension 4 over the finite field $\mathbb{F}_q$, where $q = p^m$ for a prime p. For that we start with a well-known lemma (see [Berlekamp 68], Theorems 6.69, 6.695).

Lemma 5.2. *Let $u \in \mathbb{F}_q$, where $q = 2^m$. Then the equation $X^2 + X + u$ has a solution in $\mathbb{F}_q$ if and only if*

$$\mathrm{Tr}_2(u) = \sum_{i=0}^{m-1} u^{2^i} = 0.$$

The classes M^1, M^2, M_a^3, M^4, M^5, $M_{a,b}^6$ contain 1, 1, q, 1, 1, q^2 algebras respectively.

As noted before, the class $M_{a,b}^7$ splits in three subclasses: $M_{a,a}^7$, $M_{a,0}^7$ $(a \neq 0)$, and $M_{0,b}^7$ $(b \neq 0)$. The first of these contains q elements. We have $M_{a,0}^7 \cong M_{a',0}^7$ if and only if $a = \alpha^3 a'$ for some $\alpha \in \mathbb{F}_q$. First suppose that q is odd. If $q \equiv 1 \bmod 6$, then $X^3 = 1$ has 3 solutions in $\mathbb{F}_q$. In that case $\mathbb{F}_q$ contains $(q-1)/3$ cubes and hence we get 3 algebras. If $q \not\equiv 1 \bmod 6$ then $X^3 = 1$ has 1 solution in $\mathbb{F}_q$ and hence $\mathbb{F}_q^3 = \mathbb{F}_q$ and we get only 1 algebra. Now suppose that $p = 2$, $q = 2^m$; then $\mathrm{Tr}_2(1) = m$. So by Lemma 5.2, $X^2 + X + 1$ has solutions in $\mathbb{F}_q$ if and only if m is even. This is the same as saying that $p^m \equiv 4 \bmod 6$. In the same way as above we conclude that in this case $M_{a,0}^7$ has 3 algebras. In the case $q \equiv 2 \bmod 6$, we get 1 algebra. The class $M_{0,b}^7$ contains 1 algebra if $p = 2$, and

2 algebras if $p > 2$. Summarizing

$$|M^7_{a,b}| = \begin{cases} q+5 & q \equiv 1 \bmod 6 \\ q+2 & q \equiv 2 \bmod 6 \\ q+3 & q \equiv 3 \bmod 6 \\ q+4 & q \equiv 4 \bmod 6 \\ q+3 & q \equiv 5 \bmod 6 \end{cases}.$$

From M^8 we get 1 algebra.

Now we consider M^9_a. First suppose that q is odd. We have to find the set of $a \in \mathbb{F}_q$ such that $T^2 - T - a$ has no root in $\mathbb{F}_q$. Suppose that this equation has a root α. Then the other root is $1-\alpha$ and $a = \alpha^2 - \alpha$. Let B be the set of all $\alpha^2 - \alpha$ for $\alpha \in \mathbb{F}_q$. If the equation $X^2 - X = c$ has one solution in $\mathbb{F}_q$, then it has two solutions, unless $c = -\frac{1}{4}$. This implies that $|B| = (q+1)/2$. Let A be the set of all $a \in \mathbb{F}_q$ such that $T^2 - T - a$ has no root in $\mathbb{F}_q$. Then $|A| = (q-1)/2$. Also, for $0 \neq \beta \in \mathbb{F}_q$ we define $h_\beta : \mathbb{F}_q \to \mathbb{F}_q$ by $h_\beta(x) = \beta^2(x + \frac{1}{4}) - \frac{1}{4}$. Then h_β is a bijection. It stabilizes B and hence A. Now $M^9_a \cong M^9_b$ precisely if $h_\beta(a) = b$ for some $0 \neq \beta \in \mathbb{F}_q$. There are exactly $(q-1)/2$ different h_β's. So all M^9_a for $a \in A$ are isomorphic. Hence we get 1 algebra. Secondly, suppose that q is even. Choose a, b such that $T^2 + T + a$ and $T^2 + T + b$ have no roots in $\mathbb{F}_q$. Then by Lemma 5.2, $\mathrm{Tr}_2(a) = \mathrm{Tr}_2(b) = 1$. But then $\mathrm{Tr}_2(a+b) = 0$ and $T^2 + T + a + b$ has roots in $\mathbb{F}_q$. Hence $M^9_a \cong M^9_b$. So we get one algebra in this case as well.

The classes M^{10}_a and $M^{11}_{a,b}$ are only defined for characteristic 2. For perfect fields they both have one algebra. However, the algebra in M^{10}_a disappears due to the isomorphism with M^{13}_0.

The classes M^{12} and M^{13}_a have 1 and q algebras respectively. For M^{14}_a we exclude $a = 0$, as that algebra is isomorphic to $M^7_{0,0}$. Therefore this class contains 1 algebra if $p = 2$ and 2 algebras if $p > 2$.

Now we add these numbers, and find that the total number of solvable Lie algebras over $\mathbb{F}_q$ is

$$q^2 + 3q + 9 + \begin{cases} 5 & q \equiv 1 \bmod 6 \\ 2 & q \equiv 2 \bmod 6 \\ 3 & q \equiv 3 \bmod 6 \\ 4 & q \equiv 4 \bmod 6 \\ 3 & q \equiv 5 \bmod 6 \end{cases},$$

which is slightly more than the number found in [Patera and Zassenhaus 90].

Remark 5.3. With $L_{4,i}$ as in [Patera and Zassenhaus 90] we have $L_{4,1} \cong M^1$, $L_{4,2} \cong M^5$ (for this one has to correct the table given in [Patera and Zassenhaus 90]; with the table as given in [Patera and Zassenhaus 90], we have $L_{4,2} \cong L_{4,3}$), $L_{4,3} \cong M^7_{0,0}$, $L_{4,4} \cong M^4$, $L_{4,5} \cong M^3_0$, $L_{4,6} \cong M^7_{0,\alpha}$, $L_{4,7} \cong M^6_{0,-\alpha}$, $L_{4,8} \cong M^8$, $L_{4,9} \cong M^6_{0,-\frac{1}{4}}$ (if the characteristic is not 2), $L_{4,9} \cong M^7_{0,1}$ (if the characteristic is 2), $L_{4,10} \cong M^2$, $L_{4,11} \cong M^3_\alpha$, $L_{4,12} \cong M^6_{\alpha_3,-\alpha_2}$, $L_{4,13} \cong M^7_{\alpha,\alpha}$, $L_{4,14} \cong M^7_{\alpha,0}$, $L'_{4,8} \cong M^9_{-\alpha}$, $L_{4,15} \cong M^7_{-2,3}$ (characteristic not 3) $L_{4,15} \cong M^3_1$ (characteristic 3), $L_{4,16} \cong M^6_{-2\alpha^3+\alpha^2,3\alpha^2-2\alpha}$ ($\alpha \neq \frac{1}{3}$), $L_{4,16} \cong M^3_1$ ($\alpha = \frac{1}{3}$), $L_{4,17} \cong M^6_{\frac{1}{27},-\frac{1}{3}}$ (characteristic not 3 and $\alpha \neq 0$), $L_{4,17} \cong M^7_{1,0}$ (characteristic 3, and $\alpha \neq 0$), $L_{4,17} \cong M^7_{0,0}$ ($\alpha = 0$), $L_{4,18} \cong M^{12}$, $L_{4,19} \cong M^{14}_\alpha$, $L_{4,20} \cong M^{13}_{-\alpha}$.

In [Patera and Zassenhaus 90] the algebra $M^{11}_{a,b}$ is missing. This can be explained by the circumstance that the method used in [Patera and Zassenhaus 90] relies on the derived algebra being nilpotent. Now, if $a \neq 0$ and $b \neq 1$ then the derived algebra of $M^{11}_{a,b}$ is not nilpotent.

Remark 5.4. As in the dimension-3 case, it is possible to formulate an algorithm that, for a given solvable Lie algebra K of dimension 4, finds the M^i to which it is isomorphic. First we find a three-dimensional ideal and establish to which of the L^i it is isomorphic. Then for each of the four possibilities we basically follow the classification procedure.

Remark 5.5. Of course the next step will be to describe the classification for dimension 5. However, in this case the Gröbner basis computations can be rather time consuming, and there are even instances where it did not terminate in reasonable time. So for the classification in dimension 5 a better isomorphism test has to be devised.

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