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Quadratic variations for the fractional-colored stochastic heat equation*

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Abstract

Using multiple stochastic integrals and Malliavin calculus, we analyze the quadratic variations of a class of Gaussian processes that contains the linear stochastic heat equation on $\mathbf{R}^d$ driven by a non-white noise which is fractional Gaussian with respect to the time variable (Hurst parameter H) and has colored spatial covariance of α-Riesz-kernel type. The processes in this class are self-similar in time with a parameter K distinct from H, and have path regularity properties which are very close to those of fractional Brownian motion (fBm) with Hurst parameter K (in the heat equation case, $K = H - (d-\alpha)/4$). However the processes exhibit marked inhomogeneities which cause naïve heuristic renormalization arguments based on K to fail, and require delicate computations to establish the asymptotic behavior of the quadratic variation. A phase transition between normal and non-normal asymptotics appears, which does not correspond to the familiar threshold $K = 3/4$ known in the case of fBm. We apply our results to construct an estimator for H and to study its asymptotic behavior.

Keywords: multiple stochastic integral; stochastic heat equation; fractional Brownian motion; Malliavin calculus; non-central limit theorem; quadratic variation; Hurst parameter; self-similarity; statistical estimation.
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1 Introduction

Statistical inference for stochastic equations driven by fractional Brownian motion (fBm) is a recent research direction in probability theory. It appeared only after the development of stochastic calculus with respect to fBm in the 1990's. For results on parameter estimation for finite dimensional equations we refer to, among others, [15], [37], [33], or [13]. The inference references related to infinite-dimensional stochastic

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equations driven by fBm are very limited. We mention the works [18], [19] for MLE of the drift parameter of infinite-dimensional Ornstein-Uhlenbeck process, and to [5], [6] for various types of equations with fractional, mostly additive, noise.

A common denominator of all these works is that the Hurst parameter of the fractional noise is assumed to be known. Only in [2] do the authors provide an estimator for the Hurst parameter of a fractional diffusion based on a regularization procedure. As far as we know, there are no results on the estimation of the Hurst parameter in (finite or infinite) stochastic differential systems with a Gaussian noise which behaves as a fBm.

The purpose of this paper is to make a first step in this direction. Our principal motivating example is the heat equation with linear additive noise

$$\frac{\partial u^H}{\partial t} = \frac{1}{2}\Delta u^H + \frac{\partial W^H}{\partial t}, \quad t \in [0,T], x \in \mathbb{R}^d \tag{1.1}$$

where Δ is the Laplacian on $\mathbb{R}^d$ and W^H is a Gaussian noise (a generalized-function-valued process in the space parameter) which behaves like fBm in time and has white or colored spatial covariance in space, given by a Riesz kernel of order $\alpha \in [0,d)$; the case $\alpha = 0$ is the case of white noise in space. A necessary and sufficient condition for the existence of and uniqueness of a solution to (1.1), in terms of the Hurst parameter H, the spatial dimension d, and the structure of the spatial covariance, was given in [1]. We refer to Section 2 for a precise definition of the driving noise and of the solution to equation (1.1).

We will assume that the process $u^H := u$ is observed at discrete times at a fixed space location x, and we estimate the parameter H from the observation of u. We will employ a classical method, based on the discrete quadratic variations V_N of the process u in time, for fixed x: this V_N is the centered sum of the squared increments of u over the observation intervals; see the definition of V_N in (3.3) in Section 3. In the case of fBm or of other self-similar processes, such as the Hermite processes, these statistics are used to derive strongly consistent estimators for the Hurst parameter H, and their associated normal convergence results. A detailed study can be found in [12], [14] or more recently in [7], [3], [4], [38]. The behavior of the quadratic variations V_N is used to derive asymptotic properties for the corresponding estimators.

Typically, the asymptotic analysis of V_N is the most technical part of the study. This is eminently true in our paper, and in fact, the behavior of our V_N constitutes a non-trivial result in its own right, in the sense that it could not have been predicted without an extensive calculation, and breaks with some common intuition about quadratic variations of self-similar processes. We give some heuristic elements to justify this claim here. For more details regarding the technical subtleties at work, see Remark 3.2 on page 12 in Section 3, and the paragraphs at the start of both Sections 4 and 5, before the statements of the theorems in those sections.

We prove that for u, our V_N satisfies a central limit theorem for $H < 3/4$ and has a non central behavior for $H > 3/4$. This dichotomy with a threshold at $H = 3/4$ is identical to what one obtains for V_N in the case of fBm (see summary in [38]). The analogy with fBm stops there, however, and one could argue that this is highly unexpected, since the solution u to the stochastic heat equation with Riesz-kernel spatially correlated noise with parameter α in $\mathbb{R}^d$ turns out to be Gaussian self-similar in time with parameter $H - \beta/2$ where $\beta := (d-\alpha)/2$. In particular, judging by the fBm case, this self-similarity ought to imply that, in order to get a normal (central-limit) behavior, the correct normalization of V_N should be $N^{2H-\beta-1/2}$, since for the H-self-similar fBm, the normalization is $N^{2H-1/2}$, and that the normal behavior should occur for $H - \beta/2 < 3/4$. As it turns out, that comparison with fBm yields the correct normalization, but not the correct dichotomy threshold. A “back-of-the-envelope” calculation to try to guess the correct normalization in the case $H > 3/4$ can then proceed as follows: we try to use again the analogy with fBm, but only by requiring that the normalizing power for $H > 3/4$ be independent of H; by also bravely presuming that the power ought to be continuous at $H = 3/4$ (as is known to be true for quadratic variations of all self-similar processes studied to date, see [3]) and be piecewise linear in β, one then finds that one should multiply V_N by $N^{1-\beta}$. This arguably convoluted set of heuristic choices does indeed yield the correct normalization.

The full statement of the correct normalizations, including exact formulae for the asymptotic variances, can be found in Theorem 3.8 on page 31. The asymptotic distributions can be found in Theorems 4.2 and 5.2 on pages 33 and 38 respectively. Here we give a brief summary.

For the mild solution u of the stochastic heat equation with an additive noise W^H which is H-fBm in time and with α-Riesz covariance in space in $\mathbb{R}^d$, let $\beta := (d-\alpha)/2$. Assume $0 < \beta \leq \min(2H, d/2)$ and $H \in (1/2, 1)$ [these conditions are needed for existence of u]. Then for any $x \in \mathbb{R}^d$, the centered version V_N of the quadratic variation $S_N := \sum_{i=0}^{N-1} \left| u\left(\frac{i+1}{N}, x\right) - u\left(\frac{i}{N}, x\right) \right|^2$ has the following behavior, for some constants K_1 and K_2 depending only on H and β:

- for $H < 3/4$, $N^{2H-\beta-1/2} V_N$ converges in distribution to a centered Gaussian law with variance K_1;
- for $H > 3/4$, $N^{1-\beta} V_N$ converges in distribution to a Rosenblatt law with variance K_2 and self-similarity parameter H; we proved this for $d = 1, 2, 3$ and $\alpha = 0$; we suspect it holds for all $\alpha \in [0, d)$;
- in both cases, even if $\alpha \neq 0$, the expression $\hat{H}_N := 2^{-1} \left[-\left(\log \frac{S_N}{C_0} \right) / (\log N) + \beta + 1 \right]$ is a strongly consistent estimator of H; asymptotic properties of $\hat{H}_N$ are inherited from the previous two bullet points (see a discussion of how to handle the constant C_0 on page 41).

What makes the “correct” heuristic choices given above non-obvious, is that one could have easily argued for a simpler heuristic. For instance, we mention below in connection to property (3.2) on page 6 that the solution u is $(H-\beta/2)$-Hölder-continuous in time; this adds to the fact that u is self-similar in time with the same parameter $H - \beta/2$, not to mention that u is centered and Gaussian points further to analogies with an fBm with parameter $H' := H - \beta/2$. One could be swayed by such mounting evidence, and conclude that the threshold parameter and the normalization powers should be identical to those in the case of fBm, i.e. $H' = 3/4$, with the normalizer $N^{2H'-1/2}$ for $H' < 3/4$, and the normalizer N for $H' > 3/4$. But as we said, the only part of this heuristic that ends up being correct is the power for small H (even this becomes incorrect for H close to the true threshold $3/4$). In conclusion, the threshold for the phase transition between normal and non-normal asymptotics is actually determined only by H, i.e. the time-behavior of W^H, while the normalizing powers on either side of the transition involve a subtle combination of H, the space-correlation length α, and the dimension d.

We think these complex effects, in which space and time correlation ranges interact non-trivially via the linear coupling induced by the stochastic heat equation, are partially a result of the fact that the solution u is simultaneously self-similar and highly non-stationary in time, property (3.2) notwithstanding. There could be other effects which cannot be described heuristically, and are unknown to us. Overall, the non-trivial calculations we perform in this article (see the proofs of Propositions 3.1, 3.3, and 3.6

in Section 3 particularly) are the only solid way one has of determining the asymptotic behavior of V_N. We also think that, while the non-stationary aspect of u makes the aforementioned proofs rather involved analytically, producing unexpected results, its self-similarity is crucial in allowing us to bring these proofs to fruition. We had originally thought that extensions to processes which are only roughly self-similar would be straightforward using the same tools, but we now believe other tools would have to be imployed, such as universality-type arguments (see e.g. [24, Chapter 11]). Self-similarity is also helpful in the statistical estimation application.

Our approach to prove the above results is based on the Malliavin calculus and multiple stochastic integrals. In a seminal paper [31] , Nualart and Peccati discovered a surprising central limit theorem (called the *fourth moment theorem*) for sequences of multiple stochastic integrals of a fixed order: in this context, convergence in distribution to the standard normal law is actually equivalent to convergence of only the fourth moment. Later, the work by Nualart and Ortiz-Latorre [30], gave a new proof in which they express the sufficient condition for the normal convergence of a sequence of random variables in a fixed Wiener chaos in terms of the Malliavin derivatives. A new and crucial step in this theory is the paper [26] by Nourdin and Peccati in which, by bringing together Stein's method with the Malliavin calculus, the authors were able (among other things) to associate quantitative bounds to the fourth moment theorem.

These are the tools that we employ to analyze the asymptotic behavior of the discrete centered quadratic variation V_N of the solution u to (1.1). Since the solution is Gaussian, its quadratic variation, and moments thereof, can be written as multiple integrals in fixed Wiener chaos, and it is therefore obvious that the techniques described in the monograph [26] can be used. Our work extends the results in [32] or [34] for the solution to the heat equations with space-time white noise. One the main difficulties in our work is the estimation of the joint increments of the process u. This is due to the complex covariance structure of the solution to (1.1) as alluded to above: u is self-similar but with increments that are sufficiently non-stationary to make the identification of V_N's precise asymptotics difficult. We point out that already in the space-time white noise case, the behavior of the covariance of the increments of the solution to the heat equation was tedious: see [34] for these estimates. The passage from the white-noise case to the fractional-colored case (Riesz kernel covariance, in this paper) brings new technical challenges. We will see in Section 3 that the mere norm in $L^2(\Omega)$ of the sequence V_N can be written as a sum of six terms, each of them being of the same magnitude, forcing us to do a complete analysis of all six terms.

Our paper is organized as follows. Section 2 contains a brief description of the properties of the solution to linear heat equations with fractional-colored noise. In Section 3 we compute the variance of the quadratic variations statistic of the Gaussian field that solves the heat equation. In Sections 4 and 5 we study the limit in distribution of this sequence, while Section 6 contains theoretical and numerical results concerning the self-similarity parameter estimation; the tables and figures referenced therein are at the end of this article, preceeded by an Appendix which gathers useful facts about multiple Wiener integrals.

2 Stochastic heat equation with fractional noise in time

We begin by describing the spatial covariance of the noise. Let us recall the framework from [8]. Let μ be a non-negative tempered measure on $\mathbb{R}^d$, i.e. a non-negative measure which satisfies $\int_{\mathbb{R}^d} \left(1+|\xi|^2\right)^{-\ell} \mu(d\xi) < \infty$, for some $\ell > 0$. Since the integrand is non-increasing in ℓ, we may assume that $\ell \geq 1$ is an integer. Note that $1+|\xi|^2$ behaves as a constant around 0, and as $|\xi|^2$ at ∞, and hence the above condition on μ is equivalent to $\int_{|\xi|\le 1}\mu(d\xi)<\infty$ and $\int_{|\xi|\ge 1}\mu(d\xi)\frac{1}{|\xi|^{2l}}<\infty$, for some $\ell\ge 1$. Let $f:\mathbb{R}^d\to\mathbb{R}_+$ be the Fourier transform of μ in $\mathcal{S}'(\mathbb{R}^d)$, i.e.

$$\int_{\mathbb{R}^d} f(x)\varphi(x)dx=\int_{\mathbb{R}^d}\mathcal{F}\varphi(\xi)\mu(d\xi),\quad \forall\varphi\in\mathcal{S}(\mathbb{R}^d).$$

Let the *Hurst* parameter H be fixed in $(1/2,1)$. On a complete probability space $(\Omega,\mathcal{F},P)$, we consider a zero-mean Gaussian field $W^H=\{W_t^H(A);t\ge 0,A\in\mathcal{B}_b(\mathbb{R}^d)\}$ with covariance:

$$\mathbf{E}(W_t^H(A)W_s^H(B))=R_H(t,s)\int_A\int_B f(x-y)dxdy=:\langle 1_{[0,t]\times A},1_{[0,s]\times B}\rangle_{\mathcal{H}}. \tag{2.1}$$

where R_H is the covariance of the fBm

$$R_H(t,s)=\frac{1}{2}(t^{2H}+s^{2H}-|t-s|^{2H}),\qquad s,t\ge 0.$$

This can be extended to a Gaussian noise measure on $\mathcal{B}_b(\mathbb{R}_+\times\mathbb{R}^d)$ by setting $W^H((s,t]\times A)$ $:=W_t^H(A)-W_s^H(A)$.

Let us consider the formal expression for a linear stochastic heat equation

$$u_t=\frac{1}{2}\Delta u+\dot{W}^H,\quad t\in[0,T],x\in\mathbb{R}^d \tag{2.2}$$

with $u(0,\cdot)=0$, and $(W^H(t,x))_{t\in[0,T],x\in\mathbb{R}^d}$ a (generalized) centered Gaussian noise with covariance (2.1). A *mild* interpretation of equation (2.2) has a unique solution given by

$$u(t,x)=\int_0^t\int_{\mathbb{R}^d}G(t-s,x-y)W^H(ds,dy) \tag{2.3}$$

where the above integral is a Wiener integral with respect to the Gaussian noise measure W^H and G is the standard heat kernel given by $G(t,x)=(2\pi t)^{-d/2}\exp\left(-\frac{|x|^2}{2t}\right)$ for $t>0$, $x\in\mathbb{R}^d$. See [1] for details of the above claims and a proof of the following.

Theorem 2.1. *The process u exists on $[0,T]\times\mathbb{R}^d$ and satisfies $\sup_{t\in[0,T],x\in\mathbb{R}^d}\mathbf{E}\left(u(t,x)^2\right)$ $<+\infty$ if and only if*

$$\int_{\mathbb{R}^d}\left(\frac{1}{1+|\xi|^2}\right)^{2H}\mu(d\xi)<\infty. \tag{2.4}$$

Example 2.2. *i. Suppose that the noise is white in space. In this case μ is the Lebesgue measure on $\mathbb{R}^d$ and Condition (2.4) is equivalent to*

$$d<4H.$$

This implies that, in contrast to the time-white-noise case, we are allowed to consider the spatial dimension d to be $1,2$ or 3. Recall that the stochastic heat equation with time-space white noise admit a solution if and only if $d=1$.

ii. Suppose that the noise is colored in space and the spatial covariance is given by the Riesz kernel of order α:

$$f(x)=R_\alpha(x):=\gamma_{\alpha,d}|x|^{-d+\alpha},\quad 0<\alpha<d,$$

where $\gamma_{\alpha,d}=2^{d-\alpha}\pi^{d/2}\Gamma((d-\alpha)/2)/\Gamma(\alpha/2)$. In this case, $\mu(d\xi)=|\xi|^{-\alpha}d\xi$. Then Condition (2.4) is equivalent to

$$d<4H+\alpha.$$

If the noise has spatial covariance given by the Riesz kernel of order α, the covariance of the solution u was also computed in [1], in Theorem 2.7 therein: for $0 \leq s \leq t$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} R(t,s) &:= \mathbf{E}\left(u(t,x)u(s,x)\right) \\ &= \alpha_H (2\pi)^{-d} \left(\int_{\mathbb{R}^d} \mu(d\xi) e^{-\frac{|\xi|^2}{2}} \right) \int_0^t \int_0^s |u-v|^{2H-2} \left((t+s)-(u+v)\right)^{-\frac{d-\alpha}{2}} dv du. \end{aligned} \quad (2.5)$$

Here $\alpha_H = H(2H-1)$. The special case of white noise in space is identical to the particular case of (2.5) with $\alpha = 0$, in which case μ is the Lebesgue measure, and the integral w.r.t μ in (2.5) reduces to $(2\pi)^{d/2}$.

Remark 2.3. *Formula (2.5) readily shows that for any $\alpha \in [0,d)$, for any fixed $x \in \mathbb{R}^d$, the process $u(\cdot, x)$ is self similar on $\mathbb{R}_+$, of order $H - \frac{d-\alpha}{4}$.*

3 Quadratic variation computation

This section contains the main quantitative estimates presented in this article. Our motivation is to compute the asymptotic behavior of the quadratic variations of the solution to (2.2) viewed as a process with respect to t with x fixed. It is convenient to extend this question slightly.

Throughout the remainder of this article we consider the centered Gaussian process $(U_t)_{t\in[0,1]}$ with covariance

$$R(t,s) = D \int_0^t \int_0^s |u-v|^{2H-2} \left((t+s)-(u+v)\right)^{-\beta} dv du \quad (3.1)$$

with $0 < 2\beta \leq d$ and $2\beta < 4H$, and with D a fixed positive scaling constant. Since the covariance formulas (2.5) and (3.1) have the same form, $(U_t)_{t\in[0,1]}$ and $u(\cdot, x)$ have the same law, where $u(t,x)$ is solution to (2.2), provided we set the correspondence

$$\beta = \frac{d-\alpha}{2}; \quad D = d(\alpha, H) := \alpha_H (2\pi)^{-d} \int_{\mathbb{R}^d} d\xi |\xi|^{-\alpha} e^{-\frac{|\xi|^2}{2}}.$$

In particular, by Remark 2.3, $(U_t)_{t\in[0,1]}$ is self-similar of order $H - \beta/2$.

Interestingly, the parameter $H - \beta/2$ is also the Hölder-continuity parameter of $t \mapsto U_t$. This was proved for the stochastic heat equation with white noise ($\alpha = 0$) by Swanson in [34], and later extended to all $\alpha > 0$ in the preprint [22]. A similar result was proved in [21] for a related stochastic heat equation, with applications to hitting probabilities. Since $(U_t)_{t\in[0,1]}$ has the same law as $(u(t,x))_{t\in[0,1]}$ with the above correspondence, the property extends to $(U_t)_{t\in[0,1]}$. Specifically, there exists two strictly positive constants C_1, C_2 such that for any $t, s \in [0,1]$ and for any $x \in \mathbb{R}^d$

$$C_1 |t-s|^{2H-\beta} \leq \mathbf{E}\left|U_t - U_s\right|^2 \leq C_2 |t-s|^{2H-\beta}. \quad (3.2)$$

This confirms that $H - \beta/2$ is the crucial parameter when examining U's regularity. This property (3.2) implies that U_t has the same regularity properties as fBm with parameter $H - \beta/2$; for instance, by the classical regularity theory for Gaussian fields (see e.g. [10]), via the Dudley-Fernique entropy integral upper bound, we get that $r \mapsto r^{H-\beta/2} \log^{1/2}(1/r)$ is almost surely a uniform modulus of continuity for U_t.

More can potentially be made of relation (3.2). According to the method developed in [21], property (3.2) could also be used to infer properties of hitting probabilities, if accompanied by similar regularity in the space parameter, via conditional variance estimates; the point here is that any hitting probability results would reflect the regularity exponents such as the $H - \beta/2$ in (3.2).

However, drawing analogies with fBm by looking only at such regularity parameters has its limitations, as we said in the introduction. We will find in this article that the asymptotic properties of U's quadratic variation are different from those of fBm with parameter $H-\beta/2$. In any case, we will only refer to the regularity property (3.2) for illustration purposes, not to prove any of our theorems.

Define the centered quadratic variation of the process (U_t):

$$V_N := \sum_{i=0}^{N-1}\left[\left(U_{t_{i+1}}-U_{t_i}\right)^2 - \mathbf{E}\left(U_{t_{i+1}}-U_{t_i}\right)^2\right]. \tag{3.3}$$

Let I_n denote the multiple integral with respect to the Gaussian process (U_t). Then we have

$$U_{t_{i+1}}-U_{t_i} = I_1\left(1_{(t_i,t_{i+1})}\right)$$

and thanks to the product formula (8.3), we can express the sequence V_N as a multiple integral of order 2:

$$V_N = I_2\left(\sum_{i=0}^{N-1} 1_{(t_i,t_{i+1})}^{\otimes 2}\right).$$

We consider only the even partition of the unit interval $[0,1]$: $t_i := \frac{i}{N}$ for $i=0,..,N$. We have, using the isometry of multiple stochastic integrals (8.1)

$$\mathbf{E}V_N^2 = 2!\sum_{i,j=0}^{N-1}\langle 1_{(t_i,t_{i+1})}^{\otimes 2}, 1_{(t_j,t_{j+1})}^{\otimes 2}\rangle = 2\sum_{i,j=0}^{N-1}\langle 1_{(t_i,t_{i+1})}, 1_{(t_j,t_{j+1})}\rangle^2.$$

Here $\langle\cdot,\cdot\rangle_{\mathcal{U}} := \langle\cdot,\cdot\rangle$ denotes the scalar product in the canonical Hilbert space $\mathcal{U}$ associated with the process U which is defined as the closure of the set of indicator functions $\left(1_{[0,t]}, t\in[0,T]\right)$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle = R(t,s)$$

where $R(t,s)$ is given by (3.1). Then

$$\begin{aligned}
& D^{-1}\langle 1_{(t_i,t_{i+1})}, 1_{(t_j,t_{j+1})}\rangle \\
= \;& \int_0^{\frac{i+1}{N}} du \int_0^{\frac{j+1}{N}} dv |u-v|^{2H-2}\left(\frac{i+1}{N}+\frac{j+1}{N}-(u+v)\right)^{-\beta} \\
& -\int_0^{\frac{i+1}{N}} du \int_0^{\frac{j}{N}} dv |u-v|^{2H-2}\left(\frac{i+1}{N}+\frac{j}{N}-(u+v)\right)^{-\beta} \\
& -\int_0^{\frac{i}{N}} du \int_0^{\frac{j+1}{N}} dv |u-v|^{2H-2}\left(\frac{i}{N}+\frac{j+1}{N}-(u+v)\right)^{-\beta} \\
& +\int_0^{\frac{i}{N}} du \int_0^{\frac{j}{N}} dv |u-v|^{2H-2}\left(\frac{i}{N}+\frac{j}{N}-(u+v)\right)^{-\beta}.
\end{aligned}$$

By the change of variables $\tilde{u} = uN, \tilde{v} = vN$ we get

$$\begin{aligned}
& D^{-1}\left\langle 1_{(t_i,t_{i+1})}, 1_{(t_j,t_{j+1})}\right\rangle \\
= \quad & N^{-2H+\beta}\left[\int_0^{i+1} du \int_0^{j+1} dv |u-v|^{2H-2}\left(i+1+j+1-(u+v)\right)^{-\beta}\right. \\
& - \int_0^{i+1} du \int_0^{j} dv |u-v|^{2H-2}\left(i+1+j-(u+v)\right)^{-\beta} \\
& - \int_0^{i} du \int_0^{j+1} dv |u-v|^{2H-2}\left(i+j+1-(u+v)\right)^{-\beta} \\
& \left. + \int_0^{i} du \int_0^{j} dv |u-v|^{2H-2}\left(i+j-(u+v)\right)^{-\beta}\right] \\
:= \quad & N^{-2H+\beta}\left[A(i,j)+B(i,j)+C(i,j)\right]
\end{aligned}$$

where

$$A(i,j) = \int_i^{i+1} du \int_j^{j+1} dv |u-v|^{2H-2}\left(i+j+2-(u+v)\right)^{-\beta}, \tag{3.4}$$

$$\begin{aligned}
B(i,j) = & \textstyle\int_i^{i+1} du \int_0^j dv |u-v|^{2H-2}\left[\left(i+j+2-(u+v)\right)^{-\beta} - \left(i+j+1-(u+v)\right)^{-\beta}\right] \\
& + \textstyle\int_0^i du \int_j^{j+1} dv |u-v|^{2H-2}\left[\left(i+j+2-(u+v)\right)^{-\beta} - \left(i+j+1-(u+v)\right)^{-\beta}\right]
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
& C(i,j) = \int_0^i du \int_0^j dv |u-v|^{2H-2} \cdot \\
& \left[\left(i+j+2-(u+v)\right)^{-\beta} - 2\left(i+j+1-(u+v)\right)^{-\beta} + \left(i+j-(u+v)\right)^{-\beta}\right].
\end{aligned} \tag{3.6}$$

Therefore

$$\begin{aligned}
\mathbf{E}V_N^2 \quad = \quad & 2D^2 N^{-4H+2\beta} \sum_{i,j=0}^{N-1}\left[A(i,j)+B(i,j)+C(i,j)\right]^2 \\
= \quad & 2D^2 N^{-4H+2\beta} \sum_{i,j=0}^{N-1}\left[A(i,j)^2+B(i,j)^2+C(i,j)^2\right. \\
& \left. +2A(i,j)B(i,j)+2A(i,j)C(i,j)+2B(i,j)C(i,j)\right] \\
:= \quad & 2D^2(T_{1,N}+T_{2,N}+T_{3,N}+T_{4,N}+T_{5,N}+T_{6,N}).
\end{aligned} \tag{3.7}$$

We will evaluate the asymptotic behavior, as $N \to \infty$ of the six terms from above. Actually, it happens that the six summands that appear in the decomposition of $\mathbf{E}V_N^2$ are all of them of the same magnitude. This makes our computations delicate and lengthy. There is no negligible part that can be ignored in the estimation of $\mathbf{E}V_N^2$.

We give first the behavior of the summand $T_{1,N}$ above.

Proposition 3.1. *If $H < \frac{3}{4}$, then $\lim_{N\to\infty} N^{4H-2\beta-1} T_{1,N} = K_{1,1}$ where*

$$\begin{aligned}
K_{1,1} \quad := \quad & \left(\int_0^1 du \int_0^1 dv |u-v|^{2H-2}(u+v)^{-\beta}\right)^2 \\
+ \quad & 2\sum_{k=1}^{\infty}\left(\int_0^1 du \int_0^1 dv |u-v+k|^{2H-2}(u+v)^{-\beta}\right)^2 \\
= \quad & \sum_{k=-\infty}^{\infty}\left(\int_0^1 du \int_0^1 dv |u-v+k|^{2H-2}(u+v)^{-\beta}\right)^2.
\end{aligned}$$

If $H > \frac{3}{4}$ *then* $\lim_{N\to\infty} N^{2-2\beta} T_{1,N} = K_{1,2}$ *where*

$$\begin{aligned} K_{1,2} &:= \left(\int_0^1\int_0^1 dxdy|x-y|^{4H-4}\right)\left(\int_0^1 du\int_0^1 dv\,(u+v)^{-\beta}\right)^2 \\ &= \frac{2}{(4H-3)(4H-2)}\left(\frac{2\left|2^{1-\beta}-1\right|}{|1-\beta|(2-\beta)}\right)^2. \end{aligned}$$

Proof: With the change of variables $\tilde{u} = u - i, \tilde{v} = v - j$

$$\begin{aligned} T_{1,N} &= N^{-4H+2\beta}\sum_{i,j=0}^{N-1} A(i,j)^2 \\ &= N^{-4H+2\beta}\sum_{i,j=0}^{N-1}\left(\int_0^1 du\int_0^1 dv|u-v+i-j|^{2H-2}(2-(u+v))^{-\beta}\right)^2 \\ &= N^{-4H+2\beta}\sum_{i,j=0}^{N-1}\left(\int_0^1 du\int_0^1 dv|u-v+i-j|^{2H-2}(u+v)^{-\beta}\right)^2. \end{aligned}$$

Step 1. The case $H > \frac{3}{4}$. In this case we write

$$T_{1,N} = N^{-4H+2\beta}N^{4H-4}N^2\frac{1}{N^2}\sum_{i,j=0}^{N-1}\left(\int_0^1 du\int_0^1 dv\left|\frac{u-v}{N}+\frac{i-j}{N}\right|^{2H-2}(u+v)^{-\beta}\right)^2,$$

and we seek to show that the term in which the $i-j$ appears can be treated as a general term of a Riemann sum, where we must strive to show that the term $u-v$ can be ignored.

Step 1.1. We claim that

$$T_{1,N} \simeq N^{2\beta-2}\left(\frac{1}{N^2}\sum_{0\le i\ne j\le N-1}\left|\frac{i-j}{N}\right|^{4H-4}\right)\left(\int_0^1 du\int_0^1 dv\,(u+v)^{-\beta}\right)^2 \tag{3.8}$$

where the symbol $\simeq$ means the the two sides have the same limit as $N\to\infty$.

Step 1.2. We see that we have ignored the diagonal ($i=j$) terms (since they would lead to an infinite term), so the first thing to show is that the diagonal terms in $T_{1,N}$ are of a lower order than the above. These terms are equal to

$$\begin{aligned} T_{1,N,diag} &:= N^{2\beta-2}\frac{1}{N^2}\sum_{i=0}^{N-1}\left(\int_0^1 du\int_0^1 dv\left|\frac{u-v}{N}\right|^{2H-2}(u+v)^{-\beta}\right)^2 \\ &= N^{2\beta-2}\frac{4}{N}N^{4-4H}\left(\int_0^1 du\int_0^u dv\,(u-v)^{2H-2}(u+v)^{-\beta}\right)^2. \end{aligned}$$

By the bivariate change of variable $x = u - v$ and $y = u + v$, whose Jacobian matrix has determinant 2, and which transforms a domain included in the set $0 \le v \le u \le 1$ into the domain $0 \le x \le y \le 2$, we get that

$$\begin{aligned} T_{1,N,diag} &\le N^{2\beta-2}N^{3-4H}4\frac{1}{4}\left(\int_0^2 dy\int_0^y dx\,x^{2H-2}y^{-\beta}\right)^2 \\ &= N^{2\beta-2}N^{3-4H}\left((2H-1)^{-1}(2H-\beta)^{-1}2^{2H-\beta}\right)^2. \end{aligned} \tag{3.9}$$

This last expression is an infinitesimal compared to $N^{2\beta-2}$ because $3 < 4H$ and $2\beta < 4H$ by assumption.

Step 1.3. To finish the proof of the claim in (3.8), we now need only compare $T_{1,N}$ without the diagonal terms $i=j$ and the same expression with the $u-v$ removed. In other words, it is sufficient to show that the following tends to 0 as $N\to\infty$:

$$\begin{aligned}
T_{1,N,error} \quad &:= \quad \frac{1}{N^2}\sum_{\substack{i,j=0\\ i\neq j}}^{N-1}\left(\int_0^1 du\int_0^1 dv\left|\frac{u-v}{N}+\frac{i-j}{N}\right|^{2H-2}(u+v)^{-\beta}\right)^2\\
&\quad -\frac{1}{N^2}\sum_{\substack{i,j=0\\ i\neq j}}^{N-1}\left(\int_0^1 du\int_0^1 dv\left|\frac{i-j}{N}\right|^{2H-2}(u+v)^{-\beta}\right)^2\\
&= \quad \frac{1}{N^2}\sum_{\substack{i,j=0\\ i\neq j}}^{N-1}\left(\int_0^1 du\int_0^1 dv\left[\left|\frac{u-v}{N}+\frac{i-j}{N}\right|^{2H-2}-\left|\frac{i-j}{N}\right|^{2H-2}\right](u+v)^{-\beta}\right)\\
&\quad \cdot\left(\int_0^1 du\int_0^1 dv\left[\left|\frac{u-v}{N}+\frac{i-j}{N}\right|^{2H-2}+\left|\frac{i-j}{N}\right|^{2H-2}\right](u+v)^{-\beta}\right).
\end{aligned}$$

In the last expression above, the factor which includes a difference is convenient to us, because we can use the mean value theorem on the $(2H-2)$-power function to show that it is small, but the second term, which includes a sum, needs to be controlled. Since $|u-v|\leq 1$, and $|i-j|\geq 1$, it will be convenient to us to discard another portion of the sum, those terms for which $|i-j|=1$. The corresponding terms yield a contribution that is of the same order N^{3-4H} as for the term $T_{1,N,diag}$, using a similar computation as that leading to (3.9), which is left to the reader, and we may thus ignore the terms for $|i-j|=1$. In other words we have

$$\begin{aligned}
T_{1,N,error} \quad &= \quad O\left(N^{3-4H}\right)\\
&\quad + \quad \frac{1}{N^2}\sum_{\substack{i,j=0\\ |i-j|\geq 2}}^{N-1}\left(\int_0^1 du\int_0^1 dv\left[\left|\frac{u-v}{N}+\frac{i-j}{N}\right|^{2H-2}-\left|\frac{i-j}{N}\right|^{2H-2}\right](u+v)^{-\beta}\right)\\
&\quad \cdot\left(\int_0^1 du\int_0^1 dv\left[\left|\frac{u-v}{N}+\frac{i-j}{N}\right|^{2H-2}+\left|\frac{i-j}{N}\right|^{2H-2}\right](u+v)^{-\beta}\right).
\end{aligned}$$

Now, the assumption $|i-j|\geq 2$ justifies writing $\left|\frac{u-v}{N}+\frac{i-j}{N}\right|\geq\left|\frac{i-j}{N}\right|/2$. Also, by the mean value theorem with a value $\xi\in(\frac{u-v}{N}+\frac{i-j}{N},\frac{i-j}{N})$, which thus satisfies $|\xi|\geq\left|\frac{i-j}{N}\right|/2$, we get

$$\begin{aligned}
\left|\left|\frac{u-v}{N}+\frac{i-j}{N}\right|^{2H-2}-\left|\frac{i-j}{N}\right|^{2H-2}\right| \quad &= \quad (2-2H)\left|\frac{u-v}{N}\right||\xi|^{2H-3}\\
&\leq \quad 2^{3-2H}(2-2H)\left|\frac{u-v}{N}\right|\left|\frac{i-j}{N}\right|^{2H-3}.
\end{aligned}$$

Putting these estimates together we can now write, with k_H a constant depending only on H, and also using $|u-v|\le 1$,

$$\begin{aligned}
|T_{1,N,error}| &\le O\left(N^{3-4H}\right)+\frac{k_H}{N^2}\sum_{\substack{i,j=0\\|i-j|\ge 2}}^{N-1}\left(\int_0^1 du\int_0^1 dv\left|\frac{u-v}{N}\right|\left|\frac{i-j}{N}\right|^{2H-3}(u+v)^{-\beta}\right)\\
&\quad\cdot\left(\int_0^1 du\int_0^1 dv\left|\frac{i-j}{N}\right|^{2H-2}(u+v)^{-\beta}\right)\\
&\le O\left(N^{3-4H}\right)+\frac{k_H}{N}\frac{1}{N^2}\sum_{\substack{i,j=0\\|i-j|\ge 2}}^{N-1}\left|\frac{i-j}{N}\right|^{4H-5}\left(\int_0^1 du\int_0^1 dv\,(u+v)^{-\beta}\right)^2\\
&= O\left(N^{3-4H}\right)+k_H N^{2-4H}\sum_{\substack{i,j=0\\|i-j|\ge 2}}^{N-1}|i-j|^{4H-5}\left(\int_0^1 du\int_0^1 dv\,(u+v)^{-\beta}\right)^2\\
&\le O\left(N^{3-4H}\right)+k_H N^{2-4H}N^{4H-3}\left(\int_0^1 du\int_0^1 dv\,(u+v)^{-\beta}\right)^2\\
&= O\left(N^{3-4H}\right)+O\left(N^{-1}\right).
\end{aligned}$$

In the last step, the constant $I_\beta := \int_0^1 du\int_0^1 dv\,(u+v)^{-\beta}$ is finite because $2\beta<4H<4$. Therefore $\lim_{N\to\infty}T_{1,N,error}=0$, which finishes the proof of the claim (3.8).

Step 1.4. Thanks to this claim, we now invoke a Riemann sum to immediately show that

$$T_{1,N}\simeq N^{2\beta-2}\left(\int_0^1\int_0^1 dxdy|x-y|^{4H-4}\right)\left(\int_0^1 du\int_0^1 dv\,(u+v)^{-\beta}\right)^2=K_{1,2}$$

with $K_{1,2}$ defined in the statement of the proposition. We already mentioned that the last factor above is the finite constant I_β^2; the previous factor above is a finite constant because $H>3/4$. This finishes the statement of the proposition when $H>3/4$, modulo the second expression for the constant, which is elementary.

Step 2. The case $H<\frac{3}{4}$.

Step 2.1. With change of index, we can write

$$\begin{aligned}
T_{1,N} &= 2N^{-4H+2\beta}\sum_{k=1}^{N-1}(N-k)\left(\int_0^1 du\int_0^1 dv|u-v+k|^{2H-2}(u+v)^{-\beta}\right)^2\\
&\quad+N^{-4H+2\beta+1}\left(\int_0^1 du\int_0^1 dv|u-v|^{2H-2}(u+v)^{-\beta}\right)^2.
\end{aligned}$$

The squared integral in the second term above is finite simply because it is smaller than $N^{4H-2\beta-1}T_{1,N}$ (one can also reprove this by hand, using the fact that $2H-2>-1$ and $2\beta<4H<4$). Consequently, $T_{1,N}$ will be at least of the order of a constant times $N^{-4H+2\beta+1}$. It will be convenient for us to also single out the term for $k=1$ above, in order to better pin down the asymptotics of $T_{1,N}$. Therefore we write

$$\begin{aligned}
T_{1,N} &= N^{-4H+2\beta+1}\left(\int_0^1 du\int_0^1 dv|u-v|^{2H-2}(u+v)^{-\beta}\right)^2\\
&\quad+2N^{-4H+2\beta+1}\frac{N-1}{N}\left(\int_0^1 du\int_0^1 dv|u-v+1|^{2H-2}(u+v)^{-\beta}\right)^2\\
&\quad+2N^{-4H+2\beta}\sum_{k=2}^{N-1}(N-k)\left(\int_0^1 du\int_0^1 dv|u-v+k|^{2H-2}(u+v)^{-\beta}\right)^2 \qquad (3.10)
\end{aligned}$$

The first and second terms above are evidently equivalent to constants times $N^{-4H+2\beta+1}$, and we will show that the term in line (3.10), which we call $T_{1,N,3}$ is also of this form, and find its asymptotic constant.

Step 2.2. We first prove that $N^{4H-2\beta-1}T_{1,N,3}$ is bounded as $N\to\infty$. Indeed, since $k\geq 2$, and $|u-v|\leq 1$, we get $|k+u-v|\geq k/2$, which implies, with the constant I_β from Step 1.3

$$T_{1,N,3}\leq 2N^{-4H+2\beta+1}2^{4-4H}I_\beta^2\sum_{k=2}^{N-1}k^{4H-4}.$$

Since $4H-4<-1$, the above partial sum converges, proving our claim.

Step 2.3. Next, we claim that

$$N^{4H-2\beta-1}T_{1,N,3}\simeq 2\sum_{k=2}^{N-1}\left(\int_0^1 du\int_0^1 dv|u-v+k|^{2H-2}\,(u+v)^{-\beta}\right)^2.$$

To prove this, we show that the difference between the two sides above is of a lower order than $N^{-4H+2\beta+1}$. Using again the inequality $|k+u-v|\geq k/2$ and the constant I_β we get

$$\begin{aligned}
&\left|T_{1,N,3}-2N^{-4H+2\beta+1}\sum_{k=2}^{N-1}\left(\int_0^1 du\int_0^1 dv|u-v+k|^{2H-2}\,(u+v)^{-\beta}\right)^2\right|N^{4H-2\beta-1}\\
=\;&\frac{2}{N}\sum_{k=2}^{N-1}k\left(\int_0^1 du\int_0^1 dv|u-v+k|^{2H-2}\,(u+v)^{-\beta}\right)^2\\
\leq\;&\frac{2}{N}\sum_{k=2}^{N-1}k\left(\int_0^1 du\int_0^1 dv|k/2|^{2H-2}\,(u+v)^{-\beta}\right)^2=\frac{2^{5-4H}}{N}I_\beta^2\sum_{k=2}^{N-1}k^{4H-3}.
\end{aligned}$$

The partial sum above diverges; we know it is of order N^{4H-2}. This implies that the last line in the calculation above is bounded above by a constant times N^{4H-3}, which tends to 0 as $N\to\infty$. Our claim is proved.

Step 2.4. From the result of step 2.2, the equivalent for $T_{1,N,3}$ found in step 2.3 is also equivalent to what one obtains by taking the sum to infinity. By combining this with the results of step 2.1 we have now proved that

$$\begin{aligned}
T_{1,N}N^{4H-2\beta-1}\;\simeq\;&\left(\int_0^1 du\int_0^1 dv|u-v|^{2H-2}\,(u+v)^{-\beta}\right)^2\\
&+\;2\left(\int_0^1 du\int_0^1 dv|u-v+1|^{2H-2}\,(u+v)^{-\beta}\right)^2\\
&+\;2\sum_{k=2}^{\infty}\left(\int_0^1 du\int_0^1 dv|u-v+k|^{2H-2}\,(u+v)^{-\beta}\right)^2\\
=\;&\left(\int_0^1 du\int_0^1 dv|u-v|^{2H-2}\,(u+v)^{-\beta}\right)^2\\
&+\;2\sum_{k=1}^{\infty}\left(\int_0^1 du\int_0^1 dv|u-v+k|^{2H-2}\,(u+v)^{-\beta}\right)^2=K_{1,1}.
\end{aligned}$$

This is the claim made in the proposition when $H<3/4$, which finishes the proposition's proof. ■

Remark 3.2. *We will see in Propositions 3.3 and 3.6 that $T_{3,N}$ and $T_{2,N}$ are of the same magnitude as $T_{1,N}$. We now concentrate on some remarks regarding the unexpected* *behavior of $T_{1,N}$, most of which apply also to $T_{2,N}$ and $T_{3,N}$. However, in some sense, the behavior of $T_{3,N}$ has additional unexpected features; see Remark 3.5 for additional comments on this. The term $T_{2,N}$ is in some sense a combination of the terms in $T_{1,N}$ and $T_{3,N}$; we will not comment specifically on it.*

1. *From the computations and result of Proposition 3.1, one can surmise that the sequence $T_{1,N}$ behaves like the quadratic variations of a Gaussian process which is close to the fractional Brownian motion with parameter H, except that the kernel in its covariance's representation would have an additional weight. Specifically, the comparable process would be the centered Gaussian process with covariance $R(s,t) = c\int_0^t \int_0^s |u-v|^{2H-2}(2-u-v)^{-\beta}$. It is therefore perhaps natural to expect the same renormalization of $T_{1,N}$ as for the quadratic variations of the fractional Brownian motion with parameter H; according to the known results (see [38]) this would be that*
   - *$T_{1,N}N^{4H-1}$ should converge as $N \to \infty$ for $H < 3/4$, and*
   - *$T_{1,N}N$ should converge as $N \to \infty$ for $H > 3/4$.*
2. *Proposition 3.1 does not corroborate these fact. Instead, we see that the threshold of $H = 3/4$ is indeed the correct one to determine between two different magnitudes of $T_{1,N}$, but the correct normalization in the case $H < 3/4$ is given by using the self-similarity parameter $H - \beta/2$ instead of H: $T_{1,N}N^{4H-2\beta-1}$ converges when $H < 3/4$.*
3. *Matters become even more puzzling when one moves on to the case $H > 3/4$. In the classical fBm case with parameter H, for $H > 3/4$, one can compute the new normalization for the variance of the quadratic variation by multiplying the normalization for $H < 3/4$ by the factor N^{-4H+3} (see again [38]). By analogy with this fact, and assuming we could still consider that normalization factors for $T_{1,N}$ should be computed by using the self-similarity parameter $H - \beta/2$ instead of H in this additional factor $N^{-4(H-\beta/2)+3}$, one would think that for $T_{1,N}$, the normalizing factor should be N^1; but Proposition 3.1 shows that this is not the case, and that we must base our heuristic on H rather than $H - \beta/2$ to compute this transition factor. Another interpretation of the correct normalization factor for $H > 3/4$ is that it is the one which is needed in order to obtain continuity of the normalizing factor's power at the transition $H = 3/4$. In that sense, the factor $N^{2-2\beta}$ in Proposition is not a surprise. But the heuristic to give this value requires that one admit two things: the normalization power has to be continuous at $H = 3/4$ and has to be constant for $H > 3/4$.*
4. *This all shows that the combination of having to use the threshold $H = 3/4$ along with the self-similarity parameter $H - \beta/2$ does cause one to have to abandon all intuition about the importance of regularity and/or self-similarity parameters in computing the asymptotic behavior of quadratic variations. In hindsight, the interpretation at the end of the previous item (3.) above provides a way to predict the exponents in the normalizing factors of $T_{1,N}$ to some extent: for H small, one should use the self-similarity parameter $H - \beta/2$ to find this exponent (as in the classical case), for large H, the exponent should not depend on H, and the two exponents should agree at the threshold. The surprising conclusion of Proposition 3.1, one for which we cannot find heuristics to predict, is that the threshold is at $H = 3/4$, rather that the $H = 3/4 + \beta/2$ as self-similarity would have indicated. In other words, interpreted in the representation of stochastic heat equations, the normalizations in Proposition 3.1 show how the space, time, and dimension behavior parameters determine the result via H and $\beta = (d-\alpha)/2$, but that the*

phase transition between asymptotic regimes is determined by the time parameter H only.

Let us now study the summand $T_{3,N}$.

Proposition 3.3. *Let $g_\beta(x) := x^{-\beta} - 2(x+1)^{-\beta} + (x+2)^{-\beta}$. If $H > 3/4$ then $\lim_{N\to\infty} N^{2-2\beta} T_{3,N} = K_{3,2}$ where*

$$\begin{aligned} K_{3,2} &:= \left(\int_0^1 \int_0^1 dxdy|x-y|^{4H-4}\right)\left(\int_0^\infty 2x\ g_\beta(x)\ dx\right)^2 \\ &= \frac{2}{(4H-3)(4H-2)}\left(\frac{4\left|2^{1-\beta}-1\right|}{(2-\beta)|1-\beta|}\right)^2. \end{aligned}$$

If $H < 3/4$ then $\lim_{N\to\infty} N^{4H-2\beta-1} T_{3,N} = K_{3,1}$ where

$$\begin{aligned} K_{3,1} &:= 2\sum_{m=1}^\infty \left|\int_0^\infty dx\ g_\beta(x) \int_{-x}^x dy\ \ |y-m|^{2H-2}\right|^2 + \left|\frac{1}{2H-1}\int_0^{+\infty} dx\ g_\beta(x)\ \ x^{2H-1}\right|^2 \\ &= \sum_{k=-\infty}^\infty \left|\int_0^\infty dx\ g_\beta(x)\int_{-x}^x dy\ \ |y-k|^{2H-2}\right|^2. \end{aligned}$$

Remark 3.4. *The series in the constant $K_{3,1}$ can be approximated readily, because the general term of the series is of order m^{4H-4}, so that the tail decays at the rate m^{4H-3}, and symbolic algebra software such as Maple is able to compute each term for each fixed m.*

Proof: The proof of this proposition is delicate, and will proceed in several steps.

Step 1: Setup. By symmetry, we have $C(i,j) = C(j,i)$, so that we may assume without loss of generality that $j \le i$ in the sum defining $T_{3,N}$. We will separate the estimation of $T_{3,N}$ by considering the terms with $i = j$ and those with $i > j$. By changing the variables (u,v) to $(i-u, j-v)$ in the integral, the term $C(i,j)$ can be written as

$$C(i,j) = \int_0^i du \int_0^j dv|u-v-(i-j)|^{2H-2}\left[(u+v)^{-\beta} - 2(u+v+1)^{-\beta} + (u+v+2))^{-\beta}\right].$$

The magnitude of $T_{3,N}$ is largely determined by the behavior of the integrand above near the diagonal $\{u=v\}$ of the integration domain $(u,v) \in [0,i]\times[0,j]$. For this reason, it is convenient to change variables in the definition of $C(i,j)$ to $(x,y) = (u+v, u-v)$. The change of variables formula implies an inverse Jacobian term equal to $1/2$. The price to pay for this change is that the domain of integration becomes slightly challenging. We denote it by

$$D = \left\{(x,y) \in \mathbf{R}^2 : x+y \in [0,2i]; x-y \in [0,2j]\right\}.$$

We will integrate in the (x,y) pair by starting with y for fixed x, and will need to separate this domain into several parts. Let us describe these sub-domains.

- When $i = j$, it will be sufficient to divide this domain into whether $x \le i$ or $x \ge i$. That is, we write

$$D = L \cup M := [D \cap \{x \le i\}] \cup [D \cap \{x > i\}].$$

  - The subdomain L is thus $\left\{(x,y) \in \mathbf{R}^2 : x \in [0,i],\ y \in [-x,x]\right\}$
  - The subdomain M is rather $\left\{(x,y) \in \mathbf{R}^2 : x \in [i,2i],\ y \in [x-2i, 2i-x]\right\}$.

- When $i > j$, the shape of D in the (x,y) variable is more complicated, and we use the values of j and i as thresholds for x. In other words we define

$$\begin{aligned} D &= A \cup B \cup C \\ &= [D \cap \{(x,y) : x \in [0,j]\}] \cup [D \cap \{(x,y) : x \in [j,i]\}] \\ &\quad \cup [D \cap \{(x,y) : x \in [i, j+i]\}] \end{aligned}$$

  More precisely we find that

  - $A = \left\{(x,y) \in \mathbf{R}^2 : x \in [0,j],\ y \in [-x,x]\right\}$
  - $B = \left\{(x,y) \in \mathbf{R}^2 : x \in [j,i],\ y \in [x-2j,x]\right\}$
  - $C = \left\{(x,y) \in \mathbf{R}^2 : x \in [i,i+j],\ y \in [x-2j, 2i-x]\right\}.$

Each of the five pieces, corresponding to the five different integration domains above, can be written as

$$C(i,j) = \frac{1}{2} \int_{x_-}^{x_+} dx\ g_\beta(x) \int_{y_-(x)}^{y_+(x)} dy\ |y-m|^{2H-2}$$

where $x_-, x_+, y_-(x), y_+(y)$ correspond to the integration endpoints given in the expressions above for A, B, C, L, M, where

$$m := i - j,$$

and the function g_β is defined by

$$g_\beta(x) := x^{-\beta} - 2(x+1)^{-\beta} + (x+2)^{-\beta}.$$

When x is “large”, this function is “approximately” equal to $\beta(\beta+1)x^{-\beta-2}$. We will make use of this approximation in various parts of our calculation, although it is not valid near the origin for x, and care will be taken there. It should be noted that, by the mean value theorem, there is a value $\xi \in (x, x+2)$ such that $g_\beta(x) = \beta(\beta+1)\xi^{-\beta-2}$. In particular, this is positive, and is bounded above as

$$g_\beta(x) \le \beta(\beta+1)x^{-\beta-2}. \tag{3.11}$$

The positivity shows that each term $C(i,j)$ is positive, and the upper bound will be convenient in many instances below, particularly when one only needs to prove that a term under consideration is negligible compared to the total contribution of $T_{3.N}$.

Step 2: The case $i = j$. Here $m = 0$ and in both subcases we have that $y_+(x) = -y_-(x)$, which implies

$$C(i,i) = \int_{x_-}^{x_+} dx\ g_\beta(x) \int_0^{y_+(x)} dy\ y^{2H-2} = \frac{1}{2H-1} \int_{x_-}^{x_+} dx\ g_\beta(x)\ y_+(x)^{2H-1}.$$

For the term corresponding to L we have

$$C_L(i,i) = \frac{1}{2H-1} \int_0^i dx\ g_\beta(x)\ x^{2H-1}.$$

This implies that a series of the form $\sum_{i=1}^N |C_L(i,i)|^2$ will be asymptotically equivalent to $Nc_L(H,\beta)$ where the constant c_L is defined by

$$c_L(H,\beta) = \left| \frac{1}{2H-1} \int_0^{+\infty} dx\ g_\beta(x)\ x^{2H-1} \right|^2.$$

Note that this constant is finite because the integrand is equivalent to $x^{-\beta-1+2H}$ near 0 and equivalent to $x^{-\beta-3+2H}$ near $+\infty$, both of which are integrable due to the conditions $0 \leq 2\beta < 4H < 4$.

For the term corresponding to M, since there we have $x \geq i \geq 1$, we can use the upper bound (3.11) to state that

$$\begin{aligned} C_M(i,i) &= \int_i^{2i} dx\, g_\beta(x) \int_0^{2i-x} dy\, y^{2H-2} \leq \frac{\beta(\beta+1)}{2H-1} \int_i^{2i} dx\, x^{-\beta-2}(2i-x)^{2H-1} \\ &\leq \frac{\beta(\beta+1)}{2H-1}\, i^{-\beta-1}(2i)^{2H-1}. \end{aligned}$$

When $H < 3/4$, the square of this expression is the general term of a converging series, since the power of i would be $i^{-2\beta-4+4H}$, and since $2\beta > 0$. Therefore, when $H < 3/4$, in the expression

$$\sum_{i=1}^{N} |C(ii)|^2 = \sum_{i=1}^{N} |C_L(ii) + C_M(ii)|^2,$$

which is a diverging series, only the term corresponding to C_L remains in the asymptotic behavior. This proves that when $H < 3/4$,

$$\lim_{N\to\infty} N^{-1} \sum_{i=1}^{N} |C(ii)|^2 = c_L(H,\beta) = \left| \frac{1}{2H-1} \int_0^{+\infty} dx\, g_\beta(x)\; x^{2H-1} \right|^2.$$

When $H > 3/4$, it remains true that $\sum_{i=1}^{N} |C_L(i,i)|^2$ is of order N, and by using a Riemann sum we readily check that $\sum_{i=1}^{N} |C_M(i,i)|^2$ is of order $NN^{-2\beta-4+4H} \ll N$, proving that $\sum_{i=1}^{N} |C(i,i)|^2$ is of order N.

Step 3: the case $i > j$, the term corresponding to the subdomain A. Here, according to the expression for A in Step 1, we must calculate, for $i > j$ fixed and $m = i - j$,

$$C_A(i,j) := \int_0^j dx\, g_\beta(x) \int_{-x}^{x} dy\; |y-m|^{2H-2}. \tag{3.12}$$

This expression will turn out to be of the order m^{2H-2}, and therefore it is convenient to change variables by dividing y by m, in other words

$$C_A(i,j) = m^{2H-1} \int_0^j dx\, g_\beta(x) \int_{-x/m}^{x/m} dy\; |y-1|^{2H-2} =: m^{2H-1} \int_0^j dx\, g_\beta(x)\, f_H(x/m) \tag{3.13}$$

where $f_H(x) := \int_{-x}^{x} dy\; |y-1|^{2H-2}$. This function f_H is increasing, it is equivalent to $2x$ for x small, and to $2x^{2H-1}/(2H-1)$ for x large. More precisely for x small we may write $f_H(x) = 2x + O(x^3)$, which can immediately seen by differentiating f_H thrice, yielding $f'_H(0) = 2$, $f''_H(0) = 0$, and $f'''_H(x) = (2-2H)(3-2H)$. The small and large x behavior of f_H, and its continuity, imply the global bound $f_H(x) \leq K_H x$ for all $x > 0$, for some K_H depending only on H.

As an initial estimate, it is useful to prove that all terms $m^{2-2H} C_A(i,j)$ are bounded below by a positive constant uniformly in $1 \leq j < i \leq N$. To prove this we write

$$\begin{aligned} C_A(i,j) &\geq \int_0^1 dx\, g_\beta(x) \int_{-x}^{x} dy\; |y-m|^{2H-2} \\ &= \int_0^1 dx\, g_\beta(x) \int_{-x}^{x} dy\; (m-y)^{2H-2}. \end{aligned}$$

The last expression above is a constant depending only on H, β when $m = 1$. When $m \geq 2$, we can write

$$C_A(i,j) \geq \int_0^1 dx\ g_\beta(x) \int_{-x}^{x} dy\ \ (m+1)^{2H-2} = (m+1)^{2H-2} \int_0^1 dx\ 2x\ g_\beta(x).$$

Since $m+1 \leq 2m$, our claim follows, i.e.

$$C_A(i,j) \geq K m^{2H-2} \tag{3.14}$$

for some constant $K > 0$ depending only on H and β, and for all $1 \leq j < i \leq N$. Note that K is finite since $x\ g_\beta(x)$ is of order $x^{-\beta+1}$ near 0, which is integrable since $\beta < 2$.

This estimate is important because it implies that, in the series that we are trying to estimate, namely $\sum_{i=1}^{N} \left(\sum_{m=1}^{i-1} |C_A(i,j)|^2 \right)$, the ith term is the general term of a diverging series (if $H < 3/4$, it is asymptotically at least as big as the constant $K^2 \sum_{m=1}^{\infty} m^{4H-4}$; if $H > 3/4$, it is asymptotically of order no less than i^{4H-3}). Therefore, the terms for small i can be ignored, and we may and will assume that i is large.

A similar proof as above also shows that for some constant K' depending only on H, β, for all $1 \leq j < i \leq N$,

$$C_A(i,j) \leq K' m^{2H-2}. \tag{3.15}$$

Indeed, to show this we separate the integral in $C_A(i,j)$ over $x \in [0,1]$ and $x \in [1,j]$. For the first part with $m \geq 2$, we get the upper bound $(m/2)^{2H-2} \int_0^1 dx\, 2x\, g_\beta(x)$, while for the second part, we use the facts that $g_\beta(x) \leq \beta(\beta+1)\, x^{-\beta-2}$, and $\sup_{x \in [1,\infty)} f_H(x)/x < \infty$: from the expression in (3.13), up to a multiplicative constant depending only on H, β, the integral in $C_A(i,j)$ for $x \in [1,j]$ is bounded above by $m^{2H-1} \int_1^\infty dx\ x^{-\beta-2}(x/m) = m^{2H-2} \int_1^\infty dx\ x^{-\beta-1}$. This estimate on $C_A(i,j)$ will be useful below when we attempt to ignore the largest values of m.

We will now prove the following sharper estimate, in the case where i is large, and $m \leq i - \sqrt{i}$, which is the same as $j \geq \sqrt{i}$:

$$C_A(i,j) = m^{2H-2} \int_0^\infty 2x\ g_\beta(x)\ \ dx + o\left(m^{2H-2}\right) =: m^{2H-2} K_A(\beta) + o\left(m^{2H-2}\right). \tag{3.16}$$

Note that the constant $\int_0^\infty 2x\ g_\beta(x)\ \ dx := K_A(\beta)$ is finite by the arguments used to establish the previous two bounds on $C_A(i.j)$. To prove estimate (3.16) we consider the integral in the last expression in (3.13): since $j \geq \sqrt{i}$, a fortiori, $j \geq \sqrt{m}$; therefore we can cut this integral up at the value $x = \sqrt{m}$. This means that for $x \in [0, \sqrt{m}]$, we can use the small-x asymptotics for f_H, to get $f_H(x/m) = 2x/m + \varepsilon_H(x/m)$ where $|\varepsilon_H(z)| \leq K'_H z^3$ for some constant K'_H depending only on H and all $z \in [0,1]$. This yields

$$\begin{aligned} C_A(i,j) &= m^{2H-2} \int_0^{\sqrt{m}} dx\ 2x\ g_\beta(x) + m^{2H-1} \int_0^{\sqrt{m}} dx\ \varepsilon_H(x/m)\ \ g_\beta(x) \\ &\quad + m^{2H-1} \int_{\sqrt{m}}^{j} dx\ g_\beta(x)\, f_H(x/m). \end{aligned}$$

For large m, the first term on the right-hand side above above is equivalent to $K_A(\beta)\, m^{2H-2}$, as announced. We only need to prove the other two terms are of lower orders. For the second term, the upper bound on ε_H and the estimate (3.11) give an upper bound of

$$K_H m^{2H-4} \int_0^{\sqrt{m}} x^{-\beta+1} dx = \frac{K_H}{2-\beta} m^{2H-2-\beta} = o\left(m^{2H-2}\right),$$

as announced. For the third term, from (3.11) and the global bound $f_H(x) \le K'_H x$, we get the upper bound

$$\beta(\beta+1) K_H \; m^{2H-2} \int_{\sqrt{m}}^{\infty} x^{-\beta-1} = (\beta+1) K_H \; m^{2H-2-\beta/2} = o\left(m^{2H-2}\right).$$

This finishes the proof of the estimate (3.16).

This step's purpose is to compute large-N asymptotics for $\sum_{i=1}^{N}\sum_{m=1}^{i-1}|C_A(i,j)|^2$. As explained above, the global lower bound (3.14) proves that when N is large, we can discard all terms where i is not large. Now for i large, we show that we can ignore the terms for $m \ge i - \sqrt{i}$. Indeed, from the global upper bound (3.15),

$$\sum_{m\in[i-\sqrt{i},i-1]} |C_A(i,j)|^2 \le K'^2 \sum_{m\in[i-\sqrt{i},i-1]} m^{4H-4}.$$

When $H < 3/4$, this expression is a $o(1)$ since the series in m converges, which implies that its contribution to the series over $i = 1,\cdots,N$ is $o(N)$. When $H > 3/4$, the series in m diverges, and therefore the displayed expression above is bounded above by $i^{4H-4+1/2}$, with a contribution to the series over $i = 1,\cdots,N$ that is bounded above by $N^{4H-3+1/2} = o\left(N^{4H-2}\right)$. However, the global lower bound (3.14) implies that $\sum_{i=1}^{N}\sum_{m=1}^{i-1}|C_A(i,j)|^2$ is bounded below by the order N when $H < 3/4$ and N^{4H-2} when $H > 3/4$. This proves that to compute this series, we may discard the terms with $m \ge i-\sqrt{i}$, i.e. those with $j \le \sqrt{i}$.

Finally, we can compute the asymptotics of the series in i. The easier case is that of $H > 3/4$. In this case, for large i, using the asymptotics in (3.16), the series $\sum_{m=1}^{i-\sqrt{i}}|C_A(i,j)|^2$ diverges. We can compare it to a Riemann sum by writing, for large i,

$$\begin{aligned}
\sum_{m=1}^{i-\sqrt{i}} |C_A(i,j)|^2 &= (1+o(1))\,|K_A(\beta)|^2\, i^{4H-3}\frac{1}{i}\sum_{m=1}^{i-\sqrt{i}}\left(\frac{m}{i}\right)^{4H-4} \\
&= (1+o(1))\,|K_A(\beta)|^2\, i^{4H-3}\int_0^1 x^{4H-4}dx \\
&= \frac{|K_A(\beta)|^2}{4H-3} i^{4H-3} + o\left(i^{4H-3}\right).
\end{aligned}$$

This term is also the general term of a diverging series, and again using a Riemann sum comparison, we finally get

$$\lim_{N\to\infty} N^{2-4H}\sum_{i=1}^{N}\sum_{m=1}^{i-1}|C_A(i,j)|^2 = \frac{|K_A(\beta)|^2}{(4H-3)(4H-2)}.$$

For the case $H < 3/4$, we know the asymptotics for the series in i will be of order N, and we already proved that the terms for $m \ge i-\sqrt{i}$ can be ignored, but it is not possible to find an explicit asymptotic constant, since the series in m converges, and therefore terms for small m cannot be ignored. However, using similar argument as those leading to (3.16), we can exploit the fact that i is large and $j \ge \sqrt{i}$: returning to the original expression (3.12) for $C_A(i,j)$, we can show that it is legitimate to replace j by ∞ therein. Indeed,

$$C_A(i,j) = \int_0^{\infty} dx\; g_\beta(x)\int_{-x}^{x} dy\;\; |y-m|^{2H-2} - C_A^*(i,j)$$

where, using the change of variable as in (3.13),

$$0 \le C_A^*(i,j) := \int_j^{\infty} dx\; g_\beta(x)\int_{-x}^{x} dy\;\; |y-m|^{2H-2} = m^{2H-1}\int_j^{\infty} dx\; g_\beta(x)\, f_H(x/m).$$

Now using the global estimate $f_H(x) \le K_H x$,

$$\begin{aligned} C_A^*(i,j) &\le K_H m^{2H-2}\int_j^\infty dx\ x^{-\beta-1} = \frac{K_H}{\beta} m^{2H-2}\ j^{-\beta/2} \\ &\le \frac{K_H}{\beta} m^{2H-2}\ i^{-\beta/2}. \end{aligned}$$

Therefore, up to constants depending only on H, β, the contribution from the term $C_A^*(i,j)$ to the series in i will be bounded above by

$$\sum_{i=1}^{N} i^{-\beta}\sum_{m=1}^{i-1} m^{4H-4} \le \sum_{i=1}^{N} i^{-\beta}\sum_{m=1}^{\infty} m^{4H-4}.$$

Depending on the value of β, this is either bounded, or of order $N^{1-\beta}$. In either case, it is negligible compared to $\sum_{i=1}^{N}\sum_{m=1}^{i-1}|C_A(i,j)|^2$. This proves that

$$\sum_{i=1}^{N}\sum_{m=1}^{i-1}|C_A(i,j)|^2 = (1+o(1))\sum_{i=1}^{N}\sum_{m=1}^{i-1}\left|\int_0^\infty dx\ g_\beta(x)\int_{-x}^{x} dy\ \ |y-m|^{2H-2}\right|^2.$$

Note that we also replaced $i-\sqrt{i}$ by i in the sum over m, which is easily checked by showing, similarly to what was done above, that the sum for $m \in [i-\sqrt{i}, i-1]$ does not contribute. Since the summand above does not depend on i, we can make the above expression more precise by changing the order of summation:

$$\begin{aligned} \sum_{i=1}^{N}\sum_{m=1}^{i-1}|C_A(i,j)|^2 &= (1+o(1))\sum_{m=1}^{N-1}(N-m)\left|\int_0^\infty dx\ g_\beta(x)\int_{-x}^{x} dy\ \ |y-m|^{2H-2}\right|^2 \\ &= N\sum_{m=1}^{\infty}\left|\int_0^\infty dx\ g_\beta(x)\int_{-x}^{x} dy\ \ |y-m|^{2H-2}\right|^2 + o(N) \\ &\quad -\sum_{m=1}^{N-1} m\left|\int_0^\infty dx\ g_\beta(x)\int_{-x}^{x} dy\ \ |y-m|^{2H-2}\right|^2. \end{aligned}$$

The last line above is the partial sum of a series whose general term is of order m^{4H-3}. Since $4H-3 > -1$, this series diverges, and using Riemann sums, it is seen to be of order N^{4H-2}. Since $H < 3/4$, this is a $o(N)$. We have proved that in this case,

$$\lim_{N\to\infty} N^{-1}\sum_{i=1}^{N}\sum_{m=1}^{i-1}|C_A(i,j)|^2 = \sum_{m=1}^{\infty}\left|\int_0^\infty dx\ g_\beta(x)\int_{-x}^{x} dy\ \ |y-m|^{2H-2}\right|^2.$$

where the series on the right converges because its general term is of order m^{4H-4}.

Step 4: the case $i > j$, the term corresponding to the subdomain B. Here, according to the expression for B in Step 1, we must calculate, for $i > j$ fixed and $m = i-j$,

$$C_B(i,j) := \int_j^i dx\ g_\beta(x)\int_{x-2j}^{x} dy\ \ |y-m|^{2H-2}.$$

We will find that the sum $\sum_{i=1}^{N}\sum_{j=1}^{i-1}|C_B(i,j)|^2$ is negligible compared to the same expression for A found in the previous step, and thus contributes nothing to the result of the proposition. Since $x \ge j \ge 1$, it is efficient to use the upper bound (3.11) on $g_\beta(x)$ since it is also a lower bound for it, up to a constant. Thus

$$C_B(i,j) \le \beta(\beta+1)\int_j^i dx\ x^{-\beta-2}\int_{x-2j}^{x} dy\ \ |y-m|^{2H-2}.$$

It is convenient to separate the sum into whether or not $j \leq i/2$, in order to properly calculate the integral in y.

For $j \geq i/2$ we have $m = i - j \leq i/2 \leq j \leq x$. On the other hand, $x \leq i \leq i + j$ which implies $m \geq x - 2j$. Thus for any $x \in [j, i]$, we have $m \in [x - 2j, x]$, and the integral in y must be split up at the value m. Moreover, since i and j are commensurate in this case, it is efficient to bound $x^{-\beta-2} \leq j^{-\beta-2}$. Hence

$$\begin{aligned}
\frac{C_B(i,j)}{\beta(\beta+1)} &\leq j^{-\beta-2} \int_j^i dx \left(\int_{x-2j}^m dy \ (m-y)^{2H-2} + \int_m^x dy \ (y-m)^{2H-2} \right) \\
&= \frac{j^{-\beta-2}}{2H-1} \int_j^i dx \ \left((i+j-x)^{2H-1} + (-i+j+x)^{2H-1} \right) dx \\
&= \frac{j^{-\beta-2}}{(2H-1)\,2H} \left(i^{2H} - (2j-i)^{2H} \right).
\end{aligned}$$

Thus, with a constant K depending only on H, β, we have

$$\begin{aligned}
&\sum_{i=1}^{N} \sum_{i/2 \leq j \leq i-1} |C_B(i,j)|^2 \leq K \sum_{i=1}^{N} i^{-2\beta-4} \sum_{i/2 \leq j \leq i-1} \left(i^{2H} - (2j-i)^{2H} \right)^2 \\
= \ & K \sum_{i=1}^{N} i^{-2\beta-4} \ i^{4H+1} \frac{1}{i} \sum_{i/2 \leq j \leq i-1} \left(1 - \left(2\frac{j}{i} - 1 \right)^{2H} \right)^2 \\
= \ & K(1+o(1)) \sum_{i=1}^{N} i^{-2\beta-3+4H} \int_{1/2}^{1} \left(1 - (2-x)^{2H} \right)^2 dx \\
\leq \ & cst \ N^{-2\beta-2+4H}.
\end{aligned}$$

Since $2\beta \geq 1$, we get $-2\beta - 2 + 4H \leq 4H - 3$. Thus, when $H > 3/4$, the sum above is a $o\left(N^{4H-2}\right)$, while when $H < 3/4$, it is a $o(N)$. In both cases, our claim that this sum is negligible compared to the sum with $C_A(i,j)$ is establised.

For $j \leq i/2$, whether or not we have to split the integral in y depends on whether $x \leq m = i = j$. Since in this case, $m \in [i/2, i] \subset [j, i]$, we must always consider the case $x \leq m$, in which case we always have $m \notin [x - 2j, x]$ of course, and we must always consider the case $x \geq m$, in which case we always have $m \in [x - 2j, x]$. Thus, using the upper bound (3.11) on g_β, we end up with

$$\begin{aligned}
\frac{C_B(i,j)}{\beta(\beta+1)} &\leq \int_j^{i-j} dx \ x^{-\beta-2} \left((i+j-x)^{2H-1} - (i-j-x)^{2H-1} \right) \\
&\quad + \int_{i-j}^{i} dx \ x^{-\beta-2} \left((i+j-x)^{2H-1} + (x-(i-j))^{2H-1} \right) \\
&= : C_{B1}(i,j) + C_{B2}(i,j)
\end{aligned} \tag{3.17}$$

We need only prove that for each of the two integrals in (3.17), the contribution to the sum of their squares over $\{1 \leq j \leq i/2, \ 1 \leq i \leq N\}$ is negligible compared to the terms for $C_A(i,j)$.

For the first integral in (3.17), it is efficient to use the mean value theorem to state that $0 \leq (i+j-x)^{2H-1} - (i-j-x)^{2H-1} \leq 2(2H-1) \ j \ (i-j-x)^{2H-2}$. Therefore, we have for a constant K' depending only on H, β,

$$\begin{aligned} C_{B1}(i,j) &\leq K' \, j \int_j^{i-j} dx \, x^{-\beta-2} (i-j-x)^{2H-2} \\ &= K' \, i^{-\beta-3+2H} \, j \int_{j/i}^{1-j/i} dx \, x^{-\beta-2} (1-j/i-x)^{2H-2} \\ &=: K' \, i^{-\beta-3+2H} \, j \, F(j/i), \end{aligned}$$

where we must evaluate the behavior of the function $F(z) := \int_z^{1-z} dx \, x^{-\beta-2} (1-z-x)^{2H-2}$ for $j/i = z \in [0, 1/2]$. Since $H > 1/2$, the term $(1-z-x)^{2H-2}$ is integrable near $x = 1-z$, while the term $x^{-\beta-2}$ is not integrable near $x = 0$; therefore, near 0, the function $F(z)$ behaves like its divergent integral component near 0. We leave further details of the following estimate to the reader: there is a constant K'' depending only on H, β such that for all $z \in (0, 1/2]$

$$F(z) \leq K'' z^{-\beta-1}.$$

This implies that

$$\begin{aligned} &\sum_{i=1}^{N} \sum_{1 \leq j \leq i/2} |C_{B1}(i,j)|^2 \leq (K'K'')^2 \sum_{i=1}^{N} \sum_{1 \leq j \leq i/2} \left[i^{-\beta-3+2H} \, j \, \left(\frac{j}{i}\right)^{-\beta-1} \right]^2 \\ = \quad &(K'K'')^2 \sum_{i=1}^{N} \sum_{1 \leq j \leq i/2} i^{4H-4} \, j^{-2\beta} \leq (K'K'')^2 \left(\sum_{j=1}^{\infty} j^{-2\beta} \right) \sum_{i=1}^{N} i^{4H-4}. \end{aligned}$$

If $H > 3/4$ this is of order $N^{4H-3} = o\left(N^{4H-2}\right)$, and if $H < 3/4$, this is bounded, thus a $o(N)$, as required in both cases.

Finally, for the second integral in (3.17), since $i - j \geq i/2$, it is efficient to bound $x^{-\beta-2}$ above by $(i/2)^{-\beta-2}$, and we get

$$\begin{aligned} C_{B2}(i,j) &\leq (i/2)^{-\beta-2} \int_{i-j}^{i} dx \, \left((i+j-x)^{2H-1} + (x-(i-j))^{2H-1} \right) \\ &= \frac{(i/2)^{-\beta-2}}{2H} (2j)^{2H}. \end{aligned}$$

Consequently, for some contant K''' depending only on H, β, we get

$$\sum_{i=1}^{N} \sum_{1 \leq j \leq i/2} |C_{B1}(i,j)|^2 \leq K''' \sum_{i=1}^{N} \sum_{1 \leq j \leq i/2} i^{-2\beta-4} \, j^{4H} \leq K''' \sum_{i=1}^{N} i^{-2\beta-3+4H}.$$

If $H > 3/4$, this of order $N^{4H-2-2\beta} = o\left(N^{4H-2}\right)$; if $H < 3/4$, this is bounded, thus a $o(N)$, as required in both cases. This finishes the proof that the contribution of $C_B(i,j)$ is negligible compared to that of $C_A(i,j)$.

Step 5: the case $i > j$, the term corresponding to the subdomain C. Here, as in the previous step, we will show that the contribution of $C_C(i,j)$ is negligible compared to that of $C_A(i,j)$, and thus contributes nothing to the result proposition. According to the expression for C in Step 1, we must calculate, for $i > j$ fixed and $m = i - j$,

$$C_C(i,j) := \int_i^{i+j} dx \, g_\beta(x) \int_{2i-x}^{x-2j} dy \, |y-m|^{2H-2}.$$

We first check that for any $x \in [i, i+j]$, we have $2i - x \leq m \leq x - 2j$: indeed both these inequalities are equivalent to $x \leq i + j$. Thus

$$\begin{aligned} C_C(i,j) &:= \int_i^{i+j} dx\ g_\beta(x) \int_{2i-x}^{m} dy\ \ (m-y)^{2H-2} \\ &\quad + \int_i^{i+j} dx\ g_\beta(x) \int_m^{x-2j} dy\ \ (y-m)^{2H-2} \\ &= \frac{2}{2H-1} \int_i^{i+j} dx\ g_\beta(x)\,(i+j-x)^{2H-1}. \end{aligned}$$

Since x is commensurate with i, it is efficient to use the upper bound (3.11) on g_α, and we get

$$C_C(i,j) \leq \frac{2\beta(\beta+1)}{2H-1} i^{-\beta-2} \int_i^{i+j} (i+j-x)^{2H-1}\, dx = \frac{2\beta(\beta+1)}{2H(2H-1)}\ i^{-\beta-2}\ j^{2H}.$$

Thus we have, for some constant K'''' depending only on H, β,

$$\begin{aligned} \sum_{i=1}^{N} \sum_{m=1}^{i-1} |C_C(i,j)|^2 &\leq K'''' \sum_{i=1}^{N} i^{-2\beta-4} \sum_{m=1}^{i-1} j^{4H} \\ &\leq K'''' \sum_{i=1}^{N} i^{-2\beta-3+4H}. \end{aligned}$$

This is the same estimate as in Step 4, which is the announced result.

Step 6: conclusion. Steps 4 and 5 show that the corresponding terms contribute nothing to the asymptotic behavior of $\sum_{i=1}^{N}\sum_{m=1}^{i-1}|C(i,j)|^2$. Therefore, by Step 3, when $H > 3/4$, we get

$$\lim_{N\to\infty} N^{2-4H} \sum_{i=1}^{N}\sum_{m=1}^{i-1} |C(i,j)|^2 = \frac{\left|\int_0^\infty 2x\ g_\beta(x)\ \ dx\right|^2}{(4H-3)(4H-2)},$$

while by Step 2, the sum $\sum_{i=1}^{N}|C(i,i)|^2$ is only of order N, and thus does not contribute in this case; the fact that $C(i,j) = C(j,i)$ and the definition of $T_{3,N}$ now provide the statement of the proposition when $H > 3/4$. When $H < 3/4$, we get from Step 3 that

$$\lim_{N\to\infty} N^{-1} \sum_{i=1}^{N}\sum_{m=1}^{i-1} |C(i,j)|^2 = \sum_{m=1}^{\infty} \left| \int_0^\infty dx\ g_\beta(x) \int_{-x}^{x} dy\ \ |y-m|^{2H-2} \right|^2,$$

which, again by symmetry of C, contributes twice, while the diagonal terms $\sum_{i=1}^{N}|C(i,i)|^2$ contribute the constant identified in Step 2, concluding the proof of the proposition (modulo the second expression for the constant $K_{3,2}$, which is elementary). ■

Remark 3.5. *As we pointed out in Remark 3.2, it is somewhat unexpected to find the same behavior for $T_{3,N}$ as for $T_{1,N}$. We can explain this fact as follows. The summand $T_{3,N}$ involves the increments of the Green kernel with respect to time. However – and this can be seen after a change of variable in the integral appearing in the expression of $C(i,j)$ – it is also related to the increment of a weighted fBm, and this part related with the fBm turns out to be dominant. It does not seem possible to predict the behavior of $T_{3,N}$ without the careful inspection afforded by the proof of Proposition 3.3.*

Proposition 3.6. *If $H > 3/4$ then $\lim_{N\to\infty} N^{2-2\beta} T_{2,N} = K_{2,2}$ where,*

$$K_{2,2} := \frac{64}{(1-\beta)^2} \left(\frac{\left|\beta 2^{-\beta+1} - 1\right|}{(2-\beta)} + \left|2^{-\beta+1} - 1\right| \right)^2$$

for $\beta \neq 1$, while for $\beta = 1$ we have

$$K_{2,2} := 256\,(\ln 2)^2\,.$$

If $H < 3/4$, then $\lim_{N\to\infty} N^{4H-2\beta-1} T_{2,N} = K_{2,1}$ with $h_\beta(x) = x^{-\beta} - (x+1)^{-\beta}$ and

$$K_{2,1} := \sum_{k=-\infty}^{\infty} \left(\int_0^1 du \int_0^\infty dv \left(|v-u+k|^{2H-2} + |v-u-k|^{2H-2} \right) h_\beta(u+v+1) \right)^2$$

Proof: From the definition of $B(i,j)$, we can immediately see that $B(i,j) = B(j,i)$. Therefore, we can assume that $i \geq j$. The proof techniques are similar to what we did in Proposition 3.3. For the sake of conciseness, we will indicate only the proof elements that are substantially different, in order to compute the relevant constants. By using changes of variables, we find

$$B(i,j) = -\iint_{D(j)} h_\beta(x)\,|y-n|^{2H-2}\,dy - \iint_{D(i)} h_\beta(x)\,|y+n-2|^{2H-2}\,dy$$

where $n = i+1-j$, $h_\beta(x) = x^{-\beta} - (x+1)^{-\beta}$, and the domain $D(j)$ is defined as $D(j) = A \cup B \cup C$ where

- $A = \left\{(x,y) \in \mathbf{R}^2 : x \in [0,1],\ y \in [-x,x]\right\}$
- $B = \left\{(x,y) \in \mathbf{R}^2 : x \in [1,j],\ y \in [-x,2-x]\right\}$
- $C = \left\{(x,y) \in \mathbf{R}^2 : x \in [j,j+1],\ y \in [x-2j,2-x]\right\}$.

Note that by convexity h_β is a non-negative function, so that all terms corresponding to these domains contribute positively to $-B(i,j)$; also note that $h_\beta(x) \leq \beta x^{-\beta-1}$ by the mean-value theorem, and that for large x, $h_\beta(x) \simeq \beta x^{-\beta-1}$. Throughout this proof, we will consider contributions to $-B$ rather than B, to deal with positive quantities.

Step 1: the term corresponding to domain C. We will show that this term contributes negligibly for all $H \in (1/2,1)$. We will treat the term corresponding to the domain $D(j)$ only, the one with $D(i)$ being similar. In C, we always have $n \geq 2-x$ since this is equivalent to $x \geq 1+j-i$, which is implies by the lower bound on x defining C. Therefore the term from $B(i,j)$ corresponding to $D(j)$ is

$$B_{C1}(i,j) := \int_j^{j+1} dx\ h_\beta(x) \int_{x-2j}^{2-x} (n-y)^{2H-2}\,dy.$$

Using the mean-value theorem on h_β, and the fact that $n-y \geq n-2+x \geq i-1$ (we can assume $i > 1$ since i can be large), we get

$$\begin{aligned} B_{C1}(i,j) &\leq \beta j^{-\beta-1} \int_j^{j+1} dx \int_{x-2j}^{2-x} (i-1)^{2H-2}\,dy \\ &= \beta j^{-\beta-1}(i-1)^{2H-2} \int_j^{j+1} (2-2x+2j)\,dx \leq 2\beta j^{-\beta-1}(i-1)^{2H-2}\,. \end{aligned}$$

The term $\left|j^{-\beta-1}\right|^2$ is summable. Therefore the contribution of $B_{C1}(i,j)$ to the series in i and j is bounded above by $\left(\sum_{j=1}^\infty \left|j^{-\beta-1}\right|^2\right) \sum_{i=1}^N i^{4H-4}$, which is either bounded (when $H < 3/4$), or bounded above by N^{4H-3} (when $H > 3/4$), so that in both cases, as announced, this term is negligible compared to those which contribute to the results in this Proposition, or indeed Propositions 3.1 and 3.3 .

Step 2: the case $H > 3/4$. In this case, arguing as in the proof of Proposition 3.3, we can show that only the terms with large i will contribute, and moreover that the series

$\sum_{j=1}^{i} |B(i,j)|^2$ is divergent in j for any i, so that we can discard all the terms with j too small or too close to i. In Proposition 3.3 we did this by choosing $\sqrt{i}$ as the threshold level. Here it will be preferable to choose $i^{2/3}$. Therefore, below in this Step 2, we will always assume that i is large and that $j \geq i^{2/3}$and $j \leq i - i^{2/3}$; in particular, this implies that $n := i + 1 - j \geq i^{2/3}$. We will now show that the term corresponding to the domain B, namely

$$\begin{aligned} B_B(i,j) &:= \int_1^j dx\ h_\beta(x) \int_{-x}^{2-x} |n-y|^{2H-2}\, dy + \int_1^i dx\ h_\beta(x) \int_{-x}^{2-x} |n+y-2|^{2H-2}\, dy \\ &=: B_{B1}(i,j) + B_{B2(i,j)} \end{aligned} \tag{3.18}$$

gives a non-zero contribution because for i large and n, j as above, we have

$$B_B(i,j) = (1 + o(1)) \left(4 \lim_{M\to\infty} \int_1^M dx\ h_\beta(x) \right) n^{2H-2}. \tag{3.19}$$

Below, we omit some of the details, and refer the reader instead to the techniques used in the proof of Step 3 of Proposition 3.3.

We consider first the term $B_{B1}(i,j)$ in (3.18). Since $i \geq j+1$, $n \geq 2 \geq 2-x$, there is no cutoff due to the absolute value in $B_{B1}(i,j)$, and we get

$$B_{B1}(i,j) = \int_1^j dx\ h_\beta(x) \int_{-x}^{2-x} (n-y)^{2H-2}\, dy.$$

Only the asymptotics of this term matter, for large n and j. Since $n-y \geq n+x-2 \geq n-1$ is also large, the integral in y is equivalent to $2(n+x)^{2H-2}$ for every $x \in [1,j]$. Similarly, for x large, $h_\beta(x) \simeq \beta x^{-\beta-1}$. Therefore, the integral in x then contains powers of the type $x^{-\beta+2H-3}$, and since $-\beta + 2H - 3 < -1 \iff 2H < 2 + \beta$ which is always true, the integral in x is convergent, and we have

$$B_{B1}(i,j) = (1 + o(1))\, 2 \lim_{M\to\infty} \int_1^M dx\ h_\beta(x)\, (n+x)^{2H-2}.$$

But again, since the limit in M above is finite, and n is large, the terms for $x \geq \sqrt{n}$, say, can be ignored, so that we can replace $(n+x)^{2H-2}$ by n^{2H-2}, yielding

$$B_{B1}(i,j) = (1 + o(1))\ \ 2n^{2H-2} \lim_{M\to\infty} \int_1^M dx\ h_\beta(x). \tag{3.20}$$

For the term $B_{B2}(i,j)$, the computation is considerably more involved because there are several possibilities to consider for the cutoff in the y-integral, but the contribution is actually identical to the case of $\mathrm{B}_{B1}(i,j)$. Specifically, we will show

$$\begin{aligned} B_{B2}(i,j) &:= \int_1^i dx\ h_\beta(x) \int_{-x}^{2-x} |n+y-2|^{2H-2}\, dy \\ &= (1 + o(1))\ \ 2n^{2H-2} \lim_{M\to\infty} \int_1^M dx\ h_\beta(x). \end{aligned} \tag{3.21}$$

For $x < n-2$, we have $|n+y-2| = y - (2-n)$ for all $y \in [-x, 2-x]$. For $x > n$, $|n+y-2| = 2-n-y$ for all $y \in [-x, 2-x]$. For $x \in [n-2, n]$, there is a cutoff to consider, at $y=2-n$. Overall, we get

$$\begin{aligned}
& B_{B2}(i,j) := \int_1^{n-2} dx\ h_\beta(x) \int_{-x}^{2-x} (y-2+n)^{2H-2}\, dy \\
+ & \int_{n-2}^{n} dx\ h_\beta(x) \int_{-x}^{2-n} (2-n-y)^{2H-2}\, dy + \int_{n-2}^{n} dx\ h_\beta(x) \int_{-x}^{2-x} (y-2+n)^{2H-2}\, dy \\
+ & \int_n^i dx\ h_\beta(x) \int_{-x}^{2-x} (2-n-y)^{2H-2}\, dy \\
= & \int_1^{n-2} dx\ \frac{h_\beta(x)}{2H-1} \left((n-x)^{2H-1} - (n-x-2)^{2H-1} \right) + \int_{n-2}^{n} dx\ \frac{h_\beta(x)}{2H-1} n^{2H-1} \\
+ & \int_{n-2}^{n} dx\ \frac{h_\beta(x)}{2H-1} (x+2-n)^{2H-1} + \int_n^i dx\ \frac{h_\beta(x)}{2H-1} \left((x+2-n)^{2H-1} - (x-n)^{2H-1} \right) \\
=: & \ B_{B21}(i,j) + B_{B22}(i,j) + B_{B23}(i,j) + B_{B24}(i,j).
\end{aligned}$$

We first show that the last three terms do not contribute: it is sufficient to show that for large n, they are negligible compared to n^{2H-2}. By the mean-value theorem, we have $B_{B23}(i,j) \le \beta (n-2)^{-\beta-1} n^{2H-1} = o\left(n^{2H-2}\right)$ since $\beta > 0$. We also have $B_{B22}(i,j) \le \frac{\beta(n-2)^{-\beta-1}}{2H-1} \int_{n-2}^{n} dx\ (x+2-n)^{2H-1}$ which equals $\frac{\beta(n-2)^{-\beta-1}}{2H(2H-1)} 2^{2H}$; since $\beta > 0$ and $2H > 1$, we always have $2H-2 > -\beta-1$, so that $B_{B22}(i,j) = o\left(n^{2H-2}\right)$. For the 4th term, things are more delicate. Again by the mean-value theorem, and using a change of variable, we have

$$\begin{aligned}
B_{B24}(i,j) & \le \frac{\beta}{2H-1} \int_n^i x^{-\beta-1} dx\ \left((x+2-n)^{2H-1} - (x-n)^{2H-1} \right) \\
& = \frac{\beta}{2H-1} \int_0^{j-1} (x+n)^{-\beta-1}\, dx\ \left((x+2)^{2H-1} - x^{2H-1} \right) \\
& \le O\left(n^{-\beta-1}\right) + 2\beta \int_0^{j-1} (x+n)^{-\beta-1}\, dx\ x^{2H-2} \\
& \le O\left(n^{-\beta-1}\right) + \mathbf{1}_{j \le i/2} 2\beta n^{-\beta-1} \frac{1+j^{2H-1}}{2H-1} \\
& + \left(\mathbf{1}_{j>i/2} 2\beta \int_0^n (x+n)^{-\beta-1}\, dx\ x^{2H-2} + \int_n^{j-1} (x+n)^{-\beta-1}\, dx\ x^{2H-2} \right) \\
& \le K \mathbf{1}_{j \le i/2}\ n^{-\beta-1}\ j^{2H-1} + K \mathbf{1}_{j>i/2}\ n^{-\beta-3+2H} + K \mathbf{1}_{j>i/2}\ n^{2H-2}\ n^{-\beta}
\end{aligned}$$

for some constant K depending only on β, H, which may change from line to line below. Thus the contribution of $B_{B24}(i,j)$ to the sum over i,j would be bounded above by

$$\begin{aligned}
& K \sum_{i=1}^{N} \sum_{j=1}^{i/2} (i+1-j)^{-2\beta-2}\ j^{4H-2} + K \sum_{i=1}^{N} \sum_{n=1}^{i/2} n^{4H-4-2\beta} \\
= & \ K \sum_{i=1}^{N} i^{4H-3-2\beta} \frac{1}{i} \sum_{j=1}^{i/2} \left(1 + i^{-1} - j/i\right)^{-2\beta-2}\ (j/i)^{4H-2} + NK \left(\sum_{n=1}^{\infty} n^{4H-5} \right) \\
\le & \ K \left(\int_0^{1/2} (1-x)^{-2\beta-2} x^{4H-2} dx \right) N^{4H-2-2\beta} + KN = o\left(N^{4H-2}\right).
\end{aligned}$$

Now that we proved the other three terms do not contribute, let us calculate the contribution of $B_{B21}(i,j)$. We only need its asymptotics for large i,n,j, since we can restrict to $j \ge i^{2/3}$ and $n \ge i^{2/3}$. We will split up the integration over x at the points $i^{1/4}$,

$i^{2/3}/2$, and $n-i^{2/3}/2$, yielding 4 terms. We will show that only the first one contributes. We begin with the others. Here K will denote a constant depending only on H,β which can change from line to line. The first of these three terms is

$$\begin{aligned} B_{B212}(i,j) &:= \int_{i^{1/4}}^{i^{2/3}/2} dx\, \frac{h_\beta(x)}{2H-1}\left((n-x)^{2H-1}-(n-x-2)^{2H-1}\right) \\ &\leq K\int_{i^{1/4}}^{i^{2/3}/2} dx\, x^{-\beta-1}n^{2H-2} \\ &\leq Kn^{2H-2}i^{-\beta/4}. \end{aligned}$$

This term's contribution to the sum would be less than

$$K\sum_{i=1}^{N} i^{-\beta/2}\sum_{n=1}^{i-1} n^{4H-4}$$

which is equivalent to $K\sum_{i=1}^{N} i^{-\beta/2+4H-3}$ which is less than $KN^{4H-2-\beta/2}=o\left(N^{4H-2}\right)$. The next term is

$$\begin{aligned} B_{B213}(i,j) &:= \int_{i^{2/3}/2}^{n-i^{2/3}/2} dx\, \frac{h_\beta(x)}{2H-1}\left((n-x)^{2H-1}-(n-x-2)^{2H-1}\right) \\ &\leq K\, i^{-2\beta/3-2/3}n^{2H-2}. \end{aligned}$$

The contribution to the sum over i,j of this term would be less than $K\sum_{i=1}^{N} i^{(-\beta-1)4/3+4H-3}$ and this series is less than $KN^{4H-3-\beta}=o\left(N^{4H-2}\right)$. Next, we have

$$\begin{aligned} B_{B214}(i,j) &:= \int_{n-i^{2/3}/2}^{n-2} dx\, \frac{h_\beta(x)}{2H-1}\left((n-x)^{2H-1}-(n-x-2)^{2H-1}\right) \\ &\leq K\, n^{-\beta-1}\left(\left(i^{2/3}/2\right)^{2H}-2^{2H}-\left(i^{2/3}/2-2\right)^{2H}\right) \\ &\leq K\, n^{-\beta-1}\, i^{(2H-1)2/3}. \end{aligned}$$

Its contribution to the sum would be $K\sum_{i=1}^{N} i^{(2H-1)4/3}\sum_{n=i^{2/3}}^{i-1} n^{-2\beta-2}$ which is bounded above by $K\sum_{i=1}^{N} i^{(2H-1)2/3}\, i^{(-2\beta-1)2/3}$. For this to be a $o\left(N^{4H-2}\right)$ it is sufficient to have that the power of i in this series is less than $4H-3$, which is equivalent to asking $2H+2\beta>3/2$. This is true since $2H>1$ and $2\beta\geq 1$.

Thus we have proved that for large i and for $n\geq i^{2/3}$ and $j\geq i^{2/3}$, we have

$$B_{B2}(i,j)=(1+o(1))\,B_{B211}(i,j)$$

where

$$B_{B211}(i,j):=\int_{1}^{i^{1/4}} dx\, \frac{h_\beta(x)}{2H-1}\left((n-x)^{2H-1}-(n-x-2)^{2H-1}\right).$$

Since therein, $n-x$ is large, $i^{1/4}$ is large, and $\int_1^{i^{1/4}} dx\, h_\beta(x)$ converges, we have

$$\begin{aligned} B_{B211}(i,j) &= (1+o(1))\,2\int_1^{i^{1/4}} dx\, h_\beta(x)\,(n-x)^{2H-2} \\ &= (1+o(1))\,2\int_1^{\infty} dx\, h_\beta(x)\,(n-x)^{2H-2} \\ &= (1+o(1))\ 2n^{2H-2}\int_1^{\infty} dx\, h_\beta(x) \end{aligned}$$

which is the statement of (3.21). Gathering (3.20) and (3.21) gives us (3.19).

To finish the case $H > 3/4$, we only need to evaluate the contribution of the last remaining term, namely $B_A(i,j) := B_{A1}(i,j) + B_{A2}(i,j)$ where

$$\begin{aligned} B_{A1}(i,j) &:= \int_0^1 dx\ h_\beta(x) \int_{-x}^{x} |n-y|^{2H-2}\,dy, \\ B_{A2}(i,j) &:= \int_0^1 dx\ h_\beta(x) \int_{-x}^{x} |y+n-2|^{2H-2}\,dy. \end{aligned}$$

As invoked repeatedly above, since $H > 3/4$, no individual value of n makes any contribution to $\sum_{i=1}^{N}\sum_{j=1}^{i-1} B(i,j)^2$, and we thus assume that n is large. We begin wtih $B_{A1}(i,j)$: since $0 \le x \le 1 \ll n$, we have

$$B_{A1}(i,j) = \int_0^1 dx\ h_\beta(x) \int_{-x}^{x} (n-y)^{2H-2}\,dy = (1+o(1))\, n^{2H-2} \left(\int_0^1 dx\ 2x\ h_\beta(x) \right).$$

The other term $B_{A2}(i,j)$ behaves identically: indeed, since n is large and x is not, we have

$$B_{A2}(i,j) = \int_0^1 dx\ h_\beta(x) \int_{-x}^{x} (y+n-2)^{2H-2}\,dy = (1+o(1))\, n^{2H-2} \left(\int_0^1 dx\ 2x\ h_\beta(x) \right),$$

yielding from these two estimates that

$$B_A(i,j) = (1+o(1))\ \ 4n^{2H-2} \left(\int_0^1 dx\ x\ h_\beta(x) \right). \tag{3.22}$$

Finally, by combining (3.19) and (3.22), and the fact that $B_C(i,j)$ does not contribute (Step 1), we get

$$\lim_{N\to\infty} \frac{1}{N^{4H-2}} \sum_{i=1}^{N}\sum_{j=1}^{N} B(i,j)^2 = 64 \left(\lim_{M\to\infty} \int_1^M dx\ h_\beta(x) + \int_0^1 dx\ x\ h_\beta(x) \right)^2.$$

To match the formula in the statement of the proposition, we only need to evaluate the integrals above. The second one is equal to $\frac{\left|\beta 2^{-\beta+1}-1\right|}{(2-\beta)|1-\beta|}$ when $\beta \neq 1$ and is equal to $\ln 2$ when $\beta = 1$. The first one has to be computed more carefully. We have for large M, and $\beta \neq 1$,

$$\begin{aligned} \int_1^M dx\ h_\beta(x) &= \frac{1}{1-\beta}\left(M^{-\beta+1} - 1 - (M+1)^{-\beta+1} + 2^{-\beta+1} \right) \\ &= \frac{\left|2^{-\beta+1}-1\right|}{|1-\beta|} + M^{-\beta}\,|1-\beta| + o\left(M^{-\beta}\right), \end{aligned}$$

while for $\beta = 1$ we get

$$\int_1^M dx\ h_\beta(x) = \ln M - \ln(M+1) + \ln 2 = \ln 2 + O\left(M^{-1}\right).$$

The statement of the proposition follows for $H > 3/4$.

Step 3: the case $H < 3/4$.

Here we revert to the technique used in the proof of Proposition 3.1, abandoning the variables (x,y), using (u,v) instead. By formula (3.5)

$$\begin{aligned} B(i,j) &= \int_i^{i+1} du \int_0^j dv |u-v|^{2H-2} \left[(i+j+2-(u+v))^{-\beta} - (i+j+1-(u+v))^{-\beta} \right] \\ &+ \int_0^i du \int_j^{j+1} dv |u-v|^{2H-2} \left[(i+j+2-(u+v))^{-\beta} - (i+j+1-(u+v))^{-\beta} \right]. \end{aligned}$$

We can see that both terms above are negative, and since we are ultimately only interested in the square of this term, we abuse notation by changing the entire sign of $B(i,j)$. Now, using the change of variables $\tilde{u} = i - u, \tilde{v} = j - v$, we get

$$\begin{aligned} -B(i,j) &= \int_0^1 du \int_0^j dv |i-j-u+v|^{2H-2} h_\beta (u+v+1) \\ &\quad + \int_0^i du \int_0^1 dv |i-j-u+v|^{2H-2} h_\beta (u+v+1). \end{aligned} \tag{3.23}$$

We only need to prove that $\frac{1}{N}\sum_{i,j=0}^{N-1} B(i,j)^2$ converges to a non-zero finite constant. We will show below that this series is identical to the same series where in (3.23) the integrals $\int_0^j dv$ and $\int_0^i du$ are replaced by $\int_0^\infty dv$ and $\int_0^\infty du$; this claim is proved in (3.25) below. Assuming this is true, switching the names of the letters (u,v) shows that we only need to evaluate $\frac{1}{N}\sum_{i,j=0}^{N-1} \left(B'(i-j) + B'(j-i)\right)^2$ where for every $k \in \mathbf{Z}$ we define

$$B'(k) := \int_0^1 du \int_0^\infty dv |k-u+v|^{2H-2} h_\beta (u+v+1).$$

Thus (using Claim (3.25) below), by changing variables in the sum, we get

$$\frac{1}{N}\sum_{i,j=0}^{N-1} B(i,j)^2 = \frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=-j}^{N-j-1} \left(B'(k) + B'(-k)\right)^2.$$

Switching the order of summation, we see that for each k from $-(N-1)$ to $N-1$, there are always exactly N corresponding values of j. Thus

$$\begin{aligned} \frac{1}{N}\sum_{i,j=0}^{N-1} B(i,j)^2 &= \sum_{k=-N+1}^{N-1} \left(B'(k) + B'(-k)\right)^2 \\ &= 2\left|B'(0)\right|^2 + 2\sum_{k=1}^{N-1} \left(B'(k) + B'(-k)\right)^2. \end{aligned}$$

Modulo Claim (3.25), since $B'(k)$ are positive for all values of $k \in \mathbf{Z}$, we will have proved the proposition as soon as we can show that

$$\sum_{k=0}^{\infty} \left(\left|B'(k)\right|^2 + \left|B'(-k)\right|^2\right) < \infty. \tag{3.24}$$

Let us show (3.24) first. We leave it to the reader to check that every term in both series is actually finite, which follows from the fact that $-\beta - 1 < -1$ and $2H - 2 > -1$. Then, for both terms, for $|k|$ large enough, we can bound the integral in $B'(k)$ for $(u,v) \in [0,1] \times [0,2]$ above by $2|k|^{2H-2} \int_0^1 du \int_0^2 dv \,(u+v+1)^{-\beta} = c_\beta |k|^{2H-2}$, where c_β depends only on β. Since $H < 3/4$, we get $|k|^{2H-2}$ square-summable. Thus it is enough to find a square-summable upper bound for

$$B''(k) + B''(-k) := \int_0^1 du \int_2^\infty dv \left(|k-u+v|^{2H-2} + |-k-u+v|^{2H-2}\right) h_\beta (u+v+1).$$

As it turns out, the square-summability of $B''(k)$ is easier than for $B''(-k)$. We can use the fact that by the mean-value theorem, since $\beta > 0$, $h_\beta(u+v+1) \le \beta (u+v+1)^{-\beta-1}$. Thus for $k \ge 1$, since $v - u \ge 1$, and $u \ge 0$,

$$B''(k) \le k^{2H-2} \beta \int_2^\infty dv \,(v+1)^{-\beta-1}.$$

There, the integral converges, and the term k^{2H-2} is square-summable. Now for the harder term $B''(-k)$, we must split the integral up into several pieces because for v near k, a (integrable) singularity occurs, and other complications arise. For $k \geq 4$, we will denote by c a positive finite constant depending on H and β which can change from line to line; we consider:

- $v \in [2, k/2]$: here $|-k+v-u| \geq ck^{2H-2}$ and we get
$$\begin{aligned} & \int_0^1 du \int_2^{k/2} dv |-k-u+v|^{2H-2} h_\beta(u+v+1) \\ \leq \quad & ck^{2H-2} \int_2^\infty (v+1)^{-\beta-1} dv = ck^{2H-2}, \end{aligned}$$
which is square-summable in k;
- $v \in [k/2, k]$: in this case, $|-k-u+v| = k-v+u \geq k-v \geq 0$ and $v+1 \geq ck$ so that
$$\begin{aligned} & \int_0^1 du \int_{k/2}^{k} dv |-k-u+v|^{2H-2} h_\beta(u+v+1) \\ \leq \quad & ck^{-\beta-1} \int_{k/2}^{k} dv\, (k-v)^{2H-2} = ck^{-\beta-1}k^{2H-1}; \end{aligned}$$
this is square-summable if and only if $2(-\beta-2+2H) < -1$, i.e. $2\beta > 4H-3$, which is always true for $H < 3/4$ since $\beta > 0$;
- $v \in [k, k+2]$: here
$$\begin{aligned} & \int_0^1 du \int_{k}^{k+2} dv |-k-u+v|^{2H-2} h_\beta(u+v+1) \\ \leq \quad & ck^{-\beta-1} \int_0^1 du \int_0^2 dv\, |v-u|^{2H-2} = ck^{-\beta-1} \end{aligned}$$
which is square-summable since $\beta > 0$;
- $v \in [k+2, 2k]$: in this case, we have $v-k \geq 2$ and $u \leq 1$ so that $v-k-u \geq (v-k)/2$ and we get
$$\begin{aligned} & \int_0^1 du \int_{k+2}^{2k} dv |-k-u+v|^{2H-2} h_\beta(u+v+1) \\ \leq \quad & c \int_0^1 du \int_{k+2}^{2k} dv\, (v-k)^{2H-2} (v+1)^{-\beta-1} \leq ck^{-\beta-1}k^{2H-1} \end{aligned}$$
which is the same behavior as the case $v \in [k, k+2]$ above, thus square-summable;
- $v \in [2k, \infty)$ finally: we obtain $v-k \geq v/2 \geq k$ and thus
$$\begin{aligned} & \int_0^1 du \int_{2k}^{\infty} dv |-k-u+v|^{2H-2} h_\beta(u+v+1) \\ \leq \quad & ck^{2H-2} \int_{2k}^\infty dv\ v^{-\beta-1} = ck^{2H-2-\beta}, \end{aligned}$$
which is square-summable since $4H-4 < -1$ and $-\beta < 0$. The five bullet points above establish that $B''(-k)$ is square-summable.

Finally, we are left to prove that the discrepancy between $B(i,j)$ and $B'(i-j) + B'(j-i)$ is negligible. In other words, it is sufficient to show that
$$\lim_{N\to\infty} \frac{1}{N} \sum_{i,j=0}^{N-1} (B'''(i,j))^2 = 0 \tag{3.25}$$

where

$$\begin{aligned} B'''(i,j) &:= B_1'''(i,j) + B_2'''(i,j) \\ &:= \int_0^1 du \int_j^\infty dv |i-j-(u-v)|^{2H-2} h_\beta(u+v+1) \\ &\quad + \int_0^1 dv \int_i^\infty du |i-j-(u-v)|^{2H-2} h_\beta(u+v+1). \end{aligned}$$

For conciseness, we will show this for the first part of B'''only, since the second part is treated with an identical argument. Also, we include only the calculation for $i,j \geq 2$, the remaining terms being straightforward to control. Using the bound $i-j+v-u \geq i-j+j-1 = i-1$, we obtain

$$\begin{aligned} \frac{1}{N}\sum_{i,j=2}^{N-1} \left(B_1''''(i,j)\right)^2 &\leq \frac{1}{N}\sum_{i,j=2}^{N-1} (i-1)^{4H-4} \left(\int_0^1 du \int_j^\infty dv h_\beta(u+v+1)\right)^2 \\ &\leq \frac{1}{N}\sum_{i=2}^{N-1} (i-1)^{4H-4} \beta \sum_{j=2}^{\infty} \left(\int_0^1 du \int_j^\infty dv (u+v+1)^{-\beta-1}\right)^2 \\ &\leq \left(\sum_{i=1}^{\infty} i^{4H-4}\right) \beta \frac{1}{N} \sum_{j=1}^{N} j^{-\beta}. \end{aligned}$$

The infinite series in the last line above converges since $H < 3/4$. To finish the proof of (3.25) and thus of the proposition, it is sufficient to note that if $\sum_{j=1}^\infty j^{-\beta}$ converges the term $1/N$ causes the whole expression above to go to 0, while if $\sum_{j=1}^\infty j^{-\beta}$ diverges, this could be because $\beta = 1$, in which case $\frac{1}{N}\sum_{j=1}^N j^{-\beta}$ is of order $N^{-1}\log N$, which goes to 0, while if $\beta < 1$, $\frac{1}{N}\sum_{j=1}^N j^{-\beta}$ can be rescaled into a Riemann sum for the integral $\int_0^1 x^{-\beta}dx$, and is thus of order $N^{-\beta}$. In all cases, since $\beta > 0$, claim (3.25) is established, and so is the proposition. ■

Since the terms $T_{1,N}, T_{2,N}, T_{3,N}$ have all the same order of convergence, the "mixed" terms $T_{4,N}$, $T_{5,N}$, and $T_{6,N}$ will also contribute to the asymptotic behavior of $\mathbf{E}V_N^2$. We give their limits in the next result.

Proposition 3.7. *Let $\beta = \frac{d-\alpha}{2}$ and $g_\beta(x) := x^{-\beta} - 2(x+1)^{-\beta} + (x+2)^{-\beta}$ and $h_\beta(x) = x^{-\beta} - (x+1)^{-\beta}$. For every $k \in \mathbf{Z}$, define the following constants which are positive:*

$$\begin{aligned} K_A(k) &:= \int_0^1 du \int_0^1 dv |u-v+k|^{2H-2} (u+v)^{-\beta} \\ K_B(k) &:= \int_0^1 du \int_0^\infty dv \left(|v-u+k|^{2H-2} + |v-u-k|^{2H-2}\right) h_\beta(u+v+1) \\ K_C(k) &:= \int_0^\infty dx\, g_\beta(x) \int_{-x}^x dy\ |y-k|^{2H-2} \end{aligned}$$

For every $H < \frac{3}{4}$

$$\lim_{N\to\infty} N^{4H-2\beta-1} T_{4,N} = K_{4,1}, \quad \lim_{N\to\infty} N^{4H-2\beta-1} T_{5,N} = K_{5,1}, \quad \lim_{N\to\infty} N^{4H-2\beta-1} T_{6,N} = K_{6,1}$$

where

$$K_{4,1} := -2\sum_{k=-\infty}^{\infty} K_A(k) K_B(k), \; K_{5,1} := 2\sum_{k=-\infty}^{\infty} K_A(k) K_C(k),$$
$$K_{6,1} := -2\sum_{k=-\infty}^{\infty} K_B(k) K_C(k).$$

For every $H > \frac{3}{4}$

$$\lim_{N\to\infty} N^{2-2\beta}T_{4,N} = K_{4,2},\ \lim_{N\to\infty} N^{2-2\beta}T_{5,N} = K_{5,2},\ \lim_{N\to\infty} N^{2-2\beta}T_{6,N} = K_{6,2}.$$

where, with the constants given in Propositions 3.1, 3.3, and 3.6, we define

$$K_{4,2} := -2\sqrt{K_{1,2}K_{2,2}},\ K_{5,2} := 2\sqrt{K_{1,2}K_{3,2}},\ K_{6,2} := -2\sqrt{K_{2,2}K_{3,2}}.$$

Proof: The proof follows from the computations contained in the proofs of Propositions 3.1, 3.3 and 3.6.

For instance, when $H < \frac{3}{4}$, the term $N^{4H-2\beta-1}T_{4,N} = 2N^{4H-2\beta-1}\sum_{i,j=0}^{N-1} A_{i,j}B_{i,j}$ has the same limit as the sequence

$$\frac{2}{N}\sum_{i,j=0}^{N-1}\left(\int_0^1 du\int_0^1 dv|u-v+k|^{2H-2}(u+v+2)^{-\beta}\right)$$
$$\times\left(\int_0^1 du\int_0^\infty dv\left(\left|v-u+k\right|^{2H-2}+\left|v-u-k\right|^{2H-2}\right)\left(-h_\beta(u+v+2)^{-\beta}\right)\right).$$

In fact, $N^{4H-2\beta-1}T_{4,N}$ can be written as the above sequence plus several remainder terms that converges to zero, by using the arguments in the proofs of Proposition 3.1 and 3.6. Also note that the above series converges to $K_{4,1}$, which is finite by Propositions 3.1 and 3.6, and Schwarz's inequality.

Similarly, when $H > \frac{3}{4}$, for instance, the sequence $N^{2-2\beta}T_{4,N}$ is equivalent as $N\to\infty$ (see (3.8), (3.19) and (3.22)) to $2\sum_{i,j=1;i\neq j}^{N-1}|i-j|^{2H-2}\left(\int_0^1\int_0^1 dudv(u+v+2)^{-\beta}\right)4|i-j|^{2H-2}(\lim_{M\to\infty}\int_1^M dxh_\beta(x)+\int_0^1 dxh_\beta(x))$ where h_β is defined in the proof of Proposition 3.6. ■

Theorem 3.8. *Let $K_{i,2} : i = 1,2,3$ be as defined in Propositions 3.1, 3.3, and 3.6, and $K_A(k), K_B(k), K_B(k)$ be as in Proposition 3.7. For $H < \frac{3}{4}$,*

$$\lim_{N\to\infty} N^{4H-2\beta-1}\mathbf{E}V_N^2 = 2D^2\sum_{k=-\infty}^{\infty}\left(K_A(k)-K_B(k)+K_C(k)\right)^2 =: K_1$$

and for $H > \frac{3}{4}$,

$$\lim_{N\to\infty} N^{2-2\beta}\mathbf{E}V_N^2 = 2D^2\left(\sqrt{K_{1,2}}-\sqrt{K_{2,2}}+\sqrt{K_{3,2}}\right)^2 =: K_2.$$

Proof: This is a direct consequence of the results stated in Propositions 3.1, 3.3, 3.6, and 3.7 and of the decomposition (3.7). ■

Remark 3.9. *Since Theorem 3.8 features sums of two positive and one negative constants, one cannot immediately rule out the possibility that the sum might be zero. In the case of $H > 3/4$, the expression $\sqrt{K_{1,2}}-\sqrt{K_{2,2}}+\sqrt{K_{3,2}}$ is a sum of ratios of algebraic C^1 functions of H and β; the set of pairs (H,β) for which it is zero is thus a one-dimensional manifold, and one can thus argue that models with the undesirable property that K_2 is equal or close to 0 represent unlikely parameter choices. In the sequel, when $H > 3/4$, we will assume that H,β are chosen such that $K_2 \neq 0$, in other words*

$$\left(\frac{2}{(4H-3)(4H-2)}\right)^{1/2}\frac{4\left|2^{2-\beta}-2\right|}{2-\beta}\neq 8\left(\frac{\left|\beta 2^{-\beta+1}-1\right|}{(2-\beta)}+\left|2^{-\beta+1}-1\right|\right). \tag{3.26}$$

Remark 3.10.

- *In the case of $H < 3/4$, the possibility of having to deal with a model with K_1 close to 0 is even more remote, since it would require being close to every (H, β)-manifold solution of $K_A(k) - K_B(k) + K_C(k) = 0$ for every $k \in \mathbf{Z}$. In particular, when $H < 3/4$, one can check that there is no values of H, β such that $K_1 = 0$.*
- *The non-trivial computations performed in this section are due to the fact that the increments of the process $(u(t,x), t \in [0,T])$ given by (2.3) are not stationary with respect to the time variable. An interesting question is whether, by changing the initial condition in the heat equation (1.1), one may find a solution with stationary increments in time. In the case of the white-noise in time (which corresponds to the case $H = 1/2$ for us), it was proved in [20] that, with a suitable intial condition in the heat equation, at the origin $x = 0$ in space, the solution $(u(t,0), t \in [0,T])$ has stationary increments in the time parameter. Moreover, this solution is still self-similar, and consequently it is a fBm, so that all the known results about quadratic variations of fBm apply directly (see [38]). Note that the stationarity of the increments in [20] is obtained only for the particular value $x = 0$ and one loses the fact that the covariance $\mathbf{E}u(t,x)u(s,x)$ is constant with respect to x; thus the random field in [20] is not spatially homogeneous (translation-invariant in law). This example is in sharp contrast with our solutions, which are spatially homogeneous, and do not have stationary increments in time, and for which, consequently, the quadratic variation do not follow the same behavior as for fBm, despite their self-similarity. This comparison of our results with those for the solution in [20] thus provides further insight into how sensitive quadratic variations are to non-stationarity of increments in time.*

4 Normal convergence for $H < \frac{3}{4}$

Consider the sequence $\tilde{V}_N := \sum_{i=0}^{N-1}\left[\frac{(U_{t_{i+1}}-U_{t_i})^2}{\mathbf{E}(U_{t_{i+1}}-U_{t_i})^2} - 1\right]$. Using the behavior of the increments of the process (U_t) (property (3.2)) and Theorem 3.8, we notice that when $H < 3/4$, $\mathbf{E}\left(N^{-1/2}\tilde{V}_N\right)^2$ is bounded above and below by positive constants. This suggests that $\tilde{V}_N$ might converge to a Gaussian distribution. However, as explained at the start of Section 3, property (3.2) also indicates that $U.$ has the same regularity as fBm with parameter $H - \beta/2$. If we were to draw an analogy with the quadratic variation of fBm, as in [7], we would then suspect that we should obtain normal convergence for $\tilde{V}_N$ or other equivalently normalized versions of V_N for all $H < 3/4 + \beta/2$. In this section we show normal convergence holds for $H < 3/4$, and in Section 5, we show that it does not hold for $H > 3/4$. Therefore, the regularity estimates do not provide the right insight into the behavior of U's quadratic variation. See the start of Section 5 for further discussion of the inability of property (3.2) to properly predict the behavior of U's quadratic variation.

We will show asymptotic normality for the normalization of V_N and the asymptotic constant K_1 which were identified in Theorem 3.8. Let

$$F_N := K_1^{-\frac{1}{2}} N^{2H-\beta-\frac{1}{2}} V_N. \tag{4.1}$$

From Theorem 3.8 it follows that $\lim_{N\to\infty} \mathbf{E}F_N^2 = 1$. We prove that F_N converges in law to the standard normal law. Our approach is based on Stein's method combined with the Malliavin calculus, as developed by I. Nourdin and G. Peccati, see the recent book [24]. Below we use the letter $\mathcal{D}$ to denote one of several metrics on the space of probability measures on $\mathbf{R}$, including the Kolmogorov, Wasserstein, and Total Variation metrics

d_{Kol}, d_W and d_{TV} respectively. We also abuse notation by using random variables, rather than their laws, as arguments for these metrics. For instance, $d_{Kol}(X,Y) := \sup_{z\in\mathbf{R}} |\mathbf{P}[X \le z] - \mathbf{P}[Y \le z]|$, and it is known that d_{Kol} metrizes certain convergences in distribution: if F has a cummulative distribution function that is continuous, then F_N converges to F in distribution if and only if $\lim_{N\to\infty} d_{Kol}(F_N, F) = 0$. See [24, Appendix C] for other definitions and properties. The following theorem is a consequence of [26]; see [24, Theorem 5.2.6].

Theorem 4.1. *Let $I_q(f)$ be a multiple integral of order $q \ge 1$. Assume $\mathbf{E}\left[I_q(f)^2\right] = \sigma^2$. Then*

$$\mathcal{D}\left(I_q(f), N(0,\sigma^2)\right) \le \frac{c}{q^2}\left(\mathrm{Var}\left[\frac{1}{q}\|DI_q(f)\|^2_{L^2([0,1])}\right]\right)^{\frac{1}{2}},$$

where $c = 1/\sigma^2$ when $\mathcal{D} = d_{Kol}$, and $c = 1/\sigma$ when $\mathcal{D} = d_W$, and finally $c = 2/\sigma^2$ for $\mathcal{D} = d_{TV}$.

We can apply this theorem to F_N given by (4.1) since it is a multiple integral of order 2, obtaining the following normal convergence result.

Theorem 4.2. *For $H < \frac{3}{4}$, F_N converges in law to a standard normal law. More precisely, for $\mathcal{D} = d_W, d_{Kol}$, or d_W,*

$$\lim_{N\to\infty} \mathcal{D}(F_N, N(0,1))^2 = 0.$$

Proof:

Step 1: computing the variance of the norm of the Malliavin derivative. We start by computing the Malliavin derivative of the second-chaos variable F_N and then we evaluate its norm. We have, for every s

$$D_s F_N = 2K_1^{-\frac{1}{2}} N^{2H-\beta-\frac{1}{2}} \sum_{i=0}^{N-1} I_1(1_{(\frac{i}{N},\frac{i+1}{N})})1_{(\frac{i}{N},\frac{i+1}{N})}(s)$$

and

$$\frac{1}{2}\|DF_N\|^2_{L^2([0,1])} = \frac{2}{K_1} N^{4H-2\beta-1} \sum_{i,j=0}^{N-1} I_1(1_{(\frac{i}{N},\frac{i+1}{N})}) I_1(1_{(\frac{j}{N},\frac{j+1}{N})}) \langle 1_{(\frac{i}{N},\frac{i+1}{N})}, 1_{(\frac{j}{N},\frac{j+1}{N})}\rangle_{L^2([0,1])}.$$

Note that, by the product formula (8.3), the variance of the double sum in this expression is

$$\mathcal{V}_N := \mathbf{E}\left[\left(\sum_{i,j=0}^{N-1} \langle 1_{(\frac{i}{N},\frac{i+1}{N})}, 1_{(\frac{j}{N},\frac{j+1}{N})}\rangle_{L^2([0,1])} I_2\left((1_{(\frac{i}{N},\frac{i+1}{N})} \otimes 1_{(\frac{j}{N},\frac{j+1}{N})}\right)\right)^2\right]. \tag{4.2}$$

Step 2: applying the comparison theorems. With the variance expression in the last step, using Theorem 4.1 with $q = 2$ and $\sigma^2 = \sigma_N^2 := \mathbf{E}\left[F_N^2\right]$, with $\mathcal{D}$ any of the metrics d_W, d_{Kol}, or d_W, we get

$$\begin{aligned} &\mathcal{D}(F_N, N(0,\sigma_N))^2 \\ \le\ & 4K_1^{-2} N^{8H-4\beta-2} \frac{2}{\min(\sigma_N, \sigma_N^2)} \mathcal{V}_N. \end{aligned} \tag{4.3}$$

Before analyzing this last expression, we note that we can use the triangle inequality for $\mathcal{D}$ to state that

$$\begin{aligned}\mathcal{D}(F_N, N(0,1)) &\leq \mathcal{D}(F_N, N(0,\sigma_N)) + \mathcal{D}(N(0,1), N(0,\sigma_N)) \\ &\leq \mathcal{D}(F_N, N(0,\sigma_N)) + \frac{2\left|\sigma_N^2 - 1\right|}{\min\left(1,\sigma_N^2\right)}\end{aligned}$$

where the second inequality is comes from well-known comparisons of Normal laws (see [24, Proposition 3.6.1]). By Theorem 3.8, $\lim_{N\to\infty}\sigma_N = 1$, and therefore it is sufficient for us to prove that $\mathcal{D}(F_N, N(0,\sigma_N))$, or *a fortiori* that the expression in (4.3), converges to 0 as $N\to\infty$. Since $\lim_{N\to\infty}\min\left(\sigma_N,\sigma_N^2\right) = 1$, we only need to show that

$$\lim_{N\to\infty} N^{8H-4\beta-2}\mathcal{V}_N = 0$$

Step 3: setting up the main quantitative estimate. Using the isometry property for multiple integrals, we compute the expected square $\mathcal{V}_N$ used in (4.3). With c denoting a generic strictly positive constant, we have

$$\begin{aligned}\mathcal{V}_N &= 2\sum_{i,j,i',j'=0}^{N-1}\left\langle 1_{\left(\frac{i}{N},\frac{i+1}{N}\right)}, 1_{\left(\frac{j}{N},\frac{j+1}{N}\right)}\right\rangle_{L^2([0,1])}\left\langle 1_{\left(\frac{i'}{N},\frac{i'+1}{N}\right)}, 1_{\left(\frac{j'}{N},\frac{j'+1}{N}\right)}\right\rangle_{L^2([0,1])} \\ &\quad \left\langle 1_{\left(\frac{i}{N},\frac{i+1}{N}\right)}\otimes 1_{\left(\frac{j}{N},\frac{j+1}{N}\right)}, 1_{\left(\frac{i'}{N},\frac{i'+1}{N}\right)}\otimes 1_{\left(\frac{j'}{N},\frac{j'+1}{N}\right)}\right\rangle \\ &= c\sum_{i,j,i',j'=0}^{N-1}\left\langle 1_{\left(\frac{i}{N},\frac{i+1}{N}\right)}, 1_{\left(\frac{j}{N},\frac{j+1}{N}\right)}\right\rangle_{L^2([0,1])}\left\langle 1_{\left(\frac{i'}{N},\frac{i'+1}{N}\right)}, 1_{\left(\frac{j'}{N},\frac{j'+1}{N}\right)}\right\rangle_{L^2([0,1])} \\ &\quad \left\langle 1_{\left(\frac{i}{N},\frac{i+1}{N}\right)}, 1_{\left(\frac{i'}{N},\frac{i'+1}{N}\right)}\right\rangle_{L^2([0,1])}\left\langle 1_{\left(\frac{j}{N},\frac{j+1}{N}\right)}, 1_{\left(\frac{j'}{N},\frac{j'+1}{N}\right)}\right\rangle_{L^2([0,1])}.\end{aligned}$$

We recognize from the proofs of Propositions 3.1, 3.6, 3.3 and 3.7 that each term above of the form

$$\left\langle 1_{\left(\frac{i}{N},\frac{i+1}{N}\right)}, 1_{\left(\frac{j}{N},\frac{j+1}{N}\right)}\right\rangle_{L^2([0,1])}$$

is equal to $N^{-2H+\beta}(A(i,j)+B(i,j)+C(i,j))$, where these three terms are given by (3.4), (3.5), (3.6)). We will now show that for large $|k|$ where $k := i-j$, $A(i,j)$, $-B(i,j)$, and $C(i,j)$ are all bounded above by multiples of $|i-j|^{2H-2}$ (recall that these terms are all positive). This will imply that the last series above can be compared to a Riemann integral of order 4 on $[0,1]^4$, which implies that the contributions of the terms with small k can be ignored, since they would lead to integrals over sets of Lebesgue measure 0.

Next, to estimate $A(i,j)$, $-B(i,j)$, and $C(i,j)$, a number of the computations are nearly identical to those in Propositions 3.3, and 3.6, and we will not repeat them. We simply state that for large $|k| = |i-j|$, using these propositions' proofs, we find asymtotic equivalents to these terms by taking the square root of the corresponding terms in the series expansions of the asymptotic constants in the two aforementioned propositions. In other words we have, we have that for $|k|$ large (and noting that for $A(i,j)$ there is no approximation),

$$\begin{aligned}A(i,j) &= \int_0^1 du\int_0^1 dv|u-v+k|^{2H-2}(u+v)^{-\beta} \\ -B(i,j) &= (1+o(1))\int_0^1 du\int_0^\infty dv\left(|v-u+k|^{2H-2}+|v-u-k|^{2H-2}\right)h_\beta(u+v), \\ C(i,j) &= (1+o(1))\int_0^\infty dx\, g_\beta(x)\int_{-x}^{x} dy\ |y-k|^{2H-2}, \qquad (4.4)\end{aligned}$$

where we recall that $h_\beta(x) := (x+1)^{-\beta} - (x+2)^{-\beta}$ and $g_\beta(x) = x^{-\beta} - 2(x+1)^{-\beta} + (x+2)^{-2\beta}$.

Step 4: controlling $A(i,j)$*,* $-B(i,j)$*, and* $C(i,j)$*.* In the computations in this step, c denotes any positive finite constant depending only on H and β. Moreover, the case $\beta = 1$ requires special computations because of the occurence of logarithms, which we leave to the reader, since this case falls in the middle of our possible range $(0, 2H)$, and thus does not create any real difficulties. Thus we assume $\beta \neq 1$. To avoid abusing notation, we use the letters B_∞, and C_∞ for the asymptotic expressions at the end of the previous step.

The term $A(i,j)$ is the easiest to control. Since $|u-v| \leq 1$ in the integral in $A(i,j)$, for $|k| \geq 2$, we have

$$A(i,j) \leq \int_0^1 du \int_0^1 dv |k/2|^{2H-2} (u+v)^{-\beta} = |k/2|^{2H-2} \int_0^1 \int_0^1 du dv (u+v)^{-\beta} = c|k|^{2H-2}. \tag{4.5}$$

For C_∞, it is convenient to use Fubini first, yielding

$$C_\infty(i,j) = \int_0^\infty dy \left(|y-k|^{2H-2} + |y+k|^{2H-2} \right) \int_y^\infty g_\beta(x)\, dx.$$

Using the mean-value theorem, for y large, we find $\int_y^\infty g_\beta(x)\, dx = (1+o(1))\, \beta y^{-\beta-1}$; in particular, for $y \geq 1$, we get $\int_y^\infty g_\beta(x)\, dx \leq c y^{-\beta-1}$. For $y \leq 1$, for $|k| \geq 2$, we get that $|y-k|^{2H-2} + |y+k|^{2H-2}$ is bounded above by $2|k/2|^{2H-2}$. Thus, separating the integral in C_∞ over $y \in [0,1]$ and $y \in [1,\infty)$, we obtain

$$\begin{aligned} C_\infty(i,j) &\leq 2|k/2|^{2H-2} \int_0^\infty \left(\int_y^\infty g_\beta(x)\, dx \right) dy \\ &\quad + c \int_1^\infty y^{-\beta-1} dy \left(|y-k|^{2H-2} + |y+k|^{2H-2} \right) \\ &\leq c|k|^{2H-2} + 2c \int_1^\infty y^{-\beta-1} |-|k|+y|^{2H-2} dy, \end{aligned} \tag{4.6}$$

where we noted that $y \mapsto \int_y^\infty g_\beta(x)\, dx$ is integrable over $[0,\infty)$ because for large y it is equivalent to $\beta y^{-\beta-1}$ and for y near 0, its explicit expression is only singular in the term $y^{1-\beta}/(\beta-1)$, which is integrable since $\beta < 2H < 2$. For the last integral in (4.6), we must separate the integration into the four intervals $[1, |k|/2]$, $[|k|/2, |k|]$, $[|k|, 2|k|]$, and $[2|k|, \infty)$. We have

$$\begin{aligned} \int_1^{|k|/2} y^{-\beta-1} |-|k|+y|^{2H-2} dy &\leq c|k|^{2H-2} \int_1^{|k|/2} y^{-\beta-1} dy = c|k|^{2H-2}; \\ \int_{|k|/2}^{|k|} y^{-\beta-1} |-|k|+y|^{2H-2} dy &\leq c|k|^{-\beta-1} \int_0^{|k|/2} y^{2H-2} dy = c|k|^{2H-2-\beta}; \\ \int_{|k|}^{2|k|} y^{-\beta-1} |-|k|+y|^{2H-2} dy &\leq c|k|^{-\beta-1} \int_0^{|k|} y^{2H-2} dy = c|k|^{2H-2-\beta}; \\ \int_{2|k|}^{\infty} y^{-\beta-1} |-|k|+y|^{2H-2} dy &\leq c|k|^{2H-2} \int_{2|k|}^{\infty} y^{-\beta-1} dy = c|k|^{2H-2-\beta}. \end{aligned}$$

From (4.6), this now proves that

$$C_\infty(i,j) \leq c|k|^{2H-2}. \tag{4.7}$$

The last term to look at is $B_\infty(i,j)$. For this term, it is convenient to revert from the integration variables (u,v) to $x=u+v$ and $y=u-v$, as we had for $C_\infty(i,j)$. We thus get

$$\begin{aligned} B_\infty(i,j) &= \int_0^1 h_\beta(x)\,dx \int_{-x}^{x}\left(|y+k|^{2H-2}+|y-k|^{2H-2}\right)dy \\ &\quad+\int_1^\infty h_\beta(x)\,dx\int_{x-2}^{x}\left(|y+k|^{2H-2}+|y-k|^{2H-2}\right)dy. \\ &=: B_{\infty,0}(i,j)+B_{\infty,1}(i,j). \end{aligned}$$

For the first term, for $|k|\geq 2$, we have

$$B_{\infty,0}(i,j)\leq c\,|k|^{2H-2}\int_0^1 x h_\beta(x)\,dx\leq c\,|k|^{2H-2}. \tag{4.8}$$

For the second term, it is convenient to use Fubini, to write

$$B_{\infty,1}(i,j)=\int_{-1}^{\infty}\left(|y+k|^{2H-2}+|y-k|^{2H-2}\right)dy\int_y^{y+2}h_\beta(x)\,dx.$$

For the part of this integral for $y\in[-1,0]$, and $|k|\geq 2$, we get an upper bound of $c\,|k|^{2H-2}\int_y^{y+2}h_\beta(x)\,dx$. We have

$$\begin{aligned} \int_y^{y+2}h_\beta(x)\,dx &\leq \int_y^{y+2}(x+1)^{-\beta}\,dx=\frac{1}{1-\beta}\left((y+1)^{1-\beta}-(y+2)^{1-\beta}\right) \\ &\leq c\left(|y+1|^{1-\beta}+|y+2|^{1-\beta}\right). \end{aligned}$$

These are integrable for $y\in[-1,0]$ because $1-\beta>-1$. Therefore

$$B_{\infty,1}(i,j)\leq c\,|k|^{2H-2}+\int_0^\infty\left(|y+k|^{2H-2}+|y-k|^{2H-2}\right)dy\int_y^{y+2}h_\beta(x)\,dx.$$

By the mean-value theorem applied to h_β, we get $\int_y^{y+2}h_\beta(x)\,dx\leq 2\beta\,(y+1)^{-\beta-1}$. Thus we have

$$\begin{aligned} B_{\infty,1}(i,j) &\leq c\,|k|^{2H-2}+c\int_0^\infty\left(|y+k|^{2H-2}+|y-k|^{2H-2}\right)(y+1)^{-\beta-1}\,dy \\ &= c\,|k|^{2H-2}+c\int_1^\infty\left(|y+k-1|^{2H-2}+|y-k-1|^{2H-2}\right)y^{-\beta-1}dy \\ &\leq c\,|k|^{2H-2}+c\int_1^\infty|-|k-1|+y|^{2H-2}\,y^{-\beta-1}dy. \end{aligned}$$

This expression is identical to the one we already studied in (4.6), with k replaced by $k-1$. Therefore, for $|k-1|\geq 2$, the same calculation leading to (4.6), combined with (4.8), yields

$$B_\infty(i,j)\leq c\,|k|^{2H-2}. \tag{4.9}$$

Gather (4.5), (4.7), and (4.9), and the conclusion of Step 3, we now get, for some finite positive constant c depending only on β,H, for $|i-j|\geq 3$

$$|A(i,j)+B(i,j)+C(i,j)|\leq c\,|i-j|^{2H-2}.$$

Step 5. Conclusion. Now, from the previous estimate and the identification observed in Step 3, we get

$$\left|\langle 1_{(\frac{i}{N},\frac{i+1}{N})},1_{(\frac{j}{N},\frac{j+1}{N})}\rangle_{L^2([0,1])}\right|\leq cN^{-2H+\beta}\,|i-j|^{2H-2}.$$

From the conclusion of Step 2, and the formula for $\mathcal{V}_N$ obtained in Step 3, we only need to prove that the next expression converges to 0 as $N \to \infty$:

$$\begin{aligned}
& N^{8H-4\beta-2}\mathcal{V}_N \\
= \;& N^{-2}\left(1+o\left(1\right)\right)c\sum_{i,j,i',j'=0}^{N-1}\left|i-j\right|^{2H-2}\left|i'-j'\right|^{2H-2}\left|i-i'\right|^{2H-2}\left|j-j'\right|^{2H-2} \\
= \;& N^{8H-6}\frac{1}{N^4}\sum_{i,j,i',j'=0}^{N-1}\left(\frac{|i-j|}{N}\right)^{2H-2}\left(\frac{|i'-j'|}{N}\right)^{2H-2}\left(\frac{|i-i'|}{N}\right)^{2H-2}\left(\frac{|j-j'|}{N}\right)^{2H-2}.
\end{aligned}$$

This is equivalent to $N^{8H-6}\int_{[0,1]^4}|u-v|^{2H-2}|u'-v'|^{2H-2}|u-u'|^{2H-2}|v-v'|^{2H-2}dudvdu'dv'$, which converges to 0 when $H<3/4$, finishing the proof of the theorem. ■

5 Non central convergence for $H > \frac{3}{4}$

Just as in Section 4, we can refer to property (3.2) and Theorem 3.8 to suggest that, when $H > 3/4$, a version of V_N normalized according to the rate and constant identified in Theorem 3.8 might converge in distribution. When looking at the case of fBm for comparison purposes (see [36]), one sees that for fBm with $H > 3/4$, the renormalized quadratic variation converges to a non-normal law, the so-called Rosenblatt distribution (described below in this section). Since property (3.2) implies that our time-indexed process U shares the same regularity properties as fBm with parameter $H-\beta/2$, one may suspect that a normalized version of V_N might converge to a Rosenblatt distribution when $H > 3/4+\beta/2$. The results of the previous section are not contradictory with this naive intuition, since therein Theorem 4.2 proves that for $H < 3/4$, a different normalization is needed and the limit is normal, but this does not help understand the phenomenon for U, and in particular there is no solid reason to believe that when $H > 3/4$, the correct renormalization of V_N (as given in Theorem 3.8) would give rise to a limiting Rosenblatt distribution.

In this section, we show that this is nonetheless true in some cases: a Rosenblatt-distributed limit does occur for the renormalized V_N, for the threshold $H > 3/4$, not the naive threshold $H > 3/4+\beta/2$. This shows once again that the intuition provided by the regularity estimate (3.2) does not help understand the limiting distribution of V_N.

Unlike a proof of normal convergence, a proof of convergence to a Rosenblatt distribution cannot rely on a characterization as simple as Theorem 4.1, for several reasons, one being that the shape of the Rosenblatt law depends on an additional parameter beyond its variance, and more generally speaking, the class of Rosenblatt laws is only one type of second-chaos distributions. There do exist characterizations of convergences to certain second-chaos laws which are similar in spirit to Theorem 4.1, but these are known only for convergences to Gamma laws. The original work in this direction was performed primarily for variables in fixed Wiener chaoses: see [25], [26]; a more general recent treatment is in [9], but does not go beyond Gamma laws for second chaos limits; also see Theorem 4.5 in [28] for the case of chi-squared limits.

Unfortunately, the Rosenblatt distributions are not Gamma, so none of these Malliavin-calculus-based tools are applicable for us. However, when considering a sequence of random variables that is already in the second chaos on classical Wiener space, such as our sequence V_N, one may simply check the convergence of the kernels in $L^2\left(\mathbb{R}^2\right)$ to the kernel of a second-chaos law when the latter is know; this is the case for the Rosenblatt distribution. This is the method that has been used in the past for variations of fBm and other processes, as in [3], [4], [36], [37], [38], and is closer to the method we will use here. We will invoke an additional simplification in characterizing convergences in the second chaos, based on the concept of *cumulant*.

The mth cumulant $\kappa_m(X)$ of a random variable X having all moments is defined as $m!$ times the coefficients in the Maclaurin series of $g(t) = \log \mathbf{E}e^{tX}$, $t \in \mathbb{R}$; in other words $\kappa_m(X) := g^{(m)}(0)$. The first cumulant c_1 is the expectation of X, the second one is the variance of X, and the higher ones are combinations of moments, but they present computational advantages, particularly in Wiener chaos calculations (see [24, Chapter 8]). The key fact we will use is that for random variables in the second Wiener chaos the cumulants characterizes the law. A second fact we will use is that the cummulant of a second-chaos variable can be computed easily as a multiple integral. We record these facts here (see [24], [27], and [11]).

Theorem 5.1. *Let $I_2(f)$ and $I_2(g)$ be two random variables in the classical second chaos; i.e. $f, g \in L^2(\mathbb{R}^2)$ and are symmetric. Then for every integer $m \geq 1$, the mth cumulant of $I_2(f)$ is given by*

$$\kappa_m(I_2(f)) = 2^{m-1}(m-1)! \int_{\mathbb{R}^m} f(y_1, y_2) f(y_2, y_3) \ldots . f(y_{m-1}, y_m) f(y_m, y_1) dy_1 \ldots dy_m. \quad (5.1)$$

Moreover, if $\kappa_m(I_2(f)) = \kappa_m(I_2(g))$ for every m, then $I_2(f)$ and $I_2(g)$ have the same law.

It is important to note that the law characterization in Theorem 5.1 works specifically for second-chaos laws, and is known to be false for all chaos laws of higher order.

As it turns out, even the computation of these cumulants for V_N, which is simpler than looking for the limit in $L^2(\mathbb{R}^2)$ of the kernel of V_N, is exceedingly technical, and would require calculations of the same complexity as the proofs of Propositions 3.1, 3.3, 3.6, without being able to rely directly on those proofs. To minimize the length of our presentation, we use a trick that works specifically for the cases $\beta = 1/2, 1, 3/2$.

Theorem 5.2. *Assume that $\beta \in \{1/2, 1, 3/2\}$ and $H > \frac{3}{4}$. The renormalized quadratic variation $\widehat{V}_N := K_2^{-\frac{1}{2}} N^{1-\beta} V_N$, with K_2 defined in Theorem 3.8, converges in law, as $N \to \infty$, to a Rosenblatt distribution with variance 1 and self-similarity parameter H.*

Remark 5.3. *The values of β in the previous theorem correspond to the case of the stochastic heat equation with W^H that has white noise behavior in space, for $d = 1, 2, 3$. We use this fact in the proof of the theorem. We conjecture that the result of Theorem 5.2 holds for all $\beta \in (0, 2H)$.*

Proof:

Assuming that $\beta \in \{1/2, 1, 3/2\}$, we recall that our process U_t on $[0,1]$ with covariance given by (3.1) can be represented as the mild solution $u(t,x)$ of the stochastic heat equation, as given in (2.3), with parameter choices $d = 2\beta$ and $\alpha = 0$, which implies that the differential random field $W^H(dt, dx)$ in the formula (2.3) is the differential of fBm in time with parameter H, and is white noise in space. We then invoke a transfer formula (see [16]) extending the formula by which the fBm can be written using its classical moving average representation: for every $t \in [0,T]$ and $x \in \mathbb{R}^d$

$$u(t,x) = \int_{-\infty}^{t} \int_{\mathbb{R}^d} W(ds, dy) \left(\int_0^t du G(t-u, x-y)(u-s)_+^{H-\frac{3}{2}} \right) \quad (5.2)$$

where W is a Gaussian process white noise in time and with the same spatial covariance as W^H. Since we restricted W^H to be white in space ($\alpha = 0$), we end up with our W being white in all parameters, i.e. $W(ds, dy)$ is space-time white noise in $\mathbb{R} \times \mathbb{R}^d$. In this case, we can easily compute the cumulants using formula (5.1).

With K_2 defined in Theorem 3.8 and $\beta \in \{1/2, 1, 3/2\}$, with $\widehat{V}_N$ defined in the statement of the theorem, let f_N be the kernel of $\widehat{V}_N$ in its representation as a double Wiener integral: $\widehat{V}_N = I_2(f_N)$. From formula (5.2) and the definition of V_N in (3.3), we immediately get

$$
\begin{aligned}
& f_N\left((s_1,y_1),(s_2,y_2)\right) \\
= \quad & K_2^{-\frac{1}{2}} N^{-(\beta-1)} \sum_{i=0}^{N-1} \Bigg(1_{(-\infty,t_{i+1})\times\mathbb{R}^d}(s_1,y_1) \int_0^{t_{i+1}} G(t_{i+1}-u_1,x-y_1)(u_1-s_1)_+^{H-\frac{3}{2}} du_1 \\
& - 1_{(-\infty,t_i)\times\mathbb{R}^d}(s_1,y_1) \int_0^{t_i} G(t_i-u_1,x-y_1)(u_1-s_1)_+^{H-\frac{3}{2}} du_1 \Bigg) \\
& \Bigg(1_{(-\infty,t_{i+1})\times\mathbb{R}^d}(s_2,y_2) \int_0^{t_{i+1}} G(t_{i+1}-v_1,x-y_2)(v_1-s_2)_+^{H-\frac{3}{2}} dv_1 \\
& - 1_{(-\infty,t_i)\times\mathbb{R}^d}(s_2,y_2) \int_0^{t_i} G(t_i-v_1,x-y_2)(v_1-s_2)_+^{H-\frac{3}{2}} dv_2 \Bigg).
\end{aligned}
$$

Let us now study the asymptotic behavior of the cumulants of $I_2(f_N)$. We will need the following two useful formulas. For $a+b<-1$

$$
\int_{-\infty}^{u\wedge v} (u-y)^a (v-y)^b dy = \mathrm{B}(-1-a-b,b+1) \tag{5.3}
$$

with B denoting the Beta function, and for every x,y

$$
\int_{\mathbb{R}^d} dy G(t-u,x-y)G(s-v,x-y) = (2\pi)^{-\beta}(t+s-u-v)^{-\beta}. \tag{5.4}
$$

Using relations (5.3) and (5.4), we use the notation $A(i,j), B(i,j)$ and $C(i,j)$ introduced in (3.4), (3.5), and (3.6), and obtain

$$
\begin{aligned}
& (2^{m-1}(m-1)!)^{-1}\kappa_m(I_2(f_N)) \\
= \quad & K_2^{-\frac{m}{2}} N^{m(1-\beta)}(2\pi)^{-m\beta}\beta(2H-1,H-\frac{1}{2})^m d(0,H)^m N^{m(2-2H)} \\
& \sum_{i_1,..,i_m=0}^{N-1} \left[\left(A(i_1,i_2)+B(i_1,i_2)+C(i_1,i_2)\right)\left(A(i_2,i_3)+B(i_2,i_3)+C(i_2,i_3)\right) \right. \\
& \left. \ldots \left(A(i_m,i_1)+B(i_m,i_1)+C(i_m,i_1)\right)\right].
\end{aligned}
$$

In the calculations of the proofs of Propositions 3.1, 3.3, and 3.6 when $H>3/4$, we notice that all asymptotic constants are the results of non-converging series which must therefore be compared with Riemann sums, for which only the terms with $i-j$ large need to be taken into account, and that the dominant behaviors of $A(i,j)$, $B(i,j)$, and $C(i,j)$ which contribute to the limit are of order $|i-j|^{2H-2}$. More specifically, the normalized sums of partial series with these terms are asymptotically equivalent to the same normalized series with $A(i,j)$ replaced by $\sqrt{K_{1,2}}|i-j|^{2H-2}$, $B(i,j)$ replaced by $\sqrt{K_{2,2}}|i-j|^{2H-2}$, and $C(i,j)$ replaced by $\sqrt{K_{3,2}}|i-j|^{2H-2}$. This effect can be see in the proof of claim (3.8) in the proof of Proposition 3.1 for $A(i,j)$. For $C(i,j)$, in the proof of Proposition 3.3, see the proof of claim (3.16). For $B(i,j)$, in the proof of Proposition 3.6, see the proof of the two claims (3.19) and (3.22). Therefore, we can write that

$$
\begin{aligned}
& \lim_{N\to\infty}(2^{m-1}(m-1)!)^{-1}\kappa_m(I_2(f_N)) = \lim_{N\to\infty} N^{-m}N^{m(2-2H)}K_2^{-\frac{m}{2}}(2\pi)^{-\frac{dm}{2}} \\
& \cdot\mathrm{B}(2H-1,H-\frac{1}{2})^m d(0,H)^m (\sum_{j=1}^{3}\sqrt{K_{i,2}})^m \sum_{i_1,..,i_m=0}^{N-1}\prod_{j=1}^{m}|i_j-i_{j+1}|^{2H-2} \\
= \quad & \mathrm{B}(2H-1,H-\frac{1}{2})^m \alpha(H)^m \int_{[0,1]^{2m}} du_1dv_1...du_mdv_m \int_{[0,1]^m}\prod_{j=1}^{m}|x_j-x_{j+1}|^{2H-2}
\end{aligned} \tag{5.5}
$$

with the convention $x_{m+1} := x_1$ and $v_{m+1} := v_1$, and where B is the Beta function.

By Theorem 5.1, the sequence V_N converges in distribution to the second Wiener chaos law characterized by having its mth cumulant given by the expression (5.5) for every m. We recognized that this is the mth cumulant of the Rosenblatt law with variance 1 and self-similarity parameter H (see [35] or [23]). ■

6 Statistical application: estimating H

We use a classical method to find a consistent estimator for the parameter H by using the quadratic variation (see [7]). We give the details of the method in the case $H < 3/4$; the case $H > 3/4$ uses the same method, and we omit the corresponding details. Denote by

$$S_N = \sum_{i=0}^{N-1} \left(U_{t_{i+1}} - U_{t_i}\right)^2 .$$

Using the calculations in the proof of Theorem 3.8, for N large, we can prove that

$$\mathbf{E}S_N = (1 + o\,(1))\, C_0 N^{-2H+\beta+1} \tag{6.1}$$

where the constant C_0 can be identified as the sum of the three terms in (4.4) with $k = 0$:

$$\begin{aligned} C_0 \quad := \quad & \int_0^1 du \int_0^1 dv |u-v|^{2H-2} (u+v)^{-\beta} \\ & -2 \int_0^1 du \int_0^\infty dv \, |v-u|^{2H-2} h_\beta (u+v) \\ & + \int_0^\infty dx \; g_\beta (x) \int_{-x}^x dy \;\; |y|^{2H-2} . \end{aligned} \tag{6.2}$$

Next, by estimating $\mathbf{E}S_N$ by S_N and taking the logarithm above, we find

$$-(2H - \beta - 1) \log N \simeq \log \frac{S_N}{C_0}$$

which gives the initial estimator

$$\hat{H}_N := \frac{1}{2} \left[-\frac{\log \frac{S_N}{C_0}}{\log N} + \beta + 1 \right] . \tag{6.3}$$

By (3.3), we have $S_N = V_N + \mathbf{E}S_N$. Thus S_N is asymptotically equivalent to $C_0 N^{-2H+\beta+1} + V_N$ for N large, and we get

$$\begin{aligned} \hat{H}_N - H \quad & = \quad \frac{1}{2} \left[-\frac{\log \frac{S_N}{C_0}}{\log N} + \beta + 1 \right] - H \\ & = \quad \frac{1}{2} \left[-\frac{\log(N^{-2H+\beta+1} + C_0^{-1} V_N)}{\log N} + \beta + 1 \right] - H \\ & = \quad \frac{1}{2} \left[-\frac{\log(1 + C_0^{-1} N^{2H-\beta-1} V_N)}{\log N} \right] \\ & = \quad (1 + o\,(1))\, \frac{1}{2} \left[-\frac{C_0^{-1} N^{2H-\beta-1} V_N}{\log N} \right] \end{aligned} \tag{6.4}$$

almost surely. The limit in the last line above uses the a.s. convergence of $N^{2H-\beta-1} V_N$ to zero, which we now prove, thereby obtaining strong consistence (a.s. convergence of $\hat{H}_N$ to H) as an immediate consequence.

Proposition 6.1. *Let* $\tilde{V}_N := V_N N^{2H-\beta-1}$*. Almost surely,* $\lim_{N\to\infty} \tilde{V}_N = 0$*, so that* $\hat{H}_N$ *is strongly consistent. In fact, for any* $0 < \gamma < 1/2$*, almost surely,* $\tilde{V}_N = o\left(N^{-\gamma}\right)$*. Consequently, from (6.4), we get* $\hat{H}_N - H = o\left(N^{-\gamma}\right)$ *almost surely.*

Proof: In view of (6.4), using the Borel-Cantelli lemma, it suffices to show that

$$\sum_{N\geq 1} \mathbf{P}\left(\tilde{V}_N > N^{-\gamma}\right) < \infty$$

for all $0 < \gamma < \frac{1}{2}$. By Markov's inequality and the hypercontractivity property of multiple Wiener-Itô integrals (see (8.5))

$$P\left(\tilde{V}_N > N^{-\gamma}\right) \leq N^{\gamma p}\mathbf{E}\left|\tilde{V}_N\right|^p \leq c_p N^{\gamma p}\left(\mathbf{E}\left|\tilde{V}_N\right|^2\right)^{\frac{p}{2}}$$

for every $p \geq 1$. By Theorem 3.8, $\lim_{N\to\infty} N\mathbf{E}\left|\tilde{V}_N\right|^2 = \lim_{N\to\infty} N^{4H-2\beta-1}\mathbf{E}V_N^2 = K_1$. Therefore we obtain for N large enough,

$$P\left(\tilde{V}_N > N^{-\gamma}\right) \leq c_p\left(K_1\right)^{p/2} N^{\gamma p - \frac{p}{2}}.$$

By choosing $\gamma < 1/2$ and p large enough, $\sum_{N\geq 1} P\left(\tilde{V}_N > N^{-\gamma}\right)$ converges. ∎

We now obtain $\hat{H}_N$'s asymptotic distribution.

Theorem 6.2. *Suppose that* $H > \frac{1}{2}$ *and assume that* $\hat{H}_N$ *is given by (6.3).*

- *Let* C_0 *be given as in (6.2) and* K_1 *as in Theorem 3.8; if* $H \in (\frac{1}{2}, \frac{3}{4})$*, then, in distribution,*
$$\lim_{N\to\infty} \sqrt{N}\log(N)\left(\hat{H}_N - H\right) = \mathcal{N}\left(0, \frac{K_1}{(2C_0)^2}\right);$$
- *If* $H \in (\frac{3}{4}, 1)$ *and* $\beta = 1/2$*,* 1*, or* $3/2$*, then, in distribution, for some constant* C_1*,*
$$\lim_{N\to\infty} C_1 N^{2-2H}\log(N)\left(\hat{H}_N - H\right) = Z$$
where Z *is a Rosenblatt random variable with variance* 1 *and parameter* H*.*

Proof: The normal convergence follows from Theorem 4.2 and relation (6.4). The proof of the Rosenblatt convergence is based on Theorem 5.2 and the technique leading to (6.4), with details left to the reader. ∎

The estimator one gets by replacing C_0 by 1 in (6.3) is also strongly consistent as $N \to \infty$, but its speed of convergence is logarithmic in N, and therefore it is preferable to use the term C_0. However, in practice, since C_0 depends on H, the definition of $\hat{H}_N$ is somewhat circular. This is an issue which does not occur for stationary self-similar processes, where an analogous estimator works, one in which β is replaced by 0 and C_0 is replaced by 1 (see [38]). In order to devise an implementable version of $\hat{H}_N$ based on a single trajectory of U, one may instead define the estimator $\check{H}_N$ as the solution ξ to the equation

$$\xi := \frac{1}{2}\left[\frac{\log C_0\left(\xi\right)}{\log N} - \frac{\log S_N}{\log N} + \beta + 1\right]. \tag{6.5}$$

The function $\phi : \xi \mapsto \log C_0(\xi)$ is C^1 on $(0, +\infty)$, with bounded derivative. We can use this to prove strong consistency and asymptotic normality of $\check{H}$. Indeed, with the bounded first derivative, we compute

$$\begin{aligned} \left|\check{H}_N - \hat{H}_N\right| &= \frac{1}{\log N}\left|\phi\left(\check{H}_N\right) - \phi(H)\right| \leq \frac{\|\phi'\|_\infty}{2\log N}\left|\check{H}_N - H\right| \\ &\leq \frac{\|\phi'\|_\infty}{\log N}\left(\left|\check{H}_N - \hat{H}_N\right| + \left|\hat{H}_N - H\right|\right) \end{aligned}$$

which implies, for N large enough that $1 > \frac{\|\phi'\|_\infty}{\log N}$,

$$\left|\check{H}_N - H\right| \leq \frac{2}{1 - \|\phi'\|_\infty / \log N}\left|\hat{H}_N - H\right|,$$

proving that $\lim_{N\to\infty} \check{H}_N = H$ a.s., and that the speed of convergence is the same as for $\hat{H}_N$, up to any constant greater than 2. To prove asymptotic normality, similarly, we get

$$\check{H}_N - H = \frac{1}{1 - \phi'(\xi_N) / \log N}\left(\hat{H}_N - H\right),$$

where ξ_N is a point in the interval $(\check{H}_N, H)$, and thus $\phi'(\xi_N)$ converges almost surely to $\phi'(H)$, so that $1/\left[(1 - \phi'(\xi_N)) / \log N\right]$ converges to 1 almost surely; combined with the convergence in distribution of $\sqrt{N}\log(N)\left(\hat{H}_N - H\right)$ to a normal by Theorem 6.2, the sequence $\sqrt{N}\log(N)\left(\check{H}_N - H\right)$ has the same limit in law, by Slutsky's theorem.

Solving the fixed point equation (6.5) can be done using various numerical methods. Note first that, for N large enough, the function $\phi(\xi)/2\log N$ will be a contraction, so that the solution exists and is unique. However, one can further check that ϕ is C^2 with bounded derivatives, and Newton's method provides a faster convergence than the fixed point algorithm: indeed the error ε_n in Newton's method at iteration level n satisfies $\varepsilon_{n+1} \leq M\varepsilon_n^2$ with $M = \sup_\xi\left(2^{-1}|f''(\xi)| / |f'|(\xi)\right)$ where $f(\xi) = \phi(\xi)/(2\log N) - \xi$, and for N large enough, $M = \sup|\phi''|(2\log N - \sup|\phi'|)$ is finite since ϕ' and ϕ'' are bounded.

7 Numerical implementation

This section presents a simulations-based analysis of the asymptotic behavior of the estimator $\hat{H}_N$ for H given in (6.3). We base our analysis on inspection of empirical means, standard deviations, skewness and kurtosis, boxplots with outliers, and frequency histograms, for 1000 independently-generated paths of the Gaussian process U on $[0, 1]$ with covariance (3.1), with number of time steps $N = 1000$. Numerically, the computation of $\hat{H}_N$ or $\check{H}_N$ via formula (6.3) or equation (6.5), and of these empirical statistics, based on a data stream coming from the process U, is essentially immediate. However, since our aim in this section is to work from simulated data, we must use an efficient way of generating our 1000 paths of U. We also choose to repeat this generation for a number of different values of $H \in (0, 1)$ in order to illustrate the various asymptotic distributions identified in Theorem 6.2.

To simulate the N-dimensional multivariate centered Normal distribution with covariance given by (3.1) with $s, t \in \{k/N : k = 1, \cdots, 1000\}$, the double integral in this expression needs to be computed with high accuracy. While computational software such as MATLAB should in principle handle this sort of task via deterministic numerical methods, the computational time turns out to be excessive. We proceeded instead via adaptative Monte Carlo methods, coded in C++ to increase speed.

Not surprisingly, our simulations give evidence of strong consistency (a.s. convergence) of $\hat{H}_N$ for all values of H, as in Proposition 6.1. We also checked for approximate normality of the distribution of $\hat{H}_N$ for $H < 0.75$ using the tests available in MATLAB, and found clear evidence of this normality. In addition, for $H > 0.75$, one would want to know whether the asymptotic distribution of $\hat{H}_N$ is close to the Rosenblatt law announced in Theorem 6.2; for this, since there are currently no theoretical tools to check for such laws, we used the empirical tools provided by Veillette and Taqqu in [39]. We now report the details of this empirical evidence in both the cases $H \leq 0.75$ and $H > 0.75$.

Figures 1 and 2 show the frequency histograms (sampling distribution) of the 1000 values generated for $\hat{H}_N$ for eleven values of H ranging from 0.51 to 0.99. A clear break occurs at $H = 0.80$ and above, in which histograms go from looking rather normal (unimodal, symmetric, light tails, low pointedness), to having a strong asymmetry with a long right tail and no left tail, and a strong pointedness. In Figure 3, we have also included graphs showing how well our histograms can be fitted to normal curves for $H \leq 0.75$; for the case of $H = 0.90$, we have included Figure 4 to illustrate fitting to the simulated Rosenblatt density given in [39] (the Rosenblatt density is not known explicitly, no bona fide goodness-of-fit test currently exists, since we don't have an appropriate asymptotic theory). Interestingly, the case of $H = 0.75$, which we did not treat in this article for the sake of conciseness, is expected to give rise to normal behavior with a logarithmic correction in the convergence rate, and this is consistent with our histograms and curve-fitting.

We also provide the boxplots for our data, since they are better graphical tools for detecting outliers, relative to the Gaussian case, than are histograms. They are also good measures of asymmetry. In Figures 5 and 6, the boxplots' whiskers extend to the last data points which are within 1.5 times the inter-quartile range above the 3rd quartile and below the 1st quartile (approximately 99.3% coverage in the Gaussian case). We note that the boxplots for $H \leq 0.75$ show no presence of left-tail outliers, and very few right-tail outliers, and as we move up from $H = 0.80$ to $H = 0.99$, the outliers become numerous on the right, and increasingly so, consistent with the heavier right tail and associated pronounced asymmetry. Similarly, the two middle inter-quartiles ranges are largely symmetric for $H \leq 0.75$, and strongly asymmetric for $H \geq 0.80$.

Finally, tables of summary statistics are provided. In Table 1, we report the empirical mean, standard deviation, skewness (centered 3rd moment over cubed standard deviation) and kurtosis (centered 4th moment over 4th power of standard deviation, minus 3). The bias and standard deviations increase as H increases; they are both within 0.5% for $H \leq 0.75$; biases remain this low for all H but standard deviations become quite large for large H, indicating a need for higher N when H is closer to 1. For $H = 0.51$ to 0.75, the skewness and kurtosis do not deviate from the Gaussian values in a statistically significant way that would allow us to reject the normal law, even though, closer to 0.75, as already mentioned, outliers begin to appear in the right tail. This situation changes immediately for $H = 0.80$ to 0.99: the skewness and kurtosis are significantly (and sharply) different from the normal ones. Table 2 presents the difference between skewness and kurtosis for our distributions and the corresponding simulated values in [39].

8 Appendix: Multiple Wiener integrals and Malliavin derivatives

Here we describe the elements from stochastic analysis that we will need in the paper. Consider $\mathcal{H}$ a real separable Hilbert space and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathcal{A}, P)$, which is a centered Gaussian family of random variables such that $\mathbf{E}\left(B(\varphi)B(\psi)\right) = \langle \varphi, \psi \rangle_{\mathcal{H}}$. Denote I_n the multiple stochastic integral with respect to B (see [29]). This I_n is actually an isometry between the Hilbert space $\mathcal{H}^{\odot n}$(symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{n!}}\|\cdot\|_{\mathcal{H}^{\otimes n}}$ and the Wiener chaos of order n which is defined as the closed linear span of the random variables $H_n(B(\varphi))$ where $\varphi \in \mathcal{H}, \|\varphi\|_{\mathcal{H}} = 1$ and H_n is the Hermite polynomial of degree $n \geq 1$

$$H_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n}\left(\exp\left(-\frac{x^2}{2}\right)\right), \qquad x \in \mathbb{R}.$$

The isometry of multiple integrals can be written as: for m, n positive integers,

$$\begin{aligned} \mathbf{E}\left(I_n(f)I_m(g)\right) &= n!\langle \tilde{f}, \tilde{g} \rangle_{\mathcal{H}^{\otimes n}} \quad \text{if } m = n, \\ \mathbf{E}\left(I_n(f)I_m(g)\right) &= 0 \quad \text{if } m \neq n. \end{aligned} \tag{8.1}$$

It also holds that

$$I_n(f) = I_n\big(\tilde{f}\big)$$

where $\tilde{f}(x_1, \ldots, x_n) = \frac{1}{n!}\sum_{\sigma \in \mathcal{S}_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ defines the symmetrization of f.

We recall that any square integrable random variable which is measurable with respect to the σ-algebra generated by B can be expanded into an orthogonal sum of multiple stochastic integrals

$$F = \sum_{n \geq 0} I_n(f_n) \tag{8.2}$$

where $f_n \in \mathcal{H}^{\odot n}$ are (uniquely determined) symmetric functions and $I_0(f_0) = \mathbf{E}\left[F\right]$.

Let L be the Ornstein-Uhlenbeck operator

$$LF = -\sum_{n \geq 0} n I_n(f_n)$$

if F is given by (8.2) and it is such that $\sum_{n=1}^{\infty} n^2 n! \|f_n\|^2_{\mathcal{H}^{\otimes n}} < \infty$.

For $p > 1$ and $\alpha \in \mathbb{R}$ we introduce the Sobolev-Watanabe space $\mathbb{D}^{\alpha,p}$ as the closure of the set of polynomial random variables with respect to the norm

$$\|F\|_{\alpha,p} = \|((I - L)F)^{\frac{\alpha}{2}}\|_{L^p(\Omega)}$$

where I represents the identity. We denote by D the Malliavin derivative operator that acts on smooth functions of the form $F = g(B(\varphi_1), \ldots, B(\varphi_n))$ (g is a smooth function with compact support and $\varphi_i \in \mathcal{H}$)

$$DF = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(B(\varphi_1), \ldots, B(\varphi_n))\varphi_i.$$

The operator D is continuous from $\mathbb{D}^{\alpha,p}$ into $\mathbb{D}^{\alpha-1,p}\left(\mathcal{H}\right)$.

We will need the general formula for calculating products of Wiener chaos integrals of any orders p, q for any symmetric integrands $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot q}$; it is

$$I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \otimes_r g) \tag{8.3}$$

as given for instance in D. Nualart's book [29, Proposition 1.1.3]; the contraction $f \otimes_r g$ is the element of $\mathcal{H}^{\otimes(p+q-2r)}$ defined by

$$\begin{aligned} &(f \otimes_\ell g)(s_1, \ldots, s_{n-\ell}, t_1, \ldots, t_{m-\ell}) \\ &= \textstyle\int_{[0,T]^{m+n-2\ell}} f(s_1, \ldots, s_{n-\ell}, u_1, \ldots, u_\ell) g(t_1, \ldots, t_{m-\ell}, u_1, \ldots, u_\ell) du_1 \ldots du_\ell. \end{aligned} \tag{8.4}$$

We recall the following hypercontractivity property for the L^p norm of a multiple stochastic integral (see [17, Theorem 4.1])

$$\mathbf{E}\left|I_m(f)\right|^{2m} \leq c_m \left(\mathbf{E} I_m(f)^2\right)^m \tag{8.5}$$

where c_m is an explicit positive constant and $f \in \mathcal{H}^{\otimes m}$.

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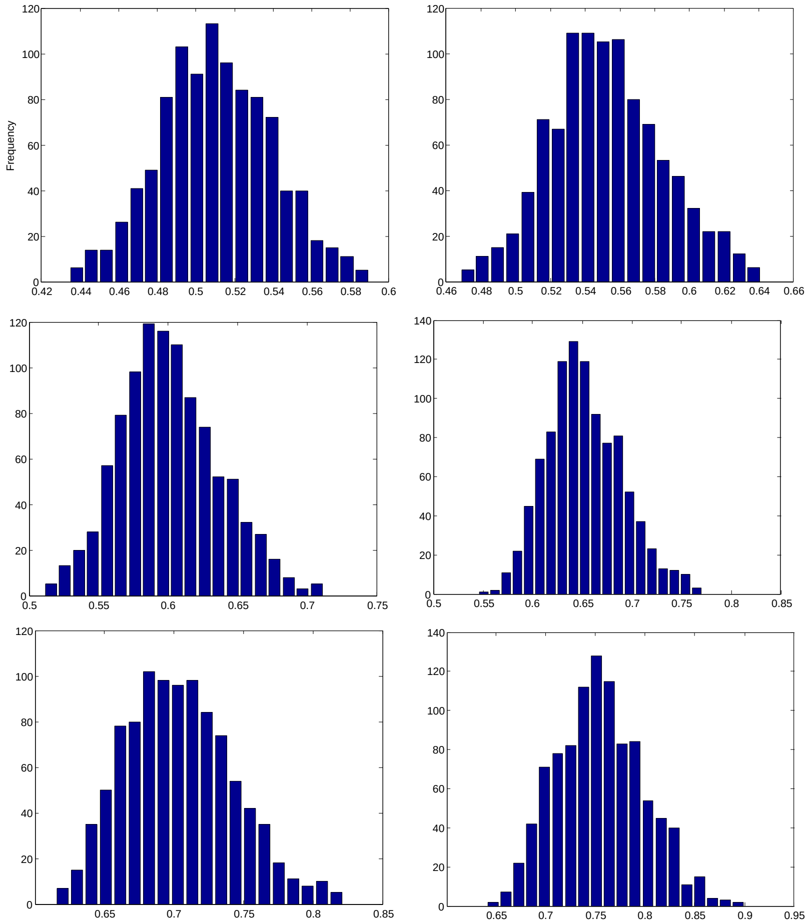

Figure 1: Frequency Histograms for $H = 0.51, 0.55, 0.60, 0.65, 0.70, 0.75$.

True Value H	Mean $\hat{H}_N$	Std. Dev.	Third Cumul.	Fourth Cumul
0.51	0.5092	0.0291	0.0601	-0.244
0.55	0.5515	0.0325	0.2159	-0.2533
0.60	0.6009	0.0357	0.3181	0.054
0.65	0.6523	0.0378	0.3732	-0.0653
0.70	0.7036	0.0391	0.3605	-0.216
0.75	0.7545	0.0440	0.2445	-0.2626
0.80	0.8045	0.0668	2.4665	7.1913
0.85	0.8516	0.0799	2.6959	10.6009
0.90	0.8874	0.3302	2.3424	9.2712
0.95	0.9490	0.6860	2.8332	10.8353
0.99	0.9936	0.0860	2.9371	11.3271

Table 1: Statistics of $\hat{H}_N$ over 1000 paths.

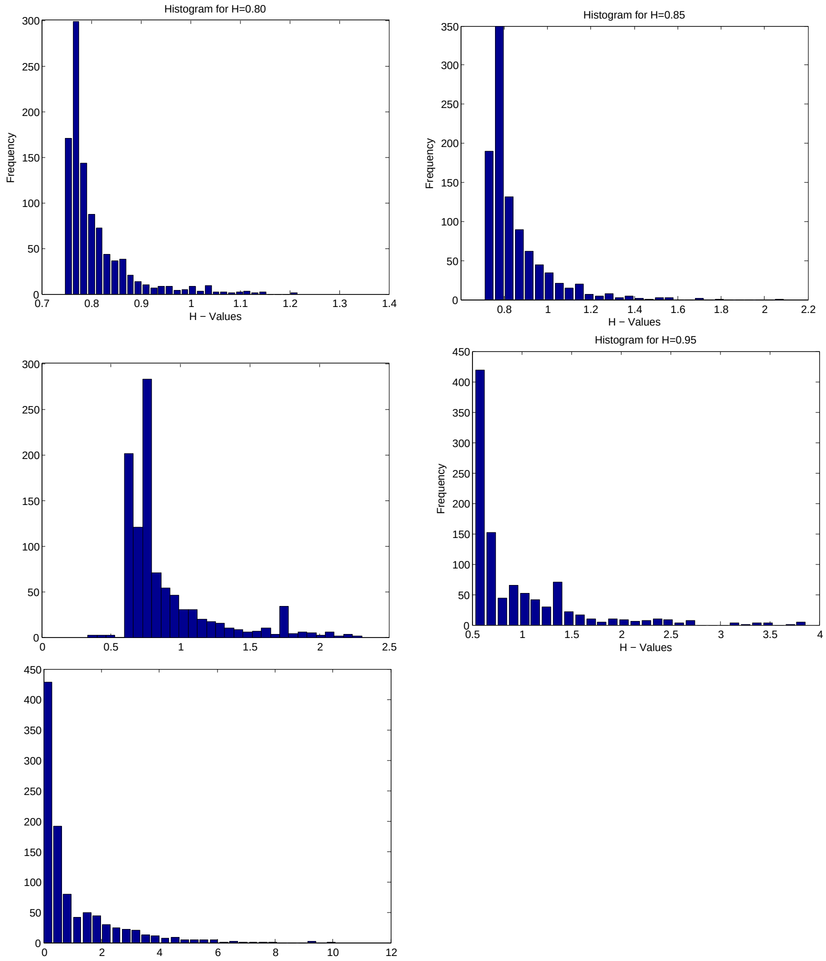

Figure 2: Frequency Histograms for $H = 0.80, 0.85, 0.90, 0.95, 0.99$.

H (true value)	Rosenblatt 3rd cumul. (simul)	Rosenblatt 4th cumul. (simul)	difference 3rd cumul.	difference 4th cumul.
0.80	2.548	10.350	-0.0815	-3.1587
0.85	2.684	11.150	0.0119	-0.5491
0.90	2.770	11.660	-0.4266	-2.3888
0.95	2.815	11.920	0.0182	-1.0847
0.99	2.828	12.0	0.1091	-0.6729

Table 2: Statistics of $\hat{H}_N$ over 1000 paths: comparison with empirical Rosenblatt cumulants from [39] for $H > 0.75$.

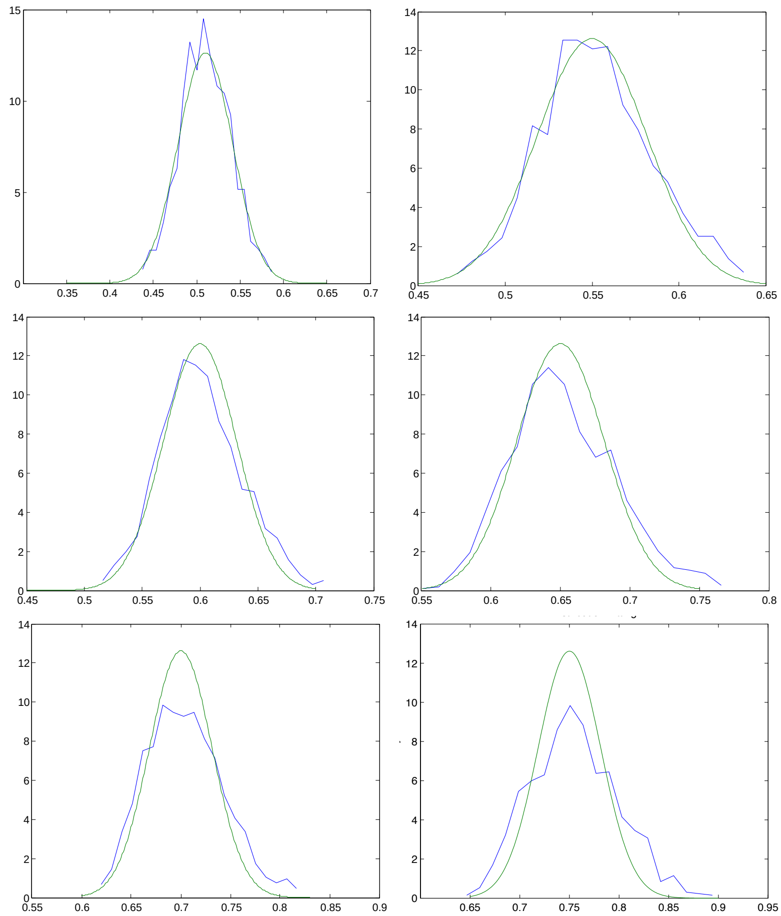

Figure 3: Fitting empirical distributions (blue) for $H = 0.51$ to $H = 0.75$ to a Gaussian density (green).

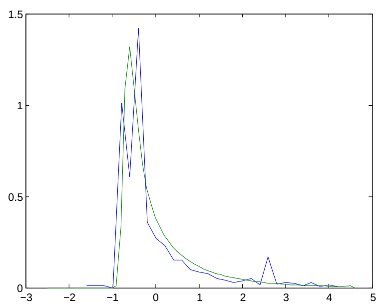

Figure 4: Fitting empirical distribution (blue) for $H = 0.90$ to a simulated Rosenblatt density (green).

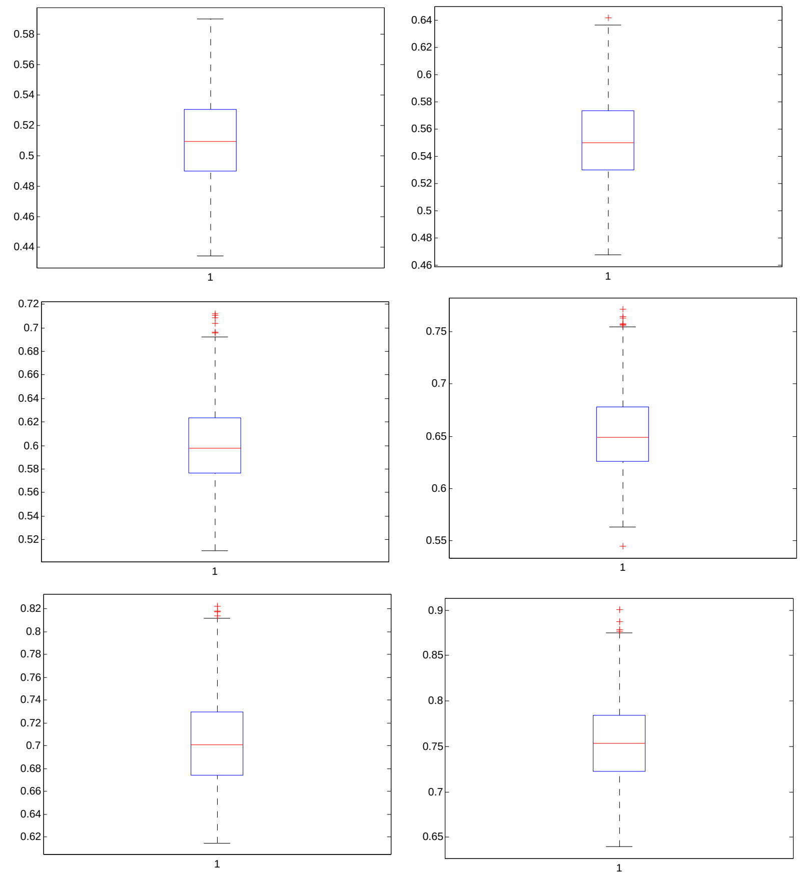

Figure 5: Box plots for $H = 0.51, 0.55, 0.60, 0.65, 0.70, 0.75$.

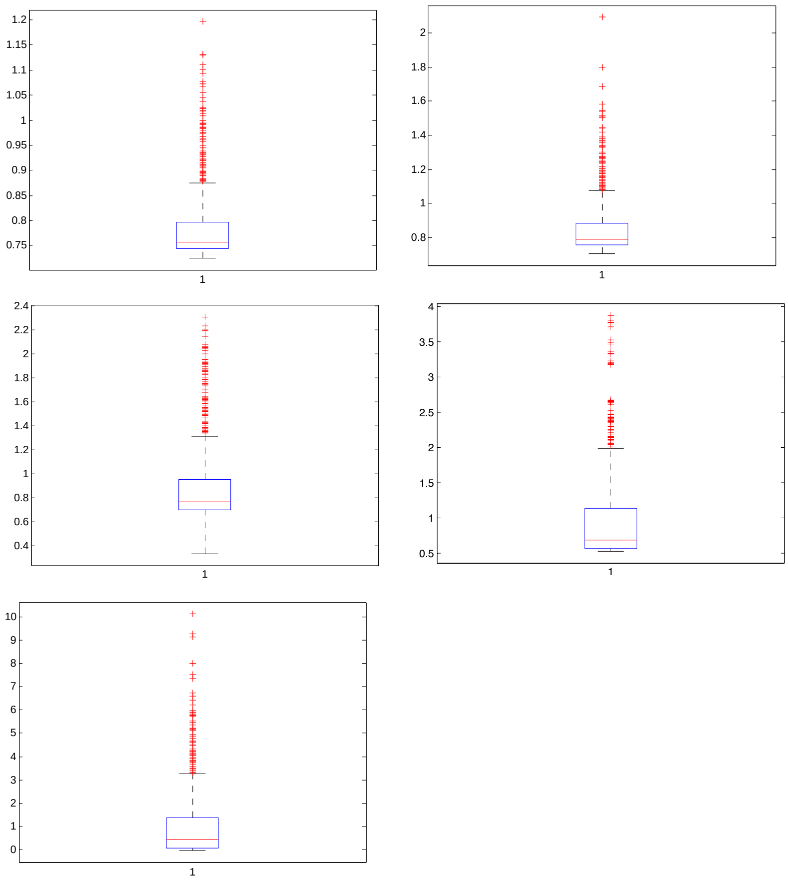

Figure 6: Box plots for $H = 0.80, 0.85, 0.90, 0.95, 0.99$.

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