

Low rank perturbations of large elliptic random matrices*

Sean O’Rourke[†] David Renfrew[‡]

Abstract

We study the asymptotic behavior of outliers in the spectrum of bounded rank perturbations of large random matrices. In particular, we consider perturbations of elliptic random matrices which generalize both Wigner random matrices and non-Hermitian random matrices with iid entries. As a consequence, we recover the results of Capitaine, Donati-Martin, and Féral for perturbed Wigner matrices as well as the results of Tao for perturbed random matrices with iid entries. Along the way, we prove a number of interesting results concerning elliptic random matrices whose entries have finite fourth moment; these results include a bound on the least singular value and the asymptotic behavior of the spectral radius.

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1 Introduction

In this note, we investigate the asymptotic behavior of outliers in the spectrum of bounded rank perturbations of large random matrices. We begin by introducing the empirical spectral distribution of a square matrix.

The *eigenvalues* of a $N \times N$ matrix M are the roots in \mathbb{C} of the characteristic polynomial $\det(M - zI)$, where I is the identity matrix. We let $\lambda_1(M), \dots, \lambda_N(M)$ denote the eigenvalues of M . In this case, the *empirical spectral measure* μ_M is given by

$$\mu_M := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i(M)}.$$

The corresponding *empirical spectral distribution* (ESD) is given by

$$F^M(x, y) := \frac{1}{N} \# \{1 \leq i \leq N : \operatorname{Re}(\lambda_i(M)) \leq x, \operatorname{Im}(\lambda_i(M)) \leq y\}.$$

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[†]Department of Mathematics, Yale University, USA. E-mail: sean.ourourke@yale.edu

[‡]Department of Mathematics, UCLA, USA. E-mail: dtrenfrew@math.ucla.edu

Here $\#E$ denotes the cardinality of the set E . If the matrix M is Hermitian, then the eigenvalues $\lambda_1(M), \dots, \lambda_N(M)$ are real. In this case the ESD is given by

$$F^M(x) := \frac{1}{N} \# \{1 \leq i \leq N : \lambda_i(M) \leq x\}.$$

Given a random $N \times N$ matrix Y_N , an important problem in random matrix theory is to study the limiting distribution of the empirical spectral measure as N tends to infinity. We will use asymptotic notation, such as O, o, Ω , under the assumption that $N \rightarrow \infty$. See Section 2.2 for a complete description of our asymptotic notation.

1.1 Random matrices with independent entries

We consider two ensembles of random matrices with independent entries. We first define a class of Hermitian random matrices with independent entries originally introduced by Wigner [52].

Definition 1.1 (Wigner random matrices). *Let ξ be a complex random variable with mean zero and unit variance, and let ζ be a real random variables with mean zero and finite variance. We say Y_N is a Wigner matrix of size N with atom variables ξ, ζ if $Y_N = (y_{ij})_{i,j=1}^N$ is a random Hermitian $N \times N$ matrix that satisfies the following conditions.*

- $\{y_{ij} : 1 \leq i \leq j \leq N\}$ is a collection of independent random variables.
- $\{y_{ij} : 1 \leq i < j \leq N\}$ is a collection of independent and identically distributed (iid) copies of ξ .
- $\{y_{ii} : 1 \leq i \leq N\}$ is a collection of iid copies of ζ .

The prototypical example of a Wigner real symmetric matrix is the *Gaussian orthogonal ensemble* (GOE). The GOE is defined by the probability distribution

$$\mathbb{P}(dM) = \frac{1}{Z_N^{(\beta)}} \exp\left(-\frac{\beta}{4} \operatorname{tr} M^2\right) dM \tag{1.1}$$

on the space of $N \times N$ real symmetric matrices when $\beta = 1$ and dM refers to the Lebesgue measure on the $N(N + 1)/2$ different elements of the matrix. Here $Z_N^{(\beta)}$ denotes the normalization constant. So for a matrix $Y_N = (y_{ij})_{i,j=1}^N$ drawn from the GOE, the elements $\{y_{ij} : 1 \leq i \leq j \leq N\}$ are independent Gaussian random variables with mean zero and variance $1 + \delta_{ij}$.

The classical example of a Wigner Hermitian matrix is the *Gaussian unitary ensemble* (GUE). The GUE is defined by the probability distribution given in (1.1) with $\beta = 2$, but on the space of $N \times N$ Hermitian matrices. Thus, for a matrix $Y_N = (y_{ij})_{i,j=1}^N$ drawn from the GUE, the N^2 different real elements of the matrix,

$$\{\operatorname{Re}(y_{ij}) : 1 \leq i \leq j \leq N\} \cup \{\operatorname{Im}(y_{ij}) : 1 \leq i < j \leq N\},$$

are independent Gaussian random variables with mean zero and variance $(1 + \delta_{ij})/2$.

A classical result for Wigner random matrices is Wigner’s semicircle law [5, Theorem 2.5].

Theorem 1.2 (Wigner’s Semicircle law). *Let ξ be a complex random variable with mean zero and unit variance, and let ζ be a real random variables with mean zero and finite variance. For each $N \geq 1$, let Y_N be a Wigner matrix of size N with atom variables ξ, ζ , and let A_N be a deterministic $N \times N$ Hermitian matrix with rank $o(N)$. Then the ESD of $\frac{1}{\sqrt{N}}(Y_N + A_N)$ converges almost surely to the semicircle distribution F_{sc} as $N \rightarrow \infty$, where*

$$F_{sc}(x) := \int_{-\infty}^x \rho_{sc}(t) dt, \quad \rho_{sc}(t) := \begin{cases} \frac{1}{2\pi} \sqrt{4 - t^2}, & \text{if } |t| \leq 2 \\ 0, & \text{if } |t| > 2. \end{cases}$$

Remark 1.3. *Wigner’s semicircle law holds in the case when the entries of Y_N are not identically distributed (but still independent) provided the entries satisfy a Lindeberg-type condition. See [5, Theorem 2.9] for further details.*

We now consider an ensemble of random matrices with iid entries.

Definition 1.4 (iid random matrices). *Let ξ be a complex random variable. We say Y_N is an iid random matrix of size N with atom variable ξ if Y_N is a $N \times N$ matrix whose entries are iid copies of ξ .*

When ξ is a standard complex Gaussian random variable, Y_N can be viewed as a random matrix drawn from the probability distribution

$$\mathbb{P}(dM) = \frac{1}{\pi^{N^2}} e^{-\text{tr}(MM^*)} dM$$

on the set of complex $N \times N$ matrices. Here dM denotes the Lebesgue measure on the $2N^2$ real entries of M . This is known as the *complex Ginibre ensemble*. The *real Ginibre ensemble* is defined analogously. Following Ginibre [28], one may compute the joint density of the eigenvalues of a random $N \times N$ matrix Y_N drawn from the complex Ginibre ensemble.

Mehta [37, 38] used the joint density function obtained by Ginibre to compute the limiting spectral measure of the complex Ginibre ensemble. In particular, he showed that if Y_N is drawn from the complex Ginibre ensemble, then the ESD of $\frac{1}{\sqrt{N}}Y_N$ converges to the *circular law* F_{circ} , where

$$F_{\text{circ}}(x, y) := \mu_{\text{circ}}(\{z \in \mathbb{C} : \text{Re}(z) \leq x, \text{Im}(z) \leq y\})$$

and μ_{circ} is the uniform probability measure on the unit disk in the complex plane. Edelman [22] verified the same limiting distribution for the real Ginibre ensemble.

For the general (non-Gaussian) case, there is no formula for the joint distribution of the eigenvalues and the problem appears much more difficult. The universality phenomenon in random matrix theory asserts that the spectral behavior of a random matrix does not depend on the distribution of the atom variable ξ in the limit $N \rightarrow \infty$. In other words, one expects that the circular law describes the limiting ESD of a large class of random matrices (not just Gaussian matrices)

The first rigorous proof of the circular law for general (non-Gaussian) distributions was by Bai [3, 5]. He proved the result under a number of assumptions on the moments and smoothness of the atom variable ξ . Important results were obtained more recently by Pan and Zhou [41] and Götze and Tikhomirov [31]. Using techniques from additive combinatorics, Tao and Vu [46] were able to prove the circular law under the assumption that $\mathbb{E}|\xi|^{2+\varepsilon} < \infty$ for some $\varepsilon > 0$. Recently, Tao and Vu [47, 48] established the law assuming only that ξ has finite variance.

For any matrix M , we denote the Hilbert-Schmidt norm $\|M\|_2$ by the formula

$$\|M\|_2 := \sqrt{\text{tr}(MM^*)} = \sqrt{\text{tr}(M^*M)}. \tag{1.2}$$

Theorem 1.5 (Tao-Vu, [48]). *Let ξ be a complex random variable with mean zero and unit variance. For each $N \geq 1$, let Y_N be a $N \times N$ matrix whose entries are iid copies of ξ , and let A_N be a $N \times N$ deterministic matrix. If $\text{rank}(A_N) = o(N)$ and $\sup_{N \geq 1} \frac{1}{N^2} \|A_N\|_2^2 < \infty$, then the ESD of $\frac{1}{\sqrt{N}}(Y_N + A_N)$ converges almost surely to the circular law F_{circ} as $N \rightarrow \infty$.*

1.2 Outliers in the spectrum

From Theorem 1.2 and Theorem 1.5, we see that the low rank perturbation A_N does not effect the limiting ESD. In other words, the majority of the eigenvalues remain distributed according to semicircle law or circular law, respectively. However, the perturbation A_N may create one or more outliers.

Let Y_N be a $N \times N$ random matrix whose entries are iid copies of ξ . When the atom variable ξ has finite fourth moment, one can compute the asymptotic behavior of the spectral radius [5, Thoerem 5.18]. We remind the reader that the spectral radius of a square matrix is the largest eigenvalue in absolute value.

Theorem 1.6 (No outliers for iid matrices). *Let ξ be a complex random variable with mean zero, unit variance, and finite fourth moment. For each $N \geq 1$, let Y_N be a $N \times N$ random matrix whose entries are iid copies of ξ . Then the spectral radius of $\frac{1}{\sqrt{N}}Y_N$ converges to 1 almost surely as $N \rightarrow \infty$.*

In [49], Tao computes the asymptotic location of the outlier eigenvalues for bounded rank perturbations of iid random matrices.

Theorem 1.7 (Outliers for small low rank perturbations of iid matrices, [49]). *Let ξ be a complex random variable with mean zero, unit variance, and finite fourth moment. For each $N \geq 1$, let Y_N be a $N \times N$ random matrix whose entries are iid copies of ξ , and let C_N be a deterministic matrix with rank $O(1)$. Let $\varepsilon > 0$, and suppose that for all sufficiently large N , there are no eigenvalues of C_N in the band $\{z \in \mathbb{C} : 1 + \varepsilon < |z| < 1 + 3\varepsilon\}$, and there are j eigenvalues $\lambda_1(C_N), \dots, \lambda_j(C_N)$ for some $j = O(1)$ in the region $\{z \in \mathbb{C} : |z| \geq 1 + 3\varepsilon\}$. Then, almost surely, for sufficiently large N , there are precisely j eigenvalues $\lambda_1(\frac{1}{\sqrt{N}}Y_N + C_N), \dots, \lambda_j(\frac{1}{\sqrt{N}}Y_N + C_N)$ of $\frac{1}{\sqrt{N}}Y_N + C_N$ in the region $\{z \in \mathbb{C} : |z| \geq 1 + 2\varepsilon\}$, and after labeling these eigenvalues properly,*

$$\lambda_i \left(\frac{1}{\sqrt{N}}Y_N + C_N \right) = \lambda_i(C_N) + o(1)$$

as $N \rightarrow \infty$ for each $1 \leq i \leq j$.

Recently, Benaych-Georges and Rochet [11] obtained an analogous result for finite rank perturbations of random matrices whose distributions are invariant under the left and right actions of the unitary group. Benaych-Georges and Rochet also study the fluctuations of the outlier eigenvalues.

Similar results have also been obtained for Wigner random matrices. When the atom variables have finite fourth moment, the asymptotic behavior of the spectral radius can be computed [5, Theorem 5.2].

Theorem 1.8 (No outliers for Wigner matrices). *Let ξ be a complex random variable with mean zero, unit variance, and finite fourth moment, and let ζ be a real random variables with mean zero and finite variance. For each $N \geq 1$, let Y_N be a Wigner matrix of size N with atom variables ξ, ζ . Then the spectral radius of $\frac{1}{\sqrt{N}}Y_N$ converges to 2 almost surely as $N \rightarrow \infty$.*

The asymptotic location of the outliers for bounded rank perturbations of Wigner matrices and other classes of self adjoint random matrices have also been determined. In fact, the fluctuations of the outlier eigenvalues can be explicitly computed. We refer the reader to [8, 9, 10, 17, 18, 19, 23, 24, 35, 36, 42, 43, 44] and references therein for further details.

Theorem 1.9 (Outliers for small low rank perturbations of Wigner matrices, [44]). *Let ξ be a real random variable with mean zero, unit variance, and finite fourth moment,*

and let ζ be a real random variables with mean zero and finite variance. For each $N \geq 1$, let Y_N be a Wigner matrix of size N with atom variables ξ, ζ . Let $k \geq 1$. For each $N \geq k$, let C_N be a $N \times N$ deterministic Hermitian matrix with rank k and nonzero eigenvalues $\lambda_1(C_N), \dots, \lambda_k(C_N)$, where $k, \lambda_1(C_N), \dots, \lambda_k(C_N)$ are independent of N . Let $S = \{1 \leq i \leq k : |\lambda_i(C_N)| > 1\}$. Then we have the following.

- For all $i \in S$, after labeling the eigenvalues of $\frac{1}{\sqrt{N}}Y_N + C_N$ properly,

$$\lambda_i \left(\frac{1}{\sqrt{N}}Y_N + C_N \right) \rightarrow \lambda_i(C_N) + \frac{1}{\lambda_i(C_N)}$$

in probability as $N \rightarrow \infty$.

- For all $i \in \{1, \dots, k\} \setminus S$, after labeling the eigenvalues of $\frac{1}{\sqrt{N}}Y_N + C_N$ properly,

$$\left| \lambda_i \left(\frac{1}{\sqrt{N}}Y_N + C_N \right) \right| \rightarrow 2$$

in probability as $N \rightarrow \infty$.

Remark 1.10. Under additional assumptions on the atom variables ξ, ζ , the convergence in Theorem 1.9 can be strengthened to almost sure convergence [17].

Non-Hermitian finite rank perturbations of random Hermitian matrices have been studied in the mathematical physics literature. We refer the reader to [25, 26, 27] and references therein for further details.

1.3 Elliptic random matrices

We consider the following class of random matrices with dependent entries that generalizes the ensembles introduced above. These so-called elliptic random matrices were originally introduced by Girko [29, 30].

Definition 1.11 (Condition **C1**). Let (ξ_1, ξ_2) be a random vector in \mathbb{R}^2 , where both ξ_1, ξ_2 have mean zero and unit variance. We set $\rho := \mathbb{E}[\xi_1 \xi_2]$. Let $\{y_{ij}\}_{i,j \geq 1}$ be an infinite double array of real random variables. For each $N \geq 1$, we define the $N \times N$ random matrix $Y_N = (y_{ij})_{i,j=1}^N$. We say the sequence of random matrices $\{Y_N\}_{N \geq 1}$ satisfies condition **C1** with atom variables (ξ_1, ξ_2) if the following hold:

- $\{y_{ii} : 1 \leq i\} \cup \{(y_{ij}, y_{ji}) : 1 \leq i < j\}$ is a collection of independent random elements,
- $\{(y_{ij}, y_{ji}) : 1 \leq i < j\}$ is a collection of iid copies of (ξ_1, ξ_2) ,
- $\{y_{ii} : 1 \leq i\}$ is a collection of iid random variables with mean zero and finite variance.

Remark 1.12. Let $\{Y_N\}_{N \geq 1}$ be a sequence of random matrices that satisfy condition **C1** with atom variables (ξ_1, ξ_2) . If $\rho := \mathbb{E}[\xi_1 \xi_2] = 1$, then $\{Y_N\}_{N \geq 1}$ is a sequence of Wigner real symmetric matrices.

Remark 1.13. Let ξ be a real random variable with mean zero and unit variance. For each $N \geq 1$, let Y_N be a $N \times N$ random matrix whose entries are iid copies of ξ . Then $\{Y_N\}_{N \geq 1}$ is a sequence of random matrices that satisfy condition **C1**.

If $\{Y_N\}_{N \geq 1}$ is a sequence of random matrices that satisfy condition **C1**, then it was shown in [40] that the limiting ESD of $\frac{1}{\sqrt{N}}Y_N$ is given by the uniform distribution on the interior of an ellipse. The same conclusion was shown to hold by Naumov [39] for elliptic random matrices whose atom variables satisfy additional moment assumptions.

For $-1 < \rho < 1$, define the ellipsoid

$$\mathcal{E}_\rho := \left\{ z \in \mathbb{C} : \frac{\operatorname{Re}(z)^2}{(1+\rho)^2} + \frac{\operatorname{Im}(z)^2}{(1-\rho)^2} \leq 1 \right\}. \tag{1.3}$$

Let

$$F_\rho(x, y) := \mu_\rho(\{z \in \mathbb{C} : \operatorname{Re}(z) \leq x, \operatorname{Im}(z) \leq y\}),$$

where μ_ρ is the uniformly probability measure on \mathcal{E}_ρ . It will also be convenient to define \mathcal{E}_ρ when $\rho = \pm 1$. For $\rho = 1$, let \mathcal{E}_1 be the line segment $[-2, 2]$, and for $\rho = -1$, we let \mathcal{E}_{-1} be the line segment $[-2, 2]\sqrt{-1}$ on the imaginary axis¹.

Theorem 1.14 (Elliptic law, [40]). *Let $\{Y_N\}_{N \geq 1}$ be a sequence of random matrices that satisfies condition **C1** with atom variables (ξ_1, ξ_2) , where $\rho = \mathbb{E}[\xi_1 \xi_2]$, and assume $-1 < \rho < 1$. For each $N \geq 1$, let A_N be a $N \times N$ matrix, and assume the sequence $\{A_N\}_{N \geq 1}$ satisfies $\operatorname{rank}(A_N) = o(N)$ and $\sup_{N \geq 1} \frac{1}{N^2} \|A_N\|_2^2 < \infty$. Then the ESD of $\frac{1}{\sqrt{N}}(Y_N + A_N)$ converges almost surely to F_ρ as $N \rightarrow \infty$.*

Remark 1.15. *A version of Theorem 1.14 holds when ξ_1, ξ_2 are complex random variables [40]. However, this note will only focus on real elliptic random matrices.*

2 Main results

In this note, we consider the outliers of perturbed elliptic random matrices. In particular, we consider versions of Theorem 1.6, Theorem 1.7, Theorem 1.8, and Theorem 1.9 for elliptic random matrices whose entries have finite fourth moment.

Definition 2.1 (Condition **C0**). *Let (ξ_1, ξ_2) be a random vector in \mathbb{R}^2 , where both ξ_1, ξ_2 have mean zero and unit variance. We set $\rho := \mathbb{E}[\xi_1 \xi_2]$. For each $N \geq 1$, let Y_N be a $N \times N$ random matrix. We say the sequence of random matrices $\{Y_N\}_{N \geq 1}$ satisfies condition **C0** with atom variables (ξ_1, ξ_2) if the following conditions hold:*

- The sequence $\{Y_N\}_{N \geq 1}$ satisfies condition **C1** with atom variables (ξ_1, ξ_2) ,
- We have

$$M_4 := \max\{\mathbb{E}|\xi_1|^4, \mathbb{E}|\xi_2|^4\} < \infty.$$

We will also define the neighborhoods

$$\mathcal{E}_{\rho, \delta} := \{z \in \mathbb{C} : \operatorname{dist}(z, \mathcal{E}_\rho) \leq \delta\}$$

for any $\delta > 0$.

We first consider a version of Theorem 1.6 and Theorem 1.8 for elliptic random matrices. Because of the elliptic shape of the limiting ESD, it is not enough to just consider the spectral radius.

Theorem 2.2 (No outliers for elliptic random matrices). *Let $\{Y_N\}_{N \geq 1}$ be a sequence of random matrices that satisfies condition **C0** with atom variables (ξ_1, ξ_2) , where $\rho = \mathbb{E}[\xi_1 \xi_2]$. Let $\delta > 0$. Then, almost surely, for N sufficiently large, all the eigenvalues of $\frac{1}{\sqrt{N}}Y_N$ are contained in $\mathcal{E}_{\rho, \delta}$.*

Theorem 1.14 and Theorem 2.2 immediately imply the following asymptotic behavior for the spectral radius of elliptic random matrices.

¹We use $\sqrt{-1}$ to denote the imaginary unit and reserve i as an index.

Corollary 2.3 (Spectral radius of elliptic random matrices). *Let $\{Y_N\}_{N \geq 1}$ be a sequence of random matrices that satisfies condition **C0** with atom variables (ξ_1, ξ_2) , where $\rho = \mathbb{E}[\xi_1 \xi_2]$. Then the spectral radius of $\frac{1}{\sqrt{N}}Y_N$ converges almost surely to $1 + |\rho|$ as $N \rightarrow \infty$.*

We now consider the analogue of Theorem 1.7 and Theorem 1.9 for elliptic random matrices. Figure 1 shows an eigenvalue plot of a perturbed elliptic random matrix as well as the location of the outlier eigenvalues predicted by the following theorem.

Theorem 2.4 (Outliers for low rank perturbations of elliptic random matrices). *Let $k \geq 1$ and $\delta > 0$. Let $\{Y_N\}_{N \geq 1}$ be a sequence of random matrices that satisfies condition **C0** with atom variables (ξ_1, ξ_2) , where $\rho = \mathbb{E}[\xi_1 \xi_2]$. For each $N \geq 1$, let C_N be a deterministic $N \times N$ matrix, where $\sup_{N \geq 1} \text{rank}(C_N) \leq k$ and $\sup_{N \geq 1} \|C_N\| = O(1)$. Suppose for N sufficiently large, there are no nonzero eigenvalues of C_N which satisfy*

$$\lambda_i(C_N) + \frac{\rho}{\lambda_i(C_N)} \in \mathcal{E}_{\rho, 3\delta} \setminus \mathcal{E}_{\rho, \delta} \quad \text{with} \quad |\lambda_i(C_N)| > 1, \tag{2.1}$$

and there are j eigenvalues $\lambda_1(C_N), \dots, \lambda_j(C_N)$ for some $j \leq k$ which satisfy

$$\lambda_i(C_N) + \frac{\rho}{\lambda_i(C_N)} \in \mathbb{C} \setminus \mathcal{E}_{\rho, 3\delta} \quad \text{with} \quad |\lambda_i(C_N)| > 1.$$

Then, almost surely, for N sufficiently large, there are exactly j eigenvalues of $\frac{1}{\sqrt{N}}Y_N + C_N$ in the region $\mathbb{C} \setminus \mathcal{E}_{\rho, 2\delta}$, and after labeling the eigenvalues properly,

$$\lambda_i \left(\frac{1}{\sqrt{N}}Y_N + C_N \right) = \lambda_i(C_N) + \frac{\rho}{\lambda_i(C_N)} + o(1)$$

for each $1 \leq i \leq j$.

Remark 2.5. *Theorem 2.4 generalizes the results of both Theorem 1.7 and Theorem 1.9. Indeed, if $\rho = 1$, then $\{Y_N\}_{N \geq 1}$ is a sequence of Wigner real symmetric matrices. In this case, Theorem 2.4 implies the almost sure convergence of the outlier eigenvalues to the locations described by Theorem 1.9. Additionally, Theorem 2.4 also deals with the case when C_N is non-Hermitian. On the other hand, if $\{Y_N\}_{N \geq 1}$ is a sequence of random matrices whose entries are iid random variables, then $\rho = 0$, and Theorem 2.4 gives precisely the results of Theorem 1.7.*

Remark 2.6. *In [17, 19], Capitaine, Donati-Martin, Féral, and Février consider spiked deformations of Wigner random matrices plus deterministic matrices. Theorem 2.4 can be viewed as a non-Hermitian extension of the results in [17, 19]. Indeed, the subordination functions appearing in [19] appears very naturally in our analysis; see Remark 5.4 for further details.*

Remark 2.7. *Theorem 2.4 requires that there are no nonzero eigenvalues of C_N which satisfy (2.1). Since δ is arbitrary, if the eigenvalues of C_N do not change with N , this condition can be ignored. This condition is analogous to the requirements of Theorem 1.7. Indeed, Theorem 1.7 requires that there are no eigenvalues of C_N in the band $\{z \in \mathbb{C} : 1 + \varepsilon < |z| < 1 + 3\varepsilon\}$.*

We now consider the case of elliptic random matrices with nonzero mean, which we write as $\frac{1}{\sqrt{N}}Y_N + \mu\sqrt{N}\varphi_N\varphi_N^*$, where $\{Y_N\}_{N \geq 1}$ is a sequence of random matrices that satisfies condition **C0** with atom variables (ξ_1, ξ_2) , μ is a fixed nonzero complex number (independent of N), and φ_N is the unit vector $\varphi_N := \frac{1}{\sqrt{N}}(1, \dots, 1)^*$. This corresponds to shifting the entries of Y_N by μ (so they have mean μ instead of mean zero). The elliptic law still holds for this rank one perturbation of $\frac{1}{\sqrt{N}}Y_N$, thanks to Theorem 1.14. In view of Theorem 2.4, we show there is a single outlier for this ensemble near $\mu\sqrt{N}$.

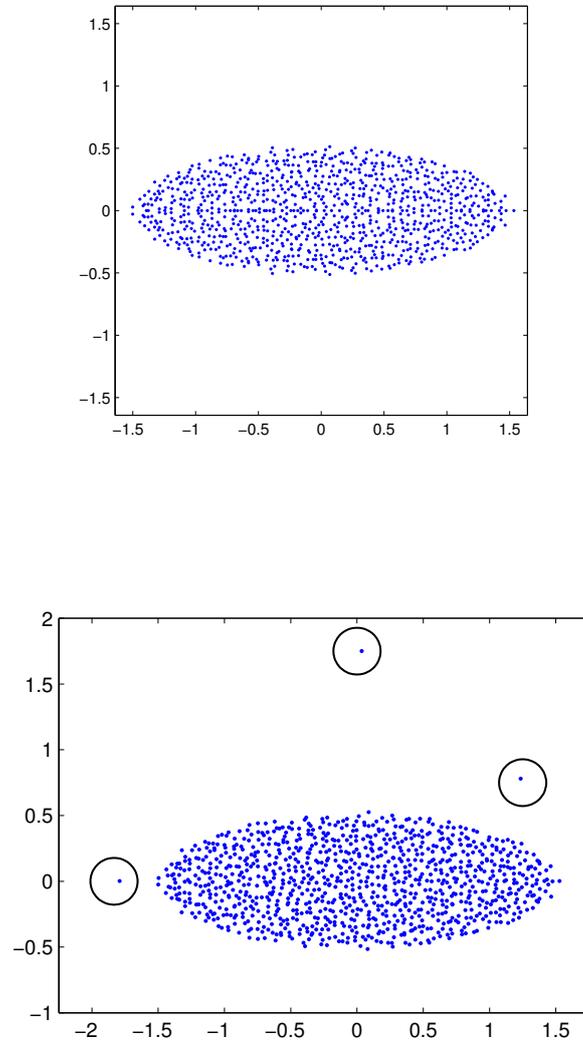


Figure 1: The plot on top shows the eigenvalues of a single $N \times N$ elliptic random matrix with Gaussian entries when $N = 1000$ and $\rho = 1/2$. The plot on bottom shows the eigenvalues of the same elliptic matrix after perturbing it by a diagonal matrix with three nonzero eigenvalues: $2\sqrt{-1}$, $-\frac{3}{2}$, and $1 + \sqrt{-1}$. The three circles are centered at $\frac{7}{4}\sqrt{-1}$, $-\frac{11}{6}$, and $\frac{5}{4} + \frac{3}{4}\sqrt{-1}$, respectively, and each have radius $N^{-1/4}$.

Theorem 2.8 (Outlier for elliptic random matrices with nonzero mean). *Let $\delta > 0$. Let $\{Y_N\}_{N \geq 1}$ be a sequence of random matrices that satisfies condition **C0** with atom variables (ξ_1, ξ_2) , where $\rho = \mathbb{E}[\xi_1 \xi_2]$, and let μ be a nonzero complex number independent of N . Then almost surely, for sufficiently large N , all the eigenvalues of $\frac{1}{\sqrt{N}}Y_N + \mu\sqrt{N}\varphi_N\varphi_N^*$ lie in $\mathcal{E}_{\rho, \delta}$, with a single exception taking the value $\mu\sqrt{N} + o(1)$.*

Remark 2.9. *A version of Theorem 2.8 was proven by Füredi and Komlós in [24] for a class of real symmetric Wigner matrices. Moreover, Füredi and Komlós study the fluctuations of the outlier eigenvalue. Tao [49] verified Theorem 2.8 when Y_N is a random matrix with iid entries.*

One of the keys to proving Theorem 2.4 and Theorem 2.8 is to control the least singular value of a perturbed elliptic random matrix. Let M be a $N \times N$ matrix. The singular values of M are the eigenvalues of $|M| := \sqrt{MM^*}$. We let $\sigma_1(M) \geq \dots \geq \sigma_N(M) \geq 0$ denote the singular values of M . In particular, the largest and smallest singular values are

$$\sigma_1(M) := \sup_{\|x\|=1} \|Mx\|, \quad \sigma_N(M) := \inf_{\|x\|=1} \|Mx\|,$$

where $\|x\|$ denotes the Euclidian norm of the vector x . We let $\|M\|$ denote the spectral norm of M . It follows that the largest and smallest singular values can be written in terms of the spectral norm. Indeed, $\sigma_1(M) = \|M\|$ and $\sigma_N(M) = 1/\|M^{-1}\|$ provided M is invertible.

We now consider a lower bound for the least singular value of perturbed elliptic random matrices of the form $\frac{1}{\sqrt{N}}Y_N - zI$, where I denotes the identity matrix. A lower bound of the form

$$\sigma_N \left(\frac{1}{\sqrt{N}}Y_N - zI \right) \geq N^{-A},$$

for some $A > 0$, was shown to hold with high probability in [39, 40]. Below, we consider only the case when z is outside the ellipse \mathcal{E}_ρ and thus obtain a constant lower bound independent of N .

Theorem 2.10 (Least singular value bound). *Let $\{Y_N\}_{N \geq 1}$ be a sequence of random matrices that satisfies condition **C0** with atom variables (ξ_1, ξ_2) , where $\rho = \mathbb{E}[\xi_1 \xi_2]$. Let $\delta > 0$. Then there exists $c > 0$ such that almost surely, for N sufficiently large,*

$$\inf_{\text{dist}(z, \mathcal{E}_\rho) \geq \delta} \sigma_N \left(\frac{1}{\sqrt{N}}Y_N - zI \right) \geq c.$$

In fact, Theorem 2.2 follows immediately from Theorem 2.10.

Proof of Theorem 2.2. We note that z is an eigenvalue of $\frac{1}{\sqrt{N}}Y_N$ if and only if

$$\det \left(\frac{1}{\sqrt{N}}Y_N - zI \right) = 0.$$

On the other hand,

$$\left| \det \left(\frac{1}{\sqrt{N}}Y_N - zI \right) \right| = \prod_{i=1}^N \sigma_i \left(\frac{1}{\sqrt{N}}Y_N - zI \right).$$

Thus, we conclude that z is an eigenvalue of $\frac{1}{\sqrt{N}}Y_N$ if and only if

$$\sigma_N \left(\frac{1}{\sqrt{N}}Y_N - zI \right) = 0.$$

The claim therefore follows from Theorem 2.10. □

The condition number $\sigma_1(M)/\sigma_N(M)$ of a $N \times N$ matrix M plays an important role in numerical linear algebra (see for example [7]). As a consequence of Theorem 2.10, we obtain the following bound for the condition number of perturbed elliptic random matrices that satisfy condition **C0**.

Corollary 2.11 (Condition number bound). *Let $\{Y_N\}_{N \geq 1}$ be a sequence of random matrices that satisfies condition **C0** with atom variables (ξ_1, ξ_2) , where $\rho = \mathbb{E}[\xi_1 \xi_2]$. Fix $z \notin \mathcal{E}_\rho$. Then there exists $C > 0$ (depending on z) such that almost surely, for N sufficiently large,*

$$\frac{\sigma_1\left(\frac{1}{\sqrt{N}}Y_N - zI\right)}{\sigma_N\left(\frac{1}{\sqrt{N}}Y_N - zI\right)} \leq C.$$

Proof. In view of Theorem 2.10, it suffices to show that almost surely

$$\sigma_1\left(\frac{1}{\sqrt{N}}Y_N - zI\right) \leq C$$

for N sufficiently large. Since

$$\sigma_1\left(\frac{1}{\sqrt{N}}Y_N - zI\right) = \left\| \frac{1}{\sqrt{N}}Y_N - zI \right\| \leq \frac{1}{\sqrt{N}}\|Y_N\| + |z|,$$

it suffices to show that almost surely

$$\frac{1}{\sqrt{N}}\|Y_N\| \leq C$$

for N sufficiently large. The claim now follow from Lemma 3.3 below. Indeed, the bound on the spectral norm of Y_N has previously been obtained in [39] and follows from [5, Theorem 5.2]. \square

2.1 Overview

In order to prove Theorem 2.4, we will make use of Sylvester’s determinant identity:

$$\det(I + AB) = \det(I + BA), \tag{2.2}$$

where A is a $N \times k$ matrix and B is a $k \times N$ matrix. In particular, the left-hand side of (2.2) is a $N \times N$ determinant, while the right-hand side is a $k \times k$ determinant.

To outline the main idea, which is based on the arguments of Benaych-Georges and Rao [8], consider the rank one perturbation $C_N = vu^*$. In order to study the outlier eigenvalues, we will need to solve the equation

$$\det\left(\frac{1}{\sqrt{N}}Y_N + C_N - zI\right) = 0 \tag{2.3}$$

for $z \notin \mathcal{E}_\rho$. Assume z is not an eigenvalue of $\frac{1}{\sqrt{N}}Y_N$, then we can rewrite (2.3) as

$$\det\left(I + \left(\frac{1}{\sqrt{N}}Y_N - zI\right)^{-1} C_N\right) = 0.$$

From (2.2), we find that this is equivalent to solving

$$1 + u^* \left(\frac{1}{\sqrt{N}}Y_N - zI\right)^{-1} v = 0.$$

Thus, the problem of locating the outlier eigenvalues reduces to studying the resolvent

$$G_N(z) := \left(\frac{1}{\sqrt{N}} Y_N - zI \right)^{-1}.$$

We develop an isotropic limit law in Section 5 to compute the limit of $u^* G_N v$; this limit law is inspired by the isotropic semicircle law developed by Knowles and Yin [35, 36] for Wigner random matrices. Namely in Theorem 5.1 we show that not only does the trace of $G_N(z)$ almost surely converge to some function $m(z)$ (defined in (4.3)) but arbitrary bilinear forms $u^* G_N v$ almost surely converge to $m(z) u^* v$.

However, instead of working with G_N directly, it will often be more convenient to work with the $2N \times 2N$ Hermitian matrix²

$$\Xi_N := \begin{bmatrix} 0 & \frac{1}{\sqrt{N}} Y_N - zI \\ \frac{1}{\sqrt{N}} Y_N^* - \bar{z}I & 0 \end{bmatrix}$$

and its resolvent $(\Xi_N - \eta I)^{-1}$. In fact, the eigenvalues of Ξ_N are given by the singular values

$$\pm \sigma_1 \left(\frac{1}{\sqrt{N}} Y_N - zI \right), \dots, \pm \sigma_N \left(\frac{1}{\sqrt{N}} Y_N - zI \right).$$

Thus, for $\text{Im}(\eta) > 0$, the matrix $\Xi_N - \eta I$ is always invertible. Moreover, when $\eta = 0$, the resolvent becomes

$$\begin{bmatrix} 0 & \left(\frac{1}{\sqrt{N}} Y_N^* - \bar{z}I \right)^{-1} \\ \left(\frac{1}{\sqrt{N}} Y_N - zI \right)^{-1} & 0 \end{bmatrix}.$$

In other words, we can recover G_N by letting η tend to zero. Similarly, we will bound the least singular value of $\frac{1}{\sqrt{N}} Y_N - zI$ and prove Theorem 2.10 by studying the eigenvalues of the resolvent $(\Xi_N - \eta I)^{-1}$ when $\text{Im}(\eta) = N^{-\beta}$ for some $\beta > 0$.

The paper is organized as follows. We present our preliminary tools in Section 3 and Section 4. In particular, Section 3 contains a standard truncation lemma; in Section 4, we study the stability of a fixed point equation which will determine the asymptotic behavior of the diagonal entries of G_N . In Section 5, we apply the truncation lemma from Section 3 to reduce both Theorem 2.4 and Theorem 2.10 to the case where we only need to consider elliptic random matrices whose entries are bounded. We also introduce an isotropic limit law for G_N and prove Theorem 2.8 in Section 5. Finally, we complete the proof of Theorem 2.10 in Section 6 and complete the proof of Theorem 2.4 in Section 7.

A number of auxiliary proofs and results are contained in the appendix. Appendix A contains a somewhat standard proof of the truncation lemma from Section 3. Appendix B contains a large deviation estimate for bilinear forms. In Appendix C, we study some additional properties of a limiting spectral measure which was analyzed in [40].

2.2 Notation

We use asymptotic notation (such as O, o, Ω) under the assumption that $N \rightarrow \infty$. We use $X \ll Y, Y \gg X, Y = \Omega(X)$, or $X = O(Y)$ to denote the bound $X \leq CY$ for all sufficiently large N and for some constant C . Notations such as $X \ll_k Y$ and $X = O_k(Y)$ mean that the hidden constant C depends on another constant k . $X = o(Y)$ or $Y = \omega(X)$ means that $X/Y \rightarrow 0$ as $N \rightarrow \infty$.

²Actually, for notational convenience we will work with Ξ_N conjugated by a permutation matrix (see Section 3.2 for complete details).

An event E , which depends on N , is said to hold with *overwhelming probability* if $\mathbb{P}(E) \geq 1 - O_C(N^{-C})$ for every constant $C > 0$. We let $\mathbf{1}_E$ denote the indicator function of the event E . E^c denotes the complement of the event E .

We let $\|M\|$ denote the spectral norm of M . $\|M\|_2$ denotes the Hilbert-Schmidt norm of M (defined in (1.2)). We let I_N denote the $N \times N$ identity matrix. Often we will just write I for the identity matrix when the size can be deduced from the context. For a square matrix M , we let $\text{tr}_N M := \frac{1}{N} \text{tr} M$.

We write a.s., a.a., and a.e. for almost surely, Lebesgue almost all, and Lebesgue almost everywhere respectively. We use $\sqrt{-1}$ to denote the imaginary unit and reserve i as an index.

We let C and K denote constants that are non-random and may take on different values from one appearance to the next. The notation K_p means that the constant K depends on another parameter p .

3 Preliminary tools and notation

In this section, we consider a number of tools we will need to prove our main results. We also introduce some new notation, which we will use throughout the paper.

Let $\{Y_N\}_{N \geq 1}$ be a sequence of random matrices that satisfies condition **C0** with atom variables (ξ_1, ξ_2) . We will work with the resolvent G_N defined by

$$G_N = G_N(z) := \left(\frac{1}{\sqrt{N}} Y_N - zI \right)^{-1}. \tag{3.1}$$

and it's trace, denoted $m_N(z)$

$$m_N(z) := \frac{1}{N} \text{tr} G_N(z). \tag{3.2}$$

In order to work with the resolvent, we will need control of the spectral norm $\|G_N\|$. We bound the spectral norm of G_N for z sufficiently large by bounding the spectral norm of $\frac{1}{\sqrt{N}} Y_N$ in the next subsection.

When working with G_N , we will take advantage of the following well known resolvent identity: for any invertible $N \times N$ matrices A and B ,

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}. \tag{3.3}$$

Suppose A is an invertible square matrix. Let u, v be vectors. If $1 + v^* A^{-1} u \neq 0$, from (3.3) one can deduce the Sherman–Morrison rank one perturbation formula (see [33, Section 0.7.4]):

$$(A + uv^*)^{-1} = A^{-1} - \frac{A^{-1}uv^*A^{-1}}{1 + v^*A^{-1}u} \tag{3.4}$$

and

$$(A + uv^*)^{-1}u = \frac{A^{-1}u}{1 + v^*A^{-1}u}. \tag{3.5}$$

From [33, Section 0.7.3], we obtain the inverse of a block matrix and Schur's complement:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{bmatrix}, \tag{3.6}$$

where A, B, C, D are matrix sub-blocks and $D, A - BD^{-1}C$ are non-singular. In the case that $A, D - CA^{-1}B$ are invertible, we obtain

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}.$$

It follows from the block matrix inversion formula that

$$\operatorname{tr} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \operatorname{tr}(A^{-1}) + \operatorname{tr}((D - CA^{-1}B)^{-1}(I + CA^{-2}B)) \tag{3.7}$$

provided $A, D - CA^{-1}B$ are invertible.

3.1 Bounds on the spectral norm

We begin with the following deterministic bound.

Lemma 3.1 (Spectral norm of the resolvent for large $|z|$). *Let M be a $N \times N$ matrix that satisfies $\|M\| \leq \mathcal{K}$. Then*

$$\|(M - zI)^{-1}\| \leq \frac{1}{\varepsilon}$$

for all $z \in \mathbb{C}$ with $|z| \geq \mathcal{K} + \varepsilon$.

Proof. By writing out the Neumann series, we obtain

$$\|(M - zI)^{-1}\| \leq \left\| \frac{1}{z} + \frac{1}{z} \sum_{k=1}^{\infty} \frac{M^k}{z^k} \right\| \leq \frac{1}{\mathcal{K} + \varepsilon} \sum_{k=0}^{\infty} \left(\frac{\mathcal{K}}{\mathcal{K} + \varepsilon} \right)^k \leq \frac{1}{\varepsilon}$$

for $|z| \geq \mathcal{K} + \varepsilon$. □

Remark 3.2. *If H is a Hermitian matrix, we have*

$$\|(H - zI)^{-1}\| \leq |\operatorname{Im}(z)|^{-1}, \tag{3.8}$$

provided $\operatorname{Im}(z) \neq 0$.

We will use the following estimate for the spectral norm. We note that the bound in Lemma 3.3 below is not sharp, but will suffice for our purposes.

Lemma 3.3 (Spectral norm bound). *Let $\{Y_N\}_{N \geq 1}$ be a sequence of random matrices that satisfies condition **C0** with atom variables (ξ_1, ξ_2) . Then a.s.*

$$\limsup_{N \rightarrow \infty} \left\| \frac{1}{\sqrt{N}} Y_N \right\| \leq 4. \tag{3.9}$$

Proof. We write

$$Y_N = \frac{Y_N + Y_N^*}{2} + \sqrt{-1} \frac{Y_N - Y_N^*}{2\sqrt{-1}}$$

and hence

$$\|Y_N\| \leq \left\| \frac{Y_N + Y_N^*}{2} \right\| + \left\| \frac{Y_N - Y_N^*}{2\sqrt{-1}} \right\|. \tag{3.10}$$

We observe that $\frac{Y_N + Y_N^*}{2}$ and $\frac{Y_N - Y_N^*}{2\sqrt{-1}}$ are both Hermitian random matrices.

Consider the matrix $\frac{Y_N + Y_N^*}{2}$. By assumption, the diagonal entries of the matrix have mean zero and finite variance. The above-diagonal entries are iid copies of $\frac{\xi_1 + \xi_2}{2}$. Thus the above-diagonal entries have mean zero and variance

$$\frac{1}{4} \mathbb{E}|\xi_1 + \xi_2|^2 \leq \frac{1}{2} (\mathbb{E}|\xi_1|^2 + \mathbb{E}|\xi_2|^2) \leq 1.$$

Moreover, the above-diagonal entries have finite fourth moment:

$$\mathbb{E} \left| \frac{\xi_1 + \xi_2}{2} \right|^4 \leq \mathbb{E}|\xi_1|^4 + \mathbb{E}|\xi_2|^4 \leq 2M_4 < \infty.$$

By [5, Theorem 5.2], we obtain a.s.

$$\limsup_{N \rightarrow \infty} \left\| \frac{Y_N + Y_N^*}{2\sqrt{N}} \right\| \leq 2.$$

Similarly, we have a.s.

$$\limsup_{N \rightarrow \infty} \left\| \frac{Y_N - Y_N^*}{2\sqrt{N}} \right\| \leq 2.$$

The claim follows from the bounds above and (3.10). □

3.2 Hermitization

In order to study the spectrum of a non-normal matrix it is often useful to instead consider the spectrum of a family of Hermitian matrices.

We define the *Hermitization* of an $N \times N$ matrix X to be an $N \times N$ matrix with entries that are 2×2 block matrices. The ij^{th} entry is the 2×2 block:

$$\begin{pmatrix} 0 & X_{ij} \\ \overline{X_{ji}} & 0 \end{pmatrix}$$

We note the Hermitization of X can be conjugated by a $2N \times 2N$ permutation matrix to

$$\begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}$$

Let $X_N := \frac{1}{\sqrt{N}}Y_N$ and define H_N to be the Hermitization of X_N . We will generally treat H_N as an $N \times N$ matrix with entries that are 2×2 blocks, but occasionally it will instead be useful to consider H_N as a $2N \times 2N$ matrix.

Additionally we define the 2×2 matrix

$$q := \begin{pmatrix} \eta & z \\ \overline{z} & \eta \end{pmatrix} \tag{3.11}$$

with $\eta = E + \sqrt{-1}t \in \mathbb{C}^+ := \{w \in \mathbb{C} : \text{Im}(w) > 0\}$ and $z \in \mathbb{C}$. We define the Hermitized resolvent

$$R_N(q) = R_N(\eta, z) := (H_N - I \otimes q)^{-1}.$$

Note that this is the usual resolvent of the Hermitization of $X_N - zI$, hence it inherits the usual properties of resolvents. For example, its operator norm is bound from above by t^{-1} . We will use the Hermitized resolvent extensively in Section 6 to estimate the least singular value of $X_N - zI$ and in Section 7.2 to estimate the expectation of bilinear forms involving $G_N(z)$.

3.3 Truncation

Let $\{Y_N\}_{N \geq 1}$ be a sequence of random matrices that satisfies condition **C0** with atom variables (ξ_1, ξ_2) . Instead of working with Y_N directly, we will work with a truncated version of this matrix. Specifically, we will work with a matrix \hat{Y}_N where the entries are truncated versions of the original entries of Y_N .

Recall that $Y_N = (y_{ij})_{i,j=1}^N$. Let $L > 0$. We define

$$\tilde{\xi}_i := \xi_i \mathbf{1}_{\{|\xi_i| \leq L\}} - \mathbb{E} [\xi_i \mathbf{1}_{\{|\xi_i| \leq L\}}]$$

for $i \in \{1, 2\}$, and

$$\tilde{\rho} := \mathbb{E} [\tilde{\xi}_1 \tilde{\xi}_2].$$

Here $\mathbf{1}_E$ denotes the indicator function of the event E . We will also define the truncated entries

$$\tilde{y}_{ij} := y_{ij} \mathbf{1}_{\{|y_{ij}| \leq L\}} - \mathbb{E} [y_{ij} \mathbf{1}_{\{|y_{ij}| \leq L\}}]$$

for $i \neq j$, and $\tilde{y}_{ii} := 0$ for all $i \geq 1$. We set $\tilde{Y}_N := (\tilde{y}_{ij})_{i,j=1}^N$.

We also define

$$\hat{\xi}_i := \frac{\tilde{\xi}_i}{\sqrt{\text{Var}(\tilde{\xi}_i)}}$$

for $i \in \{1, 2\}$, and

$$\hat{\rho} := \mathbb{E}[\hat{\xi}_1 \hat{\xi}_2].$$

We define the entries

$$\hat{y}_{ij} := \frac{\tilde{y}_{ij}}{\sqrt{\text{Var}(\tilde{y}_{ij})}}$$

for $i \neq j$, and $\hat{y}_{ii} := 0$ for all $i \geq 1$. We set $\hat{Y}_N := (\hat{y}_{ij})_{i,j=1}^N$. We also introduce the notations

$$\hat{G}_N = \hat{G}_N(z) := \left(\frac{1}{\sqrt{N}} \hat{Y}_N - zI \right)^{-1}, \tag{3.12}$$

$$\hat{m}_N(z) := \frac{1}{N} \text{tr} \hat{G}_N(z). \tag{3.13}$$

We verify the following standard truncation lemma.

Lemma 3.4 (Truncation). *Let $\{Y_N\}_{N \geq 1}$ be a sequence of random matrices that satisfies condition **C0** with atom variables (ξ_1, ξ_2) . Then there exists constants $C_0, L_0 > 0$ such that the following holds for all $L > L_0$.*

- $\{\hat{Y}_N\}_{N \geq 1}$ is a sequence of random matrices that satisfies condition **C0** with atom variables $(\hat{\xi}_1, \hat{\xi}_2)$.
- a.s., one has the bounds

$$\max_{1 \leq i, j} |\hat{y}_{ij}| \leq 4L \tag{3.14}$$

and

$$|\rho - \hat{\rho}| \leq \frac{C_0}{L}. \tag{3.15}$$

- a.s., one has

$$\limsup_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \|Y_N - \hat{Y}_N\| \leq \frac{C_0}{L} \tag{3.16}$$

and

$$\limsup_{N \rightarrow \infty} \sup_{|z| \geq 5} \|G_N(z) - \hat{G}_N(z)\| \leq \frac{C_0}{L}. \tag{3.17}$$

The proof of Lemma 3.4 follows somewhat standard arguments; we present the proof in Appendix A.

For the truncated matrices \hat{Y}_N , we have the following bound on the spectral norm.

Lemma 3.5 (Spectral norm bound for \hat{Y}_N). *Let $\{Y_N\}_{N \geq 1}$ be a sequence of random matrices that satisfies condition **C0** with atom variables (ξ_1, ξ_2) . Consider the truncated matrices $\{\hat{Y}_N\}_{N \geq 1}$ from Lemma 3.4 for any fixed $L > 0$. Let $\varepsilon > 0$. Then*

$$\frac{1}{\sqrt{N}} \|\hat{Y}_N\| \leq 4 + \varepsilon$$

with overwhelming probability.

The proof of Lemma 3.5 is almost identical to the proof of Lemma 3.3 except one applies [5, Remark 5.7] instead of [5, Theorem 5.2].

3.4 Martingale inequalities

The following standard bounds were originally proven for real random variables; the extension to the complex case is straightforward.

Lemma 3.6 (Rosenthal’s inequality, [16]). *Let $\{x_k\}$ be a complex martingale difference sequence with respect to the filtration $\{\mathcal{F}_k\}$. Then, for $p \geq 2$,*

$$\mathbb{E} \left| \sum x_k \right|^p \leq K_p \left(\mathbb{E} \left(\sum \mathbb{E}(|x_k|^2 | \mathcal{F}_{k-1}) \right)^{p/2} + \mathbb{E} \sum |x_k|^p \right).$$

Lemma 3.7 (Burkholder’s inequality, [16]). *Let $\{x_k\}$ be a complex martingale difference sequence with respect to the filtration $\{\mathcal{F}_k\}$. Then, for $p \geq 1$,*

$$\mathbb{E} \left| \sum x_k \right|^p \leq K_p \mathbb{E} \left(\sum |x_k|^2 \right)^{p/2}$$

Lemma 3.8 (Dilworth, [21]). *Let $\{\mathcal{F}_k\}$ be a filtration, $\{x_k\}$ a sequence of integrable random variables, and $1 \leq q \leq p < \infty$. Then*

$$\mathbb{E} \left(\sum |\mathbb{E}(x_k | \mathcal{F}_k)|^q \right)^{p/q} \leq \left(\frac{p}{q} \right)^{p/q} \mathbb{E} \left(\sum |x_k|^q \right)^{p/q}.$$

Lemma 3.9 (Lemma 6.11 of [5]). *Let $\{F_n\}$ be an increasing sequence of σ -fields and $\{X_n\}$ a sequence of random variables. Write $\mathbb{E}_k = \mathbb{E}(\cdot | F_k)$, $\mathbb{E}_\infty = \mathbb{E}(\cdot | F_\infty)$, $F_\infty := \bigvee_j F_j$. If $X_n \rightarrow 0$ a.s. and $\sup_n |X_n|$ is integrable, then a.s.*

$$\lim_{n \rightarrow \infty} \max_{k \leq n} E_k[X_n] = 0.$$

3.5 Concentration of bilinear forms

We establish the following large deviation estimate for bilinear forms, which is a consequence of Lemma B.1 from Appendix B.

Lemma 3.10 (Concentration of bilinear forms). *Let (x, y) be a random vector in \mathbb{C}^2 where x, y both have mean zero, unit variance, and satisfy*

- $\max\{|x|, |y|\} \leq L$ a.s.,
- $\mathbb{E}[\bar{x}y] = \rho$.

Let $(x_1, y_1), (x_2, y_2), \dots, (x_N, y_N)$ be iid copies of (x, y) , and set $X = (x_1, x_2, \dots, x_N)^T$ and $Y = (y_1, y_2, \dots, y_N)^T$. Let B be a $N \times N$ random matrix, independent of X and Y . Then for any integer $p \geq 2$, there exists a constant $K_p > 0$ such that, for any $t > 0$,

$$\mathbb{P} \left(\left| \frac{1}{N} X^* B Y - \frac{\rho}{N} \operatorname{tr} B \right| \geq t \right) \leq K_p \frac{L^{2p} \mathbb{E}(\operatorname{tr}(B B^*))^{p/2}}{N^p t^p}. \tag{3.18}$$

In particular, if $\|B\| \leq N^{1/4}$ a.s., then

$$\mathbb{P} \left(\left| \frac{1}{N} X^* B Y - \frac{\rho}{N} \operatorname{tr} B \right| \geq N^{-1/8} \right) \leq K_p \frac{L^{2p}}{N^{p/8}} \tag{3.19}$$

for any integer $p \geq 2$.

Proof. We first note that (3.19) follows from (3.18) by taking $t = N^{-1/8}$ and applying the deterministic bound

$$(\operatorname{tr}(B B^*))^{p/2} \leq N^{p/2} \|B B^*\|^{p/2} \leq N^{p/2} \|B\|^p.$$

It remains to prove (3.18). By Markov's inequality, it suffices to show

$$\mathbb{E} |X^*BY - \rho \operatorname{tr} B|^p \ll_p L^{2p} \mathbb{E} (\operatorname{tr}(BB^*))^{p/2} \tag{3.20}$$

for any integer $p \geq 2$. We will use Lemma B.1 from Appendix B to verify (3.20).

By conditioning on the matrix B (which is independent of X and Y), we apply Lemma B.1 and obtain

$$\begin{aligned} \mathbb{E} |X^*BY - \rho \operatorname{tr} B|^p &\ll_p \mathbb{E} \left[(L^4 \operatorname{tr}(BB^*))^{p/2} + L^{2p} \operatorname{tr}(BB^*)^{p/2} \right] \\ &\ll_p L^{2p} \left(\mathbb{E} (\operatorname{tr}(BB^*))^{p/2} + \mathbb{E} \operatorname{tr}(BB^*)^{p/2} \right) \\ &\ll_p L^{2p} \mathbb{E} (\operatorname{tr}(BB^*))^{p/2} \end{aligned}$$

since $\operatorname{tr}(BB^*)^{p/2} \leq (\operatorname{tr}(BB^*))^{p/2}$. □

3.6 ε -nets

We introduce ε -nets as a convenient way to discretize a compact set. Let $\varepsilon > 0$. A set X is an ε -net of a set Y if for any $y \in Y$, there exists $x \in X$ such that $\|x - y\| \leq \varepsilon$. We will need the following well-known estimate for the maximum size of an ε -net.

Lemma 3.11. *Let D be a compact subset of $\{z \in \mathbb{C} : |z| \leq M\}$. Then D admits an ε -net of size at most*

$$\left(1 + \frac{2M}{\varepsilon} \right)^2.$$

Proof. Let \mathcal{N} be maximal ε separated subset of D . That is, $|z - w| \geq \varepsilon$ for all distinct $z, w \in \mathcal{N}$ and no subset of D containing \mathcal{N} has this property. Such a set can always be constructed by starting with an arbitrary point in D and at each step selecting a point that is at least ε distance away from those already selected. Since D is compact, this procedure will terminate after a finite number of steps.

We now claim that \mathcal{N} is an ε -net of D . Suppose to the contrary. Then there would exist $z \in D$ that is at least ε from all points in \mathcal{N} . In other words, $\mathcal{N} \cup \{z\}$ would still be an ε -separated subset of D . This contradicts the maximal assumption above.

We now proceed by a volume argument. At each point of \mathcal{N} we place a ball of radius $\varepsilon/2$. By the triangle inequality, it is easy to verify that all such balls are disjoint and lie in the ball of radius $M + \varepsilon/2$ centered at the origin. Comparing the volumes give

$$|\mathcal{N}| \leq \frac{(M + \varepsilon/2)^2}{(\varepsilon/2)^2} = \left(1 + \frac{2M}{\varepsilon} \right)^2.$$

□

Similarly, if I is an interval on the real line with length $|I|$, then I admits an ε -net of size at most $1 + |I|/\varepsilon$.

4 Stability of the fixed point equation

We will study the limit of the sequence of functions $\{m_N\}_{N \geq 1}$ (defined in (3.2)). As is standard in random matrix theory, we will not compute the limit explicitly, but instead show that the limit satisfies a fixed point equation. In particular, we will show that the limiting function satisfies

$$\Delta(z) = -\frac{1}{z + \rho \Delta(z)}. \tag{4.1}$$

Remark 4.1. When $\rho > 0$, (4.1) also characterizes the Stieltjes transform of the semi-circle distribution with variance ρ (see for instance [5, Chapter 2]).

In this section, we study the stability of (4.1) for $-1 \leq \rho \leq 1$. We begin with a few preliminary results.

Lemma 4.2. For $-1 \leq \rho \leq 1$, $\pm 2\sqrt{\rho} \in \mathcal{E}_\rho$.

Proof. Let $z = \pm 2\sqrt{\rho}$. First consider the case when $0 \leq \rho \leq 1$. Then $z^2 = \operatorname{Re}(z)^2 = 4\rho$. Since $0 \leq (1 - \rho)^2 = (1 + \rho)^2 - 4\rho$, it follows that

$$\frac{z^2}{(1 + \rho)^2} = \frac{4\rho}{(1 + \rho)^2} \leq 1,$$

and hence $z \in \mathcal{E}_\rho$. A similar argument works for the case $-1 \leq \rho \leq 0$. □

Since (4.1) can be written as a quadratic polynomial, the solution of (4.1) has two branches when $\rho \neq 0$. We refer to the two branches as the solutions of (4.1).

Lemma 4.3 (Solutions of (4.1)). Consider equation (4.1). Then one has the following.

- (i) If $\rho = 0$, there exists exactly one solution of (4.1).
- (ii) If $-1 \leq \rho \leq 1$ and $\rho \neq 0$, there exists two solutions of (4.1), which are distinct and analytic outside the ellipsoid \mathcal{E}_ρ .
- (iii) For any $-1 \leq \rho \leq 1$, there exists a unique solution of (4.1), which we denote by $m(z)$, which is analytic outside \mathcal{E}_ρ and satisfies

$$\lim_{|z| \rightarrow \infty} m(z) = 0. \tag{4.2}$$

Furthermore,

$$m(z) := \begin{cases} \frac{-z + \sqrt{z^2 - 4\rho}}{2\rho} & \text{for } \rho \neq 0 \\ \frac{-1}{z} & \text{for } \rho = 0 \end{cases}, \tag{4.3}$$

where $\sqrt{z^2 - 4\rho}$ is the branch of the square root with branch cut $[-2\sqrt{\rho}, 2\sqrt{\rho}]$ for $\rho > 0$ and $[-2\sqrt{|\rho|}, 2\sqrt{|\rho|}]\sqrt{-1}$ for $\rho < 0$, and which equals z at infinity.

Proof. When $\rho = 0$, the results are trivial. Assume $\rho \neq 0$. By rewriting (4.1), we find

$$\rho\Delta(z)^2 + z\Delta(z) + 1 = 0.$$

Thus, by the quadratic equation, we have two solutions

$$m_1, m_2 = \frac{-z \pm \sqrt{z^2 - 4\rho}}{2\rho}, \tag{4.4}$$

where $\sqrt{z^2 - 4\rho}$ is the branch of the square root with branch cut $[-2\sqrt{\rho}, 2\sqrt{\rho}]$ for $\rho > 0$ and $[-2\sqrt{|\rho|}, 2\sqrt{|\rho|}]\sqrt{-1}$ for $\rho < 0$, and which equals z at infinity.

Now suppose $m_1(z) = m_2(z)$ for some $z \in \mathbb{C}$. Then we find

$$z = \pm 2\sqrt{\rho}.$$

Since $\pm 2\sqrt{\rho} \in \mathcal{E}_\rho$ by Lemma 4.2, the proof of (ii) is complete.

Finally, it is straightforward to check that

$$\frac{-z + \sqrt{z^2 - 4\rho}}{2\rho}$$

is the only solution of (4.1) that satisfies (4.2). □

For the remainder of the paper, we let $m(z)$ be the unique solution of equation (4.1) given by (4.3). For $\rho \neq 0$, we let $m_2(z)$ denote the other solution of equation (4.1) described in Lemma 4.3. Indeed, from the proof of Lemma 4.3, we have

$$m_2(z) := \frac{-z - \sqrt{z^2 - 4\rho}}{2\rho}. \tag{4.5}$$

Lemma 4.4. *Let $-1 \leq \rho \leq 1$ with $\rho \neq 0$ and let $\delta > 0$. Then*

$$|m(z) - m_2(z)| \geq \frac{\delta}{|\rho|}$$

for all $z \in \mathbb{C}$ with $\text{dist}(z, \mathcal{E}_\rho) \geq \delta$.

Proof. From (4.3) and (4.5), we have

$$|m(z) - m_2(z)|^2 = \frac{|z^2 - 4\rho|}{|\rho|^2} = \frac{|z - 2\sqrt{\rho}||z + 2\sqrt{\rho}|}{|\rho|^2}.$$

Since $\pm 2\sqrt{\rho} \in \mathcal{E}_\rho$ by Lemma 4.2, we conclude that

$$|m(z) - m_2(z)|^2 \geq \frac{\delta^2}{|\rho|^2}$$

for $\text{dist}(z, \mathcal{E}_\rho) \geq \delta$. □

Lemma 4.5. *Let $D \subset \mathbb{C}$ such that $D \subset \{z \in \mathbb{C} : \text{dist}(z, \mathcal{E}_\rho) \geq \delta, |z| \leq M\}$, for some $M, \delta > 0$. Then there exists $\varepsilon, C, c > 0$ (depending only on δ, M, ρ) such that the following holds. Suppose m' satisfies*

$$m'(z) = \frac{-1}{z + \rho m'(z) + \varepsilon_1(z)} + \varepsilon_2(z), \tag{4.6}$$

for all $z \in D$. If $|\varepsilon_1(z)|, |\varepsilon_2(z)| \leq \varepsilon$ for all $z \in D$, then:

1. $|m'(z)| \leq C$ for all $z \in D$,
2. $|\rho m'(z) + z| \geq c$ for all $z \in D$.

Proof. When $\rho = 0$, we note that

$$|\rho m'(z) + z| = |z| \geq 1 + \delta$$

for all $z \in D$. Moreover

$$|z + \varepsilon_1(z)| \geq |z| - \varepsilon \geq 1 + \delta - \varepsilon \geq \frac{1}{2}$$

for $\varepsilon < 1/2$ and all $z \in D$. Thus we obtain the bound $|m'(z)| \leq 5/2$.

Assume $\rho \neq 0$. Let C be a large positive constant such that $C > 100M$ and $C^2 > 2|\rho|$. Assume $\varepsilon > 0$ satisfies

$$\varepsilon < \frac{49}{100} \sqrt{2|\rho|}.$$

Then $\varepsilon < \frac{49}{100}C$ by construction. We will show that $|m'(z)| \leq C/|\rho|$ for all $z \in D$. Suppose to the contrary that $|m'(z)| > C/|\rho|$ for some $z \in D$. Then

$$|z + \rho m'(z) + \varepsilon(z)| \geq |\rho||m'(z)| - |z| - \varepsilon \geq C - \frac{C}{100} - \varepsilon \geq \frac{C}{2}.$$

Thus,

$$\frac{C}{|\rho|} \leq |m'(z)| \leq \frac{2}{C},$$

which contradicts the assumption that $C^2 > 2|\rho|$. We conclude that $|m'(z)| \leq C/|\rho|$ for all $z \in D$.

Using the bound above, we have

$$\frac{|\rho|}{C} \leq |z + \rho m'(z) + \varepsilon(z)| \leq |z + \rho m'(z)| + \varepsilon$$

for all $z \in D$. Thus, we have

$$|z + \rho m'(z)| \geq \frac{|\rho|}{C} - \varepsilon \geq \frac{|\rho|}{2C}$$

by taking ε sufficiently small. □

Lemma 4.6. *Let $\delta, M > 0$. Then there exists $C, c > 0$ (depending only on δ, M, ρ) such that $c \leq |m(z)| \leq C$ for all $z \in \mathbb{C}$ satisfying $\text{dist}(z, \mathcal{E}_\rho) \geq \delta$ and $|z| \leq M$.*

Proof. Since $m(z)$ satisfies (4.1), the claim follows from Lemma 4.5 by taking $\varepsilon_1(z) = \varepsilon_2(z) = 0$ (alternatively, one can derive the bounds directly from (4.1) and obtain an explicit expression for C, c in terms of δ, ρ, M). □

Lemma 4.7 (Stability). *Let $D \subset \mathbb{C}$ be connected and satisfy $D \subset \{z \in \mathbb{C} : \text{dist}(z, \mathcal{E}_\rho) \geq \delta, |z| \leq M\}$, for some $\delta, M > 0$. Then there exists $\varepsilon, C > 0$ (depending only on δ, M, ρ) such that the following holds. Let m' be a continuous function on D that satisfies (4.6) for all $z \in D$. If $|\varepsilon_1(z)|, |\varepsilon_2(z)| \leq \varepsilon$ for all $z \in D$, then exactly one of the following holds:*

1. $|m'(z) - m(z)| \leq C(|\varepsilon_1(z)| + |\varepsilon_2(z)|)$ for all $z \in D$,
2. $|m'(z) - m(z)| \geq \frac{\delta}{2|\rho|}$ for all $z \in D$.

Proof. First we consider the case $\rho = 0$. For $\varepsilon \leq 1/2$, we have that

$$|z + \varepsilon_1(z)| \geq |z| - \varepsilon \geq 1/2$$

for all $z \in D$. Thus

$$|m(z) - m'(z)| \leq \frac{|\varepsilon_1(z)|}{|z||z + \varepsilon_1(z)|} + |\varepsilon_2(z)| \leq 2[|\varepsilon_1(z)| + |\varepsilon_2(z)|].$$

Assume $-1 \leq \rho \leq 1$ with $\rho \neq 0$. By Lemma 4.5, there exists $\varepsilon, C' > 0$ such that if $|\varepsilon_1(z)|, |\varepsilon_2(z)| \leq \varepsilon/2$ for all $z \in D$, then $|m'(z)| \leq C'$ for all $z \in D$. By rearranging (4.6), we then obtain

$$\left| m'(z)^2 + \frac{z}{\rho} m'(z) + \frac{1}{\rho} \right| \leq \frac{C'}{|\rho|} |\varepsilon_1(z)| + \frac{C'|\rho| + M + \varepsilon}{|\rho|} |\varepsilon_2(z)| \leq C(|\varepsilon_1(z)| + |\varepsilon_2(z)|),$$

where C depends on M, ρ, C' . Define $\tilde{\varepsilon} := |\varepsilon_1(z)| + |\varepsilon_2(z)|$. Factoring the left-hand side yields

$$|m'(z) - m(z)||m'(z) - m_2(z)| \leq C\tilde{\varepsilon} \tag{4.7}$$

for all $z \in D$. From Lemma 4.4, we obtain

$$\frac{\delta}{|\rho|} \leq |m(z) - m_2(z)| \leq |m(z) - m'(z)| + |m'(z) - m_2(z)| \tag{4.8}$$

for all $z \in D$. Combining (4.7) and (4.8) we obtain the quadratic inequality

$$|m'(z) - m(z)|^2 - \frac{\delta}{|\rho|} |m'(z) - m(z)| + C\tilde{\varepsilon} \geq 0.$$

For

$$\tilde{\varepsilon} \leq \varepsilon < \frac{\delta^2}{4C|\rho|^2},$$

we obtain either

$$2|m'(z) - m(z)| \leq \frac{\delta}{|\rho|} - \sqrt{\frac{\delta^2}{|\rho|^2} - 4C\tilde{\varepsilon}} \leq \frac{4C|\rho|}{\delta} \tilde{\varepsilon}$$

or

$$2|m'(z) - m(z)| \geq \frac{\delta}{|\rho|} + \sqrt{\frac{\delta^2}{|\rho|^2} - 4C\tilde{\varepsilon}} \geq \frac{\delta}{|\rho|}.$$

For ε sufficiently small, the two possibilities above are distinct. Because $m' - m$ is continuous and since D is connected, a continuity argument implies that exactly one of the possibilities above holds for all $z \in D$. \square

We also verify that $m(z)$ is a continuous function of ρ .

Lemma 4.8. Fix $z \in \mathbb{C}$ with $|z| > 2$. Then $m(z)$ is a continuous function of $\rho \in [-1, 1]$.

Proof. In order to denote the dependence on ρ , we let $m_\rho(z)$ be the function defined by (4.3) for any $-1 \leq \rho \leq 1$. Fix $z \in \mathbb{C}$ with $|z| > 2$. Then $z \notin \cup_{-1 \leq \rho \leq 1} \mathcal{E}_\rho$. By definition,

$$\rho m_\rho^2(z) + z m_\rho(z) + 1 = 0 \tag{4.9}$$

for $-1 \leq \rho \leq 1$. Since the roots of a (monic) polynomial are continuous functions of the coefficients (see [20, 50]), we conclude that $m_\rho(z)$ is a continuous function of $\rho \in [-1, 1] \setminus \{0\}$. It remains to show $m_\rho(z)$ is continuous at $\rho = 0$.

Multiplying (4.9) by ρ , we see that $\rho m_\rho(z)$ is a continuous function of $\rho \in [-1, 1]$. Thus, we have

$$\lim_{\rho \rightarrow 0} \rho m_\rho(z) = 0,$$

and hence there exists $\varepsilon, c > 0$ such that

$$|\rho m_\rho(z) + z| \geq c$$

for all $|\rho| \leq \varepsilon$. By (4.1), it follows that

$$|m_\rho(z)| \leq \frac{1}{c}$$

for all $|\rho| \leq \varepsilon$. Let $m_0(z) = -1/z$ (i.e. $m_0(z)$ is given by (4.3) when $\rho = 0$). Then

$$z m_0(z) + 1 = 0.$$

Subtracting (4.9) from the equation above yields

$$|z| |m_0(z) - m_\rho(z)| = |\rho| |m_\rho(z)|^2 \leq \frac{|\rho|}{c^2}$$

for $|\rho| \leq \varepsilon$. Since $|z| > 2$, we conclude that $m_\rho(z)$ is continuous at $\rho = 0$. \square

5 Truncation arguments and the isotropic limit law

In this section, we begin the proof of Theorem 2.4 and Theorem 2.10 by reducing to the case where we only need to consider the truncated matrices $\{\hat{Y}_N\}_{N \geq 1}$.

5.1 Isotropic limit law

This subsection is devoted to Theorem 2.4. We will prove Theorem 2.4 using the following isotropic limit law, which is inspired by the isotropic semicircle law developed by Knowles and Yin [35, 36].

Theorem 5.1 (Isotropic limit law). *Let $\{Y_N\}_{N \geq 1}$ be a sequence of random matrices that satisfies condition **C0** with atom variables (ξ_1, ξ_2) , where $\rho = \mathbb{E}[\xi_1 \xi_2]$. Let $\delta > 0$. For each $N \geq 1$, let u_N and v_N be unit vectors in \mathbb{C}^N . Then a.s.*

$$\sup_{\text{dist}(z, \mathcal{E}_\rho) \geq \delta} \left| u_N^* \left(\frac{1}{\sqrt{N}} Y_N - zI \right)^{-1} v_N - m(z) u_N^* v_N \right| \rightarrow 0$$

as $N \rightarrow \infty$.

Assuming Theorem 2.10 and Theorem 5.1, we complete the proof of Theorem 2.4. By the singular value decomposition, we write $C_N = A_N B_N$, where A_N is a $N \times k$ matrix and B_N is a $k \times N$ matrix. By assumption, both A_N and B_N have operator norm $O(1)$. Based on [49, Lemma 2.1], we have the following lemma.

Lemma 5.2 (Eigenvalue criterion). *Let z be a complex number that is not an eigenvalue of $\frac{1}{\sqrt{N}} Y_N$. Then z is an eigenvalue of $\frac{1}{\sqrt{N}} Y_N + C_N$ if and only if*

$$\det(I + B_N G_N(z) A_N) = 0.$$

Proof. Clearly z is an eigenvalue of $\frac{1}{\sqrt{N}} Y_N + C_N$ if and only if

$$\det\left(\frac{1}{\sqrt{N}} Y_N + C_N - zI\right) = 0.$$

Since $\frac{1}{\sqrt{N}} Y_N - zI$ is invertible by assumption, we rewrite the above equation as

$$\det\left(I + \left(\frac{1}{\sqrt{N}} Y_N - zI\right)^{-1} A_N B_N\right) = 0.$$

The claim now follows from (2.2) and (3.1). □

Remark 5.3. *The proof of Lemma 5.2 actually reveals that*

$$\det(I + B_N G_N(z) A_N) = \frac{\det\left(\frac{1}{\sqrt{N}} Y_N + C_N - zI\right)}{\det\left(\frac{1}{\sqrt{N}} Y_N - zI\right)} \tag{5.1}$$

provided the denominator does not vanish. Versions of this identity have appeared in previous publications including [2, 8, 9, 10, 18].

Following Tao in [49], we define the functions

$$f(z) := \det(I + B_N G_N(z) A_N)$$

and

$$g(z) := \det(I + m(z) B_N A_N),$$

where $m(z)$ is defined in (4.3). Both f and g are meromorphic functions outside \mathcal{E}_ρ that are asymptotically equal to 1 at infinity. By Lemma 5.2, the zeroes of f coincide with the eigenvalues of $\frac{1}{\sqrt{N}} Y_N + C_N$ outside the spectrum of $\frac{1}{\sqrt{N}} Y_N$. Moreover, from (5.1) we see

that the multiplicity of any such eigenvalue is equal to the degree of the corresponding zero of f . It follows from (2.2) that

$$g(z) = \prod_{i=1}^k (1 + m(z)\lambda_i(C_N)),$$

where $\lambda_1(C_N), \dots, \lambda_k(C_N)$ are the non-trivial eigenvalues of C_N (some of which may be zero).

In order to study the zeroes of g , we consider the values of $z \notin \mathcal{E}_\rho$ for which

$$m(z) = -\frac{1}{\lambda}. \tag{5.2}$$

Indeed, for $0 < |\lambda| \leq 1$, there does not exist $z \notin \mathcal{E}_\rho$ which solves (5.2); for $|\lambda| > 1$, (5.2) holds if and only if

$$z = \lambda + \frac{\rho}{\lambda}. \tag{5.3}$$

This follows from (4.1) and an analytic continuation argument³.

By Theorem 2.2 (which was proved in Section 2 assuming Theorem 2.10 holds), it follows that a.s., for N sufficiently large, all the eigenvalues of $\frac{1}{\sqrt{N}}Y_N$ are contained in $\mathcal{E}_{\rho,\delta}$. By Rouché’s theorem, in order to prove Theorem 2.4, it suffices to show that a.s.

$$\sup_{\text{dist}(z, \mathcal{E}_\rho) \geq 2\delta} |f(z) - g(z)| \rightarrow 0$$

as $N \rightarrow \infty$. Since A_N, B_N, G_N a.s. have operator norm $O(1)$ (by Theorem 2.10) and k is fixed, independent of N , it suffices to show that a.s.

$$\sup_{\text{dist}(z, \mathcal{E}_\rho) \geq 2\delta} \|B_N (G_N - m(z)I) A_N\| \rightarrow 0$$

as $N \rightarrow \infty$. Since $B_N (G_N - m(z)I) A_N$ is a $k \times k$ matrix, the claim now follows from Theorem 5.1.

Remark 5.4. Equation (5.3) is similar to the formulas obtained in [19] for the location of the outlier eigenvalues. Indeed, (5.3) can be obtained using techniques from free probability. Let $\mu_{\text{sc},\rho}$ be the semicircle distribution with variance ρ and let $\mu_{\text{circ},1-\rho}$ be the uniform distribution on the disk centered at the origin in the complex plane with radius $(1 - \rho)^{1/2}$. Let S_ρ have distribution $\mu_{\text{sc},\rho}$, $C_{1-\rho}$ have distribution $\mu_{\text{circ},1-\rho}$, and E_ρ have elliptic distribution μ_ρ . Then

$$E_\rho = S_\rho + C_{1-\rho}$$

with S_ρ and $C_{1-\rho}$ free random variables. Outside of the ellipse, the Stieltjes transform of μ_ρ can be expressed as the Stieltjes transform of the circular law evaluated at the subordination function $F(z) = z + \rho m(z)$. This can be seen by adding the R -transforms

³One technical issue that arises when $\rho \neq 0$ is that the solution of (4.1) has two distinct analytic branches m_1, m_2 . In order to overcome this obstacle, we make the following observations.

- (i) From (4.1), we see that if (5.2) holds, it must be the case that $z = \lambda + \frac{\rho}{\lambda}$.
- (ii) The function $\lambda \mapsto \lambda + \frac{\rho}{\lambda}$ maps circles to ellipses and is one-to-one when restricted to the domain $\{\lambda \in \mathbb{C} : 0 < |\lambda| < |\rho|\} \cup \{\lambda \in \mathbb{C} : |\lambda| > 1\}$. Moreover $\lambda + \frac{\rho}{\lambda} \in \mathcal{E}_\rho$ if and only if $|\rho| \leq |\lambda| \leq 1$.

It follows that (5.2) has no solution outside the ellipse when $|\rho| \leq |\lambda| \leq 1$. Furthermore, for $|\lambda|$ sufficiently large, one can deduce the solution (5.3) for the branch m and then extend to the region $|\lambda| > 1$ by analytic continuation. Similarly, one can show that (5.2) has no solution outside the ellipse when $|\lambda| < |\rho|$; in fact, for $|\lambda| < |\rho|$, (5.3) is a solution of $m_2(z) = -\frac{1}{\lambda}$.

together and inverting to obtain the Stieltjes transform. The inverse function of F is $H(z) = z + \rho/z$, which is precisely the function appearing in (5.3). The function H plays the same role here as in [19]. Since we are only interested in solutions outside the ellipsoid, the domain of H is restricted to $|z| > 1$.

We now reduce the proof of Theorem 5.1 to the case where we only need to consider the truncated matrices $\{\hat{Y}_N\}_{N \geq 1}$. We let $\hat{m}(z)$ be the function given by (4.3) with ρ replaced by $\hat{\rho}$.

Theorem 5.5 (Isotropic limit law for \hat{Y}_N). *Let $\{Y_N\}_{N \geq 1}$ be a sequence of random matrices that satisfies condition **C0** with atom variables (ξ_1, ξ_2) , where $\rho = \mathbb{E}[\xi_1 \xi_2]$. Let $\varepsilon > 0$. Let $L > 0$, and consider the truncated random matrices $\{\hat{Y}_N\}_{N \geq 1}$ from Lemma 3.4. For each $N \geq 1$, let u_N and v_N be unit vectors in \mathbb{C}^N . Fix $z \in \mathbb{C}$ with $5 \leq |z| \leq 6$. Then a.s., for N sufficiently large,*

$$\left| u_N^* \left(\frac{1}{\sqrt{N}} \hat{Y}_N - zI \right)^{-1} v_N - \hat{m}(z) u_N^* v_N \right| \leq \varepsilon.$$

We now prove Theorem 5.1 assuming Theorem 2.10 and Theorem 5.5.

Proof of Theorem 5.1. Let $\varepsilon, \delta > 0$. It suffices to show that a.s., for N sufficiently large,

$$\sup_{\text{dist}(z, \mathcal{E}_\rho) \geq \delta} |u_N^* G_N(z) v_N - m(z) u_N^* v_N| \leq \varepsilon. \tag{5.4}$$

Consider the compact set $D := \{z \in \mathbb{C} : 5 \leq |z| \leq 6\}$. Since m is analytic on D , it follows from the Heine–Cantor theorem (see for instance [45, Theorem 4.19]) that m is uniformly continuous on D . Thus, there exists $\varepsilon' > 0$ such that if $z, w \in D$, then $|z - w| \leq \varepsilon'$ implies $|m(z) - m(w)| \leq \varepsilon/100$.

Set $\varepsilon'' := \min\{\varepsilon/100, \varepsilon'\}$, and let \mathcal{N} be a ε'' -net of D . By Lemma 3.11, $|\mathcal{N}| = O(1)$. By Theorem 5.5, we have a.s., for N sufficiently large,

$$\sup_{z \in \mathcal{N}} |u_N^* \hat{G}_N(z) v_N - \hat{m}(z) u_N^* v_N| \leq \frac{\varepsilon}{200}.$$

Furthermore, by Lemma 3.4 and Lemma 4.8 (taking L sufficiently large), we have a.s., for N sufficiently large,

$$\sup_{z \in \mathcal{N}} |u_N^* \hat{G}_N(z) v_N - m(z) u_N^* v_N| \leq \frac{\varepsilon}{100}. \tag{5.5}$$

We now extend this bound to all $z \in D$. By Lemma 3.1, Lemma 3.3, and (3.3), we have a.s., for N sufficiently large,

$$\|\hat{G}_N(z) - \hat{G}_N(w)\| \leq 4|z - w| \tag{5.6}$$

for all $z, w \in D$. Fix a realization in which (5.5) and (5.6) hold. Choose $w \in D$. Then there exists $z \in \mathcal{N}$ with $|z - w| \leq \varepsilon''$. Thus, from (5.6), we have

$$\left| u_N^* \hat{G}_N(z) v_N - u_N^* \hat{G}_N(w) v_N \right| \leq \|\hat{G}_N(z) - \hat{G}_N(w)\| \leq \frac{\varepsilon}{10}.$$

On the other hand, from the uniform continuity of m , we have

$$|m(z) u_N^* v_N - m(w) u_N^* v_N| \leq |m(z) - m(w)| \leq \frac{\varepsilon}{100}.$$

Combining the bounds above with (5.5), we conclude that, for N sufficiently large,

$$\sup_{z \in D} |u_N^* \hat{G}_N(z) v_N - m(z) u_N^* v_N| \leq \frac{\varepsilon}{2} \tag{5.7}$$

for any fixed realization in which (5.5) and (5.6) hold. In other words, we have a.s., for N sufficiently large, (5.7) holds.

By Lemma 3.1, Lemma 3.3, and the resolvent identity (3.3), we have a.s., for N sufficiently large,

$$\sup_{z \in D} \left| u_N^* \hat{G}_N(z) v_N - u_N G_N(z) v_N \right| \leq \frac{4}{\sqrt{N}} \|Y_N - \hat{Y}_N\|.$$

Thus, by Lemma 3.4, we obtain a.s., for N sufficiently large,

$$\sup_{z \in D} \left| u_N^* \hat{G}_N(z) v_N - u_N G_N(z) v_N \right| \leq \frac{C}{L} \leq \frac{\varepsilon}{2}$$

by taking L sufficiently large. Combining the bound above with (5.7), we conclude that a.s., for N sufficiently large,

$$\sup_{z \in D} |u_N^* G_N(z) v_N - m(z) u_N^* v_N| \leq \varepsilon.$$

Since ε is arbitrary, we in fact obtain that a.s.

$$\sup_{z \in D} |u_N^* G_N(z) v_N - m(z) u_N^* v_N| \rightarrow 0$$

as $N \rightarrow \infty$.

By definition (4.3), there exists $M > 0$ such that $|m(z)| \leq \varepsilon/100$ for all $|z| \geq M$. Moreover, from Lemma 3.1 and Lemma 3.3 (by increasing M if necessary), we have a.s., for N sufficiently large,

$$\sup_{|z| \geq M} \|G_N(z)\| \leq \frac{\varepsilon}{100}. \tag{5.8}$$

Consider the compact set $D' := \{z \in \mathbb{C} : \text{dist}(z, \mathcal{E}_\rho) \geq \delta, |z| \leq M\}$. By Theorem 2.10 and Lemma 4.6, we have a.s., for N sufficiently large, $u_N^*(G_N(z) - m(z)I)v_N$ is analytic and uniformly bounded on D' . We apply Vitali's convergence theorem to obtain a.s.

$$\sup_{z \in D'} |u_N^* G_N(z) v_N - m(z) u_N^* v_N| \rightarrow 0$$

as $N \rightarrow \infty$. This implies that a.s., for N sufficiently large,

$$\sup_{z \in D'} |u_N^* G_N(z) v_N - m(z) u_N^* v_N| \leq \varepsilon/2. \tag{5.9}$$

On the other hand, by (5.8), we have a.s., for N sufficiently large,

$$\sup_{|z| \geq M} |u_N^* G_N(z) v_N - m(z) u_N^* v_N| \leq \sup_{|z| \geq M} (\|G_N(z)\| + |m(z)|) \leq \varepsilon/2. \tag{5.10}$$

Combining (5.9) and (5.10), we obtain (5.4), and the proof is complete. □

We will prove Theorem 5.5 in Section 7.

5.2 Elliptic random matrices with nonzero mean

In this subsection, we use Theorem 5.1 to prove Theorem 2.8. The proof is based on the arguments in [49, Section 3].

Let $\{Y_N\}_{N \geq 1}$, μ , and φ_N be as in Theorem 2.8; let $\delta > 0$. It suffices to show that a.s. there exists one eigenvalue of $X_N + \mu\sqrt{N}\varphi_N\varphi_N^*$ outside $\mathcal{E}_{\rho,\delta}$, with this eigenvalue occurring within $o(1)$ of $\mu\sqrt{N}$. We begin with the following lemma.

Lemma 5.6. *Let $\{Y_N\}_{N \geq 1}$ and φ_N be as in Theorem 2.8. Then a.s.*

$$\varphi_N^* X_N \varphi_N \rightarrow 0$$

as $N \rightarrow \infty$.

Proof. By Markov's inequality and the Borel–Cantelli lemma, it suffices to show that

$$\mathbb{E} |\varphi_N^* X_N \varphi_N|^4 = O_{M_4} \left(\frac{1}{N^2} \right). \tag{5.11}$$

We write

$$\mathbb{E} |\varphi_N^* X_N \varphi_N|^4 = \frac{1}{N^6} \sum_{i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4=1}^N \mathbb{E} [y_{i_1 j_1} y_{i_2 j_2} y_{i_3 j_3} y_{i_4 j_4}].$$

We now consider the pairs

$$(i_1, j_1), (i_2, j_2), (i_3, j_3), (i_4, j_4)$$

for which $\mathbb{E} [y_{i_1 j_1} y_{i_2 j_2} y_{i_3 j_3} y_{i_4 j_4}]$ is nonzero. From Definition 1.11, we see that each pair (i_s, j_s) must correspond to some (i_r, j_r) or (j_r, i_r) for $s \neq r$. Counting all such pairs yields (5.11). \square

Define the functions

$$\begin{aligned} f(z) &:= 1 + \mu\sqrt{N}\varphi_N^* G_N(z)\varphi_N, \\ g(z) &:= 1 + \mu\sqrt{N}m(z), \\ h(z) &:= 1 - \frac{\mu\sqrt{N}}{z}. \end{aligned}$$

By (5.3), it follows that g has precisely one zero outside \mathcal{E}_ρ located at $\mu\sqrt{N} + \frac{\rho}{\mu\sqrt{N}}$. By Lemma 5.2 and Theorem 2.2, a.s. the eigenvalues of $X_N + \mu\sqrt{N}\varphi_N\varphi_N^*$ outside $\mathcal{E}_{\rho,\delta}$ correspond to the zeroes of f . From Theorem 5.1, we see that a.s.

$$f(z) = g(z) + o(\sqrt{N})$$

uniformly for $z \notin \mathcal{E}_{\rho,\delta}$. We conclude that if f has a zero outside $\mathcal{E}_{\rho,\delta}$, it must tend to infinity with N . Thus, for the remainder of the proof, we restrict our attention to the region $|z| \geq 5$. It remains to show that f a.s. has exactly one zero outside $|z| \geq 5$ taking the value $z = \mu\sqrt{N} + o(1)$.

By writing out the Neumann series and applying Lemma 3.3, we obtain a.s.

$$f(z) = h(z) - \frac{\mu\sqrt{N}}{z^2} \varphi_N^* X_N \varphi_N + O\left(\frac{\sqrt{N}}{|z|^3}\right)$$

uniformly for $|z| \geq 5$. Thus, by Lemma 5.6, we conclude that a.s.,

$$f(z) = h(z) + o\left(\frac{\sqrt{N}}{|z|^2}\right) + O\left(\frac{\sqrt{N}}{|z|^3}\right) \tag{5.12}$$

uniformly for $|z| \geq 5$. Let $\varepsilon > 0$; from Rouché's theorem, we conclude that a.s., for N sufficiently large, f has exactly one zero in the disk of radius ε centered at $\mu\sqrt{N}$.

Let z be any zero of f outside $\mathcal{E}_{\rho,\delta}$. Since z tends to infinity with N , we apply (5.12) and obtain a.s.

$$h(z) = o\left(\frac{\sqrt{N}}{|z|^2}\right).$$

Thus, $|z| = \Omega(\sqrt{N})$, and hence a.s.

$$z = \mu\sqrt{N} + o\left(\frac{\sqrt{N}}{|z|}\right) = \mu\sqrt{N} + o(1).$$

Therefore, we conclude that a.s., for N sufficiently large, f has precisely one zero outside $\mathcal{E}_{\rho,\delta}$ taking the value $z = \mu\sqrt{N} + o(1)$, and the proof is complete.

5.3 Least singular value bound

We now turn our attention to Theorem 2.10. Again, we will reduce to the case where we only need to consider the truncated matrices $\{\hat{Y}_N\}_{N \geq 1}$.

Theorem 5.7. *Let $\{Y_N\}_{N \geq 1}$ be a sequence of random matrices that satisfies condition **C0** with atom variables (ξ_1, ξ_2) , where $\rho = \mathbb{E}[\xi_1 \xi_2]$. Let $\delta > 0$. Then there exists $c > 0$ such that the following holds. Let $L > 0$ and consider the truncated random matrices $\{\hat{Y}_N\}_{N \geq 1}$ from Lemma 3.4. Then a.s., for N sufficiently large,*

$$\inf_{\text{dist}(z, \mathcal{E}_\rho) \geq \delta} \sigma_N \left(\frac{1}{\sqrt{N}} \hat{Y}_N - zI \right) \geq c.$$

Assuming Theorem 5.7, we now prove Theorem 2.10.

Proof of Theorem 2.10. In order to prove Theorem 2.10, it suffices to show that a.s., for N sufficiently large,

$$\sup_{\text{dist}(z, \mathcal{E}_\rho) \geq \delta} \|G_N(z)\| \leq C'$$

for some constant $C' > 0$. From (3.3), we obtain

$$G_N(z) = \hat{G}_N(z)[I - (\hat{X}_N - X_N)\hat{G}_N(z)]^{-1}$$

provided all the relevant matrices on the right-hand side are invertible. From Theorem 5.7, we have a.s., for N sufficiently large,

$$\sup_{\text{dist}(z, \mathcal{E}_\rho) \geq \delta} \|\hat{G}_N(z)\| \leq \frac{1}{c}.$$

It thus suffices to show that a.s., for N sufficiently large,

$$\sup_{\text{dist}(z, \mathcal{E}_\rho) \geq \delta} \| [I - (\hat{X}_N - X_N)\hat{G}_N(z)]^{-1} \| \leq C'' \tag{5.13}$$

for some constant $C'' > 0$.

From Lemma 3.4, it follows that a.s., for N sufficiently large,

$$\|\hat{X}_N - X_N\| \leq \frac{C}{L}$$

for some constant $C > 0$. Thus, by taking L sufficiently large, we conclude that

$$\sup_{\text{dist}(z, \mathcal{E}_\rho) \geq \delta} \|(\hat{X}_N - X_N)\hat{G}_N(z)\| \leq \frac{C}{Lc} < 1.$$

Thus, by the Neumann series, we obtain a.s., for N sufficiently large,

$$\begin{aligned} \sup_{\text{dist}(z, \mathcal{E}_\rho) \geq \delta} \| [I - (\hat{X}_N - X_N)\hat{G}_N(z)]^{-1} \| &\leq \sup_{\text{dist}(z, \mathcal{E}_\rho) \geq \delta} \sum_{k=0}^{\infty} \|(\hat{X}_N - X_N)\hat{G}_N(z)\|^k \\ &\leq \frac{1}{1 - \frac{C}{Lc}}, \end{aligned}$$

and the proof is complete. □

We now reduce to the case where z is fixed (as opposed to taking the infimum over an uncountable number of complex numbers). We proceed using an ε -net argument and the following theorem.

Theorem 5.8. *Let $\{Y_N\}_{N \geq 1}$ be a sequence of random matrices that satisfies condition **C0** with atom variables (ξ_1, ξ_2) , where $\rho = \mathbb{E}[\xi_1 \xi_2]$. Let $\delta > 0$. Then there exists a constant $c > 0$ such that the following holds. Let $L > 0$, and consider the truncated random matrices $\{\hat{Y}_N\}_{N \geq 1}$ from Lemma 3.4. Then for any z with $\text{dist}(z, \mathcal{E}_\rho) \geq \delta$ and $|z| \leq 6$, a.s., for N sufficiently large,*

$$\sigma_N \left(\frac{1}{\sqrt{N}} \hat{Y}_N - zI \right) \geq c.$$

We now verify Theorem 5.7 assuming Theorem 5.8.

Proof of Theorem 5.7. Let $\delta > 0$, and let $c > 0$ be the constant from Theorem 5.8. We first note that a.s., for N sufficiently large,

$$\sup_{|z| \geq 6} \|\hat{G}_N(z)\| \leq 1$$

by Lemma 3.1 and Lemma 3.3. Thus, it suffices to show that a.s., for N sufficiently large,

$$\inf_{z \in D} \sigma_N \left(\frac{1}{\sqrt{N}} \hat{Y}_N - zI \right) \geq c'$$

for some constant $c' > 0$, where $D := \{z \in \mathbb{C} : \text{dist}(z, \mathcal{E}_\rho) \geq \delta, |z| \leq 6\}$.

Let \mathcal{N} be a $c/10$ -net of the compact region D . By Lemma 3.11, $|\mathcal{N}| = O(1)$. Thus, by applying Theorem 5.8 to each $z \in \mathcal{N}$, we obtain a.s., for N sufficiently large,

$$\inf_{z \in \mathcal{N}} \sigma_N \left(\frac{1}{\sqrt{N}} \hat{Y}_N - zI \right) \geq c. \tag{5.14}$$

We now extend this bound to all $z \in D$. Fix a realization in which (5.14) holds. Choose $z \in D$. Then there exists $z' \in \mathcal{N}$ with $|z - z'| \leq c/10$. By Weyl's perturbation theorem (see for instance [12]),

$$\left| \sigma_N \left(\frac{1}{\sqrt{N}} \hat{Y}_N - zI \right) - \sigma_N \left(\frac{1}{\sqrt{N}} \hat{Y}_N - z'I \right) \right| \leq |z - z'| \leq \frac{c}{10}.$$

Thus, we conclude that

$$\inf_{z \in D} \sigma_N \left(\frac{1}{\sqrt{N}} \hat{Y}_N - zI \right) \geq \frac{c}{2}$$

for any realization in which (5.14) holds. The proof of the theorem is complete. \square

We will prove Theorem 5.8 in Section 6.

5.4 Notation

It remains to prove Theorem 5.5 and Theorem 5.8. As such, for the remainder of the paper we only consider the truncated matrices $\{\hat{Y}_N\}_{N \geq 1}$ from Lemma 3.4 for some arbitrarily large fixed constant $L > 0$. Thus, we drop the decorations from our notation and simply write Y_N, X_N, G_N for the matrices $\hat{Y}_N, \hat{X}_N, \hat{G}_N$. Similarly, we write $m_N(z)$ for the function $\hat{m}_N(z)$; we also write $m(z)$ for the function $\hat{m}(z)$.

6 Least singular value bound

This section is devoted to Theorem 5.8. For this entire section we work with fixed z satisfying the hypothesis of Theorem 5.8.

6.1 Hermitization

Recall the Hermitization H_N and its resolvent $R_N(q)$ defined in Section 3.2.

In this section, for any matrix H with entries that are 2×2 blocks, we mean $\text{tr}_N(H) = \frac{1}{N} \sum_i H_{ii}$ where H_{ii} is the i^{th} diagonal 2×2 block of H . When working with $N \times N$ matrices with entries that are 2×2 blocks, we use superscripts to refer to entries of the 2×2 blocks. Additionally, when forming an $N \times N$ matrix whose ij^{th} entry is the ab^{th} entry ($a, b \in \{1, 2\}$) of the ij^{th} 2×2 blocks we also use superscripts. For example, R^{21} is the $N \times N$ matrix formed from taking each R_{ij} block and replacing it by its (2,1)-entry.

Let $\Gamma_N(q) := \text{tr}_N(R_N)$. By the symmetry of the matrix H_N , $\sum_l R_{ll}^{22} = \sum_l R_{ll}^{11}$, i.e. $\Gamma_N^{11} = \Gamma_N^{22}$. Let $a_N(q) := \Gamma_N^{11}(q)$, $b_N(q) := \Gamma_N^{12}(q)$ and $c_N(q) := \Gamma_N^{21}(q)$.

From the calculations in [40] (see also [39]), it follows that $\Gamma_N(q)$ converges almost surely to a limit

$$\Gamma(q) := \begin{pmatrix} a(q) & b(q) \\ c(q) & a(q) \end{pmatrix}$$

for each fixed q . This block matrix Stieltjes transform satisfies the fixed point equation

$$\Gamma(q) = -(q + \Sigma(\Gamma(q)))^{-1}, \tag{6.1}$$

where Σ is the operator on 2×2 matrices defined by

$$\Sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} d & \rho c \\ \rho b & a \end{pmatrix}.$$

The fixed point equation should be viewed as a matrix version of (4.1). For more information on the use of this block matrix resolvent, we refer the reader to [14, 15] and the references within.

For a $N \times N$ matrix A , let ν_A denote the symmetric empirical measure built from the singular values of A . That is,

$$\nu_A := \frac{1}{2N} \sum_{i=1}^N (\delta_{\sigma_i(A)} + \delta_{-\sigma_i(A)}), \tag{6.2}$$

where $\sigma_1(A) \geq \dots \geq \sigma_N(A)$ are the singular values of A . The measure ν_A is also the empirical spectral measure of the Hermitization of A . It was established in [40] that $\nu_{X_N - zI}$ converges almost surely to a probability measure ν_z as $N \rightarrow \infty$. In Appendix C, we study the properties of $\Gamma(q)$ and ν_z . In particular, we will establish the following bound on the support of ν_z when z is outside the ellipsoid.

Theorem 6.1. *Fix $-1 < \rho < 1$ and let $\delta > 0$. Then there exists $c > 0$ such that*

$$\nu_z([-c, c]) = 0$$

for all $z \in \mathbb{C}$ with $\text{dist}(z, \mathcal{E}_\rho) \geq \delta$.

Remark 6.2. *Theorem 6.1 also holds when $\rho = \pm 1$. When $\rho = 1$, Y_N is a real symmetric Wigner matrix, and the singular values of $\frac{1}{\sqrt{N}}Y_N - zI$ are given by*

$$\left| \frac{1}{\sqrt{N}}\lambda_1(Y_N) - z \right|, \dots, \left| \frac{1}{\sqrt{N}}\lambda_N(Y_N) - z \right|,$$

where $\lambda_1(Y_N), \dots, \lambda_N(Y_N)$ are the eigenvalues of Y_N . In this case, a lower bound on the singular values follows from [5, Chapter 5]. The $\rho = -1$ case can be obtained by symmetry.

Remark 6.3. We give a complete proof of Theorem 6.1 in Appendix C. We quickly describe an alternative proof using techniques from free probability. From [40] and the work of Voiculescu [51], one can study the limiting measure ν_z by considering the distribution of

$$A_z := (\sqrt{\rho}S + \sqrt{1-\rho}C - zI)(\sqrt{\rho}S + \sqrt{1-\rho}C - zI)^*$$

for $0 \leq \rho \leq 1$, where S and C are free non-commutative random variables, S is a semi-circular variable, and C is a circular variable. Indeed, Biane and Lehner [13] showed that the spectrum of $\sqrt{\rho}S + \sqrt{1-\rho}C$ is the ellipsoid \mathcal{E}_ρ . Therefore, for any $z \in \mathbb{C}$ with $\text{dist}(z, \mathcal{E}_\rho) \geq \delta$, it follows that 0 is not in the spectrum of A_z , and hence 0 is not in the support of the distribution of A_z . A continuity argument then implies that for any $M > 0$, there exists some $c > 0$ such that for a.e. $z \in \mathbb{C}$ with $|z| \leq M$ and $\text{dist}(z, \mathcal{E}_\rho) \geq \delta$, we have $\nu_z([-c, c]) = 0$.

Proving Theorem 5.8 is equivalent to showing that a.s. $\nu_{X_N - zI}([0, c]) = 0$ for some $c > 0$. By Theorem 6.1, we choose c such that $\nu_z([0, 2c]) = 0$. In order to show that $\nu_{X_N - zI}([0, c]) = 0$ we will show that $a_N(q)$ is close to $a(q)$ for q as in (3.11) with $\eta = E + \sqrt{-1}t$, $E \in [0, c]$ and t sufficiently small. As $a_N(q)$ and $a(q)$ are the Stieltjes transform of $\nu_{X_N - zI}$ and ν_z , respectively, at the point η this will allow us to compare the two measures. The equations involving $a_N(q)$ depend crucially on $b_N(q)$ and $c_N(q)$ so it is actually more straightforward to show $\Gamma_N(q)$ is close to $\Gamma(q)$. We should note that the empirical spectral measure, μ_{X_N} , can be recovered by the formula $-\pi\mu_{X_N} = \lim_{\eta=\sqrt{-1}t \rightarrow 0} \partial_z b_N$. This formula only uses purely imaginary η . We consider more general η and a connection to the empirical spectral measure does not seem to be available.

In order to show that almost surely there are no singular values of $X_N - zI$ less than c , we follow the ideas of Bai and Silverstein [4]. First we prove an a priori bound on $\Gamma_N(q) - \Gamma(q)$, then use martingale inequalities to bound $\Gamma_N(q) - \mathbb{E}[\Gamma_N(q)]$, and finally bound $\mathbb{E}[\Gamma_N(q)] - \Gamma(q)$. Because of the correlations between X_{ij} and X_{ji} we don't directly study the Stieltjes transform of the empirical spectral measure of $(X_N - zI)^*(X_N - zI)$, but instead consider the linearized problem and study $\Gamma_N(q)$. Similar linearization tricks have been used to study eigenvalues of polynomials of Wigner matrices (e.g. see [1, 34]).

Since the vector space of 2×2 matrices is finite dimensional, all norms on it are equivalent. Therefore the use of $\|\cdot\|$ in this section can be any norm, but the reader might find it useful to think of it as the max of each entry of the matrix. In order to show a 2×2 matrix converges, it suffices to show that each entry of the matrix converges. We will often employ this strategy.

We conclude this section with some useful matrix identities and notation.

We write H_i to be the i^{th} column (of 2×2 blocks) of H_N and $H_i^{(i)}$ to be the i^{th} column of H_N with the i^{th} block removed. We let $R_N^{(i)}$ be the resolvent of H_N where the i^{th} row and i^{th} column of H_N (viewed as an $N \times N$ matrix of 2×2 blocks) have been removed. Finally $\Gamma_N^{(i)}(q) := \frac{1}{N} \sum_{j \neq i} R_{jj}^{(i)}$.

6.2 A priori estimate

Following the ideas of [4] we begin with an a priori bound on $\Gamma_N(q) - \Gamma(q)$ for $\eta = E + \sqrt{-1}t_N$, with t_N going to zero polynomially and $E \in [0, c]$. This gives an upper bound on the number of singular values of $X_N - zI$ less than c . In the second and third steps we use this bound to show that a.s. $\mathbb{E}[\Gamma_N(q)] - \Gamma_N(q) = o((t_N N)^{-1})$ and $\mathbb{E}[\Gamma_N(q)] - \Gamma(q) = O(N^{-1})$

By Schur's Complement, the diagonal entries of the resolvent are

$$\begin{aligned} R_{ii} &= -(q + H_i^{(i)} R_N^{(i)} H_i^{(i)})^{-1} \\ &= -(q + \Sigma(\Gamma_N) - \Sigma(\Gamma_N) + \Sigma(\Gamma_N^{(i)}) - \Sigma(\Gamma_N^{(i)}) + H_i^{(i)*} R_N^{(i)} H_i^{(i)})^{-1} \end{aligned}$$

Recall that the diagonal elements of X_N and hence H_N have been set to zero. Let

$$\widehat{\gamma}_N^{(i)} := H_i^{(i)*} R_N^{(i)} H_i^{(i)} - \Sigma(\Gamma_N^{(i)}). \tag{6.3}$$

Summing over i gives the trace:

$$\Gamma_N(q) = \sum_i -(q + \Sigma(\Gamma_N(q)) - \Sigma(\Gamma_N(q)) + \Sigma(\Gamma_N^{(i)}(q)) + \widehat{\gamma}_N^{(i)})^{-1} \tag{6.4}$$

Let \mathcal{S}_N be a N^{-1} -net of the interval $[0, c]$. Clearly $|\mathcal{S}_N| = O(N)$.

Lemma 6.4. *There exist some $\alpha, \beta > 0$ such that if q is as in (3.11) with $t_N \geq N^{-\beta}$, then almost surely*

$$\sup_{1 \leq i \leq N, E \in \mathcal{S}_N} \|\Sigma(\Gamma_N(q)) - \Sigma(\Gamma_N^{(i)}(q)) - \widehat{\gamma}_N^{(i)}\| = O(N^{-\alpha}).$$

The proof will show that we can take $\alpha = 1/3$ and $\beta = 1/16$; these values are not optimal, but are sufficient for our purposes. We will require that $\alpha + \beta < 1/2$ and $\beta < \alpha$.

Proof. We begin by showing

$$\|\Sigma(\Gamma_N - \Gamma_N^{(i)})\| = O((Nt_N)^{-1}). \tag{6.5}$$

Since Σ is a bounded operator it suffices to show $\|\Gamma_N - \Gamma_N^{(i)}\| = O((Nt_N)^{-1})$.

We define the modified resolvent $\check{R}_N^{(i)}(q) := (q - (H_N - e_i H_i^* - H_i e_i^*))^{-1}$. Where e_i is the $N \times 1$ vector whose i^{th} 2×2 block is the identity matrix and whose other entries are zero. The difference between the trace of $\check{R}_N^{(i)}(q)$ and the trace of $R_N^{(i)}(q)$ is q^{-1} . The difference between the trace of $R_N(q)$ and of $\check{R}_N^{(i)}(q)$ is bounded by the operator norm of $R_N(q) - \check{R}_N^{(i)}(q)$ times its rank.

The matrix $R_N(q) - \check{R}_N^{(i)}(q)$ has rank at most 4 (viewed as a $2N$ by $2N$ matrix). Indeed, by the resolvent identity,

$$R_N(q) - \check{R}_N^{(i)}(q) = R_N(q)(e_i H_i^* + H_i e_i^*) \check{R}_N^{(i)}(q).$$

The trivial bound on the resolvent, (3.8), shows the operator norm of the difference is bounded by $2t_N^{-1}$.

Thus, we obtain the estimate

$$\|\Gamma_N - \Gamma_N^{(i)}\| = \frac{1}{N} \|\text{tr}(R_N(q)) - \text{tr}(\check{R}_N^{(i)}(q)) + \text{tr}(\check{R}_N^{(i)}(q)) - \text{tr}(R_N^{(i)}(q))\| = O((Nt_N)^{-1}).$$

Since we assume $t_N \geq N^{-\beta}$, this term is deterministically bounded by $CN^{\beta-1}$ uniformly for $E \in [0, c]$ and $1 \leq i \leq N$.

We now bound $\|\widehat{\gamma}_N^{(i)}\|$ by applying the bound on quadratic forms (Lemma B.1) to each entry of this block.

$$\begin{aligned} \mathbb{E}[|\widehat{\gamma}_N^{(i)ab}|^p] &\leq \frac{K_p}{N^p} \mathbb{E} \left[(\text{tr}(R^{(i)a'b'}(R^{(i)a'b'})^*))^{p/2} \right] \\ &\leq \frac{K_p t_N^{-p}}{N^{p/2}} \end{aligned} \tag{6.6}$$

with a and b either 1 or 2, and $a' = a + 1 \pmod{2}$, $b' = b + 1 \pmod{2}$. The final estimate uses that N times the operator norm of a self adjoint matrix bounds its trace. The trivial bound shows the operator norm is bounded by t_N^{-2} .

Then by Chebyshev's inequality and the union bound

$$\begin{aligned} \mathbb{P}(\max_{1 \leq i \leq N, E_j \in \mathcal{S}_N} \|\widehat{\gamma}_N^{(i)}\| \geq N^{-\alpha}) &\leq \sum_{1 \leq i \leq N, E_j \in \mathcal{S}_N} N^{p\alpha} \mathbb{E}(\|\widehat{\gamma}_N^{(i)}\|^p) \\ &\leq K_p N^{2+p(\alpha+\beta-1/2)} \end{aligned}$$

In order for this term to converge to zero we require that $\alpha + \beta < 1/2$. Then, p can be chosen large enough to make the right-hand side summable. An application of the Borel-Cantelli lemma implies almost sure convergence.

Since $\alpha + \beta < 1/2$ implies that $\beta - 1 < -\alpha$, we conclude the proof of the lemma. \square

Now we state and prove our a priori bound.

Lemma 6.5. *Let q be as in (3.11) with $t_N \geq N^{-\beta}$ and β as in Lemma 6.4. Then almost surely*

$$\sup_{E \in [0, c]} \|\Gamma_N(q) - \Gamma(q)\| = o(N^{-\beta})$$

Proof. First note that it is sufficient to prove the estimate on \mathcal{S}_N . If $|E - E'| \leq \delta$, then $\|\Gamma_N(q) - \Gamma_N(q')\| = \|\Gamma_N(q)(q - q')\Gamma_N(q')\| \leq \delta t_N^{-2}$. Therefore showing that $\|\Gamma_N(q)\| = O(N^{-\beta})$ for $E \in \mathcal{S}_N$ with $t_N > N^{-\beta}$ implies the bound $\|\Gamma_N(q')\| = O(N^{-\beta})$ for all q' with $E' \in [0, c]$.

We introduce the notation

$$\epsilon_N^i := -\Sigma(\Gamma_N(q)) + \Sigma(\Gamma_N^{(i)}(q)) + \widehat{\gamma}_N^{(i)}$$

and

$$\epsilon_N := \frac{1}{N} \sum_{i=1}^N (q + \Sigma(\Gamma_N(q)))^{-1} (\epsilon_N^i) (q + \Sigma(\Gamma_N(q)) + \epsilon_N^i)^{-1}.$$

Let Λ_N be the event that $\max_{1 \leq i \leq N} \|\epsilon_N^i\| \leq N^{-\alpha}$. By Lemma 6.4, $\mathbf{1}_{\Lambda_N} = 1$ almost surely. With this notation we rewrite (6.4) as

$$\Gamma_N(q) = -(q + \Sigma(\Gamma_N(q)))^{-1} + \epsilon_N. \tag{6.7}$$

For sufficiently large N we have the bound,

$$\|1/N \sum_i (\epsilon_N^i) (q + \Sigma(\Gamma_N(q)) + \epsilon_N^i)^{-1}\| \mathbf{1}_{\Lambda_N} = \|1/N \sum_{i=1}^N (\epsilon_N^i) R_{ii}\| \mathbf{1}_{\Lambda_N} = O(N^{-\alpha} t_N^{-1}) < 1/2.$$

Thus, we can solve for $-(q + \Sigma(\Gamma_N(q)))^{-1} \mathbf{1}_{\Lambda_N}$:

$$-(q + \Sigma(\Gamma_N(q)))^{-1} \mathbf{1}_{\Lambda_N} = \Gamma_N(q) (1 - 1/N \sum_{i=1}^N (\epsilon_N^i) R_{ii})^{-1} \mathbf{1}_{\Lambda_N},$$

and conclude

$$\|(q + \Sigma(\Gamma_N(q)))^{-1}\| \mathbf{1}_{\Lambda_N} \leq C t_N^{-1}.$$

This leads to the bound

$$\|\epsilon_N\| \mathbf{1}_{\Lambda_N} = \left\| \frac{1}{N} \sum_{i=1}^N (q + \Sigma(\Gamma_N(q)))^{-1} (\epsilon_N^i) R_{ii} \right\| \mathbf{1}_{\Lambda_N} \leq C N^{-\alpha} t_N^{-2}.$$

The following lemma will allow us to complete the proof.

Lemma 6.6. *Let $\tilde{q} = \tilde{q}_N := q + \Sigma(\epsilon_N)$ with q as in (3.11), $E \in [0, c]$, and $t \geq N^{-\beta}$. Then almost surely*

$$\Gamma_N(q) = \Gamma(\tilde{q}) + \epsilon_N.$$

We defer the proof of this lemma until the end of the current proof.

Now, assuming Lemma 6.6, almost surely, we can replace $\Gamma(q)$ with $\Gamma_N(q - \Sigma(\epsilon_N)) - \epsilon_N$ and conclude

$$\begin{aligned} \|\Gamma_N(q) - \Gamma(q)\| &= \|\Gamma(\tilde{q}) + \epsilon_N - \Gamma(q)\| \\ &\leq \|\Gamma(\tilde{q}) (\Sigma(\epsilon_N)) \Gamma(q)\| + \|\epsilon_N\| \\ &\leq \|\Gamma(\tilde{q})\| \|\epsilon_N\| \|\Gamma(q)\| + \|\epsilon_N\| \\ &= O(t_N^{-4} N^{-\alpha}). \end{aligned}$$

Choosing $\alpha = 1/3$ and $\beta = 1/16$ gives the almost sure bound $\sup_{E \in \mathcal{S}_N} \|\Gamma_N(q) - \Gamma(q)\| = o(N^{-\beta})$. □

Proof of Lemma 6.6. Applying the resolvent identity to the difference between (6.1) and (6.7) leads to

$$\begin{aligned} \Gamma_N(q) - \Gamma(\tilde{q}) - \epsilon_N &= -(q + \Sigma(\Gamma_N(q)))^{-1} + (\tilde{q} + \Sigma(\Gamma(\tilde{q})))^{-1} \\ &= -(q + \Sigma(\Gamma_N(q)))^{-1} (\tilde{q} - q + \Sigma(\Gamma(\tilde{q}) - \Gamma_N(q))) (\tilde{q} + \Sigma(\Gamma(\tilde{q})))^{-1} \\ &= (q + \Sigma(\Gamma_N(q)))^{-1} (\Sigma(\Gamma_N(q) - \Gamma(\tilde{q}) - \epsilon_N)) (\tilde{q} + \Sigma(\Gamma(\tilde{q})))^{-1} \end{aligned}$$

From the (C.11) and Remark C.4 there exist a K such that

$$\|(q + \Sigma(\Gamma(q)))^{-1}\| = \|\Gamma(q)\| \leq K$$

for all q with $t > 0$.

Since $\tilde{q} \mathbf{1}_{\Lambda_N} = (q + \Sigma(\epsilon_N)) \mathbf{1}_{\Lambda_N}$ and $\|\Sigma(\epsilon_N)\| \mathbf{1}_{\Lambda_N}$ converges to zero, we can choose N sufficiently large such that the imaginary part of the diagonal entries of \tilde{q} are almost surely greater than zero, yielding the almost sure bound $\|\Gamma(\tilde{q})\| \leq K$.

Then using that $\|\Gamma_N(q)\| \leq t_N^{-1}$, we obtain almost surely

$$\|\Gamma_N(q) - \Gamma(\tilde{q}) - \epsilon_N\| \leq K |t_N|^{-1} \|\Gamma_N(q) - \Gamma(\tilde{q}) - \epsilon_N\|.$$

We conclude for η such that $1 > K |t_N|^{-1}$, $E \in [0, c]$ and then for all η with $t_N > N^{-\beta}$ and $E \in [0, c]$ by analytic continuation that almost surely

$$\Gamma_N(q) = \Gamma(\tilde{q}) + \epsilon_N.$$

□

Now we define the Stieltjes transforms of the measure $\nu_{X_N - zI}$ and ν_z restricted to $[-2c, 2c]$ and its complement to be

$$\begin{aligned} a_N^{in}(q) &= \int_{[-2c, 2c]^c} \frac{d\nu_{X_N - zI}(x)}{x - \eta}, & a^{in}(q) &= \int_{[-2c, 2c]^c} \frac{d\nu_z(x)}{x - \eta} \\ a_N^{out}(q) &= \int_{[-2c, 2c]} \frac{d\nu_{X_N - zI}(x)}{x - \eta}, & a^{out}(q) &= \int_{[-2c, 2c]} \frac{d\nu_z(x)}{x - \eta} = 0. \end{aligned}$$

Note that $t_N^{-1} \text{Im}(a_N^{in}(q)) = \int_{[-2c, 2c]^c} \frac{d\nu_{X_N - zI}(x)}{(x - E)^2 + t_N^2}$, and observe that $\frac{1}{(x - E)^2 + t_N^2}$ forms a uniformly bounded, equicontinuous family as a function of $E \in [0, c]$ for $x \notin [-2c, 2c]$.

Furthermore, since $\nu_{X_N - zI}$ converges almost surely to ν_z by the calculations in [40] (see also [39]), we can conclude that a.s.

$$\sup_{E \in \mathcal{S}_N} t_N^{-1} (\text{Im}(a_N^{in}(q)) - \text{Im}(a^{in}(q))) \longrightarrow 0.$$

Combining this estimate with Lemma 6.5 gives that a.s.

$$\sup_{E \in \mathcal{S}_N} t_N^{-1} \text{Im}(a_N^{out}(q) - a^{out}(q)) \longrightarrow 0.$$

We conclude this section with a bound on the number of singular values less than c and then turn this into a bound on the trace of the resolvent. Let \mathcal{T}_N be an t_N -net of $[0, c]$. Using the inequality $\mathbf{1}_{[E_j - t_N, E_j + t_N]}(x) \leq -2t_N \text{Im}(1/(x - E_j + \sqrt{-1}t_N))$, we obtain

$$\begin{aligned} \nu_{X_N - zI}([0, c]) &\leq \sum_{E_j \in \mathcal{T}_N} \int_{E_j - t_N}^{E_j + t_N} d\nu_{X_N - zI}(x) \\ &\leq \sum_{E_j \in \mathcal{T}_N} \int_0^c \frac{2t_N^2}{(E_j - x)^2 + t_N^2} d\nu_{X_N - zI}(x) \\ &\leq Ct_N^{-1} t_N \sup_{E_j \in \mathcal{T}_N} |\text{Im}(a_N^{out}(q) - a^{out}(q))| = o(N^{-\beta}) \quad \text{a.s.} \end{aligned}$$

So we conclude on the almost sure event Λ_N that there are $o(N^{1-\beta})$ eigenvalues in the interval $[0, c]$. We will require a similar a priori bound on the number of small eigenvalues for the $(N - 1) \times (N - 1)$ submatrices $X_N^{(i)}$, defined by removing the i^{th} row and column of X_N . Thus, we define the event Λ_i that $\nu_{X_N^{(i)} - zI}([0, c]) = o(N^{-\beta})$. By the interlacing theorem, $\Lambda_N \subset \Lambda_i$, so Λ_i also occurs almost surely.

For $k = 1, \dots, N$, let \mathbb{E}_k be averaging with respect to the first k rows and columns of X_N , and let \mathbb{E}_0 be the identity. Since $\nu_{X_N - zI}([0, c])$ is bounded, almost surely $o(N^{-\beta})$ and $\mathbb{E}_k[\nu_{X_N - zI}([0, c])]$ forms a martingale, we can apply Lemma 3.9 to obtain the almost sure estimate

$$\max_{k \leq N} \mathbb{E}_k[\nu_{X_N - zI}([0, c])] = o(N^{-\beta}). \tag{6.8}$$

Repeating the argument shows this estimate also holds for $\max_{k \leq N} \mathbb{E}_k[\nu_{X_N^{(i)} - zI}([0, c])]$.

Now we use the spectral theorem to turn this bound on the number of singular values into a bound on the trace of powers of the resolvent. In order for this bound to be useful we will increase the imaginary part of η .

Lemma 6.7. *Let q be as in (3.11) with $E \in [0, c]$, $t = N^{-\beta/4}$ and β as in Lemma 6.4. Then*

$$\begin{aligned} \max_{k \leq N} \mathbb{E}_k[\text{tr}_N(|R_N^{ab}|^2) \mathbf{1}_{\Lambda_N}] &= O(1), & \max_{k \leq N} \mathbb{E}_k[\text{tr}_N(|(R_N R_N)^{ab}|^2) \mathbf{1}_{\Lambda_N}] &= O(1) \\ \max_{i, k \leq N} \mathbb{E}_k[\text{tr}_N(|R_N^{(i)ab}|^2) \mathbf{1}_{\Lambda_i}] &= O(1), & \max_{i, k \leq N} \mathbb{E}_k[\text{tr}_N(|(R_N^{(i)} R_N^{(i)})^{ab}|^2) \mathbf{1}_{\Lambda_i}] &= O(1). \end{aligned} \tag{6.9}$$

Proof. Let $X_N - z$ have singular value decomposition $U_N D_N V_N$ with $D_{ii} = \sigma_i = \sigma_i(X_N - z)$, then the block matrix

$$\begin{pmatrix} R_N^{11} & R_N^{12} \\ R_N^{21} & R_N^{22} \end{pmatrix}^p = \frac{1}{2} \begin{pmatrix} U_N & U_N \\ V_N^* & -V_N^* \end{pmatrix} \begin{pmatrix} (D_N - \eta)^{-1} & 0 \\ 0 & (-D_N - \eta)^{-1} \end{pmatrix}^p \begin{pmatrix} U_N^* & V_N \\ U_N^* & -V_N \end{pmatrix}$$

$$\begin{aligned} \max_{k \leq N} \mathbb{E}_k [\text{tr}_N (|(R_N^p)^{11}|^2)] \mathbf{1}_{\Lambda_N} &= \max_{k \leq N} \mathbb{E}_k \left[\sum_{i=1}^N \sum_{j=1}^N U_{ij} \left| \left(\frac{1}{\sigma_j - \eta} \right)^p + \left(\frac{1}{-\sigma_j - \eta} \right)^p \right|^2 U_{ji}^* \right] \mathbf{1}_{\Lambda_N} \\ &\leq \max_{k \leq N} \mathbb{E}_k \left[\frac{C}{N} \sum_{|\sigma_j| \leq c} \frac{1}{|\eta - \sigma_j|^{2p}} + \frac{C}{N} \sum_{|\sigma_j| \geq c} \frac{1}{|\eta - \sigma_j|^{2p}} \right] \mathbf{1}_{\Lambda_N} \\ &\leq C(N^{-1}(N^{1-\beta})t_N^{-2p} + N^{-1}Nc^{-1}). \end{aligned}$$

$$\begin{aligned} \max_{k \leq N} \mathbb{E}_k [\text{tr}_N (|R_N^{12}|^p)] \mathbf{1}_{\Lambda_N} &= \max_{k \leq N} \mathbb{E}_k \left[\sum_{i=1}^N \sum_{j=1}^N U_{ij} \left| \left(\frac{1}{\sigma_j - \eta} \right)^p - \left(\frac{1}{-\sigma_j - \eta} \right)^p \right|^2 V_{ji} \right] \mathbf{1}_{\Lambda_N} \\ &\leq \max_{k \leq N} \mathbb{E}_k \left[\sum_{j=1}^N \left| \frac{\sigma_j}{\eta^2 - \sigma_j^2} \right|^{2p} \|U_{\cdot j}\|_2 \|V_{j \cdot}\|_2 \right] \mathbf{1}_{\Lambda_N} \\ &\leq C(N^{-1}(N^{1-\beta})t_N^{-2p} + N^{-1}Nc^{-1}). \end{aligned}$$

This term is $O(1)$ if $p = 1, 2$ because $t_N = N^{-\beta/4}$. The same argument bounds the 21 and 22 term. The above computation also verifies the lemma when a row or column has been removed. □

6.3 Estimating $\Gamma_N - \mathbb{E}[\Gamma_N]$

Theorem 6.8. *Let q be as in (3.11) with $t = N^{-\beta/4}$ and β as in Lemma 6.4. Then almost surely*

$$\sup_{E \in [0, c]} Nt_N (\Gamma_N(q) - \mathbb{E}[\Gamma_N(q)]) \rightarrow 0.$$

Before proceeding with the proof, we define the relevant notation and give a lemma containing crude estimates.

Applying (3.7) to R_N (which we view as an $N \times N$ matrix) yields

$$\Gamma_N - \Gamma_N^{(i)} = -\frac{1}{N} (q + H_i^{(i)*} R_N^{(i)} H_i^{(i)})^{-1} (I_2 + H_i^{(i)*} R_N^{(i)} R_N^{(i)} H_i^{(i)}) \tag{6.10}$$

Note that by Schur's Complement (3.6) the first term is an entry of the resolvent:

$$R_{ii} = -(q + H_i^{(i)*} R_N^{(i)} H_i^{(i)})^{-1}.$$

We define

$$\zeta_N^{(i)} := (I_2 + H_i^{(i)*} R_N^{(i)} R_N^{(i)} H_i^{(i)}).$$

In order to study R_{ii} we introduce the non-random 2×2 matrix

$$\widehat{R}_{ii} := -(q + \mathbb{E}[\zeta_N^{(i)}])^{-1}.$$

Note that this is not actually an entry of a resolvent. In order to control the fluctuations of R_{ii} , we use the resolvent identity to compare the R_{ii} with \widehat{R}_{ii} :

$$R_{ii} - \widehat{R}_{ii} = \widehat{R}_{ii} (H_i^{(i)*} R_N^{(i)} H_i^{(i)} - \mathbb{E}[\zeta_N^{(i)}]) R_{ii}. \tag{6.11}$$

This motivates the definition

$$\gamma_N^{(i)} := H_i^{(i)*} R_N^{(i)} H_i^{(i)} - \mathbb{E}[\zeta_N^{(i)}]. \tag{6.12}$$

We remind the reader

$$\widehat{\gamma}_N^{(i)} := H_i^{(i)*} R_N^{(i)} H_i^{(i)} - \Sigma(\Gamma_N^{(i)}).$$

Redefine \mathcal{S}_N to be a N^{-2} -net of the interval $[0, c]$. Once again it suffices to prove the theorem for $E \in \mathcal{S}_N$.

Lemma 6.9. For $a, b \in \{1, 2\}$, and $p \geq 2$:

$$\mathbb{E}[|\Gamma_N^{(i)ab}(q) - \mathbb{E}[\Gamma_N^{(i)ab}(q)]|^p] \leq K_p N^{-p/2} t_N^{-p} \tag{6.13}$$

$$\mathbb{E}[|\widehat{\gamma}_N^{(i)ab}|^p] \leq \frac{K_p t_N^{-p}}{N^{p/2}}. \tag{6.14}$$

There exist some K such that for all large N ,

$$\|\widehat{R}_{ii}\| \leq K. \tag{6.15}$$

We note that part of the use of the first inequality is the equality $\gamma_N^{(i)} - \widehat{\gamma}_N^{(i)} = \Sigma(\Gamma_N^{(i)}(q) - \mathbb{E}[\Gamma_N^{(i)}(q)])$.

Proof. Using the martingale inequality, (3.7), and the bound on a trace of a matrix and a submatrix, (6.5), we can bound any entry as

$$\begin{aligned} \mathbb{E}[|\Gamma_N^{(i)ab}(q) - \mathbb{E}[\Gamma_N^{(i)ab}(q)]|^p] &= \mathbb{E}\left[\left|\sum_{j \neq i} (\mathbb{E}_{j-1} - \mathbb{E}_j)(\Gamma_N^{(i)ab} - \Gamma_N^{(i,j)ab})\right|^p\right] \\ &\leq K_p \mathbb{E}\left[\sum_{j \neq i} |(\mathbb{E}_i - \mathbb{E}_{i-1})(\Gamma_N^{(i)ab} - \Gamma_N^{(i,j)ab})|^2\right]^{p/2} \\ &\leq K_p N^{-p/2} t_N^{-p}. \end{aligned}$$

Combining this estimate with (6.6) leads to

$$\mathbb{E}[|\gamma_N^{(i)ab}|^p] = \mathbb{E}[|\widehat{\gamma}_N^{(i)ab} + \Sigma(\Gamma_N^{(i)ab}(q)) - \Sigma(\mathbb{E}[\Gamma_N^{(i)ab}(q)])|^p] \leq \frac{K_p t_N^{-p}}{N^{p/2}}$$

To bound \widehat{R}_{ii} , we begin with

$$\mathbb{E}[R_{ii}] = \mathbb{E}[\Gamma_N].$$

Since $\|\mathbb{E}[\Gamma_N] - \Gamma\| = o(t_N)$ and Γ is bounded by some constant K , so is $\mathbb{E}[R_{ii}]$. Combining this estimate with the trivial bound, $|R_{ii}| \leq |t_N|^{-1}$, leads to:

$$\begin{aligned} \|\widehat{R}_{ii} - \mathbb{E}[R_{ii}]\| &= \left\| \mathbb{E}\left[\widehat{R}_{ii} \gamma_N^{(i)} \mathbb{E}[R_{ii}]\right] \right\| \\ &\leq \text{Im}(\eta)^{-1} \mathbb{E}\left[\left\|\gamma_N^{(i)}\right\|\right] K \\ &\leq \text{Im}(\eta)^{-3} N^{-1/2} K \end{aligned}$$

So

$$\|\widehat{R}_{ii}\| \leq K + \text{Im}(\eta)^{-3} N^{-1/2} K$$

The last term is bounded for η in our domain, and the proof is complete. □

Proof of Theorem 6.8. To control $\Gamma_N(q) - \mathbb{E}[\Gamma_N(q)]$, we rewrite it as a sum of martingale differences. Using the equality $\mathbb{E}_i[\Gamma_N^{(i)}(q)] = \mathbb{E}_{i-1}[\Gamma_N^{(i)}(q)]$ and the formula for the differences of traces of submatrices (6.10), we have

$$\Gamma_N - \mathbb{E}[\Gamma_N] = \sum_i (\mathbb{E}_{i-1} - \mathbb{E}_i)\Gamma_N = \sum_i (\mathbb{E}_{i-1} - \mathbb{E}_i)(\Gamma_N - \Gamma_N^{(i)}) = \frac{1}{N} \sum_i (\mathbb{E}_{i-1} - \mathbb{E}_i)R_{ii}\zeta_N^{(i)}. \tag{6.16}$$

To complete the proof it suffices to show that for arbitrary $\epsilon > 0$,

$$\mathbb{P}(\max_{E \in \mathcal{S}_N} Nt_N \|\Gamma_N - \mathbb{E}[\Gamma_N]\| \geq \epsilon, \text{ i.o.}) = 0.$$

Recalling that

$$\mathbb{P}(\cup_{i=1}^N \{\mathbf{1}_{\Lambda_i} = 0\}, \text{ i.o.}) = 0,$$

leads to the estimate

$$\begin{aligned} & \mathbb{P}(\max_{E \in \mathcal{S}_N} Nt_N \|\Gamma_N - \mathbb{E}[\Gamma_N]\| > \epsilon, \text{ i.o.}) \\ &= \mathbb{P}((\max_{E \in \mathcal{S}_N} Nt_N \|\Gamma_N - \mathbb{E}[\Gamma_N]\| > \epsilon) \cap_{i=1}^N [\mathbf{1}_{\Lambda_i} = 1], \text{ i.o.}) \\ &\leq \mathbb{P}(\max_{E \in \mathcal{S}_N} t_N \|\sum_{i=1}^N (\mathbb{E}_{i-1} - \mathbb{E}_i)R_{ii}\zeta_N^{(i)}\| \mathbf{1}_{\Lambda_i} > \epsilon, \text{ i.o.}). \end{aligned} \tag{6.17}$$

To estimate (6.17) we iteratively apply (6.11) leading to:

$$\begin{aligned} R_{ii}\zeta_N^{(i)} &= (R_{ii} - \widehat{R}_{ii})\zeta_N^{(i)} + \widehat{R}_{ii}\zeta_N^{(i)} \\ &= \widehat{R}_{ii}\gamma_N^{(i)}R_{ii}\zeta_N^{(i)} + \widehat{R}_{ii}\zeta_N^{(i)} \\ &= \widehat{R}_{ii}\gamma_N^{(i)}\widehat{R}_{ii}(I_2 + \Sigma(\text{tr}_N(R_N^{(i)}R_N^{(i)}))) \\ &\quad + \widehat{R}_{ii}\gamma_N^{(i)}\widehat{R}_{ii}(H_i^{(i)*}R_N^{(i)}R_N^{(i)}H_i^{(i)} - \Sigma(\text{tr}_N(R_N^{(i)}R_N^{(i)}))) \\ &\quad + \widehat{R}_{ii}\gamma_N^{(i)}\widehat{R}_{ii}\gamma_N^{(i)}R_{ii}\zeta_N^{(i)} + \widehat{R}_{ii}\zeta_N^{(i)}. \end{aligned}$$

After applying this expansion to (6.17), it suffices to show that each entry of the 2×2 block converges to zero almost surely. By the triangle inequality, it suffices to bound an arbitrary product of entries of the blocks in the expansion. For the remainder of this section, we use lower case superscripts starting with the beginning of the alphabet to denote the values 1 or 2.

To bound $\widehat{R}_{ii}\gamma_N^{(i)}\widehat{R}_{ii}(I_2 + \Sigma(\text{tr}_N(R_N^{(i)}R_N^{(i)})))$ we apply Rosenthal's inequality (Lemma 3.6), the bound on moments of quadratic forms (Lemma B.1), the bound on \widehat{R}_{ii} (6.15), and the a priori bound (6.9). We obtain

$$\begin{aligned} & \mathbb{E}[|t_N \sum_{i=1}^N (\mathbb{E}_{i-1} - \mathbb{E}_i)(\widehat{R}_{ii}^{ab}\gamma_N^{(i)bc}\widehat{R}_{ii}^{cd}(I_2 + \Sigma(\text{tr}_N(R_N^{(i)}R_N^{(i)})))^{de})\mathbf{1}_{\Lambda_i}|^p] \\ &\leq K_p t_N^p \mathbb{E}[(\sum_{i=1}^N \mathbb{E}_i |\widehat{\gamma}_N^{(i)bc}\mathbf{1}_{\Lambda_i}|^2)^{p/2} + \sum_{i=1}^N |\widehat{\gamma}_N^{(i)bc}\mathbf{1}_{\Lambda_i}|^p] \\ &\leq K_p t_N^p \mathbb{E}[(\sum_{i=1}^N N^{-2} \mathbb{E}_i [\text{tr}(R^{b'c'(i)}R^{b'c'(i)*})\mathbf{1}_{\Lambda_i}])^{p/2}] + K_p N N^{-p/2} t_N^{-p} \\ &\leq K_p t_N^p (C + N^{1-p/2} t_N^{-p}), \end{aligned}$$

which is summable for large p . Recall that $b' = b + 1 \pmod{2}$. The same estimates are used to bound the $\widehat{R}_{ii}\zeta_N^{(i)}$ term.

In order to bound $\widehat{R}_{ii}\gamma_N^{(i)}\widehat{R}_{ii}\gamma_N^{(i)}R_{ii}\zeta_N^{(i)}$, we begin with Burkholder’s inequality (Lemma 3.7) and then apply $R_{ii}\zeta_N^{(i)} = N(\Gamma_N(q) - \Gamma_N^{(i)}(q)) \leq Kt_N^{-1}$ by (6.5) and the bound on \widehat{R}_{ii} given in (6.15) along with the Cauchy-Schwarz inequality and the estimate on quadratic forms (6.14).

$$\begin{aligned} & \mathbb{E}\left[|t_N \sum_{i=1}^N (\mathbb{E}_{i-1} - \mathbb{E}_i) \widehat{R}_{ii}^{ab} \gamma_N^{(i)bc} \widehat{R}_{ii}^{cd} \gamma_N^{(i)de} (R_{ii}\zeta_N^{(i)})^{ef} \mathbf{1}_{\Lambda_i}|^p\right] \\ & \leq K_p t_N^p t_N^{-p} \left(\sum_{i=1}^N \mathbb{E}[|\gamma_N^{(i)bc} \gamma_N^{(i)de} \mathbf{1}_{\Lambda_i}|^2] \right)^{p/2} \\ & \leq K_p \left(\sum_{i=1}^N (\mathbb{E}[|\gamma_N^{(i)bc}|^4] \mathbb{E}[|\gamma_N^{(i)de}|^4])^{1/2} \right)^{p/2} \\ & \leq K_p N^{p/2} N^{-p} t_N^{-2p} = K_p N^{-p/2} t_N^{-2p}. \end{aligned}$$

The estimate of the $\widehat{R}_{ii}\gamma_N^{(i)}\widehat{R}_{ii} \left(H_i^{(i)*} R_N^{(i)} R_N^{(i)} H_i^{(i)} - \Sigma(\text{tr}_N(R_N^{(i)} R_N^{(i)})) \right)$ term is done the same way.

Choosing p large enough in the above estimates to make the right-hand sides summable and an application of Borel-Cantelli shows that almost surely

$$\max_{E \in \mathcal{S}_N} N t_N \|\Gamma_N - \mathbb{E}[\Gamma_N]\| \rightarrow 0$$

□

6.4 Estimating $\mathbb{E}[\Gamma_N(q)] - \Gamma(q)$

We now show that for q as in (3.11) with $t_N = N^{-\beta/4}$ and β as in Lemma 6.4

$$\sup_{E \in [0, c]} \|\mathbb{E}[\Gamma_N(q)] - \Gamma(q)\| = O(N^{-1}). \tag{6.18}$$

We begin in a similar fashion to the a priori estimates with Schur’s Complement,

$$\mathbb{E}[\Gamma_N(q)] = \mathbb{E}[R_{11}] = -\mathbb{E}[(q + H_1^{(1)*} R_N^{(1)} H_1^{(1)})^{-1}],$$

from which we will subtract $-(q + \Sigma(\mathbb{E}[\Gamma_N(q)]))^{-1}$. We will apply the resolvent identity, add and subtract $\Sigma(\mathbb{E}[\Gamma_N^{(1)}(q)])$, and repeatedly apply the identity

$$\begin{aligned} \mathbb{E}[\Gamma_N(q)] + (q + \Sigma(\mathbb{E}[\Gamma_N^{(1)}(q)]))^{-1} &= \mathbb{E}[-(q + H_1^{(1)*} R_N^{(1)} H_1^{(1)})^{-1} + (q + \Sigma(\mathbb{E}[\Gamma_N^{(1)}(q)]))^{-1}] \\ &= \mathbb{E}[\Gamma_N(q) \gamma_N^{(1)} (q + \Sigma(\mathbb{E}[\Gamma_N^{(1)}(q)]))^{-1}]. \end{aligned}$$

Leading to the expansion

$$\begin{aligned} & \mathbb{E}[\Gamma_N(q) + (q + \Sigma(\mathbb{E}[\Gamma_N(q)]))^{-1}] \\ &= \mathbb{E}[\Gamma_N(z) \left(H_1^{(1)*} R_N^{(1)} H_1^{(1)} - \Sigma(\mathbb{E}[\Gamma_N(q)]) \right) (q + \Sigma(\mathbb{E}[\Gamma_N(q)]))^{-1}] \\ &= \mathbb{E}[\Gamma_N(z) \left(\Sigma(\mathbb{E}[\Gamma_N^{(1)}]) - \Sigma(\mathbb{E}[\Gamma_N(q)]) \right) (q + \Sigma(\mathbb{E}[\Gamma(q)]))^{-1}] + \mathbb{E}[\Gamma_N(z) \left(\gamma_N^{(1)} \right) (q + \Sigma(\mathbb{E}[\Gamma_N(q)]))^{-1}] \\ &= \mathbb{E}[\Gamma_N(z) \left(\Sigma(\mathbb{E}[\Gamma_N^{(1)}]) - \Sigma(\mathbb{E}[\Gamma_N(q)]) \right) (q + \Sigma(\mathbb{E}[\Gamma(q)]))^{-1}] \\ & \quad + \mathbb{E}[\widehat{R}_{11} \left(\gamma_N^{(1)} \right) (q + \Sigma(\mathbb{E}[\Gamma_N(q)]))^{-1}] \\ & \quad + \mathbb{E}[\widehat{R}_{11} \left(\gamma_N^{(1)} \right) \widehat{R}_{11} \left(\gamma_N^{(1)} \right) (q + \Sigma(\mathbb{E}[\Gamma_N(q)]))^{-1}] \\ & \quad + \mathbb{E}[\Gamma_N(q) \left(\gamma_N^{(1)} \right) \widehat{R}_{11} \left(\gamma_N^{(1)} \right) \widehat{R}_{11} \left(\gamma_N^{(1)} \right) (q + \Sigma(\mathbb{E}[\Gamma_N(q)]))^{-1}]. \end{aligned}$$

Note that the third line is zero, because $\mathbb{E}[\gamma_N^{(1)}] = 0$ and the other terms are non-random. To bound the rest of the terms we need the following lemma:

Lemma 6.10. *Let q be as in (3.11) with $E \in [0, c]$, $t_N = N^{-\beta/4}$, and β as in Lemma 6.4. Then*

$$\|\mathbb{E}[\Gamma_N(q) - \Gamma_N^{(1)}(q)]\| = O(N^{-1}), \tag{6.19}$$

$$\mathbb{E}[|\hat{\gamma}^{(1)ab}|^2] \leq K(\mathbb{E}[|\hat{\gamma}^{(1)ab}|^2] + \mathbb{E}[|\hat{\gamma}^{(1)ab} - \gamma^{(1)ab}|^2]) = O(N^{-1}), \tag{6.20}$$

$$\|(q + \Sigma(\mathbb{E}[\Gamma_N(q)]))^{-1}\| = O(1). \tag{6.21}$$

Before proving the lemma note that (6.18) will follow from a straight forward application of this lemma, the triangle inequality, Hölder’s inequality and the estimates $\|\Gamma_N\| \leq t_N^{-1}$, $\|\hat{R}_{ii}\| \leq K$, and the estimate on quadratic forms (6.14).

Proof. Using the formula for the difference between traces, (6.10), the bound on the trace of the resolvent, (6.9), and the bound on quadratic forms, (3.18), we obtain

$$\begin{aligned} |\mathbb{E}[\Gamma_N(q)^{ab} - \Gamma_N^{(1)ab}(q)]| &= \frac{1}{N} |\mathbb{E}[(R_{11}\zeta_N^{(1)})^{ab}]| \\ &= \frac{1}{N} |\mathbb{E}[(\hat{R}_{11}\zeta_N^{(1)})^{ab}] + \frac{1}{N} \mathbb{E}[(\hat{R}_{11}\gamma_N^{(1)} R_{11}\zeta_N^{(1)})^{ab}]| \\ &\leq \frac{K}{N} \mathbb{E}[1 + \|\text{tr}_N((R_N^{(1)})^2)\|] \\ &\quad + \frac{K}{N} \sum_{c,d=1}^2 \mathbb{E}[|\gamma_N^{(1)cd}|] K t_N^{-1} \\ &\leq \frac{K}{N} (1 + \frac{t_N^{-2}}{N^{1/2}}). \end{aligned}$$

The first term of (6.20) is bounded from a direct calculation and the second term uses the martingale difference decomposition and the expansions of the previous estimates.

$$\begin{aligned} &\mathbb{E}[|\hat{\gamma}^{(1)ab} - \gamma^{(1)ab}|^2] \\ &= \mathbb{E}[|\Gamma_N^{(1)ab}(q) - \mathbb{E}[\Gamma_N^{(1)ab}(q)]|^2] = \sum_{i=2}^N \mathbb{E}[|(\mathbb{E}_i - \mathbb{E}_{i-1})(\Gamma_N^{(1)}(q) - \Gamma_N^{(1,i)}(q))^{ab}|^2] \\ &= \frac{1}{N^2} \sum_{i=2}^N \mathbb{E}[|(\mathbb{E}_i - \mathbb{E}_{i-1})(R_{ii}^{(1)}\zeta_N^{(1,i)})^{ab}|^2] \\ &\leq \frac{1}{N^2} K \sum_{i=2}^N \mathbb{E}[|(\mathbb{E}_i - \mathbb{E}_{i-1})(\hat{R}_{ii}^{(1)}\zeta_N^{(1,i)})^{ab}|^2] \\ &\quad + K \frac{1}{N^2} \sum_{i=2}^N \mathbb{E}[|(\mathbb{E}_i - \mathbb{E}_{i-1})(\hat{R}_{ii}^{(1)}\gamma_N^{(i,1)} R_{ii}^{(1)}\zeta_N^{(1,i)})^{ab}|^2] \\ &\leq \frac{K}{N^2} \sum_{i=2}^N \left(\sum_{c=1}^2 \mathbb{E}[\text{tr}_N(|R^{(1,i)c'b'}(q)|^4)] + \sum_{c,d=1}^2 2K t_N^{-2} \mathbb{E}[|\gamma_N^{(i,1)cd}|^2] \right) \\ &\leq \frac{1}{N}. \end{aligned}$$

The final estimate follows from

$$\mathbb{E}[\Gamma_N(q) + (q + \Sigma(\mathbb{E}[\Gamma_N^{(1)}(q)]))^{-1}] = \mathbb{E}[\Gamma_N(q) \left(H_1^{(1)*} R_N^{(1)} H_1^{(1)} - \Sigma(\mathbb{E}[\Gamma_N^{(1)}(q)]) \right) (q + \Sigma(\mathbb{E}[\Gamma_N^{(1)}(q)]))^{-1}]$$

and the boundedness of $\mathbb{E}[\Gamma_N(q)]$. The proof of the lemma is complete. □

To estimate $\|\mathbb{E}[\Gamma_N(q)] - \Gamma(q)\|$, we proceed as in the end of Section 6.2. Let

$$\epsilon_N := \mathbb{E}[\Gamma_N(q)] + (q + \Sigma(\mathbb{E}[\Gamma_N(q)]))^{-1}$$

and

$$\tilde{q} := \tilde{q}_N = q + \Sigma(\epsilon_N).$$

Then the arguments of Lemma 6.6 can be repeated to prove (6.18).

6.5 No small singular values

Following Bai and Silverstein in [4], we construct a uniformly bounded, equicontinuous family of functions, and then use weak convergence of $\nu_{X_N - zI}$ to show a.s. there are no singular values outside the support of the limiting distribution.

The arguments of this section can be repeated to show for any fixed k , and $t_N = \sqrt{k}N^{-\beta/4}$, one has a.s.

$$\sup_{E \in [0, c]} |a_N(q) - a(q)| = o((Nt_N)^{-1}). \tag{6.22}$$

Taking the imaginary part of the above equation for any $k = 1, \dots, 2/\beta$ leads to

$$\sup_{k=1, \dots, 2/\beta; E \in [0, c]} \left| \int \frac{d(\nu_{X_N - zI} - \nu_z)(x)}{(E - x)^2 + kt_N^2} \right| = o(N^{-1}t_N^{-2}). \tag{6.23}$$

Taking differences for different k_1 and k_2 gives

$$\sup_{k_1 \neq k_2; E \in [0, c]} \left| \int \frac{t_N^2 d(\nu_{X_N - zI} - \nu_z)(x)}{((E - x)^2 + k_1 t_N^2)((E - x)^2 + k_2 t_N^2)} \right| = o(N^{-1}t_N^{-2}). \tag{6.24}$$

Repeating this for all values of k and splitting the integral over two regions leads to:

$$\sup_{E \in [0, c]} \left| \int \frac{I_{[0, c+\epsilon]^c} d(\nu_{X_N - z} - \nu_z)(\lambda)}{\prod_{k=1}^{2/\beta} (E - \lambda)^2 + kt_N^2} + \sum_{\lambda_j \in [0, c+\epsilon]} \frac{t_N^{4/\beta}}{\prod_{k=1}^{2/\beta} (E - \lambda_j)^2 + kt_N^2} \right| = o(1).$$

The first integrand forms a uniformly bounded, equicontinuous family so the integral converges to zero by the weak convergence of $\nu_{X_N - z}$ to ν_z . The summand is uniformly bounded away from zero when evaluated at a singular value in the interval $[0, c]$. So we conclude that almost surely there are no singular values in the interval $[0, c]$.

7 Isotropic limit law

This section is devoted to Theorem 5.5. We divide the proof of Theorem 5.5 into the following three steps.

1. Showing that the diagonal entries of $G_N(z)$ convergence uniformly to $m(z)$.
2. Establishing a rate of convergence of the off-diagonal entries of $G_N(z)$ to zero.
3. Establishing a concentration inequality for $u_N^* G_N(z) v_N$.

7.1 Diagonal entries

Define the event

$$\Omega_N := \{\|X_N\| \leq 4.5\}.$$

By Lemma 3.5, it follows that Ω_N holds with overwhelming probability. We establish the following convergence result for the diagonal entries of $G_N(z)$.

Lemma 7.1 (Diagonal entries). *Let $\varepsilon > 0$. Then, for N sufficiently large,*

$$\sup_{5 \leq |z| \leq 6} \sup_{1 \leq i \leq N} \mathbb{E} |(G_N(z))_{ii} - m(z)| \mathbf{1}_{\Omega_N} \leq \varepsilon.$$

Proof. We introduce the following notation. For any $1 \leq i \leq N$, we let $Y_N^{(i)}$ be the $(N - 1) \times (N - 1)$ matrix formed from Y_N by removing the i -th row and i -th column. We let r_i denote the i -th row of Y_N with the i -th entry removed; let c_i denote the i -th column of Y_N with the i -th entry removed. We define

$$G_N^{(i)}(z) := \left(\frac{1}{\sqrt{N}} Y_N^{(i)} - zI \right)^{-1}$$

and $m_N^{(i)}(z) := \frac{1}{N} \text{tr} G_N^{(i)}(z)$. We let $\check{Y}_N^{(i)}$ denote the $N \times N$ matrix formed from Y_N by setting the entries in the i -row and i -th column to zero. We define

$$\check{G}_N^{(i)}(z) := \left(\frac{1}{\sqrt{N}} \check{Y}_N^{(i)} - zI \right)^{-1}$$

and $\check{m}_N^{(i)}(z) := \frac{1}{N} \text{tr} \check{G}_N^{(i)}(z)$.

Since $Y_N^{(i)}$ and $\check{Y}_N^{(i)}$ are formed from Y_N , it follows that

$$\sup_{1 \leq i \leq N} \left\| \frac{1}{\sqrt{N}} Y_N^{(i)} \right\| \leq 4.5, \quad \sup_{1 \leq i \leq N} \left\| \frac{1}{\sqrt{N}} \check{Y}_N^{(i)} \right\| \leq 4.5$$

on the event Ω_N . By Lemma 3.1, we have

$$\sup_{|z| \geq 5} \|G_N(z)\| \leq 2, \quad \sup_{|z| \geq 5} \sup_{1 \leq i \leq N} \|G_N^{(i)}(z)\| \leq 2, \quad \sup_{|z| \geq 5} \sup_{1 \leq i \leq N} \|\check{G}_N^{(i)}(z)\| \leq 2 \quad (7.1)$$

on the event Ω_N . It follows that

$$\sup_{|z| \geq 5} |m_N(z)| \leq 2, \quad \sup_{|z| \geq 5} \sup_{1 \leq i \leq N} |m_N^{(i)}(z)| \leq 2, \quad \sup_{|z| \geq 5} \sup_{1 \leq i \leq N} |\check{m}_N^{(i)}(z)| \leq 2 \quad (7.2)$$

and

$$\inf_{|z| \geq 5} |z + \rho m_N(z)| \geq 3 \quad (7.3)$$

on the event Ω_N .

Fix $1 \leq i \leq N$ and $z \in \mathbb{C}$ with $|z| \geq 5$. Let $\varepsilon > 0$. By the Schur complement (since the diagonal entries of Y_N are zero), we have that

$$(G_N(z))_{ii} = -\frac{1}{z + \rho m_N(z) + \varepsilon_N(z)}, \quad (7.4)$$

where

$$\varepsilon_N(z) := \rho m_N^{(i)}(z) - \rho m_N(z) + \frac{1}{N} r_i G_N^{(i)}(z) c_i - \rho m_N^{(i)}(z).$$

We observe that (r_i^T, c_i) and $G_N^{(i)}$ are independent. Thus, by conditioning on the event $\{\|G_N^{(i)}(z)\| \leq 2\}$ (which holds with overwhelming probability), we apply Lemma 3.10 and obtain

$$\left| \frac{1}{N} r_i G_N^{(i)}(z) c_i - \rho m_N^{(i)}(z) \right| \leq \varepsilon$$

with overwhelming probability. Since the eigenvalues of $\check{Y}_N^{(i)}$ are the eigenvalues of $Y_N^{(i)}$ with an additional eigenvalue of zero, it follows that

$$|m_N^{(i)}(z) - \check{m}_N^{(i)}(z)| \leq \frac{1}{5N}$$

on Ω_N because $|z| \geq 5$. Observe that $\check{Y}_N^{(i)} - Y_N$ is at most rank 2. By the resolvent identity, we have

$$\begin{aligned} \left| \check{m}_N^{(i)}(z) - m_N(z) \right| &= \frac{1}{N} \left| \operatorname{tr} \left[G_N(z) \frac{1}{\sqrt{N}} \left(Y_N - \check{Y}_N^{(i)} \right) \check{G}_N^{(i)}(z) \right] \right| \\ &\leq \frac{2}{N} \|G_N(z)\| \frac{1}{\sqrt{N}} \left(\|Y_N\| + \|\check{Y}_N^{(i)}\| \right) \|\check{G}_N^{(i)}(z)\| \\ &\leq \frac{C}{N} \end{aligned}$$

on the event Ω_N . We conclude that, for N sufficiently large,

$$\varepsilon_N(z) = O(\varepsilon)$$

with overwhelming probability. By (7.3) and (7.4), it follows that

$$(G_N(z))_{ii} = -\frac{1}{z + \rho m_N(z)} + O(\varepsilon) \tag{7.5}$$

with overwhelming probability.

Let D be a compact, connected set that satisfies

$$\{z \in \mathbb{C} : 5 \leq |z| \leq 6\} \subset D \subset \{z \in \mathbb{C} : |z| \geq 5\}.$$

If $\rho \neq 0$, we additionally assume that there exists $z_0 \in D$ with

$$|m(z_0)| \leq \frac{\delta}{100|\rho|}, \quad |z_0| \geq 5 + \frac{100|\rho|}{\delta}. \tag{7.6}$$

Such a choice of z_0 in (7.6) always exists by (4.2).

We now extend (7.5) to all $z \in D$. Let \mathcal{N} be an ε -net of D . By Lemma 3.11, $|\mathcal{N}| = O(1)$. Thus, by the union bound, we have

$$\sup_{z \in \mathcal{N}} \left| (G_N(z))_{ii} + \frac{1}{z + \rho m_N(z)} \right| \leq C\varepsilon, \tag{7.7}$$

with overwhelming probability. Fix a realization in the event Ω_N such that (7.7) holds. By (3.3) and (7.1),

$$\|G_N(z) - G_N(z')\| \leq 4|z - z'|, \quad |m_N(z) - m_N(z')| \leq 4|z - z'|$$

for all $z, z' \in D$. Let $z' \in D$. Then there exists $z \in \mathcal{N}$ with $|z - z'| \leq \varepsilon$. Thus, we have

$$\begin{aligned} \left| (G_N(z'))_{ii} + \frac{1}{z' + \rho m_N(z')} \right| &\leq (4 + C)\varepsilon + \left| \frac{1}{z' + \rho m_N(z')} - \frac{1}{z + \rho m_N(z)} \right| \\ &\leq (5 + C)\varepsilon \end{aligned}$$

by (7.3). Therefore, we conclude that

$$\sup_{z \in D} \left| (G_N(z))_{ii} + \frac{1}{z + \rho m_N(z)} \right| \leq C\varepsilon$$

with overwhelming probability.

By the union bound, we have

$$\sup_{1 \leq i \leq N} \sup_{z \in D} \left| (G_N(z))_{ii} + \frac{1}{z + \rho m_N(z)} \right| \leq C\varepsilon \tag{7.8}$$

with overwhelming probability. Thus, with overwhelming probability,

$$\sup_{z \in D} \left| m_N(z) + \frac{1}{z + \rho m_N(z)} \right| \leq C\varepsilon. \tag{7.9}$$

If $\rho = 0$, we conclude that

$$\sup_{z \in D} |m_N(z) - m(z)| \leq C\varepsilon \tag{7.10}$$

with overwhelming probability. We now obtain this bound in the case that $\rho \neq 0$ by applying Lemma 4.7. In view of (7.6) and Lemma 3.1, we have

$$|m_N(z_0) - m(z_0)| \leq |m_N(z_0)| + |m(z_0)| \leq \frac{\delta}{50|\rho|}$$

on the event Ω_N . Thus, by Lemma 4.7, we conclude that (7.10) holds with overwhelming probability for any $-1 \leq \rho \leq 1$.

By (7.8), (7.9), and (7.10), we obtain

$$\sup_{1 \leq i \leq N} \sup_{z \in D} |(G_N(z))_{ii} - m(z)| \leq C\varepsilon$$

with overwhelming probability. Since Ω_N holds with overwhelming probability, we have

$$\sup_{1 \leq i \leq N} \sup_{z \in D} |(G_N(z))_{ii} - m(z)| \mathbf{1}_{\Omega_N} \leq C\varepsilon$$

with overwhelming probability. By Lemma 4.6 and (7.1), we conclude that, for N sufficiently large,

$$\sup_{1 \leq i \leq N} \sup_{z \in D} \mathbb{E} |(G_N(z))_{ii} - m(z)| \mathbf{1}_{\Omega_N} \leq C\varepsilon.$$

The proof of the lemma is complete. □

7.2 Off-diagonal entries

Let H_N be the Hermitization of X_N as in Section 6. Once again we will view H_N as a $N \times N$ matrix of 2×2 blocks. We reuse the notation from Section 6 with $q = q(z, \eta)$. Let $R_N^{21}(\eta, z)$ be the $N \times N$ matrix with $(R_N^{21}(\eta, z))_{ij} = R_{ij}^{21}(\eta, z)$. We begin by noting that when defined, $u_N^* G_N(z) v_N = u_N^* (R_N^{21}(0, z)) v_N$. Just as in Section 6 when we only needed to control R_{ii}^{11} but found it easier to instead control the block R_{ii} , here we will estimate the 2×2 block R_{ij} for $i \neq j$ in order to control R_{ij}^{21} . We should note that many of our estimates will involve the norm $\|R_N(z, \eta)\|$, but on the event that there are no eigenvalues outside the ellipse this norm is $O(1)$.

Lemma 7.2 (Off-diagonal entries). *Fix $z, \eta \in \mathbb{C}$ with $5 \leq |z| \leq 6$ and $\text{Im}(\eta) > 0$. Then*

$$\sup_{i \neq j} \|\mathbb{E}[(R_N)_{ij}]\| = O(N^{-3/2}).$$

Proof. We begin with Schur's complement, (3.6), with A being the upper 1 by 1 block, D being the lower $N - 1$ by $N - 1$ block, and B and C being the corresponding off-diagonal blocks. Then for $i \neq 1$

$$R_{1i} = -R_{11}(H_1^{(1)*} R^{(1)})_i \tag{7.11}$$

and

$$R_{i1} = -(R^{(1)} H_1^{(1)})_i R_{11}. \tag{7.12}$$

Other elements of R_N can be computed by permuting the rows and columns of H_N before applying Schur's complement.

Combining the identities (generalized to an arbitrary element) for $i \neq j$ leads to

$$R_{ij} = R_{ii}^{(j)}(H_{ij} + H_i^{(i,j)*} R^{(i,j)} H_j^{(i,j)}) R_{jj}. \tag{7.13}$$

Let

$$\xi_N^{(i,j)} := H_{ij} + H_i^{(i,j)*} R^{(i,j)} H_j^{(i,j)}.$$

Additionally, recall the diagonal entries of the resolvent are

$$R_{ii} = (-q - H_i^{(i)*} R^{(i)} H_i^{(i)})^{-1},$$

the definition

$$\widehat{R}_{ii} = (q - \mathbb{E}[\Sigma(\Gamma^{(i)})])^{-1}$$

and

$$R_{ii} - \widehat{R}_{ii} = \widehat{R}_{ii}(\gamma_N^{(i)}) R_{ii}.$$

We begin with (7.11) and then apply (6.11) two times and finally (7.12) to obtain

$$\begin{aligned} \mathbb{E}[R_{ij}] &= \sum_{l \neq i} \mathbb{E}[-R_{ii}(H_{il} R_{lj}^{(i)})] \\ &= \sum_{l \neq i} \mathbb{E}[-\widehat{R}_{ii}(H_{il} R_{lj}^{(i)}) - \widehat{R}_{ii}(\gamma_N^{(i)}) \widehat{R}_{ii}(H_{il} R_{lj}^{(i)}) - \widehat{R}_{ii}(\gamma_N^{(i)}) \widehat{R}_{ii}(\gamma_N^{(i)}) R_{ii}(H_{il} R_{lj}^{(i)})] \\ &= \mathbb{E}[\sum_{l \neq i} (-\widehat{R}_{ii}(H_{il} R_{lj}^{(i)}) - \widehat{R}_{ii}(\gamma_N^{(i)}) \widehat{R}_{ii}(H_{il} R_{lj}^{(i)})) \\ &\quad - \widehat{R}_{ii}(\gamma_N^{(i)}) \widehat{R}_{ii}(\gamma_N^{(i)}) R_{ii}(H_{ij} - \sum_{l,k \neq i,j} H_{il} R_{lk}^{(i,j)} H_{kj}) R_{jj}^{(i)}]. \end{aligned}$$

The first term is zero because $\mathbb{E}[H_{il}] = 0$. We estimate the other terms as in Section 6 and bound each entry of the 2×2 blocks. Each entry is a sum of products of entries from the blocks. Thus, by the triangle inequality, it suffices to bound arbitrary products of each block's entries. As before lower case superscripts from the beginning of the alphabet are all either 1 or 2.

To bound the third term we apply Hölder's inequality and directly compute the moments:

$$\begin{aligned} &\mathbb{E}[\widehat{R}_{ii}^{ab}(\gamma_N^{(i)})^{bc} \widehat{R}_{ii}^{cd}(\gamma_N^{(i)})^{de} R_{ii}^{ef}(\xi_N^{(i,j)})^{fg} R_{jj}^{(i)gh}] \\ &\leq K \mathbb{E}[|(\gamma_N^{(i)})^{bc}|^4]^{1/4} \mathbb{E}[|(\gamma_N^{(i)})^{de}|^4]^{1/4} \mathbb{E}[|(\xi_N^{(i,j)})^{fg}|^2]^{1/2} \\ &= O(N^{-3/2}). \end{aligned} \tag{7.14}$$

We begin estimating the second term by averaging over the i^{th} row and column of H . Let $\mu_3 = \max\{\mathbb{E}[|H_{12}|^3], \mathbb{E}[|H_{21}|^3]\}$. Then

$$\begin{aligned} &|\sum_{l,m,n \neq i} \mathbb{E}[\widehat{R}_{ii}^{ab}(H_{im} R_{mn}^{(i)} H_{ni} - \mathbb{E}[\text{tr}_N(R_N^{(i)})])^{bc} \widehat{R}_{ii}^{cd}(H_{il} R_{lj}^{(i)})^{de}]| \\ &\leq \sum_{l \neq i} \frac{\mu_3}{N^{3/2}} |\mathbb{E}[\widehat{R}_{ii}^{ab}(R_{il}^{(i)})^{bc} \widehat{R}_{ii}^{cd}(R_{lj}^{(i)})^{de}]| \end{aligned} \tag{7.15}$$

We now apply the Cauchy-Schwarz inequality with (6.15) to get a weaker bound than desired. Once the weaker bound is proven, we will return to (7.15) and prove the

desired bound.

$$\begin{aligned} & \sum_{l \neq i} \frac{\mu_3}{N^{3/2}} |\mathbb{E}[\widehat{R}_{ii}^{ab}(R_{ll}^{(i)})^{bc} \widehat{R}_{ii}^{cd}(R_{lj}^{(i)})^{de}]| \\ & \leq \frac{K}{N^{3/2}} (\mathbb{E}[\sum_{l \neq i} |R_{ll}^{(i)}|^2])^{1/2} (\mathbb{E}[\sum_{l \neq i} |R_{lj}^{(i)}|^2])^{1/2} \\ & \leq \frac{K}{N^{3/2}} (\mathbb{E}[\sum_{l \neq i} |R_{ll}^{(i)}|^2])^{1/2} \|R_N^{(i)}\| \\ & = O(N^{-1}), \end{aligned}$$

which combined with (7.14) implies $\|\mathbb{E}[R_{ij}]\| = O(N^{-1})$. Returning to (7.15), applying (6.11) and (7.13) leads to:

$$\begin{aligned} & \sum_{l \neq i} \frac{\mu_3}{N^{3/2}} |\mathbb{E}[\widehat{R}_{ii}^{ab}(\widehat{R}_{ll}^{(i)})^{bc} \widehat{R}_{ii}^{cd}(R_{lj}^{(i)})^{de}]| \\ & + \sum_{l \neq i} \frac{\mu_3}{N^{3/2}} |\mathbb{E}[\widehat{R}_{ii}^{ab}(\widehat{R}_{ll}^{(i)} \gamma_N^{(l,i)} R_{ll}^{(i)})^{bc} \widehat{R}_{ii}^{cd}(R_{ll}^{(i,j)} \xi_N^{(l,j,i)} R_{jj}^{(i)})^{de}]| \\ & = O(N^{-3/2}), \end{aligned}$$

where $\xi_N^{(l,j,i)} := H_{lj} + H_l^{(l,j,i)*} R^{(l,j,i)} H_j^{(l,j,i)}$ and $\gamma_N^{(l,i)} = H_l^{(l,i)*} R_N^{(l,i)} H_i^{(l,i)} - \mathbb{E}[\Sigma(\Gamma_N^{(l,i)})]$. The first term uses the just verified $O(N^{-1})$ bound and the second uses the Cauchy-Schwarz inequality and a direct computation. □

7.3 Concentration of $u_N^* G_N v_N$

We now establish the following concentration result.

Lemma 7.3 (Concentration of bilinear forms). *Let $\varepsilon > 0$. Fix $z \in \mathbb{C}$ with $5 \leq |z| \leq 6$. Then a.s., for N sufficiently large,*

$$|u_N^* G_N(z) \mathbf{1}_{\Omega_N} v_N - \mathbb{E} u_N^* G_N(z) \mathbf{1}_{\Omega_N} v_N| \leq \varepsilon.$$

Proof. The proof below is based on the arguments of Bai and Pan [6]. Let $\varepsilon > 0$ and fix $5 \leq |z| \leq 6$. We will drop the dependence on z and simply write G_N to denote the matrix $G_N(z)$. We introduce the following notation. Let $X_N^{(k)}$ be the matrix obtained from X_N by replacing all elements in the k -th column and k -th row with zero. Define $G_N^{(k)} := (X_N^{(k)} - zI)^{-1}$. Let r_k be the k -th row of X_N ; let c_k be the k -th column of X_N . Let \mathbb{E}_k denote the conditional expectation given $r_{k+1}, \dots, r_N, c_{k+1}, \dots, c_N$. Let e_1, \dots, e_N denote the standard basis of \mathbb{C}^N . Let $u_N = (u_{N,i})_{i=1}^N$ and $v_N = (v_{N,i})_{i=1}^N$.

We will take advantage of the fact that all the elements of the k -th column and k -th row of $G_N^{(k)}$ are zero except that the (k, k) -th element is $-1/z$. Thus,

$$e_k^* G_N^{(k)} e_k = -\frac{1}{z}, \quad e_k^* G_N^{(k)} v_N = -\frac{v_{N,k}}{z}, \quad u_N^* G_N^{(k)} e_k = -\frac{\bar{u}_{N,k}}{z}, \tag{7.16}$$

$$e_k^* G_N^{(k)} c_k = 0, \quad r_k G_N^{(k)} e_k = 0. \tag{7.17}$$

It follows from the definitions above that

$$X_N = X_N^{(k)} + c_k e_k^* + e_k r_k.$$

We define

$$X_N^{(k1)} := X_N^{(k)} + e_k r_k, \quad X_N^{(k2)} := X_N^{(k)} + c_k e_k^*,$$

and

$$G_N^{(kj)} := \left(X_N^{(kj)} - zI \right)^{-1}, \quad j = 1, 2.$$

Define the events

$$\Omega_N^{(k)} := \left\{ \|X_N^{(k)}\| \leq 4.5 \right\}.$$

We let $\mathbf{1}_{\Omega_N^{(k)}}$ denote the indicator function of the event $\Omega_N^{(k)}$. Since $\Omega_N \subset \Omega_N^{(k)}$, it follows that $\mathbf{1}_{\Omega_N} \leq \mathbf{1}_{\Omega_N^{(k)}}$. By Lemma 3.1, we have

$$\|G_N\| \leq 2, \quad \sup_{1 \leq k \leq N} \|G_N^{(k)}\| \leq 2, \quad \sup_{1 \leq k \leq N} \|G_N^{(kj)}\| \leq 2, \quad j = 1, 2$$

on the event Ω_N . Moreover, $\|G_N^{(k)}\| \leq 2$ on the event $\Omega_N^{(k)}$.

Set

$$\begin{aligned} \alpha_N^{(k)} &:= \frac{1}{1 + z^{-1} r_k G_N^{(k)} \mathbf{1}_{\Omega_N} c_k}, \\ \gamma_N^{(k)} &:= \frac{1}{1 + z^{-1} \frac{\rho}{N} \left(\text{tr} G_N^{(k)} + 1/z \right) \mathbf{1}_{\Omega_N}}, \\ \xi_N^{(k)} &:= r_k G_N^{(k)} c_k - \frac{\rho}{N} \left(\text{tr} G_N^{(k)} + \frac{1}{z} \right), \end{aligned}$$

and

$$\eta_N^{(k)} := r_k G_N^{(k)} v_N u_N^* G_N^{(k)} c_k - \frac{\rho}{N} \left(u_N^* G_N^{(k)2} v_N - z^{-2} u_{N,k} v_{N,k} \right).$$

We now collect a variety of preliminary calculations and bounds we will need to complete the proof.

(i) By (3.4) and (7.17), we have

$$e_k^* G_N^{(k1)} c_k = e_k^* G_N^{(k)} c_k - \frac{e_k^* G_N^{(k)} e_k r_k G_N^{(k)} c_k}{1 + r_k G_N^{(k)} e_k} = z^{-1} r_k G_N^{(k)} c_k,$$

and hence

$$\frac{1}{1 + e_k^* G_N^{(k1)} \mathbf{1}_{\Omega_N} c_k} = \alpha_N^{(k)}.$$

Similarly, we obtain

$$\frac{1}{1 + r_k G_N^{(k2)} \mathbf{1}_{\Omega_N} e_k} = \alpha_N^{(k)}.$$

By the Schur complement, we have that

$$(G_N)_{kk} = -\frac{1}{z + r_k G_N^{(k)} c_k}.$$

Thus, on the event Ω_N , we have

$$|\alpha_N^{(k)}| = \left| \frac{z}{z + r_k G_N^{(k)} c_k} \right| \leq 6 \|G_N^{(k)}\| \leq 12.$$

On the event Ω_N^c , $\alpha_N^{(k)} = 1$. Therefore, we conclude that a.s.,

$$|\alpha_N^{(k)}| \leq 12.$$

Similarly, we have a.s.,

$$|\gamma_N^{(k)}| \leq \frac{|z|}{|z| - \|G_N^{(k)}\| \mathbf{1}_{\Omega_N} - \frac{1}{|z|N}} \leq 3.$$

(ii) By the Burkholder inequality (Lemma 3.7), for any $p > 2$, we have

$$\begin{aligned} & \mathbb{E} \left| \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) u_N^* G_N^{(k)} v_N \left(\mathbf{1}_{\Omega_N^{(k)}} - \mathbf{1}_{\Omega_N} \right) \right|^p \\ & \leq C_p \mathbb{E} \left(\sum_{k=1}^N |u_N^* G_N^{(k)} v_N|^2 \mathbf{1}_{\Omega_N^{(k)} \cap \Omega_N^c} \right)^{p/2} \\ & \leq C_p N^{p/2} \mathbb{P}(\Omega_N^c). \end{aligned}$$

(iii) By (3.5) and (i), we have

$$u_N^* G_N c_k \mathbf{1}_{\Omega_N} = \frac{u_N^* G_N^{(k1)} c_k \mathbf{1}_{\Omega_N}}{1 + e_k^* G_N^{(k1)} c_k} = \frac{u_N^* G_N^{(k1)} c_k \mathbf{1}_{\Omega_N}}{1 + e_k^* G_N^{(k1)} c_k \mathbf{1}_{\Omega_N}} = u_N^* G_N^{(k1)} c_k \mathbf{1}_{\Omega_N} \alpha_N^{(k)}.$$

Similarly,

$$u_N^* G_N e_k \mathbf{1}_{\Omega_N} = u_N^* G_N^{(k2)} e_k \mathbf{1}_{\Omega_N} \alpha_N^{(k)}.$$

(iv) By (3.4) and (7.17), we have

$$G_N^{(k1)} = G_N^{(k)} - G_N^{(k)} e_k r_k G_N^{(k)}$$

and

$$G_N^{(k2)} = G_N^{(k)} - G_N^{(k)} c_k e_k^* G_N^{(k)}.$$

(v) By definition of $\alpha_N^{(k)}$, we have

$$z^{-1} (\mathbb{E}_{k-1} - \mathbb{E}_k) r_k G_N^{(k)} c_k \alpha_N^{(k)} = -(\mathbb{E}_{k-1} - \mathbb{E}_k) (\alpha_N^{(k)}).$$

(vi) By definition of $\alpha_N^{(k)}, \gamma_N^{(k)}, \xi_N^{(k)}$, we have

$$\alpha_N^{(k)} - \gamma_N^{(k)} = -z^{-1} \alpha_N^{(k)} \gamma_N^{(k)} \xi_N^{(k)} \mathbf{1}_{\Omega_N}.$$

(vii) We note that the entries of r_k and c_k have mean zero, variance $1/N$, and are a.s. bounded by $4L/\sqrt{N}$. Moreover, (r_k^T, c_k) and $G_N^{(k)} \mathbf{1}_{\Omega_N^{(k)}}$ are independent. Thus, by Lemma B.1 in Appendix B, for any $p \geq 2$, we have

$$\sup_{1 \leq k \leq N} \mathbb{E}_k \left| \xi_N^{(k)} \right|^p \mathbf{1}_{\Omega_N} \leq \sup_{1 \leq k \leq N} \mathbb{E}_k \left| \xi_N^{(k)} \right|^p \mathbf{1}_{\Omega_N^{(k)}} = O_{L,p}(N^{-p/2}).$$

Similarly,

$$\sup_{1 \leq k \leq N} \mathbb{E}_k \left| \eta_N^{(k)} \right|^p \mathbf{1}_{\Omega_N} \leq \sup_{1 \leq k \leq N} \mathbb{E}_k \left| \eta_N^{(k)} \right|^p \mathbf{1}_{\Omega_N^{(k)}} = O_{L,p}(N^{-p/2}).$$

(viii) By the bounds in (i), we have

$$\sup_{1 \leq k \leq N} \left| \gamma_N^{(k)} \mathbf{1}_{\Omega_N} - \frac{1}{1 + z^{-1} \frac{\rho}{N} \left(\text{tr} G_N^{(k)} + \frac{1}{z} \right) \mathbf{1}_{\Omega_N^{(k)}}} \mathbf{1}_{\Omega_N^{(k)}} \right| \leq C \mathbf{1}_{\Omega_N^c}.$$

Thus, by the Burkholder inequality, for any $p \geq 2$, we have

$$\begin{aligned} & \mathbb{E} \left| \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) \bar{u}_{N,k} v_{N,k} \left(\gamma_N^{(k)} \mathbf{1}_{\Omega_N} - \frac{1}{1 + z^{-1} \frac{\rho}{N} \left(\text{tr} G_N^{(k)} + \frac{1}{z} \right) \mathbf{1}_{\Omega_N^{(k)}}} \mathbf{1}_{\Omega_N^{(k)}} \right) \right|^p \\ & \leq C_p \mathbb{P}(\Omega_N^c) \left(\sum_{k=1}^N |u_{N,k}|^2 |v_{N,k}|^2 \right)^{p/2} \\ & \leq C_p \mathbb{P}(\Omega_N^c). \end{aligned}$$

We now complete the proof of the lemma. Indeed, it suffices to show that, for any $p > 2$,

$$\mathbb{E} |u_N^* G_N v_N \mathbf{1}_{\Omega_N} - \mathbb{E} u_N^* G_N v_N \mathbf{1}_{\Omega_N}|^p = O_{L,p}(N^{-p/2}).$$

We begin by decomposing $u_N^* G_N v_N \mathbf{1}_{\Omega_N} - \mathbb{E} u_N^* G_N v_N$ as a martingale difference sequence. Since

$$\mathbb{E}_k u_N^* G_N^{(k)} v_N \mathbf{1}_{\Omega_N^{(k)}} = \mathbb{E}_{k-1} u_N^* G_N^{(k)} v_N \mathbf{1}_{\Omega_N^{(k)}},$$

we have

$$\begin{aligned} u_N^* G_N v_N \mathbf{1}_{\Omega_N} - \mathbb{E} u_N^* G_N v_N \mathbf{1}_{\Omega_N} &= \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) u_N^* G_N v_N \mathbf{1}_{\Omega_N} \\ &= \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) u_N^* \left(G_N \mathbf{1}_{\Omega_N} - G_N^{(k)} \mathbf{1}_{\Omega_N^{(k)}} \right) v_N. \end{aligned}$$

In view of (ii) and the fact that Ω_N holds with overwhelming probability, it suffices to show that, for any $p > 2$,

$$\mathbb{E} |\phi_N|^p = O_{L,p}(N^{-p/2}),$$

where

$$\phi_N := \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) u_N^* \left(G_N - G_N^{(k)} \right) v_N \mathbf{1}_{\Omega_N}.$$

By the resolvent identity, we have

$$\begin{aligned} \phi_N &= - \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) u_N^* G_N (c_k e_k^* + e_k r_k) G_N^{(k)} v_N \mathbf{1}_{\Omega_N} \\ &=: -(\phi_{N1} + \phi_{N2}). \end{aligned}$$

By (iii), (iv), (v), and (7.16), we decompose

$$\begin{aligned} \phi_{N1} &:= \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) u_N^* G_N c_k e_k^* G_N^{(k)} v_N \mathbf{1}_{\Omega_N} \\ &= - \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) z^{-1} u_N^* G_N^{(k1)} c_k v_{N,k} \alpha_N^{(k)} \mathbf{1}_{\Omega_N} \\ &= - \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) z^{-1} v_{N,k} \alpha_N^{(k)} \left(u_N^* G_N^{(k)} c_k + z^{-1} \bar{u}_{N,k} r_k G_N^{(k)} c_k \right) \mathbf{1}_{\Omega_N} \\ &= - \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) z^{-1} \left(u_N^* G_N^{(k)} c_k v_{N,k} - \bar{u}_{N,k} v_{N,k} \right) \alpha_{N,k} \mathbf{1}_{\Omega_N} \\ &=: \phi_{N11} + \phi_{N12}. \end{aligned}$$

Similarly, by (iii), (iv), and (7.16), we have

$$\begin{aligned} \phi_{N2} &:= \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) u_N^* G_N e_k r_k G_N^{(k)} v_N \mathbf{1}_{\Omega_N} \\ &= \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) u_N^* G_N^{(k2)} e_k r_k G_N^{(k)} v_N \alpha_N^{(k)} \mathbf{1}_{\Omega_N} \\ &= - \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) z^{-1} \left(\bar{u}_{N,k} - u_N^* G_N^{(k)} c_k \right) r_k G_N^{(k)} v_N \alpha_N^{(k)} \mathbf{1}_{\Omega_N} \\ &=: \phi_{N21} + \phi_{N22}. \end{aligned}$$

Therefore, in order to complete the proof, it suffices to show that, for any $p > 2$,

$$\mathbb{E}|\phi_{N11}|^p + \mathbb{E}|\phi_{N12}|^p + \mathbb{E}|\phi_{N21}|^p + \mathbb{E}|\phi_{N22}|^p = O_{L,p}(N^{-p/2}).$$

We bound each term individually.

By Rosenthal's inequality (Lemma 3.6), we have, for any $p > 2$,

$$\begin{aligned} \mathbb{E}|\phi_{N11}|^p &= \mathbb{E} \left| \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) z^{-1} u_N^* G_N^{(k)} c_k v_{N,k} \alpha_N^{(k)} \mathbf{1}_{\Omega_N} \right|^p \\ &\leq \frac{C_p}{|z|^p} \left[\mathbb{E} \left(\sum_{k=1}^N \mathbb{E}_k \left| (\mathbb{E}_{k-1} - \mathbb{E}_k) u_N^* G_N^{(k)} c_k v_{N,k} \alpha_N^{(k)} \mathbf{1}_{\Omega_N} \right|^2 \right)^{p/2} \right. \\ &\quad \left. + \sum_{k=1}^N \mathbb{E} \left| (\mathbb{E}_{k-1} - \mathbb{E}_k) u_N^* G_N^{(k)} c_k v_{N,k} \alpha_N^{(k)} \mathbf{1}_{\Omega_N} \right|^p \right] \\ &\leq C_p \left[\mathbb{E} \left(\sum_{k=1}^N |v_{N,k}|^2 \mathbb{E}_k \left| u_N^* G_N^{(k)} c_k \right|^2 \mathbf{1}_{\Omega_N^{(k)}} \right)^{p/2} \right. \\ &\quad \left. + \sum_{k=1}^N |v_{N,k}|^p \mathbb{E} \left| u_N^* G_N^{(k)} c_k \right|^p \mathbf{1}_{\Omega_N^{(k)}} \right] \\ &= O_{L,p}(N^{-p/2}). \end{aligned}$$

Here we used Lemma B.3 from Appendix B to verify that, for any $p \geq 2$,

$$\mathbb{E}_k \left| u_N^* G_N^{(k)} \mathbf{1}_{\Omega_N^{(k)}} c_k \right|^p = \mathbb{E}_k \left| c_k^* \left(G_N^{(k)} \right)^* u_N u_N^* G_N^{(k)} c_k \right|^{p/2} \mathbf{1}_{\Omega_N^{(k)}} = O_{L,p}(N^{-p/2})$$

uniformly for $1 \leq k \leq N$.

Similarly, by another application of Rosenthal's inequality, one obtains, for any $p > 2$,

$$\mathbb{E}|\phi_{N21}|^p = O_{L,p}(N^{-p/2}).$$

By (vi), we have

$$\begin{aligned} \phi_{N12} &:= \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) z^{-1} \bar{u}_{N,k} v_{N,k} \alpha_N^{(k)} \mathbf{1}_{\Omega_N} \\ &= \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) z^{-1} \bar{u}_{N,k} v_{N,k} \left(\gamma_N^{(k)} - z^{-1} \alpha_N^{(k)} \gamma_N^{(k)} \xi_N^{(k)} \right) \mathbf{1}_{\Omega_N} \\ &=: \phi_{N121} + \phi_{N122}. \end{aligned}$$

Since

$$(\mathbb{E}_{k-1} - \mathbb{E}_k) \frac{1}{1 + z^{-1} \frac{\rho}{N} \left(\text{tr} G_N^{(k)} + \frac{1}{z} \right) \mathbf{1}_{\Omega_N^{(k)}}} \mathbf{1}_{\Omega_N^{(k)}} = 0,$$

we apply (viii) to obtain $\mathbb{E}|\phi_{N121}|^p = O_{L,p}(N^{-p/2})$ for any $p \geq 2$. By (i), (vii), and Rosenthal's inequality, we have, for any $p > 2$,

$$\begin{aligned} \mathbb{E}|\phi_{N122}|^p &\leq C_p \left(\mathbb{E} \left(\sum_{k=1}^N |u_{N,k}|^2 |v_{N,k}|^2 \mathbb{E}_k \left| \xi_N^{(k)} \right|^2 \right)^{p/2} + \sum_{k=1}^N |u_{N,k}|^p |v_{N,k}|^p \mathbb{E} \left| \xi_N^{(k)} \right|^p \right) \\ &= O_{L,p}(N^{-p/2}). \end{aligned}$$

By definition of $\eta_N^{(k)}$, we have

$$\begin{aligned} \phi_{N22} &:= \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) z^{-1} u_N^* G_N^{(k)} c_k r_k G_N^{(k)} v_N \mathbf{1}_{\Omega_N} \alpha_N^{(k)} \\ &= \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) z^{-1} \alpha_N^{(k)} \mathbf{1}_{\Omega_N} \left(\eta_N^{(k)} + \frac{\rho}{N} \left(u_N^* G_N^{(k)} v_N - z^{-2} \bar{u}_{N,k} v_{N,k} \right) \right). \end{aligned}$$

From (i), we have

$$\left| z^{-1} \alpha_N^{(k)} \mathbf{1}_{\Omega_N} \frac{\rho}{N} \left(u_N^* G_N^{(k)} v_N - z^{-2} \bar{u}_{N,k} v_{N,k} \right) \right|^2 = O(N^{-2}),$$

and thus, by the Burkholder inequality, we have, for any $p \geq 2$,

$$\mathbb{E} \left| \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) z^{-1} \alpha_N^{(k)} \mathbf{1}_{\Omega_N} \frac{\rho}{N} \left(u_N^* G_N^{(k)} v_N - z^{-2} \bar{u}_{N,k} v_{N,k} \right) \right|^p = O_p(N^{-p/2}).$$

On the other hand, by (vii) and Rosenthal's inequality, for any $p > 2$, we conclude that

$$\mathbb{E} \left| \sum_{k=1}^N (\mathbb{E}_{k-1} - \mathbb{E}_k) z^{-1} \alpha_N^{(k)} \mathbf{1}_{\Omega_N} \eta_N^{(k)} \right|^p = O_{L,p}(N^{-p/2}).$$

The proof of the lemma is complete. □

7.4 Proof of Theorem 5.5

We are now ready to prove Theorem 5.5 using the results of the previous subsections.

Proof of Theorem 5.5. Let $0 < \varepsilon < 1/2$ and fix $5 \leq |z| \leq 6$. Let $\eta := \sqrt{-1}t$, where $t := \varepsilon/100$. By Lemma 3.1 and Lemma 3.3, it follows that

$$\|G_N(z)\| \leq 2 \tag{7.18}$$

on the event Ω_N . Moreover, since the eigenvalues of $R_N(\eta, z)$ are given by

$$\frac{1}{\pm \sigma_i(X_N - zI) - \eta}, \quad i = 1, \dots, N,$$

it follows that

$$\|R_N(\eta, z)\| \leq 4$$

on the event Ω_N .

Since Ω_N holds with overwhelming probability, it suffices to show that a.s., for N sufficiently large,

$$|u_N^* G_N(z) v_N \mathbf{1}_{\Omega_N} - m(z) u_N^* v_N \mathbf{1}_{\Omega_N}| \leq \varepsilon.$$

By the triangle inequality, we have

$$\begin{aligned} |u_N^* G_N(z) v_N \mathbf{1}_{\Omega_N} - m(z) u_N^* v_N \mathbf{1}_{\Omega_N}| &\leq |u_N^* G_N(z) v_N \mathbf{1}_{\Omega_N} - \mathbb{E} u_N^* G_N(z) v_N \mathbf{1}_{\Omega_N}| \\ &\quad + |\mathbb{E} u_N^* G_N(z) v_N \mathbf{1}_{\Omega_N} - \mathbb{E} u_N^* R_N^{21}(\eta, z) v_N \mathbf{1}_{\Omega_N}| \\ &\quad + |\mathbb{E} u_N^* R_N^{21}(\eta, z) v_N \mathbf{1}_{\Omega_N} - m(z) u_N^* v_N \mathbf{1}_{\Omega_N}|. \end{aligned}$$

The first term is a.s. less than $\varepsilon/8$ by Lemma 7.3. The second term is bounded by noting that $R_N^{21}(0, z) = G_N(z)$ and using (3.3) to conclude that

$$\|R_N(\eta, z) - R_N(0, z)\| \mathbf{1}_{\Omega_N} \leq 8|\eta| \leq \frac{\varepsilon}{8}. \tag{7.19}$$

Thus, it suffices to show that

$$\left| \mathbb{E} u_N^* R_N^{21}(\eta, z) v_N \mathbf{1}_{\Omega_N} - m(z) u_N^* v_N \mathbf{1}_{\Omega_N} \right| \leq \varepsilon/2. \tag{7.20}$$

We will verify (7.20) by considering the diagonal entries and off-diagonal entries of $R_N^{21}(\eta, z)$ separately. For the diagonal terms we write

$$\begin{aligned} & \left| \mathbb{E} \sum_{i=1}^N \bar{u}_i R_{ii}^{21}(\eta, z) v_i \mathbf{1}_{\Omega_N} - m(z) u_N^* v_N \right| \\ & \leq \sum_{i=1}^N |u_i| |v_i| \max_{1 \leq i \leq N} \mathbb{E} |R_{ii}^{21}(\eta, z) - m(z)| \mathbf{1}_{\Omega_N} \\ & \leq \max_{1 \leq i \leq N} \mathbb{E} |R_{ii}^{21}(\eta, z) - m(z)| \mathbf{1}_{\Omega_N} \end{aligned}$$

by the Cauchy-Schwarz inequality. By (7.19) and Lemma 7.1, we have

$$\max_{1 \leq i \leq N} \mathbb{E} |R_{ii}^{21}(\eta, z) - m(z)| \mathbf{1}_{\Omega_N} \leq \frac{\varepsilon}{8} + \max_{1 \leq i \leq N} \mathbb{E} |(G_N(z))_{ii} - m(z)| \mathbf{1}_{\Omega_N} \leq \frac{\varepsilon}{4}.$$

Thus, it suffices to show that

$$\mathbb{E} \sum_{i \neq j} \bar{u}_i R_{ij}^{21}(\eta, z) v_j \mathbf{1}_{\Omega_N} = o(1).$$

Since Ω_N holds with overwhelming probability, we have (say)

$$\mathbb{E} \sum_{i \neq j} \bar{u}_i R_{ij}^{21}(\eta, z) v_j \mathbf{1}_{\Omega_N} = \mathbb{E} \sum_{i \neq j} \bar{u}_i R_{ij}^{21}(\eta, z) v_j + O(N^{-100})$$

by the deterministic bound $\|R_N(\eta, z)\| \leq \text{Im}(\eta)^{-1}$. Thus, it suffices to show that

$$\mathbb{E} \sum_{i \neq j} \bar{u}_i R_{ij}^{21}(\eta, z) v_j = o(1).$$

From Lemma 7.2 and the Cauchy-Schwarz inequality, we see that

$$\begin{aligned} \left| \mathbb{E} \sum_{i \neq j} \bar{u}_i R_{ij}^{21}(\eta, z) v_j \right| & \leq \sum_{i \neq j} |u_i| |v_j| \max_{i \neq j} |\mathbb{E} R_{ij}^{21}(\eta, z)| \\ & \leq N \max_{i \neq j} |\mathbb{E} R_{ij}^{21}(\eta, z)| = o(1), \end{aligned}$$

and the proof is complete. □

A Truncation of elliptic random matrices

In this appendix, we establish Lemma 3.4.

Proof of Lemma 3.4. We begin by noting that, for $i \in \{1, 2\}$,

$$\text{Var}(\tilde{\xi}_i) \leq \mathbb{E} |\xi_i|^2 \mathbf{1}_{\{|\xi_i| \leq L\}} \leq 1 \tag{A.1}$$

and, by the dominated convergence theorem,

$$\lim_{L \rightarrow \infty} \text{Var}(\tilde{\xi}_i) = 1.$$

We take $L_0 > 1$ sufficiently large such that, for each $i \in \{1, 2\}$, $\text{Var}(\tilde{\xi}_i) \geq 1/2$ for all $L > L_0$. Assume $L > L_0$. Then (3.14) follows by an application of the triangle inequality.

Moreover, $\hat{\xi}_1, \hat{\xi}_2$ have mean zero and unit variance by construction. Thus, $\{\hat{Y}_N\}_{N \geq 1}$ is a sequence of random matrices that satisfies condition **C0** with atom variables $(\hat{\xi}_1, \hat{\xi}_2)$.

We now make use of the following bounds: if ψ is a random variable with finite fourth moment, then

$$|\mathbb{E}\psi \mathbf{1}_{\{|\psi| > L\}}| \leq \frac{\mathbb{E}|\psi|^4}{L^3} \quad \text{and} \quad \mathbb{E}|\psi|^2 \mathbf{1}_{\{|\psi| > L\}} \leq \frac{\mathbb{E}|\psi|^4}{L^2}. \tag{A.2}$$

We note that

$$\begin{aligned} \rho &= \mathbb{E}[\xi_1 \xi_2 \mathbf{1}_{\{|\xi_1| \leq L\}} \mathbf{1}_{\{|\xi_2| \leq L\}}] + \mathbb{E}[\xi_1 \xi_2 \mathbf{1}_{\{|\xi_1| > L\}} \mathbf{1}_{\{|\xi_2| \leq L\}}] \\ &\quad + \mathbb{E}[\xi_1 \xi_2 \mathbf{1}_{\{|\xi_1| \leq L\}} \mathbf{1}_{\{|\xi_2| > L\}}] + \mathbb{E}[\xi_1 \xi_2 \mathbf{1}_{\{|\xi_1| > L\}} \mathbf{1}_{\{|\xi_2| > L\}}]. \end{aligned}$$

Thus, by the Cauchy-Schwarz inequality and (A.2), we obtain

$$|\rho - \tilde{\rho}| \leq \frac{C}{L} \tag{A.3}$$

for some constant $C > 0$ depending on M_4 . Similarly, we have

$$|1 - \text{Var}(\hat{\xi}_i)| \leq \mathbb{E}|\xi_i|^2 \mathbf{1}_{\{|\xi_i| > L\}} + |\mathbb{E}\xi_i \mathbf{1}_{\{|\xi_i| > L\}}|^2 \leq \frac{C}{L^2} \tag{A.4}$$

for $i \in \{1, 2\}$.

By (A.1), we have

$$\begin{aligned} |\tilde{\rho} - \rho| &\leq |\tilde{\rho}| \left| \frac{1}{\sqrt{\text{Var}(\tilde{\xi}_1) \text{Var}(\tilde{\xi}_2)}} - 1 \right| \\ &\leq 2|\tilde{\rho}| \left| \sqrt{\text{Var}(\tilde{\xi}_1) \text{Var}(\tilde{\xi}_2)} - 1 \right| \\ &\leq 2|\tilde{\rho}| \left| \text{Var}(\tilde{\xi}_1) \text{Var}(\tilde{\xi}_2) - 1 \right| \\ &\leq 2|\tilde{\rho}| \left(\left| 1 - \text{Var}(\tilde{\xi}_1) \right| + \left| 1 - \text{Var}(\tilde{\xi}_2) \right| \right). \end{aligned}$$

From (A.4), we conclude that $|\tilde{\rho} - \rho| \leq \frac{C}{L^2}$ for some constant $C > 0$ depending on M_4 . Combining this bound with (A.3) yields (3.15).

It remains to prove (3.16) and (3.17). By Lemma 3.3, it follows that a.s.

$$\limsup_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \|Y_N\| \leq 4 \quad \text{and} \quad \limsup_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \|\hat{Y}_N\| \leq 4.$$

By Lemma 3.1, we have (say) a.s.

$$\limsup_{N \rightarrow \infty} \sup_{|z| \geq 5} \|G_N(z)\| \leq 2 \quad \text{and} \quad \limsup_{N \rightarrow \infty} \sup_{|z| \geq 5} \|\hat{G}_N(z)\| \leq 2.$$

Thus, by the resolvent identity (3.3), we have a.s.

$$\begin{aligned} \limsup_{N \rightarrow \infty} \sup_{|z| \geq 5} \|G_N(z) - \hat{G}_N(z)\| &\leq 4 \limsup_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \|Y_N - \hat{Y}_N\| \\ &\leq 4 \limsup_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \left(\|Y_N - \tilde{Y}_N\| + \|\tilde{Y}_N - \hat{Y}_N\| \right). \end{aligned}$$

Therefore, in order to prove (3.16) and (3.17), it suffices to show that a.s.

$$\limsup_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \left(\|Y_N - \tilde{Y}_N\| + \|\tilde{Y}_N - \hat{Y}_N\| \right) \leq \frac{C}{L}$$

for some constant $C > 0$ depending on M_4 . Consider the second term on the left-hand side. We write

$$\|\tilde{Y}_N - \hat{Y}_N\| \leq \left\| \frac{(\tilde{Y}_N - \hat{Y}_N) + (\tilde{Y}_N - \hat{Y}_N)^*}{2} \right\| + \left\| \frac{(\tilde{Y}_N - \hat{Y}_N) - (\tilde{Y}_N - \hat{Y}_N)^*}{2\sqrt{-1}} \right\|.$$

We now apply [5, Theorem 5.2] as in the proof of Lemma 3.3. By (A.4), we obtain a.s.

$$\limsup_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \|\tilde{Y}_N - \hat{Y}_N\| \leq \frac{C}{L}$$

for some constant $C > 0$ depending on M_4 . Similarly, by another application of [5, Theorem 5.2] and (A.2), we have a.s.

$$\limsup_{N \rightarrow \infty} \frac{1}{\sqrt{N}} \|Y_N - \tilde{Y}_N\| \leq \frac{C}{L}.$$

The proof of the lemma is complete. □

B Large Deviation Estimates

This section is devoted to proving a large deviation estimate for bilinear forms. Throughout this section, we let K_p denote a constant that depends only on p . These constants are non-random and may take on different values from one appearance to the next.

Lemma B.1 (Concentration of bilinear forms). *Let $(x_1, y_1), (x_2, y_2), \dots, (x_N, y_N)$ be iid random vectors in \mathbb{C}^2 such that*

$$\mathbb{E}[x_1] = \mathbb{E}[y_1] = 0, \quad \mathbb{E}|x_1|^2 = \mathbb{E}|y_1|^2 = 1, \quad \mathbb{E}[\bar{x}_1 y_1] = \rho.$$

Let $\mu_p = \max\{\mathbb{E}|x_1|^p, \mathbb{E}|y_1|^p\}$ for $p \geq 4$. Let $B = (b_{ij})$ be a deterministic complex $N \times N$ matrix and write $X = (x_1, x_2, \dots, x_N)^T$ and $Y = (y_1, y_2, \dots, y_N)^T$. Then, for any $p \geq 2$,

$$\mathbb{E}|X^* B Y - \rho \operatorname{tr} B|^p \leq K_p \left((\mu_4 \operatorname{tr}(B B^*))^{p/2} + \mu_{2p} \operatorname{tr}(B B^*)^{p/2} \right).$$

The proof of Lemma B.1 is based on the proof of [4, Lemma 2.7]. In fact, when $\rho = 1$, we recover [4, Lemma 2.7].

We will need the following results.

Lemma B.2 ((3.3.41) of [32]). *For $N \times N$ Hermitian $A = (a_{ij})$ with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$, and f convex, we have*

$$\sum_{i=1}^N f(a_{ii}) \leq \sum_{i=1}^N f(\lambda_i).$$

Lemma B.3 (Lemma A.1 of [4]). *For $X = (x_1, x_2, \dots, x_N)^T$ iid standardized complex entries, B $N \times N$ Hermitian nonnegative definite matrix, we have, for any $p \geq 1$,*

$$\mathbb{E}|X^* B X|^p \leq K_p \left((\operatorname{tr} B)^p + \mathbb{E}|x_1|^{2p} \operatorname{tr} B^p \right).$$

We are now ready to prove Lemma B.1.

Proof of Lemma B.1. Let $\{\mathcal{F}_i\}_{i=0}^N$ denote the sequence of increasing σ -algebras defined by

$$\mathcal{F}_i = \sigma(x_1, y_1, x_2, y_2, \dots, x_i, y_i)$$

for $i = 1, 2, \dots, N$. Following the usual convention, we let \mathcal{F}_0 denote the trivial σ -algebra. We will continually make use of this filtration throughout the proof.

We begin by writing

$$X^*BY - \rho \operatorname{tr} B = \sum_{i=1}^N (\bar{x}_i y_i - \rho) b_{ii} + \sum_{i=2}^N \bar{x}_i \sum_{j<i} y_j b_{ij} + \sum_{j=2}^N y_j \sum_{i<j} \bar{x}_i b_{ij}$$

and hence

$$\begin{aligned} & \mathbb{E} |X^*BY - \rho \operatorname{tr} B|^p \\ & \leq K_p \left(\mathbb{E} \left| \sum_{i=1}^N (\bar{x}_i y_i - \rho) b_{ii} \right|^p + \mathbb{E} \left| \sum_{i=2}^N \bar{x}_i \sum_{j<i} y_j b_{ij} \right|^p + \mathbb{E} \left| \sum_{j=2}^N y_j \sum_{i<j} \bar{x}_i b_{ij} \right|^p \right). \end{aligned} \quad (\text{B.1})$$

We will bound each of the three terms on the right-hand side of (B.1) separately. We begin with the first term. By Lemma 3.6,

$$\begin{aligned} \mathbb{E} \left| \sum_{i=1}^N (\bar{x}_i y_i - \rho) b_{ii} \right|^p & \leq K_p \left(\left(\sum_{i=1}^N \mathbb{E} |\bar{x}_i y_i - \rho|^2 |b_{ii}|^2 \right)^{p/2} + \sum_{i=1}^N \mathbb{E} |\bar{x}_i y_i - \rho|^p |b_{ii}|^p \right) \\ & \leq K_p \left((\mu_4 \operatorname{tr}(BB^*))^{p/2} + \mu_{2p} \sum_{i=1}^N |b_{ii}|^p \right). \end{aligned}$$

Here we have used

$$(\mathbb{E} |\bar{x}_1 y_1 - \rho|^p)^{1/p} \leq (\mathbb{E} |x_1|^{2p})^{1/p} + (\mathbb{E} |y_1|^{2p})^{1/p} + 1 \leq 3\mu_{2p}^{1/p}.$$

From Lemma B.2, we have

$$\sum_{i=1}^N |b_{ii}|^p \leq \sum_{i=1}^N (BB^*)_{ii}^{p/2} \leq \sum_{i=1}^N \lambda_i (BB^*)^{p/2} = \operatorname{tr}(BB^*)^{p/2},$$

where $\lambda_1(BB^*), \dots, \lambda_N(BB^*)$ denote the eigenvalues of BB^* . Combining the bounds above yields

$$\mathbb{E} \left| \sum_{i=1}^N (\bar{x}_i y_i - \rho) b_{ii} \right|^p \leq K_p \left((\mu_4 \operatorname{tr} BB^*)^{p/2} + \mu_{2p} \operatorname{tr}(BB^*)^{p/2} \right).$$

We now consider the second term on the right-hand side of (B.1). By Lemma 3.6,

$$\mathbb{E} \left| \sum_{i=2}^N \bar{x}_i \sum_{j<i} y_j b_{ij} \right|^p \leq K_p \left(\mathbb{E} \left(\sum_{i=2}^N \left| \sum_{j<i} y_j b_{ij} \right|^2 \right)^{p/2} + \mu_p \mathbb{E} \sum_{i=2}^N \left| \sum_{j<i} y_j b_{ij} \right|^p \right).$$

We will bound each of the terms on the right-hand side separately. For the first term, we write

$$\mathbb{E} \left(\sum_{i=2}^N \left| \sum_{j<i} y_j b_{ij} \right|^2 \right)^{p/2} = \mathbb{E} \left(\sum_{i=2}^N \left| \sum_{j=1}^N \mathbb{E}(y_j b_{ij} | \mathcal{F}_{i-1}) \right|^2 \right)^{p/2}.$$

Applying Lemma 3.8 and Lemma B.3, we have

$$\begin{aligned} \mathbb{E} \left(\sum_{i=2}^N \left| \sum_{j=1}^N \mathbb{E}(y_j b_{ij} | \mathcal{F}_{i-1}) \right|^2 \right)^{p/2} &\leq K_p \mathbb{E} \left(\sum_{i=2}^N \left| \sum_{j=1}^N y_j b_{ij} \right|^2 \right)^{p/2} \\ &\leq K_p \mathbb{E} (Y^* B^* B Y)^{p/2} \\ &\leq K_p \left((\text{tr} B B^*)^{p/2} + \mu_{2p} \text{tr}(B B^*)^{p/2} \right) \\ &\leq K_p \left((\mu_4 \text{tr} B B^*)^{p/2} + \mu_{2p} \text{tr}(B B^*)^{p/2} \right). \end{aligned}$$

For the second term, we apply Lemma 3.6 and obtain

$$\begin{aligned} \mathbb{E} \sum_{i=2}^N \left| \sum_{j<i} y_j b_{ij} \right|^p &\leq K_p \sum_{i=2}^N \left(\left(\sum_{j<i} |b_{ij}|^2 \right)^{p/2} + \mu_p \sum_{j<i} |b_{ij}|^p \right) \\ &\leq K_p (1 + \mu_p) \sum_{i=2}^N \left(\sum_{j<i} |b_{ij}|^2 \right)^{p/2}. \end{aligned}$$

We now note that

$$\sum_{i=2}^N \left(\sum_{j<i} |b_{ij}|^2 \right)^{p/2} \leq \sum_{i=1}^N ((B B^*)_{ii})^{p/2} \leq \text{tr}(B B^*)^{p/2}$$

by Lemma B.2. Thus

$$\mu_p \mathbb{E} \sum_{i=2}^N \left| \sum_{j<i} y_j b_{ij} \right|^p \leq K_p \mu_{2p} \text{tr}(B B^*)^{p/2}$$

since $\mu_p(1 + \mu_p) \leq 2\mu_p^2 \leq 2\mu_{2p}$. Combining the two bounds above, we obtain

$$\mathbb{E} \left| \sum_{i=2}^N \bar{x}_i \sum_{j<i} y_j b_{ij} \right|^p \leq K_p \left((\mu_4 \text{tr} B B^*)^{p/2} + \mu_{2p} \text{tr}(B B^*)^{p/2} \right).$$

The third term on the right-hand side of (B.1) is similarly bounded. The proof of the lemma is complete. \square

C Properties of the limiting measure

This section is devoted to studying the limiting distribution of the singular values of $\frac{1}{\sqrt{N}} Y_N - zI$, where $z \in \mathbb{C}$ and $\{Y_N\}_{N \geq 1}$ is a sequence of random matrices that satisfy condition **C0** with atom variables (ξ_1, ξ_2) . In particular, this section contains the proof of Theorem 6.1. Throughout this section, we fix $\rho := \mathbb{E}[\xi_1 \xi_2]$ with $-1 < \rho < 1$. Let \mathcal{E}_ρ be the ellipsoid defined in (1.3). We let $\sqrt{-1}$ denote the imaginary unit and reserve i as an index.

Remark C.1. *Many of the results in this section also hold when $\rho = \pm 1$ (although the proofs are different). In particular, Theorem 6.1 holds when $\rho = \pm 1$; see Remark 6.2 for further details.*

Let $a_N(\eta, z)$ be the Stieltjes transform of $\nu_{\frac{1}{\sqrt{N}}Y_N - zI}$ (defined in (6.2)). That is, for each $z \in \mathbb{C}$,

$$a_N(\eta, z) := \int_{\mathbb{R}} \frac{1}{u - \eta} \nu_{\frac{1}{\sqrt{N}}Y_N - zI}(du)$$

for $\eta \in \mathbb{C}^+ := \{w \in \mathbb{C} : \text{Im}(w) > 0\}$. We study the limiting distribution of the singular values by characterizing the limiting Stieltjes transform. We begin with the following lemma.

Lemma C.2 (Self-consistent equation). *Let $z, \eta \in \mathbb{C}$ with $\text{Im}(\eta) > 0$. Fix $-1 < \rho < 1$. If $a, b, c \in \mathbb{C}$ satisfy*

$$\begin{bmatrix} a & b \\ c & a \end{bmatrix} = \begin{bmatrix} -\eta - a & -\rho c - z \\ -\rho b - \bar{z} & -\eta - a \end{bmatrix}^{-1}, \tag{C.1}$$

then

$$\frac{1}{(a + \eta)a} + 1 = \frac{\text{Re}(z)^2}{(\eta + (1 + \rho)a)^2} + \frac{\text{Im}(z)^2}{(\eta + (1 - \rho)a)^2}. \tag{C.2}$$

Proof. We rewrite (C.1) as the system of equations

$$\frac{a}{a^2 - bc} = -\eta - a, \quad \frac{-b}{a^2 - bc} = -\rho c - z, \quad \frac{-c}{a^2 - bc} = -\rho b - \bar{z}. \tag{C.3}$$

Since $\text{Im}(\eta) > 0$, the first equation implies $a \neq 0$. Thus, we obtain

$$\begin{aligned} -b \left(\frac{\eta + a}{a} \right) &= \rho c + z \\ -c \left(\frac{\eta + a}{a} \right) &= \rho b + \bar{z}. \end{aligned}$$

Solving these equations for b and c yields

$$bq = a^2 \rho \bar{z} - a(\eta + a)z \tag{C.4}$$

$$cq = a^2 \rho z - a(\eta + a)\bar{z}, \tag{C.5}$$

where $q := (\eta + a)^2 - \rho^2 a^2$. Thus, we have

$$bcq^2 = a^2 |z|^2 (a^2 \rho^2 + (\eta + a)^2) - a^3 (\eta + a) \rho (z^2 + \bar{z}^2),$$

and hence

$$(a^2 - bc)q^2 = a^2 q^2 - a^2 |z|^2 (a^2 \rho^2 + (\eta + a)^2) + a^3 (\eta + a) \rho (z^2 + \bar{z}^2).$$

Equation (C.2) can now be obtained by combining the calculation above with the first equation from (C.3) and noting that

$$\begin{aligned} |z|^2 (a^2 \rho^2 + (\eta + a)^2) - a(\eta + a) \rho (z^2 + \bar{z}^2) \\ = \text{Re}(z)^2 (\eta + (1 - \rho)a)^2 + \text{Im}(z)^2 (\eta + (1 + \rho)a)^2. \end{aligned}$$

The proof of the lemma is complete. □

Remark C.3. *One can also use (C.4) and (C.5) to solve for b and c . Indeed, from (C.4) it follows that*

$$b = a \left(\frac{a \rho \bar{z} - (\eta + a)z}{q} \right) = -a \left(\frac{\text{Re}(z)}{\eta + a(1 + \rho)} + \sqrt{-1} \frac{\text{Im}(z)}{\eta + a(1 - \rho)} \right).$$

Similarly, from (C.5), we have

$$c = -a \left(\frac{\text{Re}(z)}{\eta + a(1 + \rho)} - \sqrt{-1} \frac{\text{Im}(z)}{\eta + a(1 - \rho)} \right).$$

Remark C.4. Fix $z \in \mathbb{C}$. If (C.1) holds for all η with $\text{Im}(\eta) > 0$, then a, b, c can be viewed as functions of η . In this case, an upper bound for a can be obtained (see (C.11)). In fact, in view of Lemma C.7, a can be uniformly bounded from above for all $\text{Im}(\eta) > 0$. Thus, one can use (C.2) and Remark C.3 to obtain uniform upper bounds on b, c for all $\text{Im}(\eta) > 0$.

We will also need the following lemma for Stieltjes transforms of probability measures on the real line.

Lemma C.5. Let ν be a probability measure on the real line. Let m be the Stieltjes transform of ν . That is,

$$m(\eta) = \int_{\mathbb{R}} \frac{1}{u - \eta} \nu(du), \quad \eta \in \mathbb{C}^+.$$

Then

$$\lim_{y \rightarrow \infty} \sup_{x \in \mathbb{R}} |m(x + \sqrt{-1}y)| = 0 \tag{C.6}$$

and

$$\lim_{y \rightarrow \infty} \sup_{|x| \leq \sqrt{y}} |(x + \sqrt{-1}y)m(x + \sqrt{-1}y) + 1| = 0. \tag{C.7}$$

Proof. Equation (C.6) follows from the trivial bound $|m(\eta)| \leq |\text{Im}(\eta)|^{-1}$. We now prove (C.7). We note that

$$(x + \sqrt{-1}y)m(x + \sqrt{-1}y) + 1 = \int_{\mathbb{R}} \frac{u}{(u - x) + \sqrt{-1}y} \nu(du)$$

and hence

$$\begin{aligned} & |(x + \sqrt{-1}y)m(x + \sqrt{-1}y) + 1| \\ & \leq \int_{|u| \leq 2\sqrt{y}} \frac{|u|}{\sqrt{(u - x)^2 + y^2}} \nu(du) + \int_{|u| > 2\sqrt{y}} \frac{|u - x| + |x|}{\sqrt{(u - x)^2 + y^2}} \nu(du) \\ & \leq \frac{2}{\sqrt{y}} + \nu((-\infty, -2\sqrt{y}) \cup (2\sqrt{y}, \infty)) + \frac{|x|}{y}. \end{aligned}$$

Thus, we have

$$\sup_{|x| \leq \sqrt{y}} |(x + \sqrt{-1}y)m(x + \sqrt{-1}y) + 1| \leq \frac{3}{\sqrt{y}} + \nu((-\infty, -2\sqrt{y}) \cup (2\sqrt{y}, \infty))$$

and the claim follows. □

We will use Lemma C.2 and Lemma C.6 to study the limit of $a_N(\eta, z)$ for each $z \in \mathbb{C}$ and $\eta \in \mathbb{C}^+$. Indeed, it follows from the calculations in [40] (see also [39]) that $a_N(\eta, z)$ converges almost surely as $N \rightarrow \infty$ to a solution of

$$\frac{1}{a(\eta, z)(\eta + a(\eta, z))} + 1 = \frac{\text{Re}(z)^2}{(\eta + (1 + \rho)a(\eta, z))^2} + \frac{\text{Im}(z)^2}{(\eta + (1 - \rho)a(\eta, z))^2}. \tag{C.8}$$

Lemma C.6 (Existence and uniqueness). Fix $-1 < \rho < 1$. For each $z \in \mathbb{C}$, there exists a unique probability measure ν_z on the real line such that

$$a(\eta, z) := \int \frac{1}{u - \eta} \nu_z(du) \tag{C.9}$$

is a solution of (C.8) for all $\eta \in \mathbb{C}^+$.

Proof. Fix $z \in \mathbb{C}$. Since almost surely $\|\frac{1}{\sqrt{N}}Y_N - zI\| = O_z(1)$ by Lemma 3.3, the sequence of measures

$$\left\{ \nu_{\frac{1}{\sqrt{N}}Y_N - zI} \right\}_{N \geq 1} \tag{C.10}$$

is almost surely tight. Existence now follows from a subsequence argument and by applying [5, Theorem B.9] and Lemma C.2.

We now prove uniqueness. Suppose ν_z and ν'_z are two probability measures on the real line whose Stieltjes transforms

$$\begin{aligned} a(\eta, z) &= \int \frac{1}{u - \eta} \nu_z(du), \\ s(\eta, z) &= \int \frac{1}{u - \eta} \nu'_z(du) \end{aligned}$$

satisfy (C.8) for all $\eta \in \mathbb{C}^+$. Seeking a contradiction, assume $\nu_z \neq \nu'_z$. Since $a(\eta, z)$ and $s(\eta, z)$ are analytic functions of η in the upper half plane, it follows from [5, Theorem B.8] that the set

$$E := \{ \eta \in \mathbb{C}^+ : m(\eta, z) = s(\eta, z) \}$$

has no accumulation point in \mathbb{C}^+ . Define the set $Q \subset \mathbb{C}^+ \setminus E$ such that $\eta \in Q$ if and only if

$$\frac{|\eta + s(\eta, z) + a(\eta, z)|}{|a(\eta, z)(\eta + a(\eta, z))s(\eta, z)(\eta + s(\eta, z))|} \geq 16 \frac{R^2 (|\eta| + |a(\eta, z)| + |s(\eta, z)|)}{| \eta + ra(\eta, z) |^2 | \eta + rs(\eta, z) |^2}$$

for each $r \in \{1 + \rho, 1 - \rho\}$ and $R := \max\{\text{Re}(z)^2, \text{Im}(z)^2\}$. By taking $\text{Im}(\eta)$ sufficiently large, it follows from Lemma C.5 that Q contains an open disk D of radius $\varepsilon > 0$. Thus, by analytic continuation (and [5, Theorem B.8]), it suffices to show that $a(\eta, z) = s(\eta, z)$ for all $\eta \in D$.

Indeed, consider (C.8) for both functions $a(\eta, z)$ and $s(\eta, z)$. We will subtract one equation from the other. We first note that

$$\begin{aligned} & \frac{1}{a(\eta, z)(\eta + a(\eta, z))} - \frac{1}{s(\eta, z)(\eta + s(\eta, z))} \\ &= (s(\eta, z) - a(\eta, z)) \frac{\eta + s(\eta, z) + a(\eta, z)}{a(\eta, z)(\eta + a(\eta, z))s(\eta, z)(\eta + s(\eta, z))}. \end{aligned}$$

We also have

$$\begin{aligned} & \left| \frac{1}{(\eta + ra(\eta, z))^2} - \frac{1}{(\eta + rs(\eta, z))^2} \right| \\ & \leq 4|s(\eta, z) - a(\eta, z)| \frac{|\eta| + |s(\eta, z)| + |a(\eta, z)|}{|\eta + ra(\eta, z)|^2 |\eta + rs(\eta, z)|^2} \end{aligned}$$

for each $r \in \{1 + \rho, 1 - \rho\}$. Thus, for $\eta \in D$, we obtain

$$\begin{aligned} & |s(\eta, z) - a(\eta, z)| \frac{|\eta + s(\eta, z) + a(\eta, z)|}{|a(\eta, z)(\eta + a(\eta, z))s(\eta, z)(\eta + s(\eta, z))|} \\ & \leq \frac{|s(\eta, z) - a(\eta, z)|}{2} \frac{|\eta + s(\eta, z) + a(\eta, z)|}{|a(\eta, z)(\eta + a(\eta, z))s(\eta, z)(\eta + s(\eta, z))|}. \end{aligned}$$

Since $\text{Im}(\eta + s(\eta, z) + a(\eta, z)) > 0$, we conclude that

$$|s(\eta, z) - a(\eta, z)| \leq \frac{|s(\eta, z) - a(\eta, z)|}{2}$$

for all $\eta \in D$, and the claim follows. □

For the remainder of the section, we fix $-1 < \rho < 1$ and let ν_z denote the unique probability measure from Lemma C.6. Let $a(\eta, z)$ be its Stieltjes transform defined by (C.9) for all $\eta \in \mathbb{C}^+$. It follows from Lemma C.2, Lemma C.6, and the calculations in [40] (see also [39]) that $a_N(\eta, z)$ converges almost surely to $a(\eta, z)$ as $N \rightarrow \infty$ for each fixed $z \in \mathbb{C}$ and $\eta \in \mathbb{C}^+$. By [5, Theorem B.9], the sequence of measures given in (C.10) converge almost surely to ν_z for each fixed $z \in \mathbb{C}$. We now derive some properties of ν_z .

Lemma C.7 (Properties of ν_z). *Fix $-1 < \rho < 1$. For each $z \in \mathbb{C}$, ν_z is compactly supported and has continuous, bounded density.*

Proof. Fix $z \in \mathbb{C}$. Since almost surely $\|\frac{1}{\sqrt{N}}Y_N - zI\| = O_z(1)$ by Lemma 3.3, it follows that ν_z is compactly supported.

We now verify that ν_z has bounded density. Consider the Stieltjes transform $a(\eta, z)$ as a solution of (C.8). We claim that for any $C' > 0$ there exists a corresponding $C > 0$ such that

$$\sup_{\eta \in \mathbb{C}^+, |\eta| \leq C'} |a(\eta, z)| \leq C. \tag{C.11}$$

Indeed, suppose there exists $\eta \in \mathbb{C}^+$ such that $|\eta| \leq C'$ and $|a(\eta, z)| \geq C$ for some sufficiently large constant $C > 0$. We take C large enough to satisfy

$$rC - C' \geq \frac{rC}{2} \tag{C.12}$$

and

$$\frac{4|z|^2}{r^2C^2} + \frac{2}{rC^2} < 1, \tag{C.13}$$

where $r := \min\{1 - \rho, 1 + \rho\}$. From (C.12), we obtain the bounds

$$\begin{aligned} \frac{1}{|\eta + (1 + \rho)a(\eta, z)|} &\leq \frac{2}{rC}, & \frac{1}{|\eta + (1 - \rho)a(\eta, z)|} &\leq \frac{2}{rC}, \\ \frac{1}{|\eta + a(\eta, z)|} &\leq \frac{2}{rC}. \end{aligned}$$

Applying the bounds above to (C.8) yields

$$1 \leq \frac{4|z|^2}{r^2C^2} + \frac{2}{rC^2}.$$

This contradicts our choice of C in (C.13), and (C.11) follows.

Choose C' sufficiently large such that ν_z is supported on $[-C'/2, C'/2]$. Let $C > 0$ be the corresponding constant such that (C.11) holds. For any finite interval $I \subset \mathbb{R}$, it follows from [5, Theorem B.8] that

$$\nu_z(I) \leq 2C|I|. \tag{C.14}$$

Here we used the fact that the continuity points of the function $x \mapsto \nu_z((-\infty, x])$ are dense in \mathbb{R} . It follows from (C.14) that ν_z has bounded density. As the roots of a polynomial depend continuously on the coefficients (see [20, 50]), (C.8) and [5, Theorem B.10] imply that ν_z has continuous density. \square

Lemma C.8. *Fix $-1 < \rho < 1$ and $z \notin \mathcal{E}_\rho$. Then there exists $c > 0$ such that*

$$\nu_z([-c, c]) = 0.$$

Proof. Fix $-1 < \rho < 1$ and $z \notin \mathcal{E}_\rho$. Since ν_z is the almost sure limit of the measures in (C.10), it suffices to show that there exists $c > 0$ such that $\nu_z([0, c]) = 0$.

From (C.8), $a(\eta, z)$ can be continuously extended to the closed upper plane $\{\eta \in \mathbb{C} : \text{Im}(\eta) \geq 0\}$. We claim that $a(0, z) = 0$. Suppose to the contrary. Taking the sequence $\eta = \sqrt{-1}y$ with $y \searrow 0$, we obtain

$$1 + \lim_{y \searrow 0} (a(\sqrt{-1}y, z))^2 = \frac{\text{Re}(z)^2}{(1 + \rho)^2} + \frac{\text{Im}(z)^2}{(1 - \rho)^2}. \tag{C.15}$$

However, since $\text{Re}(a_N(\sqrt{-1}y, z)) = 0$ for all $y > 0$, it follows that

$$\lim_{y \searrow 0} (a(\sqrt{-1}y, z))^2 \leq 0.$$

Thus, (C.15) contradicts the assumption that $z \notin \mathcal{E}_\rho$. We conclude that $a(0, z) = 0$.

From Lemma C.7, ν_z has bounded, continuous density p_z . We now derive some properties of p_z . From (C.8), we obtain

$$\frac{1}{\frac{a(\eta, z)}{\eta} (1 + \frac{a(\eta, z)}{\eta})} + \eta^2 = \frac{\text{Re}(z)^2}{\left(1 + (1 + \rho)\frac{a(\eta, z)}{\eta}\right)^2} + \frac{\text{Im}(z)^2}{\left(1 + (1 - \rho)\frac{a(\eta, z)}{\eta}\right)^2} \tag{C.16}$$

for $\eta \in \mathbb{C}^+$. We now claim that $\eta^{-1}a(\eta, z)$ is bounded for all $\eta \in \mathbb{C}^+$. In order to reach a contradiction, assume $\eta^{-1}a(\eta, z)$ is not bounded. From (C.11) and Lemma C.7, it must be the case that $|\eta^{-1}a(\eta, z)|$ tends to infinity as η tends to zero through the upper half plane. Thus, by multiplying (C.16) by $\eta^{-2}a^2(\eta, z)$ and taking the limit $\eta \rightarrow 0$, we obtain

$$1 = \frac{\text{Re}(z)^2}{(1 + \rho)^2} + \frac{\text{Im}(z)^2}{(1 - \rho)^2}$$

since $a(0, z) = 0$. This is clearly a contradiction for $z \notin \mathcal{E}_\rho$. We conclude that $\eta^{-1}a(\eta, z)$ is bounded for all $\eta \in \mathbb{C}^+$. Thus, there exists $C > 0$ such that

$$\text{Im}(a(\sqrt{-1}y, z)) \leq Cy$$

for all $y > 0$. Equivalently, for all $y > 0$, we have

$$\int_{\mathbb{R}} \frac{1}{u^2 + y^2} p_z(u) du \leq C.$$

By a change of variables, we obtain

$$\int_0^\infty \frac{1}{u + y^2} q_z(u) du \leq C \tag{C.17}$$

for all $y > 0$, where q_z is the probability density given by

$$q_z(u) := \begin{cases} \frac{1}{\sqrt{u}} p_z(\sqrt{u}), & u > 0 \\ 0, & u \leq 0 \end{cases}.$$

Let μ_z be the probability measure with density q_z . Let $s(\eta, z)$ be the Stieltjes transform of μ_z . That is,

$$s(\eta, z) := \int_{\mathbb{R}} \frac{1}{u - \eta} \mu_z(du) = \int_{\mathbb{R}} \frac{1}{u - \eta} q_z(u) du, \quad \eta \in \mathbb{C}^+.$$

By definition of μ_z , it follows that $a(\eta, z) = \eta s(\eta^2, z)$ for all $z \in \mathbb{C}$ and $\eta \in \mathbb{C}^+$ with $\text{Im}(\eta^2) \neq 0$. Thus, $a(\eta, z)$ satisfies

$$\frac{1}{s(\eta, z)(1 + s(\eta, z))} + \eta = \frac{\text{Re}(z)^2}{(1 + (1 + \rho)s(\eta, z))^2} + \frac{\text{Im}(z)^2}{(1 + (1 - \rho)s(\eta, z))^2} \tag{C.18}$$

for $\eta \in \mathbb{C}^+$. By (C.16) and the argument above for $\eta^{-1}a(\eta, z)$, it follows that, for $z \notin \mathcal{E}_\rho$, $s(\eta, z)$ is bounded for all $\eta \in \mathbb{C}^+$. By [5, Theorem B.8], we conclude that the density q_z is bounded. Moreover, from (C.18), it follows that $s(\eta, z)$ can be continuously extended to $\{\eta \in \mathbb{C} : \text{Im}(\eta) \geq 0, \eta \neq 0\}$.

By (C.17) and Lemma C.11 below, we conclude that there exists a sequence of positive real numbers $\{x_k\}_{k \geq 1}$ with $\lim_{k \rightarrow \infty} x_k = 0$ such that $\lim_{k \rightarrow \infty} q_z(x_k) = 0$. By [5, Theorem B.10], we equivalently have

$$\lim_{k \rightarrow \infty} \text{Im}(s(x_k, z)) = 0. \tag{C.19}$$

In order to prove the lemma, it suffices to show that there exists $c > 0$ such that $\mu_z([0, c]) = 0$. In order to reach a contradiction, suppose for all $c > 0$, $\mu_z([0, c]) > 0$. We write $s(\eta, z) = g(\eta) + \sqrt{-1}h(\eta)$, for real-valued functions g, h . Choose $\eta = x \in \mathbb{R}$ with $h(x) > 0$. We now compare the real and imaginary parts for both sides of (C.18) and let x approach the boundary of the support (which we have assumed to be 0). By allowing x to approach 0 along the subsequence in (C.19), we have that $h(x) \rightarrow 0$. Since $s(\eta, z)$ is bounded, we conclude (by possibly taking a further subsequence) that $g(x) \rightarrow g \in \mathbb{R}$ as x approaches the boundary. We obtain

$$\frac{1}{g(1+g)} = \frac{\text{Re}(z)^2}{(1+(1+\rho)g)^2} + \frac{\text{Im}(z)^2}{(1+(1-\rho)g)^2} \tag{C.20}$$

$$\frac{1+2g}{g^2(1+g)^2} = \frac{2\text{Re}(z)^2(1+\rho)}{(1+(1+\rho)g)^3} + \frac{2\text{Im}(z)^2(1-\rho)}{(1+(1-\rho)g)^3}. \tag{C.21}$$

Since μ_z is supported on the non-negative real line, it follows that $g > 0$. Suppose $\rho = 0$. From (C.20) and (C.21), we obtain

$$\frac{1+2g}{g^2(1+g)} = \frac{2}{g(1+g)},$$

a contradiction. For the remainder of the proof, we assume $\rho \neq 0$. Returning to (C.20) and (C.21), we have

$$\frac{2}{g(1+g)} = \frac{2\text{Re}(z)^2}{(1+(1+\rho)g)^3} + \frac{2\text{Im}(z)^2}{(1+(1-\rho)g)^3} + \frac{1+2g}{g(1+g)^2}$$

and hence

$$\frac{1}{g(1+g)^2} = \frac{2\text{Re}(z)^2}{(1+(1+\rho)g)^3} + \frac{2\text{Im}(z)^2}{(1+(1-\rho)g)^3}. \tag{C.22}$$

Combining this equation with (C.21), we obtain

$$\frac{1}{g^2(1+g)} = 2\rho \left[\frac{\text{Re}(z)^2}{(1+(1+\rho)g)^3} - \frac{\text{Im}(z)^2}{(1+(1-\rho)g)^3} \right]. \tag{C.23}$$

From (C.22) and (C.23), we have

$$\begin{aligned} \frac{1}{g^2(1+g)^2} &= 4\rho \frac{\text{Re}(z)^2}{(1+(1+\rho)g)^4} \\ \frac{-1}{g^2(1+g)^2} &= 4\rho \frac{\text{Im}(z)^2}{(1+(1-\rho)g)^4}. \end{aligned}$$

Since $\rho \neq 0$, we conclude that

$$0 = \frac{\text{Re}(z)^2}{(1+(1+\rho)g)^4} + \frac{\text{Im}(z)^2}{(1+(1-\rho)g)^4}.$$

This implies $z = 0 \in \mathcal{E}_\rho$, a contradiction. The proof of the lemma is complete. □

Remark C.9. The measure μ_z , defined in the proof of Lemma C.8 above, is the almost sure limit of the empirical spectral measures built from the eigenvalues of $(\frac{1}{\sqrt{N}}Y_N - zI)^*(\frac{1}{\sqrt{N}}Y_N - zI)$. In fact, equation (C.18) has appeared in the work of Girko (see for instance [30, Lemma 8.1]). Girko discusses the support of μ_z in [30, Section 10].

Remark C.10. In the case $\rho = 0$, the exact interval of support of the measure ν_z can be computed for all $z \in \mathbb{C}$. See [31, Remark 3.1] for further details.

Lemma C.11. Let $C > 0$. Suppose $f : \mathbb{R} \rightarrow [0, \infty)$ is a probability density function that satisfies

$$\int_0^\infty \frac{1}{x + \varepsilon} f(x) dx \leq C$$

for all $\varepsilon > 0$. Then there exists a sequence of positive real numbers $\{x_k\}_{k \geq 1}$ with $\lim_{k \rightarrow \infty} x_k = 0$ such that

$$\lim_{k \rightarrow \infty} f(x_k) = 0.$$

Lemma C.11 follows from a simple indirect proof; we leave the details to the reader. Using Lemma C.8, we now verify Theorem 6.1.

Proof of Theorem 6.1. Since ν_z is the almost sure limit of the measures in (C.10), it suffices to show that there exists $c > 0$ such that

$$\nu_z([0, c]) = 0$$

for all $z \in \mathbb{C}$ with $\text{dist}(z, \mathcal{E}_\rho) \geq \delta$. For each $z \in \mathbb{C}$, we define

$$x_z := \sup\{x \geq 0 : \nu_z([0, x]) = 0\}.$$

By Lemma C.7 the set above is nonempty, and hence $x_z \geq 0$ for all $z \in \mathbb{C}$.

We remind the reader that the least singular value of $\frac{1}{\sqrt{N}}Y_N - zI$ is trivially bounded almost surely by Lemma 3.1 for $|z|$ sufficiently large because we have the almost sure bound $\|\frac{1}{\sqrt{N}}Y_N\| = O(1)$ from Lemma 3.3. Thus, it suffices to prove the theorem for all z in a compact set $D \subset \{z \in \mathbb{C} : \text{dist}(z, \mathcal{E}_\rho) \geq \delta\}$.

We now claim that x_z is continuous in z . Indeed, since ν_z is the almost sure limit of the measures in (C.10), we obtain almost surely

$$\nu_{\frac{1}{\sqrt{N}}Y_N - zI}([0, x_z]) \longrightarrow \nu_z([0, x_z]) = 0$$

as $N \rightarrow \infty$. Thus, by Weyl's perturbation bound (see for instance [12]), for $|z - z'| \leq x_z$, we have almost surely

$$\nu_{\frac{1}{\sqrt{N}}Y_N - z'I}([0, x_z - |z - z'|]) \longrightarrow 0$$

as $N \rightarrow \infty$. We conclude that $\nu_{z'}([0, x_z - |z - z'|]) = 0$ and hence

$$x_{z'} \geq x_z - |z - z'|. \tag{C.24}$$

We note that (C.24) trivially holds when $|z - z'| > x_z$. Repeating the argument with z and z' reversed, we obtain

$$|x_z - x_{z'}| \leq |z - z'|.$$

We conclude that x_z is continuous in z . Since D is compact, it suffices to show that $x_z > 0$ for all $z \in D$. The claim now follows from Lemma C.8. \square

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