

The limiting process of N -particle branching random walk with polynomial tails*

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Abstract

We consider a system of N particles on the real line that evolves through iteration of the following steps: 1) every particle splits into two, 2) each particle jumps according to a prescribed displacement distribution supported on the positive reals and 3) only the N right-most particles are retained, the others being removed from the system. This system has been introduced in the physics literature as an example of a microscopic stochastic model describing the propagation of a front. Its behavior for large N is now well understood – both from a physical and mathematical viewpoint – in the case where the displacement distribution admits exponential moments. Here, we consider the case of displacements with regularly varying tails, where the relevant space and time scales are markedly different. We characterize the behavior of the system for two distinct asymptotic regimes. First, we prove convergence in law of the rescaled positions of the particles on a time scale of order $\log N$ and give a construction of the limit based on the records of a space-time Poisson point process. Second, we determine the appropriate scaling when we let first the time horizon, then N go to infinity.

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1 Introduction

1.1 Definitions

The N -BRW. Let X be a random variable taking values in \mathbb{R}_+ and define $h(x)$ by

$$\mathbb{P}(X > x) = 1/h(x), \quad x \geq 0. \tag{1.1}$$

We assume throughout the paper that the function $h(x)$ is regularly varying at $+\infty$ with index $\alpha > 0$ (see Section 6). For every integer $N \geq 1$, we define the following

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N -particle system. At the beginning, all N particles are located at the origin. At each time step, each of the N particles branches into two, and all $2N$ particles then perform independent jumps according to the law of X . After this, only the N particles at the maximal positions are retained. We call this system the N -branching random walk or N -BRW.

More formally, we define a sequence $(\mathcal{X}(n))_{n \geq 0}$ of N -tuples of real numbers that represent the successive populations of N particles, with

$$\mathcal{X}(n) = \{\mathcal{X}_1(n) \leq \dots \leq \mathcal{X}_N(n)\}.$$

Let $(X_{n,i})_{n \geq 0, i \in [2N]}$ denote i.i.d. random variables distributed as X . Initially, one sets $\mathcal{X}_i(0) = 0$ for all $i \in [N] := \{1, \dots, N\}$. Then, for each integer $n \geq 0$, one inductively defines $\mathcal{X}(n+1) = \{\mathcal{X}_1(n+1) \leq \dots \leq \mathcal{X}_N(n+1)\}$ to be the N largest numbers from the collection $(\mathcal{X}_i(n) + X_{n,2i+j})_{i \in [N], j \in \{0,1\}}$, sorted in ascending order.

The stairs process. We now define the *stairs process*, a real-valued stochastic process which will be shown to approximate the N -BRW when N is large. We first define a *stairs measure* to be a non-zero measure μ on $(0, \infty)$ such that $\mu([a, +\infty)) < +\infty$ for every $a > 0$ (in particular, μ is σ -finite). The μ -stairs process then is the real-valued stochastic process $(\mathcal{R}(t))_{t \geq 0}$ defined as follows: Given a Poisson point process on $\{(t, x) : t, x > 0\}$ with intensity $dt \otimes \mu$, define a process $(\xi_t)_{t \geq 0}$ by $\xi_t = x$ if (t, x) is an atom of the process and $\xi_t = 0$ otherwise. Now define $\mathcal{R}(t)$ inductively as follows: For $t \leq 0$, $\mathcal{R}(t) = 0$. For integer $n = 0, 1, 2, \dots$, knowing the value of $\mathcal{R}(t)$ for $t \leq n$, define $\mathcal{R}(t)$ for $t \in (n, n+1]$ by

$$\mathcal{R}(t) = \max_{s \in [0,1]} (\mathcal{R}(t-s-1) + \xi_{t-s}). \tag{1.2}$$

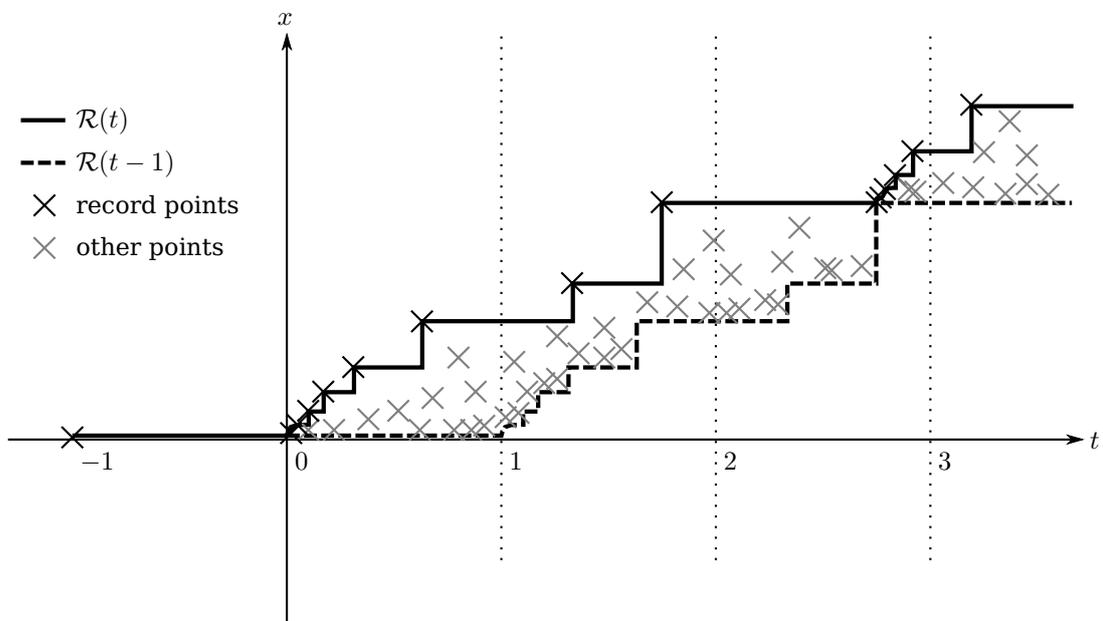


Figure 1: Graphical representation of the stairs process.

This definition is equivalent to the following construction: Suppose $\mathcal{R}(t)$ is defined for $t \leq n \in \mathbb{N}$. Now generate points in the interval $(n, n+1]$ according to the above

Poisson process and translate every atom (t, x) by $\mathcal{R}(t - 1)$ in the x -direction (note that the graph of $\mathcal{R}(t - 1)$ is the graph of $\mathcal{R}(t)$ shifted by 1 to the right). Then define $(\mathcal{R}(t))_{t \in (n, n+1]}$ to be the record process of these points. See Figure 1 for a graphical representation.

One easily verifies that \mathcal{R} is a non-decreasing càdlàg process and that the following representation holds for $t \geq 0$:

$$\mathcal{R}(t) = \max \left\{ \sum_{i=1}^k \xi_{t_i} : k \in \mathbb{N}, 0 \leq t_1 < \dots < t_k \leq t, |t_i - t_{i-1}| \geq 1 \forall i \right\} \quad (1.3)$$

Remark. The process $(\max_{s \in [0, t]} \xi_s)_{t \geq 0}$ is known in the literature as a *Poisson paced record process* [11]. The stairs process can be interpreted as a self-interacting version of it. Its long-term behaviour is quite different: While a Poisson paced record process usually grows logarithmically in t , the stairs process grows like a random walk, due to the existence of regeneration times (see Section 2).

1.2 Statements of the results

Define μ_α to be the measure on $(0, \infty)$ defined by $\mu_\alpha([x, \infty)) = x^{-\alpha}$ and let \mathcal{R}^α denote a realisation of the μ_α -stairs process. Define a sequence $(c_N)_{N \geq 1}$ by $c_N = h^{-1}(2N \log_2 N)$, where h^{-1} is the generalized inverse of h . Note that by the regular variation of h , we have $h(c_N) \sim 2N \log_2 N$ as $N \rightarrow \infty$.

Our first theorem gives convergence in law of the maximum and the minimum of the N -BRW to a certain stairs process, after rescaling of space and time. For a definition of Skorokhod's J_1 and SM_1 topologies appearing in the statement of the theorem, see [29, Chapter 12]. Note that the former topology is also commonly called *the* Skorokhod topology.

Theorem 1.1. *We have the following convergences in law, as $N \rightarrow \infty$:*

$$\begin{aligned} (c_N^{-1} \mathcal{X}_N(\lfloor t \log_2 N \rfloor))_{t \geq 0} &\Longrightarrow (\mathcal{R}^\alpha(t))_{t \geq 0}, && \text{in the } J_1\text{-topology} \\ (c_N^{-1} \mathcal{X}_N(\lfloor t \log_2 N \rfloor), c_N^{-1} \mathcal{X}_1(\lfloor t \log_2 N \rfloor))_{t \geq 0} &\Longrightarrow (\mathcal{R}^\alpha(t), \mathcal{R}^\alpha(t - 1))_{t \geq 0}, && \text{in the } SM_1\text{-topology.} \end{aligned}$$

Remark. One cannot expect convergence of $c_N^{-1} \mathcal{X}_1(\lfloor t \log_2 N \rfloor)_{t \geq 0}$ in the J_1 -topology, since for some values of N , it may have two consecutive macroscopic jumps.

For a random variable Y , denote by $\mathcal{L}(Y)$ the law of Y . Denote by $d(\cdot, \cdot)$ the Prokhorov metric on the space of probability measures on \mathbb{R} . For two positive sequences a_N and b_N , write $a_N \sim b_N$ if $a_N/b_N \rightarrow 1$ as $N \rightarrow \infty$.

The next theorem is our main theorem. It studies the limiting behaviour of the N -BRW, when we let first the time horizon, then N go to infinity.

Theorem 1.2. *We distinguish between the following cases:*

- $\alpha > 1$: *The limit $v_N = \lim_{n \rightarrow \infty} \mathcal{X}_N(n)/n = \lim_{n \rightarrow \infty} \mathcal{X}_1(n)/n$ exists almost surely and in L^1 and satisfies $v_N \sim \rho_\alpha c_N / \log_2 N$. Here, the limit $\rho_\alpha = \lim_{t \rightarrow \infty} \mathcal{R}^\alpha(t)/t$ exists almost surely and in L^1 and is a positive and finite constant.*
- $\alpha = 1, \mathbb{E}[X] < \infty$: *The limit $v_N = \lim_{n \rightarrow \infty} \mathcal{X}_N(n)/n = \lim_{n \rightarrow \infty} \mathcal{X}_1(n)/n$ exists almost surely and in L^1 and satisfies $v_N \sim (c_N / \log_2 N) \int_1^\infty h(c_N)/h(c_N x) dx$.*
- $\alpha = 1, \mathbb{E}[X] = \infty$: *Set $b_n^N = \int_1^{h^{-1}(n)} h(c_N)/h(c_N x) dx$. Then, for $i \in \{1, N\}$,*

$$\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} d \left(\mathcal{L} \left(\frac{\log_2 N}{c_N} \frac{\mathcal{X}_i(n)}{nb_n^N} \right), \delta_1 \right) = 0.$$

- $0 < \alpha < 1$: Let W_α be a random variable with Laplace transform

$$\mathbb{E}[e^{-\lambda W_\alpha}] = \exp(-\alpha \int_0^\infty (1 - e^{-\lambda x})x^{-\alpha-1} dx). \quad (1.4)$$

Then, for $i \in \{1, N\}$,

$$\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} d \left(\mathcal{L} \left((2N)^{-1/\alpha} \frac{\mathcal{X}_i(n)}{h^{-1}(n)} \right), \mathcal{L}(W_\alpha) \right) = 0.$$

1.3 Heuristics and proof strategy

The global heuristic picture of the N -BRW is the following: At a typical time and viewed on the space scale c_N , the particles are divided into one big “tribe” located near the left-most particle and containing all but $o(N)$ particles, the remaining particles to the right being split into a $O(1)$ number of smaller tribes. At each time step, the number of particles in every small tribe is multiplied by two, which eventually leads to extinction of the big tribe and another tribe taking over. Furthermore, new tribes are formed by particles performing (rightward) jumps of magnitude c_N out of the big tribe. The value of c_N has been chosen so that these events occur on the time scale $\log_2 N$, which is precisely the time it takes for a new tribe to grow to size N .

As $N \rightarrow \infty$, the jumps leading to new tribes are described by a space-time Poisson point process shifted in space by the position of the big tribe. This leads to the definition of the stairs process from Section 1.1 and illustrated in Figure 1. Note that the record points of that process exactly correspond to the creation of tribes which will eventually take over the population, the other points representing tribes which get extinct before reaching a size of order N .

In order to render this picture rigorous, we will couple the N -BRW with a discretized version of the stairs process, see (3.1), in such a way that the error is bounded in L^p for $p < 2\alpha$. Ideally, we would have liked to use a single coupling between both processes to derive Theorems 1.1 and 1.2 and say something about the empirical measure of the N -BRW. However, it turned out to be more convenient to use separate couplings for upper and lower bounds for the positions of the right- and left-most particles. The trickier part here consisted in the upper bound, the key to which is Proposition 3.2. An important ingredient consists of large deviation estimates for sums of iid variables with regularly varying tails.

Theorem 1.2 is then derived from the above coupling and Theorem 2.5 below. In order to prove the latter, we define a regeneration structure for the stairs process, which permits to use classical results on random walks with regularly varying tails.

Note that analogues of the above theorems should remain valid if one allows X to take on negative values or if one considers more general reproduction laws. Since this increases the technical difficulties without leading to new phenomena, we chose to keep to our setting for simplicity.

1.4 Discussion

For light-tailed displacement distributions, i.e. satisfying an exponential moment assumption, the N -BRW model has been studied in the physics literature as a microscopic stochastic model describing the propagation of a front, along with several variants [7, 8, 9, 10]. In the limit as $N \rightarrow \infty$, the dynamics of the model is described by an analog of the Fisher–Kolmogorov–Petrovskii–Piskounov (FKPP) equation, which is a prototypical example of a reaction-diffusion equation admitting travelling wave solutions [21, 23] (see [18] for a rigorous result of this type). One is then interested in understanding how the behaviour of the model for large but finite N reflects that of the limiting equation. The main results are the following:

- The cloud of particles propagates with a finite asymptotic velocity v_N . As $N \rightarrow \infty$, v_N converges to the velocity v of a travelling wave solution to the corresponding FKPP-type equation, which is also the speed of the right-most particle in the BRW without selection. Moreover, this convergence takes place at the unusually slow rate of $(\log N)^{-2}$ [6, 2].
- The relevant time scale both for macroscopic fluctuations of the cloud and for coalescence of ancestral lineages is $(\log N)^3$ [9, 10, 3, 24]. The genealogy is asymptotically described by the Bolthausen–Sznitman coalescent [3].

By contrast, in our heavy-tailed setting, the following happens:

- The particles propagate either linearly or superlinearly (but still at most polynomially) in time. In both cases, the scaling factor due to the number of particles in the system grows roughly polynomially in N . Without selection (i.e. for the classical BRW), the right-most particle would propagate exponentially fast in time [16] due to the exponential growth of the number of particles¹. Note that a similar behavior is observed in reaction-diffusion equations with fractional Laplacian or other non-local operators [12], which play the role of the FKPP equation in this context.
- The relevant time scale both for macroscopic fluctuations of the cloud and for coalescence of ancestral lineages is $\log N$. The genealogy is trivial². Moreover, the fluctuations come from large jumps of single particles, in contrast to the more complex mechanism leading to the fluctuations in the light-tailed setting [24].

A natural question and possibility for future research is now to investigate what happens in between the two scenarios of light-tailed displacement distributions (satisfying an exponential moment assumption) and the polynomial tails considered in the present article.

1.5 Organisation of the paper

In Section 2, we first derive some basic properties of the stairs process and prove Theorem 2.5, which gives its long-time behaviour. In Section 3, we bound the N -BRW from below and from above by a discretised version of the stairs process. The main work here lies in the upper bound, which is contained in Proposition 3.2. In the short Section 4, we couple the stairs process with its discretised version. Sections 2, 3 and 4 can be read independently of one another. Section 5 contains the proofs of Theorems 1.1 and 1.2, relying on the results obtained in the previous sections. In the appendix (Sections 6 and 7), we recall some known results about regularly varying functions (in particular, Potter’s bounds) and large deviations of random walks with regularly varying tails.

2 Properties of the stairs process

Throughout this section, we assume that μ is a stairs measure as defined in the last section (i.e., μ is a non-zero measure on $(0, \infty)$ such that $\mu([a, +\infty]) < +\infty$ for every $a > 0$). We regard μ as an element in the space of σ -finite measures on $(0, \infty)$ endowed with the vague topology, i.e. the weak topology with respect to the space of continuous functions supported on a compact subset of $(0, \infty)$. Statements such as “as μ varies” always refer to this topology. We further denote by $(\mathcal{R}(t))_{t \geq 0}$ a realization of the μ -stairs process.

¹For the stretched exponential case, see [22].

²Here, *trivial* means that the genealogy is given by the “star-shaped coalescent”. This follows from the heuristic picture described above, although we haven’t worked out a full proof of this fact.

We define a regeneration structure for $(\mathcal{R}(t))_{t \geq 0}$ as follows: Let $\tau_0 = 0$ and for $n \geq 1$, let $\tau_n = \inf\{t > \tau_{n-1} + 1 : \mathcal{R}(t) = \mathcal{R}(t - 1)\}$. By the definition of the stairs process, the random times $(\tau_n)_{n \in \mathbb{N}}$ are regeneration times, i.e. the collection of pairs $(\mathcal{R}(\tau_n) - \mathcal{R}(\tau_{n-1}), \tau_n - \tau_{n-1})_{n \geq 1}$ are i.i.d. Furthermore, we have the following lemma:

Proposition 2.1. *There exists a constant $C > 0$ (depending on μ), such that*

$$\mathbb{P}(\tau_1 > t) \leq C^{-1}e^{-Ct}, \text{ for all } t > 0.$$

This constant can be chosen to vary continuously with μ .

Proof. By definition of μ , there exists $m \in (0, \infty)$, such that $\mu([m, \infty)) < \infty$ and such that $\mu((2m, \infty)) > 0$. Let $M_n^1 \geq M_n^2 \geq \dots$ be the x -coordinates of the atoms of the Poisson process from the definition of the μ -stairs process in the time-interval $[n, n + 1)$, arranged in decreasing order. Furthermore, define for every $t \geq 0$, $\mathcal{F}_t = \sigma(\xi_s; s \in [0, t])$, where $(\xi_t)_{t \geq 0}$ is the process used in the definition of \mathcal{R} . We then have for each $n \geq 0$,

$$\mathbb{P}(\tau_1 \leq n + 3 | \mathcal{F}_n) \geq \mathbb{P}(M_n^1 < m, M_{n+1}^1 > 2m, M_{n+1}^2 < m, M_{n+2}^1 < m), \quad (2.1)$$

because the event on the right-hand side ensures that there is a time $T \in [n + 1, n + 2)$ with $\mathcal{R}(T) - \mathcal{R}(T-) > m$ and with $\mathcal{R}(T + s) = \mathcal{R}(T)$ for all $s \in [0, 1]$. This implies $\tau_1 \leq T + 1 \leq n + 3$. Now, the right-hand side of (2.1) is positive, independent of n and continuous in μ . The proposition is immediate. \square

Proposition 2.2. *Suppose $\int_1^\infty x\mu(dx) < \infty$. Then the limit $\rho = \lim_{t \rightarrow \infty} \mathcal{R}(t)/t$ exists almost surely and in L^1 and satisfies $\rho = \mathbb{E}[\mathcal{R}(\tau_1)]/\mathbb{E}[\tau_1] > 0$.*

Remark 2.3. *Proposition 2.2 implies in particular that for every $\alpha > 1$, the limit $\rho_\alpha = \lim_{t \rightarrow \infty} \mathcal{R}^\alpha(t)/t$ exists almost surely and in L^1 and is a positive and finite constant.*

Proof of Proposition 2.2. Let ξ_t be as in the definition of the μ -stairs process. For $n \geq 0$, define the process $\xi_t^n = \xi_t \mathbb{1}_{t > n}$ and let $(\mathcal{R}(n, t))_{t \in \mathbb{R}}$ be the μ -stairs process defined from ξ^n as in (1.2). One easily checks from the definition or by (1.3) that $\mathcal{R}(0, m) \leq \mathcal{R}(0, n) + \mathcal{R}(n, m)$ for every $n \leq m$. Moreover, by the hypothesis on μ , $\mathcal{R}(1)$ is positive and integrable, since $\mathbb{P}(\mathcal{R}(1) > x) = 1 - \exp(-\mu((x, \infty)))$. Kingman’s subadditive ergodic theorem [17, Theorem 6.6.1], whose remaining conditions are readily verified, now yields almost sure and L^1 -convergence of $\mathcal{R}(n)/n$ to a non-negative, finite constant ρ , and by the monotonicity of the process \mathcal{R} , this convergence also holds for $\mathcal{R}(t)/t$.

Now note that $\mathbb{E}[\tau_1] < \infty$ by Proposition 2.1, so that τ_n/n converges almost surely to $\mathbb{E}[\tau_1]$ by the law of large numbers. The almost sure convergence of $\mathcal{R}(t)/t$ established above then yields almost sure convergence of $\mathcal{R}(\tau_n)/n$ to $\mathbb{E}[\tau_1]\rho$. By the converse to the law of large numbers, this implies that $\mathbb{E}[\mathcal{R}(\tau_1)]$ is finite and that $\rho = \mathbb{E}[\mathcal{R}(\tau_1)]/\mathbb{E}[\tau_1]$. Moreover, $\mathbb{E}[\mathcal{R}(\tau_1)]$ is positive, because the measure μ is non-zero by definition and $\mathcal{R}(\tau_1) \geq \mathcal{R}(1)$. This shows $\rho > 0$.

We remark that we could have proven Proposition 2.2 without applying Kingman’s subadditive ergodic theorem; using only the regeneration structure and an argument involving Fatou’s lemma and the converse to the law of large numbers to get finiteness of $\mathbb{E}[\mathcal{R}(\tau_1)]$. \square

Proposition 2.4. *Let μ_N be a sequence of stairs measures converging to μ and denote by \mathcal{R}^N and $(\tau_n^N)_{n \geq 0}$ the corresponding stairs process and regeneration times. Then the following hold:*

1. *The sequence of processes $(\mathcal{R}^N(t))_{t \geq 0}$ converges in law to $(\mathcal{R}(t))_{t \geq 0}$ w.r.t. the J_1 -topology (“Skorokhod’s topology”), as $N \rightarrow \infty$.*

2. The sequences of random variables τ_1^N and $\mathcal{R}^N(\tau_1^N)$ converge in law to τ_1 and $\mathcal{R}(\tau_1)$, respectively, as $N \rightarrow \infty$.
3. The sequence τ_1^N is uniformly integrable in N .

Proof. We can assume w.l.o.g. that μ has no atoms and has full support in $(0, \infty)$. Moreover, we can assume that $\mu_N((0, \infty)) \leq \mu((0, \infty))$ for every N , otherwise we truncate μ_N near the origin. Let ξ_t^N and ξ_t be the processes used to construct \mathcal{R}^N and \mathcal{R} . Define the functions $F, F_N : \mathbb{R}_+^* \rightarrow \mathbb{R}_+$ by $F(x) = \mu([x, \infty))$ and $F_N(x) = \mu_N([x, \infty))$ and let $F_N^{-1}(x) = \inf\{y \geq 0 : F_N(y) \leq x\}$ be the generalised inverse of F_N . Defining the function $f_N := F_N^{-1} \circ F$, we can then couple the processes ξ_t^N and ξ_t by setting $\xi_t^N = f_N(\xi_t)$ (here we implicitly used the above-mentioned assumptions on μ). Note that for every $N \geq 1$, the function f_N is non-decreasing and the sequence $(f_N)_{N \geq 1}$ converges pointwise to the identity function on $[0, \infty)$. Using (1.3), we can now show that almost surely,

$$\forall T \geq 0 : \lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} |\mathcal{R}^N(t) - \mathcal{R}(t)| = 0, \quad \lim_{N \rightarrow +\infty} \tau_1^N = \tau_1, \quad \lim_{N \rightarrow +\infty} \mathcal{R}^N(\tau_1^N) = \mathcal{R}(\tau_1).$$

This proves the first two claims. The third follows from Proposition 2.1. □

For the next results, we suppose that μ_N is a sequence of stairs measures given by

$$\mu_N([x, \infty)) = -\gamma_N \log(1 - h(c_N x)^{-1}),$$

where $(\gamma_N)_{N \geq 1}$ is a sequence such that $\gamma_N / (2N \log_2 N) \rightarrow 1$ as $N \rightarrow \infty$. Here, h is the function given in the introduction, in particular, $h(x)$ varies regularly at infinity with index $\alpha > 0$. The sequence μ_N therefore converges to μ_α .

The following theorem gives the long-time behaviour of the μ_N -stairs process. It will be important for the proof of Theorem 1.2.

Theorem 2.5. *We distinguish among the following cases:*

- $\alpha > 1$: The limits $\rho_N = \lim_{t \rightarrow \infty} \mathcal{R}^N(t)/t$ and $\rho_\alpha = \lim_{t \rightarrow \infty} \mathcal{R}^\alpha(t)/t$ exist almost surely and in L^1 . Moreover, $\rho_N \rightarrow \rho_\alpha$ as $N \rightarrow \infty$.
- $\alpha = 1, \mathbb{E}[X] < \infty$: The limit $\rho_N = \lim_{t \rightarrow \infty} \mathcal{R}^N(t)/t$ exists almost surely and in L^1 and satisfies $\rho_N \sim \int_1^\infty h(c_N)/h(c_N x) dx$, as $N \rightarrow \infty$.
- $\alpha = 1, \mathbb{E}[X] = \infty$: Set $b_t^N = \int_1^{h^{-1}(t)} h(c_N)/h(c_N x) dx$. Then, for every N ,

$$\mathcal{R}^N(t)/tb_t^N \rightarrow 1, \quad \text{in probability as } t \rightarrow \infty.$$

- $0 < \alpha < 1$: Let W_α be a random variable with Laplace transform given by (1.4). Then,

$$\lim_{N \rightarrow \infty} \limsup_{t \rightarrow \infty} d \left(\mathcal{L} \left(\frac{c_N}{h(c_N)^{1/\alpha}} \frac{\mathcal{R}^N(t)}{h^{-1}(t)} \right), \mathcal{L}(W_\alpha) \right) = 0$$

A few remarks on Theorem 2.5: In the case $\alpha > 1$, its proof is almost immediate from Propositions 2.2 and 2.4 and Remark 2.3. Indeed, since $\rho_N = \mathbb{E}[\mathcal{R}^N(\tau_1^N)]/\mathbb{E}[\tau_1^N]$ and $\rho_\alpha = \mathbb{E}[\mathcal{R}^\alpha(\tau_1^\alpha)]/\mathbb{E}[\tau_1^\alpha]$, it remains to show that the sequence of random variables $\mathcal{R}^N(\tau_1^N)$ is uniformly integrable in N , which can easily be done through fractional moment estimates using Hölder's inequality and Proposition 2.1. It will also follow directly from Proposition 2.6 below. This proposition yields a precise estimate of the tail of $\mathcal{R}^N(\tau_1^N)$, which is the key to proving Theorem 2.5 in the more delicate case $\alpha \leq 1$. Indeed, armed with Proposition 2.6, the theorem directly follows from classic results on random walks applied to $(\mathcal{R}^N(\tau_n^N))_{n \geq 0}$.

We furthermore remark that the uniformity in N in the statement of Proposition 2.6 is only needed in the case $\mathbb{E}[X] < \infty$. In the case $\mathbb{E}[X] = \infty$, estimates not uniform in N would suffice.

Proposition 2.6. *For every $\varepsilon > 0$, there exists $x_0 = x_0(\varepsilon)$, such that for all N ,*

$$\sup_{x \geq x_0} \left| \frac{\mathbb{P}(\mathcal{R}^N(\tau_1^N) > x)}{\mathbb{E}[\tau_1^N] \mu_N([x, \infty))} - 1 \right| < \varepsilon.$$

Write $\Delta \mathcal{R}^N(t) = \mathcal{R}^N(t) - \mathcal{R}^N(t-)$ for the jump of \mathcal{R}^N at time t . As part of the proof of Proposition 2.6, we will show that if $\mathcal{R}^N(\tau_1^N)$ is large, then it is approximately equal to $\Delta \mathcal{R}^N(\tau_1^N - 1)$. We therefore first prove the following lemma:

Lemma 2.7. *For every $\varepsilon > 0$, there exists $x_0 = x_0(\varepsilon)$, such that for all N ,*

$$\sup_{x \geq x_0} \left| \frac{\mathbb{P}(\Delta \mathcal{R}^N(\tau_1^N - 1) > x)}{\mathbb{E}[\tau_1^N] \mu_N([x, \infty))} - 1 \right| < \varepsilon.$$

Proof. For better readability, write $\mu = \mu_N$, $\mathcal{R} = \mathcal{R}^N$, $\tau_1 = \tau_1^N$, $\Delta \mathcal{R}(t) = \mathcal{R}(t) - \mathcal{R}(t-)$, $\xi_t = \xi_t^N$. We further set $M_t = \max_{s \in (t-1, t]} \xi_s$ and $M_{t-} = \max_{s \in (t-1, t)} \xi_s$.

Bounding $\mathbb{P}(\Delta \mathcal{R}(\tau_1 - 1) > x)$ from above is easy: Let \mathcal{P} be the Poisson point process used to construct \mathcal{R} . We have

$$\mathbb{P}(\Delta \mathcal{R}(\tau_1 - 1) > x) = \mathbb{E} \left[\sum_{(t, \xi_t) \in \mathcal{P}} \mathbb{1}_{\tau_1 = t+1, \Delta \mathcal{R}(t) > x} \right] \leq \mathbb{E} \left[\sum_{(t, \xi_t) \in \mathcal{P}} \mathbb{1}_{\tau_1 \geq t, \xi_t > x} \right].$$

Now, since τ_1 is a stopping time for \mathcal{R} , the event $\{\tau_1 \geq t\} = \{\tau_1 < t\}^c$ is measurable w.r.t. the σ -field generated by $\mathcal{P} \cap \{[0, t) \times \mathbb{R}_+\}$ for every $t \geq 0$. By the projection theorem for Poisson processes [5, Theorem VIII.T3], the previous equation now yields

$$\mathbb{P}(\Delta \mathcal{R}(\tau_1 - 1) > x) \leq (1 - e^{-\mu([x, \infty))}) \int_0^\infty \mathbb{P}(\tau_1 \geq t) dt = (1 - e^{-\mu([x, \infty))}) \mathbb{E}[\tau_1],$$

which proves the upper bound of $\mathbb{P}(\Delta \mathcal{R}(\tau_1 - 1) > x)$.

As for the lower bound, we note that for every $t \geq 0$, on the event $\{\Delta \mathcal{R}(t) > 0\}$, we have $\tau_1 \geq t + 1$ iff $\tau_1 \geq t$. This gives for every $\varepsilon > 0$, $x > 0$ and $t \geq 0$,

$$\begin{aligned} \{\tau_1 = t + 1, \Delta \mathcal{R}(t) > x\} &\supset \{\tau_1 \geq t, M_{t+1} \leq x, \Delta \mathcal{R}(t) > x\} \\ &\supset \{\tau_1 \geq t, M_{t+1} \leq x, \xi_t > (1 + \varepsilon)x, M_{t-} \leq \varepsilon x\}. \end{aligned}$$

By the independence properties of the Poisson process and the previously mentioned projection theorem [5, Theorem VIII.T3], this gives,

$$\begin{aligned} \mathbb{P}(\Delta \mathcal{R}(\tau_1 - 1) > x) &\geq \mathbb{E} \left[\sum_{(t, \xi_t) \in \mathcal{P}} \mathbb{1}_{\tau_1 \geq t, M_{t+1} \leq x, \xi_t > (1 + \varepsilon)x, M_{t-} \leq \varepsilon x} \right] \\ &\geq \mathbb{P}(M_1 \leq x) \mathbb{E} \left[\sum_{(t, \xi_t) \in \mathcal{P}} (\mathbb{1}_{\tau_1 \geq t, \xi_t > (1 + \varepsilon)x} - \mathbb{1}_{\tau_1 \geq t-1, \xi_t > (1 + \varepsilon)x, M_{t-} > \varepsilon x}) \right] \\ &= \mathbb{P}(M_1 \leq x) (1 - e^{-\mu([(1 + \varepsilon)x, \infty))}) (\mathbb{E}[\tau_1] - (1 + \mathbb{E}[\tau_1]) \mathbb{P}(M_1 > \varepsilon x)). \end{aligned}$$

(in the second line, we used the fact that $\tau_1 \geq t$ trivially implies $\tau_1 \geq t - 1$). Now, since μ_N converges to μ_α , the variables $M_1 = M_1^N$ converge in law as well. Together with the fact that $\mathbb{E}[\tau_1] \geq 1$ by definition, this gives for $x \geq x_0(\varepsilon)$, for every N ,

$$\mathbb{P}(\Delta \mathcal{R}^N(\tau_1^N - 1) > x) \geq (1 - \varepsilon) (1 - e^{-\mu_N([(1 + \varepsilon)x, \infty))}).$$

Now, by the regular variation of $h(x)$, we have for every N , for large x , $\mu_N([(1 + \varepsilon)x, \infty)) \geq (1 + 2\varepsilon)^{-\alpha} \mu_N([x, \infty))$. Since ε was arbitrary, this proves the lemma. \square

Proof of Proposition 2.6. For better readability, we use the same notation as in the proof of Lemma 2.7. Note that we trivially have $\mathbb{P}(\Delta\mathcal{R}(\tau_1 - 1) > x) \leq \mathbb{P}(\mathcal{R}(\tau_1) > x)$, such that the lower bound on $\mathbb{P}(\mathcal{R}(\tau_1) > x)$ directly follows from Lemma 2.7.

For the upper bound, fix $\varepsilon > 0$. By the definition of the process \mathcal{R} , for every $x > 0$, the event $\{\Delta\mathcal{R}(\tau_1 - 1) \leq (1 - \varepsilon)x, \mathcal{R}(\tau_1) > x\}$ implies the event $\{\mathcal{R}^-(\tau_1 - 1) > \varepsilon x\}$, such that

$$\mathbb{P}(\mathcal{R}(\tau_1) > x) \leq \mathbb{P}(\Delta\mathcal{R}(\tau_1 - 1) > (1 - \varepsilon)x) + \mathbb{P}(\mathcal{R}^-(\tau_1 - 1) > \varepsilon x). \tag{2.2}$$

In order to bound the second term on the right-hand side, let C be large enough, such that with $L_x = \lceil C \log x \rceil$, we have $\mathbb{P}(\tau_1 > L_x) \leq \mu([x, \infty))^2$ for large x and every N (this is possible by Proposition 2.1 and Potter's bounds for regularly varying functions, see Section 6). We then have for large x , with $\mathcal{R}^-(t) = \mathcal{R}(t-)$,

$$\mathbb{P}(\mathcal{R}^-(\tau_1 - 1) > x) \leq \mu([x, \infty))^2 + \mathbb{P}(\mathcal{R}^-(\tau_1 - 1) > x, \tau_1 \leq L_x) \tag{2.3}$$

We furthermore split the second term on the right-hand side of (2.3) into two, according to whether $\max_{t < \tau_1 - 1} \xi_t > x/2$ or not. Now first note that if $\xi_t > x/2$ for some $t < \tau_1 - 1$, then necessarily $\xi_{t'} > x/4$ for some $t' < t + 1$, $t' \neq t$, otherwise $\tau_1 \leq t + 1$, which is a contradiction (this is the same reasoning as that used in the proof of Proposition 2.1). As a consequence, for some numerical constant c we have for large x ,

$$\mathbb{P}\left(\max_{t < \tau_1 - 1} \xi_t > x/2, \tau_1 \leq L_x\right) \leq c(L_x \mu([x, \infty)))^2. \tag{2.4}$$

On the other hand, by (1.3),

$$\mathbb{P}(\mathcal{R}^-(\tau_1 - 1) > x, \max_{t < \tau_1 - 1} \xi_t \leq x/2, \tau_1 < L_x) \leq \mathbb{P}(M_1 + \dots + M_{L_x} > x, \forall i : M_i \leq x/2),$$

and by Corollary 7.2 in Section 7, the last quantity is less than $K_\alpha x^{-3\alpha/2}$ for a constant K_α and for large x . Together with (2.3) and (2.4) and Potter's bounds, this gives for large x ,

$$\mathbb{P}(\mathcal{R}^-(\tau_1 - 1) > x) \leq \mu([x, \infty))x^{-\alpha/3},$$

and the regular variation of $h(x)$ then yields existence of x_0 (depending on ε), such that $\mathbb{P}(\mathcal{R}^-(\tau_1 - 1) > \varepsilon x) \leq \varepsilon \mu([x, \infty))$ for $x \geq x_0$. Together with (2.2) and Lemma 2.7, as well as the regular variation of $h(x)$ and the fact that $\mathbb{E}[\tau_1] \geq 1$ by definition, this finishes the proof. \square

Proof of Theorem 2.5. Case $\alpha \geq 1$: Here, $\int_1^\infty x \mu_\alpha(dx) < \infty$ and for every N , by Potter's bounds, $\int_1^\infty x \mu_N(dx) < \infty$. By Proposition 2.2, the limits $\rho_N = \lim_{t \rightarrow \infty} \mathcal{R}^N(t)/t$ and $\rho_\alpha = \lim_{t \rightarrow \infty} \mathcal{R}^\alpha(t)/t$ exist almost surely and in L^1 and satisfy $\rho_N = \mathbb{E}[\mathcal{R}^N(\tau_1^N)]/\mathbb{E}[\tau_1^N]$ and $\rho_\alpha = \mathbb{E}[\mathcal{R}^\alpha(\tau_1^\alpha)]/\mathbb{E}[\tau_1^\alpha]$. Furthermore, by Propositions 2.1 and 2.4, $\mathbb{E}[\tau_1^N]$ converges to $\mathbb{E}[\tau_1^\alpha]$ and $\mathcal{R}^N(\tau_1^N)$ converges in law to $\mathcal{R}^\alpha(\tau_1^\alpha)$, as $N \rightarrow \infty$. In order to show that $\rho_N \rightarrow \rho_\alpha$ as $N \rightarrow \infty$, it therefore remains to show that the sequence of random variables $\mathcal{R}^N(\tau_1^N)$ is uniformly integrable in N . But this follows from Proposition 2.6 and the fact that the restrictions of the measures μ_N to $[1, \infty)$ are uniformly integrable in N , again by Potter's bounds. This finishes the proof of the case $\alpha > 1$.

Case $\alpha = 1, \mathbb{E}[X] < \infty$: As in the previous case, the limit $\rho_N = \lim_{t \rightarrow \infty} \mathcal{R}^N(t)/t$ exists almost surely and in L^1 by Proposition 2.2 and satisfies $\rho_N = \mathbb{E}[\mathcal{R}^N(\tau_1^N)]/\mathbb{E}[\tau_1^N]$. Furthermore, since the measures μ_N converge as $N \rightarrow \infty$ to $\mu_1(dx) = x^{-2} dx$ which satisfies $\int_1^\infty x \mu_1(dx) = \infty$, we have $\int_1^\infty x \mu_N(dx) \rightarrow \infty$, as $N \rightarrow \infty$, by Fatou's lemma. Proposition 2.6 now yields as $N \rightarrow \infty$ (note that here we need the fact that x_0 in the statement of Proposition 2.6 is independent of N),

$$\rho_N \sim \int_1^\infty \mu_N([x, \infty)) dx \sim \int_1^\infty \frac{h(c_N)}{h(c_N x)} dx,$$

which yields the theorem in the case $\alpha = 1$, $\mathbb{E}[X] < \infty$.

Case $\alpha = 1$, $\mathbb{E}[X] = \infty$: For each $N \in \mathbb{N}$, define the random variable

$$S^N = \mathcal{R}^N(\tau_1^N)/\mathbb{E}[\tau_1^N],$$

and set $a_n^N = \inf\{x : \mathbb{P}(S^N > x) < n^{-1}\}$. We introduce the relation $\alpha_n^N \asymp \bar{\alpha}_n^N$ between two positive double sequences meaning that $\lim_{N \rightarrow \infty} \limsup_{n \rightarrow \infty} |\alpha_n^N/\bar{\alpha}_n^N - 1| = 0$. By Proposition 2.6,

$$a_n^N \asymp (c_N \mathbb{E}[\tau_1^N])^{-1} h^{-1}(\mathbb{E}[\tau_1^N] h(c_N) n) \asymp c_N^{-1} h(c_N) h^{-1}(n),$$

where the last relation follows from the fact that h^{-1} is regularly varying with index 1 [4, Theorem 1.5.12]. Now define $\beta_n^N = \mathbb{E}[S^N \mathbb{1}_{S^N \leq a_n^N}]$. By Proposition 2.6, and the ‘‘boundary case’’ of Karamata’s theorem [4, Proposition 1.5.9a], we have

$$\begin{aligned} \beta_n^N &= \int_0^{a_n^N} \mathbb{P}(S^N > x) dx - a_n^N \mathbb{P}(S^N > a_n^N) \\ &\sim \int_1^{a_n^N} \frac{h(c_N)}{h(c_N x)} dx \sim \int_1^{h^{-1}(n)} \frac{h(c_N)}{h(c_N x)} dx, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Furthermore, again by [4, Proposition 1.5.9a], we have $n\beta_n^N/a_n^N \rightarrow \infty$ and $\beta_n^N \sim b_{c_n}^N$ for every constant $c > 0$, as $n \rightarrow \infty$. Now, if $(S_n^N)_{n \geq 0}$ is a random walk with increments distributed according to S^N , then standard results on random walks (see e.g. [17, Theorem 2.7.7]), note that $\mathbb{P}(S^N > x)$ is regularly varying for every N , by Proposition 2.6) imply that $(S_n^N - n\beta_n^N)/a_n^N$ converges in law to a non-degenerate random variable, as $n \rightarrow \infty$. With the above, this implies that $S_n^N/n\beta_n^N \rightarrow 1$ in probability, as $n \rightarrow \infty$. Together with the fact that $\tau_n^N/n \rightarrow \mathbb{E}[\tau_1^N]$ almost surely as $n \rightarrow \infty$ and the monotonicity of $\mathcal{R}^N(t)$, this readily yields the theorem in the case $\alpha = 1$, $\mathbb{E}[X] = \infty$.

Case $\alpha \in (0, 1)$: This case is similar to the previous one. Let $(S_n^N)_{n \geq 0}$ be a random walk with increments distributed as $\mathcal{R}^N(\tau_1^N)$ and set $a_n^N = \inf\{x : \mathbb{P}(S_1^N > x) > n^{-1}\}$. Again by standard results on random walks (see e.g. [20, Theorem XVII.5.3] or [17, Theorem 2.7.7]), since, by Proposition 2.6, $\mathbb{P}(S^N > x)$ is regularly varying for every N , the sequence S_n^N/a_n^N converges in law to W_α , as $n \rightarrow \infty$, for every N . By Proposition 2.6,

$$a_n^N \asymp \frac{h(c_N)^{1/\alpha}}{c_N} h^{-1}(\mathbb{E}[\tau_1^N] n).$$

Together with the fact that $\tau_n^N/n \rightarrow \mathbb{E}[\tau_1^N]$ almost surely as $n \rightarrow \infty$ and the monotonicity of $\mathcal{R}^N(t)$, this finishes the proof. \square

We finish this section with an easy lemma which will be used in the proof of Theorem 1.1.

Lemma 2.8. *The process $(\mathcal{R}(t))_{t \geq 0}$ is stochastically continuous. Furthermore, the processes $(\mathcal{R}(t))_{t \geq 0}$ and $(\mathcal{R}(t+1))_{t \geq 0}$ almost surely do not have common jumps.*

Proof. For every $t \geq 0$ and $\delta > 0$, $\mathcal{R}(t + \delta)$ is by definition stochastically dominated by $\mathcal{R}(t) + \mathcal{R}'(\delta)$, for an independent copy \mathcal{R}' of \mathcal{R} . This easily yields the first claim. For the second claim, we note that for every stopping time T of the process \mathcal{R} , by the independence properties of the Poisson process, $T + 1$ is almost surely not a jump time of \mathcal{R} . Fix $\varepsilon > 0$ and define T_k to be the time of the k -th jump of size greater than ε . Then $T_k \rightarrow \infty$ almost surely as $k \rightarrow \infty$ and almost surely, $T_k + 1$ is not a jump time of \mathcal{R} for every k . Letting $\varepsilon \rightarrow 0$ yields the lemma. \square

3 Coupling the BRW with a discretised stairs process

Recall the definition of X , $(X_{n,i})_{n \geq 0, i \in [2N]}$ and $\mathcal{X}(n) = \{\mathcal{X}_1(n) \leq \dots \leq \mathcal{X}_N(n)\}$ from the introduction and define the rescaled variables $Y := c_N^{-1}X$, $Y_{n,i} := c_N^{-1}X_{n,i}$ and $\mathcal{Y}(n) := c_N^{-1}\mathcal{X}(n)$. Set $\mathcal{Y}_i(-n) = 0$ for all $i \in [N]$ and $n > 0$. For $n \geq 1$, let \mathcal{F}_n be the sigma-algebra generated by the variables $Y_{k,i}$ for $0 \leq k \leq n-1$, $i \geq 1$ and set \mathcal{F}_{-n} to the trivial σ -field for all $n \geq 0$. Note that $(\mathcal{X}(n))_{n \geq 0}$ and $(\mathcal{Y}(n))_{n \geq 0}$ are adapted to the filtration $(\mathcal{F}_n)_{n \geq 0}$. For integers $N, \ell \geq 1$, the (N, ℓ) -discretised stairs process (DSP) is the process $(R^{N,\ell}(n))_{n \in \mathbb{Z}}$ defined inductively by $R^{N,\ell}(n) = 0$ for all $n \leq 0$ and

$$R^{N,\ell}(n+1) = R^{N,\ell}(n) \vee \max_{i=1, \dots, 2N} (R^{N,\ell}(n-\ell) + Y_{n,i}). \tag{3.1}$$

3.1 Lower bound

Proposition 3.1. *Let $\ell_N = \lceil \log_2 N \rceil$. Then $R^{N,\ell_N}(n-\ell_N) \leq \mathcal{Y}_1(n)$ and $R^{N,\ell_N}(n) \leq \mathcal{Y}_N(n)$ for all $n \geq 0$.*

Proof. Writing $R(n) = R^{N,\ell_N}(n)$ for short, define the random time

$$\tau = \inf\{n \geq 0 : \mathcal{Y}_1(n) < R(n-\ell_N) \text{ or } \mathcal{Y}_N(n) < R(n)\}.$$

We shall prove by contradiction that τ is infinite almost surely. Assume that τ is finite for some realisation of the variables $Y_{n,i}$. By definition of τ , one has $\mathcal{Y}_1(\tau-1) \geq R(\tau-1-\ell_N)$, so that by the definitions of $\mathcal{Y}(n)$ and $R(n)$, we then have $\mathcal{Y}_N(\tau) \geq R(\tau)$. We therefore must have $\mathcal{Y}_1(\tau) < R(\tau-\ell_N)$. But now, by the definition of τ , we have $\mathcal{Y}_N(\tau-\ell_N) \geq R(\tau-\ell_N)$. Hence, at time τ , there are no particles below $R(\tau-\ell_N)$, otherwise the particle at position $\mathcal{Y}_N(\tau-\ell_N)$ at time $\tau-\ell_N$ would have $2^{\ell_N} \geq N$ descendants at time τ , all above $R(\tau-\ell_N)$, whence the total number of particles would be larger than N , which is a contradiction. Thus, $\mathcal{Y}_1(\tau) \geq R(\tau-\ell_N)$, but this contradicts the definition of τ . \square

3.2 Upper bound

The main result in this section is Proposition 3.2. One should not be fooled by its apparent simplicity, its proof is more intricate than it looks at first sight (and took us quite some time to come up with). Let δ_N be a positive sequence which tends to zero as $N \rightarrow \infty$ but such that $\delta_N N^\varepsilon \rightarrow \infty$ for all $\varepsilon > 0$. Set $m_N = \log_2 N + \log_2 \delta_N$ and assume that $m_N \in \mathbb{N}$ and $m_N \geq 1$ for all N . Define the process $(\theta_n)_{n \geq 0}$ by $\theta_0 = 0$ and

$$\theta_n = \max_{0 \leq k \leq n} \{\mathcal{Y}_N(k) - R^{N,m_N}(k)\}.$$

Proposition 3.2. *For all $\varepsilon > 0$, all large enough N and $x \geq m_N - N^{(1 \vee \alpha^{-1}) + \varepsilon} / c_N$, for all $n \geq 0$,*

$$\mathbb{P}(\theta_{n+1} - \theta_n > x \mid \mathcal{F}_{n-m_N}) \leq Nm_N 2^{m_N+1} / h(c_N x)^2.$$

Proof. Let $n \in \mathbb{N}$ and $x > 0$. Set $\mathcal{Y}_i^{(n)}(k) = \mathcal{Y}_i(k) - \theta_n$, $k = 0, 1, \dots$ and note that by definition $\mathcal{Y}_N^{(n)}(k) \leq R^{N,m_N}(k)$ for all $0 \leq k \leq n$. We now claim that the event $\theta_{n+1} - \theta_n > x$ implies that there exists $i \in [N]$ and $j \in \{0, 1\}$, such that on the one hand $Y_{n,2i+j} > x$ and on the other hand $\mathcal{Y}_i^{(n)}(n) > x + R^{N,m_N}(n-m_N) \geq x + \mathcal{Y}_N^{(n)}(n-m_N)$, whence $\mathcal{Y}_i(n) > x + \mathcal{Y}_N(n-m_N)$. Indeed, note first that by definition of θ_n , we have

$$\begin{aligned} \theta_{n+1} - \theta_n &= \max(\mathcal{Y}_N(n+1) - R^{N,m_N}(n+1), \theta_n) - \theta_n \\ &= \left(\mathcal{Y}_N^{(n)}(n+1) - R^{N,m_N}(n+1)\right) \vee 0. \end{aligned} \tag{3.2}$$

Furthermore, recall that by definition,

$$\begin{aligned} \mathcal{Y}_N^{(n)}(n+1) &= \max_{i \in [N], j \in \{0,1\}} \mathcal{Y}_i^{(n)}(n) + Y_{n,2i+j} \\ R^{N,m_N}(n+1) &= R^{N,m_N}(n) \vee \left(R^{N,m_N}(n-m_N) + \max_{r \in [2N]} Y_{n,r} \right). \end{aligned}$$

Now let $i \in [N]$. Since $\mathcal{Y}_i^{(n)}(n) \leq R^{N,m_N}(n)$ as noted above, we have on the one hand,

$$\mathcal{Y}_N^{(n)}(n+1) - R^{N,m_N}(n+1) \leq \mathcal{Y}_i^{(n)}(n) + \max_{j \in \{0,1\}} Y_{n,2i+j} - R^{N,m_N}(n) \leq \max_{j \in \{0,1\}} Y_{n,2i+j}. \quad (3.3)$$

On the other hand, we have

$$\begin{aligned} \mathcal{Y}_N^{(n)}(n+1) - R^{N,m_N}(n+1) &\leq \mathcal{Y}_i^{(n)}(n) + \max_{j \in \{0,1\}} Y_{n,2i+j} - R^{N,m_N}(n-m_N) - \max_{r \in [2N]} Y_{n,r} \\ &\leq \mathcal{Y}_i^{(n)}(n) - R^{N,m_N}(n-m_N). \end{aligned} \quad (3.4)$$

Equations (3.2), (3.3) and (3.4) then yield the above claim.

Now if we denote by $M_x(n)$ the number of particles above $x + \mathcal{Y}_N(n - m_N)$ at time n (i.e. $M_x(n) = \#\{i \in [N] : \mathcal{Y}_i(n) > x + \mathcal{Y}_N(n - m_N)\}$), then a union bound gives that

$$\mathbb{P}(\theta_{n+1} - \theta_n > x \mid \mathcal{F}_{n-m_N}) \leq \mathbb{E}(M_x(n) \mid \mathcal{F}_{n-m_N}) \mathbb{P}(Y > x) \leq \mathbb{E}[M_x(m_N)] / h(c_N x). \quad (3.5)$$

Let $(S_n)_{n \geq 0}$ be a random walk with increments distributed according to Y . Bounding $M_x(m_N)$ by the number of particles above x at time m_N in N branching random walks *without* selection, we get

$$\mathbb{E}[M_x(m_N)] \leq N 2^{m_N} \mathbb{P}(S_{m_N} > x). \quad (3.6)$$

Note that from the definition of m_N , one has that $(1/2) \log_2 N \leq m_N \leq 2 \log_2 N$ for all large enough N . Now, for all $\varepsilon > 0$, for all large enough N and $x \geq m_N^{(1 \vee \alpha^{-1}) + \varepsilon} / c_N$, we have $\mathbb{P}(S_{m_N} > x) \leq 2m_N \mathbb{P}(Y > x)$ (see Corollary 7.1 in Section 7). The statement follows. \square

Corollary 3.3. *Let $p \in [0, 2\alpha)$. Then for every $0 < \varepsilon \leq (2\alpha - p)/2$, there exists N_ε , such that for $N > N_\varepsilon$ and $n \geq 0$, we have*

$$\mathbb{E}[(\theta_{n+1} - \theta_n)^p \mid \mathcal{F}_{n-m_N}] \leq \left(1 + \frac{4p}{2\alpha - p}\right) \left(\frac{\delta_N}{\log_2 N}\right)^{\frac{p}{2\alpha + \varepsilon}}.$$

Proof. Write $\mathbb{E}_n = \mathbb{E}[\cdot \mid \mathcal{F}_{n-m_N}]$ and set $\delta'_N = \delta_N / \log_2 N$. By Proposition 3.2, we have for every $x > \gamma m_N / c_N$ and every $n \geq 0$:

$$\mathbb{E}_n[(\theta_{n+1} - \theta_n)^p] \leq x + \frac{Nm_N 2^{m_N+1}}{h(c_N)^2} \int_x^\infty \frac{h(c_N)^2}{h(c_N y^{1/p})^2} dy.$$

By definition, we have $2^{m_N} = N\delta_N$ and $h(c_N) \sim 2N \log_2 N$ as $N \rightarrow \infty$. By Potter's bounds, there now exists x_ε , such that for $c_N \geq x_\varepsilon$ and $x^{1/p} c_N \geq x_\varepsilon$, we have

$$\mathbb{E}_n[(\theta_{n+1} - \theta_n)^p] \leq x + \delta'_N I_x, \quad I_x = \int_x^\infty y^{-2\alpha/p} \max(y^{\varepsilon/p}, y^{-\varepsilon/p}) dy. \quad (3.7)$$

By the hypothesis on ε , we have for $x \leq 1$: $I_x \leq 4(p/(2\alpha - p))x^{1-(2\alpha+\varepsilon)/p}$. Setting now $x = x_N = (\delta'_N)^{p/(2\alpha+\varepsilon)}$ in (3.7) yields the lemma (note that $x_N^{1/p} c_N \geq x_\varepsilon$ and $x_N > m_N^{(1 \vee \alpha^{-1}) + \varepsilon} / c_N$ for large N , by the hypothesis on δ_N). \square

4 Coupling the discretised stairs process and the stairs process

Let $N, \ell \in \mathbb{N}$ and define the measure $\mu_{N,\ell}$ on \mathbb{R}_+ by

$$\mu_{N,\ell}([x, \infty)) = -2N\ell \log(1 - h(c_N x)^{-1}).$$

Let $\mathcal{R}^{\mu_{N,\ell}}$ be the $\mu_{N,\ell}$ -stairs process as defined in the introduction. Furthermore, let $R^{N,\ell}$ be the (N, ℓ) -DSP defined in (3.1).

Proposition 4.1. *We have $(\mathcal{R}^{\mu_{N,\ell}}(n/\ell))_{n \geq 0} \stackrel{st}{\geq} (R^{N,\ell}(n))_{n \geq 0} \stackrel{st}{\geq} (\mathcal{R}^{\mu_{N,\ell}}(n/(\ell + 1)))_{n \geq 0}$, where $(X(n))_{n \geq 0} \stackrel{st}{\geq} (Y(n))_{n \geq 0}$ means that there exists a coupling, such that $X(n) \geq Y(n)$ for all $n \geq 0$.*

Proof. Let ξ_t be the function used to define the process $\mathcal{R}^{\mu_{N,\ell}}$. By the definition of the measure $\mu_{N,\ell}$, the variables $\max\{\xi_t : t \in [0, 1/\ell]\}$ and $\max\{Y_{1,i} : i \in [2N]\}$ have the same distribution, a fact which is also known as Serfling’s coupling after [27] (see also [26]). We can therefore construct the process $R^{N,\ell}$ using ξ_t by

$$R^{N,\ell}(n + 1) = R^{N,\ell}(n) \vee \max_{t \in [0, 1/\ell]} (R^{N,\ell}(n - \ell) + \xi_{(n+1)/\ell - t}). \tag{4.1}$$

In comparison, the processes $(\mathcal{R}^{\mu_{N,\ell}}(\frac{n}{\ell}))_{n \in \mathbb{Z}}$ and $(\mathcal{R}^{\mu_{N,\ell}}(\frac{n}{\ell+1}))_{n \in \mathbb{Z}}$ satisfy by definition

$$\mathcal{R}^{\mu_{N,\ell}}(\frac{n+1}{\ell}) = \mathcal{R}^{\mu_{N,\ell}}(\frac{n}{\ell}) \vee \max_{t \in [0, 1/\ell]} (\mathcal{R}^{\mu_{N,\ell}}(\frac{n+1-\ell}{\ell} - t) + \xi_{(n+1)/\ell - t}). \tag{4.2}$$

$$\mathcal{R}^{\mu_{N,\ell}}(\frac{n+1}{\ell+1}) = \mathcal{R}^{\mu_{N,\ell}}(\frac{n}{\ell+1}) \vee \max_{t \in [0, 1/(\ell+1)]} (\mathcal{R}^{\mu_{N,\ell}}(\frac{n-\ell}{\ell+1} - t) + \xi_{(n+1)/(\ell+1) - t}). \tag{4.3}$$

Equations (4.1) and (4.2) and the monotonicity of $\mathcal{R}^{\mu_{N,\ell}}$ now directly yield the first inequality in the statement of the proposition. As for the second inequality, if we take (4.3) as the definition of the process $(\mathcal{R}^{\mu_{N,\ell}}(\frac{n}{\ell+1}))_{n \geq 0}$, then exchanging $\xi_{(n+1)/(\ell+1) - t}$ by $\xi_{(n+1)/\ell - t}$ does not change its law, and we obviously have $\max_{t \in [0, 1/(\ell+1)]} \xi_{(n+1)/(\ell+1) - t} \leq \max_{t \in [0, 1/\ell]} \xi_{(n+1)/\ell - t}$ for every n . Together with (4.1) and the monotonicity of $\mathcal{R}^{\mu_{N,\ell}}$, this yields the statement. \square

5 Proof of Theorems 1.1 and 1.2

The following lemma will be needed in the proof of Theorem 1.1.

Lemma 5.1. *We have for every $n_0 \geq 0$ and $\varepsilon > 0$,*

$$\mathbb{P}(\mathcal{Y}_1(n_0 + \lfloor (1 - \varepsilon) \log_2 N \rfloor) < \mathcal{Y}_N(n_0) + \varepsilon \mid \mathcal{F}_{n_0}) \rightarrow 1, \quad \text{as } N \rightarrow \infty.$$

Proof. Since we can bound the configuration of particles at time n_0 from above by moving all particles to the position of the maximum $\mathcal{Y}_N(n_0)$, it is clearly enough to show the lemma for $n_0 = 0$. Let $\varepsilon > 0$ and set $\gamma_N = \varepsilon / \log_2 N$. Denote by J_n the number of particles which jump by at least γ_N between times n and $n + 1$. Then $\mathbb{E}[J_n] = 2N\mathbb{P}(Y > \gamma_N) = 2N/h(c_N \gamma_N)$, such that $\mathbb{E}[J_n] \leq N^{\varepsilon/2}$ for large N , by Potter’s bounds). Now, if a particle is at a position strictly greater than $n\gamma_N$ at a time n , it must have an ancestor which has jumped by more than γ_N between times $k - 1$ and k for some $k \leq n$. This ancestor then has at most 2^{n-k} descendants at time n . Altogether, this gives for large N ,

$$\mathbb{E}[\#\{i : \mathcal{Y}_i(n) > n\gamma_N\}] \leq \sum_{k=1}^n 2^{n-k} \mathbb{E}[J_k] \leq 2^{n+1} N^{\varepsilon/2},$$

which now implies with $n_N = \lfloor (1 - \varepsilon) \log_2 N \rfloor$, for large N ,

$$\mathbb{P}(\mathcal{Y}_1(n_N) \geq \varepsilon) \leq \mathbb{P}(\#\{i : \mathcal{Y}_i(n_N) > n_N \gamma_N\} \geq N) \leq 2N^{-\varepsilon/2},$$

by Markov’s inequality. This yields the lemma. \square

Proof of Theorem 1.1. Set $(\mathcal{Y}'_N(t))_{t \geq 0} = (\mathcal{Y}_N(\lfloor t \log_2 N \rfloor))_{t \geq 0}$. We will first show that the finite-dimensional distributions of $(\mathcal{Y}'_N(t), \mathcal{Y}'_1(t))_{t \geq 0}$ converge to those of $(\mathcal{R}^\alpha(t), \mathcal{R}^\alpha(t - 1))_{t \geq 0}$. Recall the definitions of ℓ_N and m_N from Sections 3.1 and 3.2 and note that $\ell_N \sim m_N \sim \log_2 N$ as $N \rightarrow \infty$. For an upper bound, let $p = 1$ if $\alpha \geq 1$ and $p \in (\alpha, \min(1, 2\alpha))$ if $\alpha < 1$. Write $x_+ = \max(x, 0)$ for $x \in \mathbb{R}$. We then have by Corollary 3.3, for every $\varepsilon > 0$ and some $c > 0$, for large N ,

$$\forall n \geq 0 : \mathbb{E}[(\mathcal{Y}_N(n) - R^{N, m_N}(n))_+^p] \leq \mathbb{E}[\theta_n^p] \leq \sum_{k=1}^n \mathbb{E}[(\theta_k - \theta_{k-1})^p] \leq cn \left(\frac{\delta_N}{\log_2 N} \right)^{p/(2\alpha+\varepsilon)}. \tag{5.1}$$

Fix $K > 0$. If we choose $\delta_N = o((\log_2 N)^{1-(2\alpha+\varepsilon)/p})$, then for all $n \leq K \log_2 N$, the right-hand side in the last inequality tends to zero as $N \rightarrow \infty$. Together with Propositions 2.4 and 4.1 as well as Lemmas 2.8 and 5.1, this shows that the finite-dimensional distributions of $(\mathcal{Y}'_N(t), \mathcal{Y}'_1(t))$ are tight in N and all limit points are dominated by the finite-dimensional distributions of $(\mathcal{R}^\alpha(t), \mathcal{R}^\alpha(t - 1))$.

For a lower bound, note that by Propositions 3.1 and 4.1, we have for every $n \geq 0$,

$$(\mathcal{Y}_N(n), \mathcal{Y}_1(n)) \stackrel{\text{st}}{\geq} (R^{N, \ell_N}(n), R^{N, \ell_N}(n - \ell_N)) \stackrel{\text{st}}{\geq} (\mathcal{R}^{\mu_N, \ell_N}(\frac{n}{\ell_N+1}), \mathcal{R}^{\mu_N, \ell_N}(\frac{n-\ell_N}{\ell_N+1})), \tag{5.2}$$

with the coordinate-wise order on \mathbb{R}^2 (i.e. $(x, y) \leq (v, w)$ iff $x \leq v$ and $y \leq w$). Together with the first part of Proposition 2.4 and Lemma 2.8, this proves that as $N \rightarrow \infty$, every limit point of the finite-dimensional distributions of $(\mathcal{Y}'_N(t), \mathcal{Y}'_1(t))_{t \geq 0}$ dominates those of $(\mathcal{R}^\alpha(t), \mathcal{R}^\alpha(t - 1))_{t \geq 0}$. Together with the upper bound established above, this proves the convergence.

In order to prove tightness of $(\mathcal{Y}'_N(t))_{t \geq 0}$ in Skorokhod's J_1 -topology, we will use Aldous' criterion [1, Theorem 1]: Let T_N be a sequence of stopping times for \mathcal{Y}'_N . Suppose for simplicity that T_N only takes on values which are multiples of $(\log_2 N)^{-1}$. Let ε_N be a sequence of positive numbers converging to 0. We then have for every $x > 0$,

$$\mathbb{P}(\mathcal{Y}'_N(T_N + \varepsilon_N) - \mathcal{Y}'_N(T_N) > x) \leq \mathbb{P}(\mathcal{Y}'_N(\varepsilon_N) > x),$$

because we can bound the configuration of particles at time $T_N \log_2 N$ from above by moving all particles to the position of the maximum. The right-hand side of the last inequality now converges to 0 by the convergence in finite-dimensional distributions established above together with the monotonicity of $\mathcal{Y}'_N(t)$ and Lemma 2.8. By Aldous' criterion, this yields tightness in Skorokhod's J_1 -topology.

As for the convergence of $(\mathcal{Y}'_N(t), \mathcal{Y}'_1(t))_{t \geq 0}$ in the SM_1 -topology, we note that by Skorokhod's representation theorem for stochastic processes [28, §3.1.2] and the convergence of the finite-dimensional distributions established above, we can transfer the processes \mathcal{Y}'_N , \mathcal{Y}'_1 and \mathcal{R}^α onto a common probability space, such that almost surely, $(\mathcal{Y}'_N(t), \mathcal{Y}'_1(t)) \rightarrow (\mathcal{R}^\alpha(t), \mathcal{R}^\alpha(t - 1))$ for every $t \in \mathbb{Q}_+$. The monotonicity of \mathcal{Y}'_N and \mathcal{Y}'_1 then implies that almost surely, both \mathcal{Y}'_N and \mathcal{Y}'_1 converge w.r.t. the SM_1 -topology [29, Corollary 12.5.1]. Convergence of the pair now follows from the second part of Lemma 2.8, by [29, Theorem 12.6.1]. \square

Proof of Theorem 1.2. We first cover the case $\mathbb{E}[X] < \infty$ (which includes the case $\alpha > 1$). The existence of the limit $v_N = \lim_{N \rightarrow \infty} \mathcal{X}_N(n)/n = \lim_{N \rightarrow \infty} \mathcal{X}_1(n)/n$ is easily proven using subadditivity (see [2, Proposition 2]) with the convergence holding almost surely and in L^1 . The asymptotic for v_N now easily follows from (5.2) and (5.1), together with Theorem 2.5 and Proposition 4.1. Indeed, (5.2) immediately gives a lower bound on v_N and for the upper bound, we note that with $\delta_N = o(\log N)^{1-2\alpha-\varepsilon}$, the right-hand side of 5.1, multiplied by $(\log_2 N)/n$, vanishes in the limit as N goes to infinity.

In the case $\mathbb{E}[X] = \infty$, set $\beta_n = nb_n^N$ if $\alpha = 1$ and $\beta_n = h^{-1}(n)$ if $\alpha < 1$ and let $p = 1$ if $\alpha = 1$ and $p \in (\alpha, \min(1, 2\alpha))$ if $\alpha < 1$. Then $n/\beta_n^p \rightarrow 0$ as $n \rightarrow \infty$, by Potter's bounds) and the fact that $h^{-1}(n)$ is regularly varying with index $1/\alpha$ [4, Theorem 1.5.12]. Letting δ_N be any sequence satisfying the hypotheses of Corollary 3.3, we get for every N and every $\varepsilon > 0$, for some constant C_N , by (5.1),

$$\mathbb{P}(\mathcal{Y}_N(n) - R^{N,m_N}(n) > \varepsilon\beta_n) \leq \varepsilon^{-p}\beta_n^{-p}\mathbb{E}[(\mathcal{Y}_N(n) - R^{N,m_N}(n))_+^p] \leq C_N\varepsilon^{-p}n/\beta_n^p \rightarrow 0,$$

as $n \rightarrow \infty$. This, together with Theorem 2.5 and Propositions 3.1 and 4.1, implies the statement about $\mathcal{X}_N(n)$. The statement about $\mathcal{X}_1(n)$ follows from the fact that $\mathcal{X}_N(n - \lceil \log_2 N \rceil) \leq \mathcal{X}_1(n) \leq \mathcal{X}_N(n)$ for all n . \square

6 Appendix A: Regular variation

A function $f : \mathbb{R} \rightarrow (0, \infty)$ is said to *vary regularly (at $+\infty$) with index $\alpha \in \mathbb{R}$* if for every $y > 0$,

$$\frac{f(xy)}{f(x)} \rightarrow y^\alpha, \quad \text{as } x \rightarrow \infty.$$

The classic reference to regular variation and the only one that we use in this article is the book by Bingham, Goldie and Teugels [4]. We will particularly often use a result known as Potter's bounds, which we reproduce here for convenience.

Theorem. (Potter's bounds, [4, Theorem 1.5.6]) *If f is regularly varying of index α , then for every $C > 1$ and $\delta > 0$ there exists $x_0 = x_0(C, \delta)$, such that*

$$\frac{f(y)}{f(x)} \leq C \max((y/x)^{\alpha+\delta}, (y/x)^{\alpha-\delta}), \quad \text{for all } x \geq x_0, y \geq x_0.$$

7 Appendix B: Large deviations for sums of i.i.d. random variables with regularly varying tails

A substantial body of literature is devoted to the following problem: given a family of i.i.d. random variables $(X_i)_{i \geq 1}$ distributed according to X , and a sequence $(x_n)_{n \geq 1}$ such that $\lim_{n \rightarrow +\infty} x_n = +\infty$, find conditions on the distribution of X and on the sequence $(x_n)_{n \geq 1}$ under which one has

$$\lim_{n \rightarrow +\infty} \sup_{x \geq x_n} \left| \frac{\mathbb{P}(S_n > x)}{n\mathbb{P}(X > x)} - 1 \right| = 0. \tag{7.1}$$

In what follows, we assume that X is non-negative and $\mathbb{P}(X > x) = 1/h(x)$, where $h(x)$ is regularly varying at $+\infty$ with index $\alpha > 0$. The specific result we use in this paper is the following corollary of [13, Theorem 3.3]):

Corollary 7.1. (corollary to [13, Theorem 3.3]) *For a random variable X as above, property (7.1) holds for any sequence $(x_n)_{n \geq 0}$ of the form $x_n := n^{(1 \vee \alpha^{-1})+\varepsilon}$, where $\varepsilon > 0$.*

The as-yet-unpublished paper [13] quoted above both covers the whole range of values of α and provides easy-to-read statements. The reader may consult [19, Section 8.6], [25] and [14] for additional references to the classical literature, as well as [15], which is rarely cited in this context.

Implicit in proofs of results of the above type are results about random walks with truncated jumps, such as the following. It is a direct consequence of the lemma stated on p. 168 of [16], whose author refers to the proof of [15, Lemma 3].

Corollary 7.2. (corollary to [16, Lemma on p. 168]) For a random variable X as above, for every $\varepsilon > 0$, $r \in (0, 1)$, there exists a constant C such that for large n ,

$$\mathbb{P}(S_n > x, \max_{i=1, \dots, n} X_i \leq rx) \leq C(n/x^{\alpha-\varepsilon/2})^{1/r}, \quad \text{for all } x \geq x_n = n^{(1 \vee \alpha^{-1})+\varepsilon}.$$

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