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A Williams decomposition for spatially dependent super-processes*

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Abstract

We present a genealogy for super-processes with a non-homogeneous quadratic branching mechanism, relying on a weighted version of the super-process introduced by Engländer and Pinsky and a Girsanov theorem. We then decompose this genealogy with respect to the last individual alive (Williams' decomposition). Letting the extinction time tend to infinity, we get the Q-process by looking at the super-process from the root, and define another process by looking from the top. Examples including the multitype Feller diffusion (investigated by Champagnat and Roelly) and the super-diffusion are provided.

Keywords: Spatially dependent super-process; Williams' decomposition; genealogy; h-transform; Q-process.
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1 Introduction

Even if super-processes with very general branching mechanisms are known, most of the works devoted to the study of their genealogy are concerned with homogeneous branching mechanisms modeling populations with identical individuals. Four distinct approaches have been proposed for describing these genealogies. When there is no spatial motion, super-processes are reduced to continuous state branching processes, whose genealogy can be understood by a flow of subordinators, see Bertoin and Le Gall [5], or by growing discrete trees, see Duquesne and Winkel [12]. With a spatial motion, the description of the genealogy can be done using the lookdown process of Donnelly and Kurtz [10] or the snake process of Le Gall [20]. Works that are concerned with a generalization of both constructions to non-homogeneous branching mechanisms are the following: Kurtz and Rodriguez [19] recently extended the lookdown process in this direction whereas Dhersin and Serlet proposed in [9] modifications of the snake.

Let X be a non-homogeneous super-process. It models the evolution of a large population, where the location of the individuals is allowed to affect their reproduction law. We assume the extinction time $H_{\max}$ of this population is finite. We are interested in the two following conditionings on the genealogical structure of X:

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1. The distribution $X^{(h_0)}$ of X conditioned on $H_{\max} = h_0$: we derive it using a spinal decomposition involving the ancestral lineage of the last individual alive (Williams' decomposition).
2. The convergence of the distribution of $X^{(h_0)}$ as h_0 goes to ∞. This convergence is studied from two viewpoints. On the one hand, we obtain a convergence result for $(X_s^{(h_0)}, s \in [0,t])$ towards the Q-process. On the other hand, we find a convergence result for the backward process from its extinction time, namely $(X_{h_0+s}^{(h_0)}, s \in [-t,0])$. We reduce both convergences to the convergence of the ancestral lineage of the last individual alive thanks to Williams' decomposition.

Concerning the first conditioning, we stress on the following difference between super-processes with homogeneous and non-homogeneous branching mechanisms, which explains our interest in the latter model. For homogeneous branching mechanisms, the spatial motion is independent of the genealogical structure. As a consequence, the law of the ancestral lineage of the last individual alive does not distinguish from the original motion. Therefore, in this setting, the description of $X^{(h_0)}$ may be deduced from Abraham and Delmas [2] where no spatial motion is taken into account. For non-homogeneous branching mechanisms on the contrary, the law of the ancestral lineage of the last individual alive should depend on the distance to the extinction time h_0. This fact will be precised by the second conditioning.

A few lines about the terminology "Williams' decompositions" are in order: Williams [28] decomposed the Brownian excursion with respect to its maximum. After Aldous recognized in [3] the genealogy of a branching process in this excursion, this name also refers to decompositions of branching processes with respect to their height.

Our second conditioning exemplifies the interest of Williams' decomposition for investigating the process conditioned on extinction in remote time. The convergence of the super-process conditioned on extinction in remote time essentially reduces to the convergence of the ancestral lineage of the last individual alive thanks to Williams' decomposition. Also, we may consider the limit either on the fixed time interval $[0,t]$ to get the corresponding Q-process, either on the moving time interval $[h_0 - t, h_0]$. For non-homogeneous branching mechanisms, we expect a different behavior on these two time intervals for the ancestral lineage of the last individual: far away from the extinction time, it should favor the fittest types; near the extinction time, it should select the weakest ones. It is well-known how to perform such a conditioning in the homogeneous branching mechanisms: it goes back to Serlet [27] for quadratic branching mechanism; for more general branching mechanisms, it reduces to the corresponding decomposition for continuous state branching process, see Chen and Delmas [7] and the references therein. For non-homogeneous branching, a first construction of the Q-process (without genealogy) has been given in Champagnat and Roelly [6] in the particular case of a multitype Feller diffusion.

A rigorous analysis of the ancestral lineage of the last individual alive requires the introduction of a genealogy for the super-process, since the ancestral lineages are not immediately identifiable in the context of measure-valued processes. We found out that the previous genealogies defined for non-homogeneous branching mechanisms, see [19] and [9], were not suited to our need. In particular, the description in Dhersin and Serlet [9] preserves neither the extinction time nor the last individual alive, and thus does not allow to disintegrate the law of the super-process with respect to its extinction time. We thus provide a new description of the genealogy through another, more effective modification of the Brownian snake (at least regarding our purposes). More precisely, starting with non-homogeneous branching mechanism, we go back to an homogeneous one via two transformations:

- The first transformation relies on the non-linear h transform, or reweighting of super-processes, introduced in Engländer and Pinsky [16].
- The second transformation is based on a Girsanov change of measure on the law of super-processes, as described in Chapter IV of Perkins [23].

Reversing the procedure, we define the genealogy associated to a non-homogeneous branching mechanism from the one associated to an homogeneous one, see Proposition 3.12. We then obtain our two conditionings by "transfer", using the previous knowledge for homogeneous branching mechanisms. The drawback of this approach is that we have to restrict ourselves to quadratic branching mechanisms with bounded and smooth parameters.

The rest of the introduction is devoted to a presentation of a selection of our results. Let $(X_t, t \geq 0)$ be an $(\mathcal{L}, \beta, \alpha)$ super-process over a Polish space S. The underlying spatial motion $(Y_t, t \geq 0)$ is a Markov process with infinitesimal generator $\mathcal{L}$ started at x under P_x. The non-homogeneous branching mechanism $\psi(x, \lambda)$ satisfies

$$\psi(x, \lambda) = \beta(x)\lambda + \alpha(x)\lambda^2,$$

and we further assume that the functions β and α satisfy conditions $(H2)$ and $(H3)$, see Section 2. Notice in particular that smooth functions α and β, with α positive, satisfy these conditions. Branching mechanisms with constant functions α and β are called homogeneous. Let $\mathbb{P}_\nu$ be the distribution of X started from the finite measure ν on S, and $\mathbb{N}_x$ be the corresponding canonical measure of X with initial state x. The process X under $\mathbb{P}_\nu$ is distributed as $\sum_{i\in\mathcal{I}} X^i$, where $\sum_{i\in\mathcal{I}} \delta_{X^i}(dX)$ is a Poisson Point measure with intensity $\nu(dx)\mathbb{N}_x(dX)$. We define $H_{\max} = \inf\{t > 0, X_t = 0\}$ as the extinction time of X, and we will assume that X satisfies almost sure extinction, that is $\mathbb{N}_x[H_{\max} = \infty] = 0$ (this assumption is denoted by $(H1)$ in Section 2). We also define the function $v_h(x) = \mathbb{N}_x[X_h \neq 0]$, and we introduce a family of probability measures by setting:

$$\forall\, 0 \leq t < h, \quad \frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{\partial_h v_{h-t}(Y_t)}{\partial_h v_h(x)}\, \mathrm{e}^{-\int_0^t ds\ \partial_\lambda \psi(Y_s, v_{h-s}(Y_s))},$$

where $\mathcal{D}_t = \sigma(Y_s, 0 \leq s \leq t)$ is the natural filtration of Y, see Lemma 4.10.

The following Theorem solves the conditioning problem (1) raised above. It gives a Williams decomposition of X with respect to its extinction time $H_{\max}$: sub-trees given below by $(X^j, j \in J)$ are grafted on a spine given by Y under $\mathrm{P}^{(h_0)}_x$.

Theorem (Corollary 4.13). *(Williams' decomposition under $\mathbb{N}_x$) Assume that the $(\mathcal{L}, \beta, \alpha)$ super-diffusion X satisfies the almost sure extinction property. Let $x \in S$ and $Y_{[0,h_0)}$ be distributed according to $\mathrm{P}^{(h_0)}_x$. Consider the Poisson point measure $\mathcal{N} = \sum_{j\in J} \delta_{(s_j, X^j)}$ on $[0, h_0) \times \Omega$ with intensity:*

$$2\, \mathbf{1}_{[0,h_0)}(s) ds\ \mathbf{1}_{\{H_{\max}(X) < h_0 - s\}} \alpha(Y_s)\ \mathbb{N}_{Y_s}[dX].$$

Conditionally on $\{H_{\max} = h_0\}$, the $(\mathcal{L}, \beta, \alpha)$ super-diffusion X under $\mathbb{N}_x$ is distributed as $X^{(h_0)} = (X^{(h_0)}_t, t \geq 0)$ defined for all $t \geq 0$ by:

$$X^{(h_0)}_t = \sum_{j\in J,\, s_j < t} X^j_{t-s_j}.$$

This also implies the existence of a measurable family $(\mathbb{N}^{(h_0)}_x, h_0 > 0)$ of probabilities such that $\mathbb{N}^{(h_0)}_x$ is the distribution of X under $\mathbb{N}_x$ conditionally on $\{H_{\max} = h_0\}$.

We shall consider from now on the case where Y is a diffusion on $S = \mathbb{R}^K$ or a finite state space Markov process. The generator $\mathcal{L}$ of a diffusion is defined as follows: let a_{ij} and b_i be in $\mathcal{C}^{1,\mu}(S)$, the usual Hölder space of order $\mu \in [0,1)$, which consists of functions whose first order derivatives are locally Hölder continuous with exponent μ, for each i,j in $\{1,\dots,K\}$. The functions $a_{i,j}$ are chosen such that the matrix $(a_{ij})_{(i,j)\in\{1\dots K\}^2}$ is positive definite. In that case, the following elliptic operator:

$$\mathcal{L}(u) = \sum_{i=1}^K b_i\ \partial_{x_i} u + \frac{1}{2}\sum_{i,j=1}^K a_{ij}\ \partial_{x_i,x_j} u.$$

defines the generator of a diffusion on S. The super-processes associated with a diffusion is called a super-diffusion. The generator $\mathcal{L}$ of a Markov process with finite state space $S = \{1,\dots,K\}$ for K integer is given by a square matrix $(q_{ij})_{1\leq i,j\leq K}$ of size K with lines summing up to 0, and q_{ij} gives the transition rate from i to j for $i \neq j$:

$$\mathcal{L}(u)(i) = \sum_{j\neq i} q_{ij}[u(j) - u(i)].$$

This Markov process will be assumed irreducible. The super-process associated with a finite state space Markov process is called a multitype Feller diffusion.

The generalized eigenvalue λ_0 of the operator $\beta - \mathcal{L}$ is defined in Pinsky [24] for a diffusion on $\mathbb{R}^K$ and, for finite state space, it reduces to the Perron Frobenius eigenvalue, see Seneta [26]. In both cases, we have:

$$\lambda_0 = \sup\{\ell \in \mathbb{R}, \exists u \in \mathcal{D}(\mathcal{L}), u > 0 \text{ such that } (\beta - \mathcal{L})u = \ell\ u\}\cdot$$

We assume that the space of positive harmonic functions for $(\beta - \lambda_0) - \mathcal{L}$ is one dimensional, generated by a function ϕ_0. From this assumption, we also have that the space of positive harmonic functions of the adjoint of $(\beta - \lambda_0) - \mathcal{L}$ is one dimensional, and we denote by $\tilde{\phi}_0$ a generator of this space (see [24] for diffusions). The operator $(\beta - \lambda_0) - \mathcal{L}$ is said product-critical when $\int_S dx\, \phi_0(x)\, \tilde{\phi}_0(x) < \infty$, in which case the probability measure P^{ϕ_0}, given by:

$$\forall t \geq 0, \quad \frac{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{\phi_0(Y_t)}{\phi_0(Y_0)}\, \mathrm{e}^{-\int_0^t ds\ (\beta(Y_s)-\lambda_0)},$$

defines a recurrent Markov process (in the sense given by (5.10)). We precise in Proposition 5.10 (multitype Feller diffusion) and Proposition 5.11 (super-diffusion) the generators associated with $\mathrm{P}^{(h)}$ and P^{ϕ_0}. We shall assume also that ϕ_0 is bounded from below and from above by two positive constants. Together with the non-negativity of λ_0, this implies the almost sure extinction of the associated super-process, see Lemma 5.2.

The following Theorem states the weak convergence of the probability measures $(\mathbb{N}_x^{(h_0)}, h_0 > 0)$, and partly solves the conditioning problem (2). Notice the limiting object consists of sub-trees $(X^j, j \in J)$ grafted on a spine given by Y under $\mathrm{P}_x^{\phi_0}$.

Theorem (Corollary 5.19). *(Q-process under $\mathbb{N}_x$) Assume that $\lambda_0 \geq 0$, that ϕ_0 is bounded from below and above by positive constants and that the operator $(\beta - \lambda_0) - \mathcal{L}$ is product critical.*
Let Y be distributed according to $\mathrm{P}_x^{\phi_0}$, and, conditionally on Y, let $\mathcal{N} = \sum_{j\in\mathcal{I}} \delta_{(s_j, X^j)}$ be a Poisson point measure with intensity:

$$2\mathbf{1}_{\mathbb{R}^+}(s) ds\ \alpha(Y_s)\mathbb{N}_{Y_s}[dX].$$

Consider the process $X^{(\infty)} = (X_t^{(\infty)}, t \geq 0)$, which is defined for all $t \geq 0$ by:

$$X_t^{(\infty)} = \sum_{j\in J,\, s_j<t} X^j_{t-s_j},$$

and denote by $\mathbb{N}_x^{(\infty)}$ its distribution. Then, the process $(X_s^{(h_0)}, s \in [0,t])$ weakly converges to $(X_s^{(\infty)}, s \in [0,t])$ as h_0 goes to infinity.

We also prove that $\mathbb{N}_x^{(\infty)}$ actually is the law of the Q-process, defined as the weak limit of the probability measures $\mathbb{N}_x^{(\geq h_0)} = \mathbb{N}_x\left[\,|H_{\max} \geq h_0\right]$, see Lemma 5.14.

Remark 1.1. *As noticed by Gorostiza and Roelly [17] and Li [21], the multitype Dawson-Watanabe super-process can be understood as a single non-homogeneous super-process on an extended space. The above Theorem provides a genealogical construction of the Q-process associated to a multitype Feller diffusion considered in [6], and this construction gives a precise meaning to "the interactive immigration" introduced in Remark 2.8 of [6].*

Remark 1.2. *In Engländer and Kyprianou [15], the spinal decomposition of a Doob h-transform of the super-diffusion is provided, and it is "suggest[ed]" this process is the law of the "the sum of two independent processes" , see the Discussion 2.2 of [15]. This Theorem, or more precisely third item of Corollary 5.19 together with Lemma 5.14, prove that the process they considered actually is the Q-process.*

Remark 1.3. *Interestingly, the law of the spine P^{ϕ_0} is quite different from that of the backbone formed by the infinite ancestral lineages in a super-critical Dawson-Watanabe super-process investigated in Engländer and Pinsky [16], see Remark 5.20.*

We may also prove weak convergence of the probability measures $(\mathbb{N}_x^{(h_0)}, h_0 > 0)$ backward from the extinction time. Let us denote by $\mathrm{P}^{(-h)}$ the distribution of Y under $\mathrm{P}^{(h)}$ shifted by h:

$$\mathrm{P}^{(-h)}\left((Y_s, s \in [-h,0]) \in \bullet\right) = \mathrm{P}^{(h)}\left((Y_{h+s}, s \in [-h,0]) \in \bullet\right).$$

The product criticality assumption yields the existence of a probability measure denoted by $\mathrm{P}^{(-\infty)}$ such that for all $x \in S$, $t \geq 0$:

$$\mathrm{P}_x^{(-h)}\left((Y_s, s \in [-t,0]) \in \bullet\right) \xrightarrow[h \to +\infty]{} \mathrm{P}^{(-\infty)}\left((Y_s, s \in [-t,0]) \in \bullet\right).$$

The following result corresponds to second item of Theorem 5.25. It completes the answer to the conditioning problem (2). Notice the limiting object below corresponds to sub-trees $(X^j, j \in J)$ grafted on a spine given by Y under $\mathrm{P}^{(-\infty)}$.

Theorem (Theorem 5.25)**.** *Assume that $\lambda_0 > 0$, that ϕ_0 is bounded from below and above by positive constants and that the operator $(\beta - \lambda_0) - \mathcal{L}$ is product critical.*
Let Y be distributed according to $\mathrm{P}^{(-\infty)}$, and, conditionally on Y, let $\sum_{j \in J} \delta_{(s_j, X^j)}$ be a Poisson point measure with intensity:

$$2\,\mathbf{1}_{\{s<0\}}\alpha(Y_s)\,ds\,\mathbf{1}_{\{H_{\max}(X)<-s\}}\,\mathbb{N}_{Y_s}[dX].$$

Consider the process $(X_s^{(-\infty)}, s \leq 0)$, which is defined for all $s \leq 0$ by:

$$X_s^{(-\infty)} = \sum_{j \in J,\, s_j < s} X^j_{s-s_j}.$$

Then the process $(X_{h_0+s}^{(h_0)}, s \in [-t,0])$ weakly converges to $(X_s^{(-\infty)}, s \in [-t,0])$ as h_0 goes to infinity.

Remark 1.4. *Considering a super-process with homogeneous branching mechanism, the Q-process may be easily defined from the well known Q-process associated with the continuous state branching process, see [7] for instance. Thus the recurrence condition imposed on the spatial motion is not necessary for the Corollary 5.19 to hold. This condition seems more natural in the setting of Theorem 5.25.*

Remark 1.5. *It is possible to obtain Corollary 5.19 for more general processes (without using the generalized eigenvalue nor the product criticality assumption), as soon as the key Lemmas 5.6 and 5.9 may be established. In this framework, the distribution* P^{ϕ_0} *is defined as a limit from Lemma 5.9. The same remark holds for Theorem 5.25, with Lemma 5.23 in the rôle of the key Lemma.*

Finally, we comment on the effect of conditioning by the event $\{H_{\max} = h\}$ on the spatial motion of the ancestral lineages. For h finite, the ancestral lineage at time t of the most persistent lineage follows the path of an inhomogeneous h-transform of the original motion, as seen from the definition of $\mathrm{P}^{(h)}$. This means that this ancestral lineage is attracted towards the region where the branching process has high probability of dying $h-t$ times later. What happens when $h-t \to \infty$? We shall prove in Lemma 5.7 that there is a constant C such that:

$$\lim_{h\to\infty} -\partial_h v_h(x)\, \mathrm{e}^{\lambda_0 h} = C\phi_0(x).$$

This implies that, viewed from the root, the ancestral lineage of the most persistent particle of the super-process conditioned on extinction in remote time, that is conditioned on $\{H_{\max} = h\}$ and $h \to \infty$, follows an h-transform for $h(x) = \phi_0(x)$, see the definition of P^{ϕ_0}. This h-transform is an homogeneous spatial motion, and the resulting measure valued Q-process inherits from it the homogeneous Markov property. On the contrary, when $h \to \infty$ with $h-t$ fixed, the most persistent ancestral lineage viewed from the top always follows the same inhomogeneous h-transform driven by $-\partial_h v_{h-t}(x)$. Finally, an ancestral lineage chosen at random among those present at time t follows the path of a penalized motion, with the penalization given by $\exp\left(-\int_0^t ds\ \beta(Y_s)\right)$, as seen from the definition (4.11) of $\mathrm{P}^{(B,t)}$. We say penalization, and not h-transform, since the distributions of $\mathrm{P}^{(B,t)}$ do not consistently define a process as t varies. As $t \to \infty$, we shall see in Lemma 5.12 that $\mathrm{P}^{(B,t)}$ weakly converges to P^{ϕ_0} on $\mathcal{F}_s$, $s \geq 0$.

Outline. We give some background on super-processes with a non-homogeneous branching mechanism in Section 2. Section 3 begins with the definition of the h-transform in the sense of Engländer and Pinsky, Definition 3.4, goes on with a Girsanov Theorem, Proposition 3.7, and ends up with the definition of the genealogy, Proposition 3.12, by combining both tools. Section 4 is mainly devoted to the proof of Williams' decomposition, Theorem 4.12. We also take the opportunity to give a decomposition with respect to a randomly chosen individual, also known as a Bismut decomposition, in Proposition 4.2. Section 5 gives some applications of Williams' decomposition in two particular cases, the super-diffusion and the finite state space super-process. We first prove the convergence of the spine seen from the root, in the Williams setting first, in the Bismut setting then, see Sections 5.3 and 5.4. We then deduce the convergence of the super-process conditioned to extinct *at* a remote time, see Section 5.5. The same limit is shown to prevail also for the super-process conditioned to extinct to extinct *after* a remote time, also known as the Q-process. We then look at the convergence of the spine from the top in Section 5.6. The convergence of the super-process seen from the top follows, see Section 5.7.

2 Notations and definitions

This section, based on the lecture notes of Perkins [23], provides us with basic material about super-processes, relying on their characterization via the Log Laplace equation.

We first introduce some definitions:

- (S, δ) is a Polish space, $\mathcal{B}$ its Borel sigma-field.

- $\mathcal{S}$ is the set of real valued measurable functions and $b\mathcal{S} \subset \mathcal{S}$ the subset of bounded functions.
- $\mathcal{C}(S, \mathbb{R})$, or simply $\mathcal{C}$, is the set of continuous real valued functions on S, $\mathcal{C}_b \subset \mathcal{C}$ the subset of continuous bounded functions.
- $D(\mathbb{R}^+, S)$, or simply D, is the set of càdlàg paths of S equipped with the Skorokhod topology, $\mathcal{D}$ is the Borel sigma field on D, and $\mathcal{D}_t$ the canonical right continuous filtration on D.
- For each set of functions, the superscript $.^+$ will denote the subset of the non-negative functions: For instance, $b\mathcal{S}^+$ stands for the subset of non-negative functions of $b\mathcal{S}$.
- $\mathcal{M}_f(S)$ is the space of finite measures on S. The standard inner product notation will be used: for $g \in \mathcal{S}$ integrable with respect to $M \in \mathcal{M}_f(S)$, $M(g) = \int_S M(dx)g(x)$.

We now introduce the two main ingredients which enter in the definition of a super-process, the spatial motion and the branching mechanism:

- Assume $Y = (D, \mathcal{D}, \mathcal{D}_t, Y_t, \mathrm{P}_x)$ is a Borel strong Markov process. "Borel" means that $x \to \mathrm{P}_x(A)$ is $\mathcal{B}$ measurable for all $A \in \mathcal{B}$. Let E_x denote the expectation operator, and $(P_t, t \geq 0)$ the semi-group defined by: $P_t(f)(x) = \mathrm{E}_x[f(Y_t)]$. We impose the additional assumption that $P_t : \mathcal{C}_b \to \mathcal{C}_b$. In particular the process Y has no fixed discontinuities. The generator associated to the semi-group will be denoted $\mathcal{L}$. Remember f belongs to the domain $\mathcal{D}(\mathcal{L})$ of $\mathcal{L}$ if $f \in \mathcal{C}_b$ and for some $g \in \mathcal{C}_b$,
$$f(Y_t) - f(x) - \int_0^t ds\ g(Y_s) \text{ is a } P_x \text{ martingale for all } x \text{ in } S, \tag{2.1}$$
in which case $g = \mathcal{L}(f)$.
- The functions α and β being elements of $\mathcal{C}_b$, with α bounded from below by a positive constant, the non-homogeneous quadratic branching mechanism $\psi^{\beta,\alpha}$ is defined by:
$$\psi^{\beta,\alpha}(x, \lambda) = \beta(x)\lambda + \alpha(x)\lambda^2, \tag{2.2}$$
for all $x \in S$ and $\lambda \in \mathbb{R}$. We will just write ψ for $\psi^{\beta,\alpha}$ when there is no possible confusion. If α and β are constant functions, we will call the branching mechanism (and by extension, the corresponding super-process) homogeneous.

The mild form of the log-Laplace equation is given by the integral equation, for $\phi, f \in b\mathcal{S}^+$, $t \geq 0$, $x \in S$:

$$u_t(x) + \mathrm{E}_x\left[\int_0^t ds\ \psi(Y_s, u_{t-s}(Y_s))\right] = \mathrm{E}_x\left[f(Y_t) + \int_0^t ds\ \phi(Y_s)\right]\cdot \tag{2.3}$$

Theorem 2.1. *([23], Theorem II.5.11) Let $\phi, f \in b\mathcal{S}^+$. There is a unique jointly (in t and x) Borel measurable solution $u_t^{f,\phi}(x)$ of equation (2.3) such that $u_t^{f,\phi}$ is bounded on $[0,T] \times S$ for all $T > 0$. Moreover, $u_t^{f,\phi} \geq 0$ for all $t \geq 0$.*

We shall write u^f for $u^{f,0}$ when ϕ is null.

We introduce the canonical space of continuous applications from $[0, \infty)$ to $\mathcal{M}_f(S)$, denoted by $\Omega := \mathcal{C}(\mathbb{R}^+, \mathcal{M}_f(S))$, endowed with its Borel sigma field $\mathcal{F}$, and the canonical right continuous filtration $\mathcal{F}_t$. Notice that $\mathcal{F} = \mathcal{F}_\infty$.

Theorem 2.2. *([23], Theorem II.5.11) Let $u_t^{f,\phi}(x)$ denote the unique jointly Borel measurable solution of equation (2.3) such that $u_t^{f,\phi}$ is bounded on $[0,T]\times S$ for all $T>0$. There exists a unique Markov process $X=(\Omega,\mathcal{F},\mathcal{F}_t,X_t,(\mathbb{P}_\nu^{(\mathcal{L},\beta,\alpha)},\nu\in\mathcal{M}_f(S)))$ such that:*

$$\forall\phi,f\in b\mathcal{S}^+,\qquad \mathbb{E}_\nu^{(\mathcal{L},\beta,\alpha)}\left[\mathrm{e}^{-X_t(f)-\int_0^t ds\; X_s(\phi)}\right]=\mathrm{e}^{-\nu(u_t^{f,\phi})}\,. \tag{2.4}$$

The process X in the previous theorem is called the $(\mathcal{L},\beta,\alpha)$-super-process. We now state the existence theorem of the canonical measures.

Theorem 2.3. *([23], Theorem II.7.3) There exists a measurable family of σ-finite measures $(\mathbb{N}_x^{(\mathcal{L},\beta,\alpha)},x\in S)$ on $(\Omega,\mathcal{F})$ which satisfies the following properties: If $\sum_{j\in\mathcal{J}}\delta_{(x^j,X^j)}$ is a Poisson point measure on $S\times\Omega$ with intensity $\nu(dx)\;\mathbb{N}_x^{(\mathcal{L},\beta,\alpha)}$, then $\sum_{j\in\mathcal{J}}X^j$ is an $(\mathcal{L},\beta,\alpha)$-super-process started at ν.*

We will often abuse notation by denoting $\mathbb{P}_\nu$ (resp. $\mathbb{N}_x$) instead of $\mathbb{P}_\nu^{(\mathcal{L},\beta,\alpha)}$ (resp. $\mathbb{N}_x^{(\mathcal{L},\beta,\alpha)}$), and $\mathbb{P}_x$ instead of $\mathbb{P}_{\delta_x}$ when starting from δ_x the Dirac mass at point x.

Let X be a $(\mathcal{L},\beta,\alpha)$-super-process. The exponential formula for Poisson point measures yields the following equality:

$$\forall f\in b\mathcal{S}^+,\; \mathbb{N}_{x_0}\left[1-\mathrm{e}^{-X_t(f)}\right]=-\log\mathbb{E}_{x_0}\left[\mathrm{e}^{-X_t(f)}\right]=u_t^f(x_0), \tag{2.5}$$

where u_t^f is (uniquely) defined by equation (2.4).

Denote $H_{\max}$ the extinction time of X:

$$H_{\max}=\inf\{t>0;\; X_t=0\}. \tag{2.6}$$

Definition 2.4 (Global extinction)**.** *The super-process X satisfies global extinction if $\mathbb{P}_\nu(H_{\max}<\infty)=1$ for all $\nu\in\mathcal{M}_f(S)$.*

We will need the the following assumption:

$(H1)$ **The $(\mathcal{L},\beta,\alpha)$-super-process satisfies the global extinction property.**

We shall be interested in the function

$$v_t(x)=\mathbb{N}_x[H_{\max}>t]. \tag{2.7}$$

We set $v_\infty(x)=\lim_{t\to\infty}\downarrow v_t(x)$. The global extinction property is easily stated using v_∞.

Lemma 2.5. *The global extinction property holds if and only if $v_\infty=0$.*

See also Lemma 4.9 for other properties of the function v.

Proof. The exponential formula for Poisson point measures yields:

$$\mathbb{P}_\nu(H_{\max}\leq t)=\mathrm{e}^{-\nu(v_t)}\,.$$

To conclude, let t go to infinity in the previous equality to get:

$$\mathbb{P}_\nu(H_{\max}<\infty)=\mathrm{e}^{-\nu(v_\infty)}\,.$$

□

For homogeneous super-processes (α and β constant), the function v is easy to compute and the global extinction holds if and only β is non-negative. Then, using a stochastic domination argument, one gets that a $(\mathcal{L},\beta,\alpha)$-super-process, with β non-negative, exhibits global extinction (see [15] p.80 for details).

3 A genealogy for the non-homogeneous super-processes

We first recall (Section 3.1) the h-transform for super-process introduced in [16] and then (Section 3.2) a Girsanov theorem previously introduced in [23] for interactive super-processes. Those two transformations allow us to give the Radon-Nikodym derivative of the distribution of a super-process with non-homogeneous branching mechanism with respect to the distribution of a super-process with an homogeneous branching mechanism. The genealogy of the super-process with an homogeneous branching mechanism can be described using a Brownian snake, see [11]. Then, in Section 3.3, we use the Radon-Nikodym derivative to transport this genealogy and get a genealogy for the super-process with non-homogeneous branching mechanism.

3.1 The h-transform for super-processes

We first introduce a new probability measure on $(D, \mathcal{D})$ using the next Lemma.

Lemma 3.1. *Let g be a positive function of $\mathcal{D}(\mathcal{L})$ such that g is bounded from below by a positive constant. Then, the process $\left(\frac{g(Y_t)}{g(x)}\,\mathrm{e}^{-\int_0^t ds\ (\mathcal{L}g/g)(Y_s)}, t \geq 0\right)$ is a positive martingale under P_x.*

We set $\mathcal{D}_g(\mathcal{L}) = \{v \in \mathcal{C}_b,\ gv \in \mathcal{D}(\mathcal{L})\}$.

Proof. Let g be as in Lemma 3.1 and $f \in \mathcal{D}_g(\mathcal{L})$. The process:

$$\left((fg)(Y_t) - (fg)(x) - \int_0^t ds\ \mathcal{L}(fg)(Y_s),\ t \geq 0\right)$$

is a P_x martingale by definition of the generator $\mathcal{L}$. Thus, the process:

$$\left(\frac{(fg)(Y_t)}{g(x)} - f(x) - \int_0^t ds\ \frac{\mathcal{L}(fg)(Y_s)}{g(x)},\ t \geq 0\right)$$

is a P_x martingale. We set:

$$\begin{aligned} M_t^{f,g} &= \mathrm{e}^{-\int_0^t ds\ (\mathcal{L}g/g)(Y_s)}\,\frac{(fg)(Y_t)}{g(x)} - f(x) \\ &\quad - \int_0^t ds\ \mathrm{e}^{-\int_0^s dr\ (\mathcal{L}g/g)(Y_r)} \left[\frac{\mathcal{L}(fg)(Y_s)}{g(x)} - \frac{\mathcal{L}(g)(Y_s)}{g(Y_s)}\,\frac{(fg)(Y_s)}{g(x)}\right]. \end{aligned} \tag{3.1}$$

Itô's lemma then yields that the process $(M_t^{f,g}, t \geq 0)$ is another P_x martingale. Notice this is a true martingale since it is bounded on bounded time intervals from our assumptions on f and g. Remark also that the constant function equal to 1 is in $\mathcal{D}_g(\mathcal{L})$. This choice of f yields the result. □

Let P_x^g denote the probability measure on $(D, \mathcal{D})$ defined by:

$$\forall t \geq 0, \quad \frac{d\mathrm{P}^g_{x\,|\mathcal{D}_t}}{d\mathrm{P}_{x\,|\mathcal{D}_t}} = \frac{g(Y_t)}{g(x)}\,\mathrm{e}^{-\int_0^t ds\ (\mathcal{L}g/g)(Y_s)}\,. \tag{3.2}$$

Note that in the case where g is harmonic for the linear operator $\mathcal{L}$ (that is $\mathcal{L}g = 0$), the probability distribution P^g is the usual Doob h-transform of P for $h = g$.

We also introduce the generator $\mathcal{L}^g$ of the canonical process Y under P^g and the expectation operator E^g associated to P^g.

Lemma 3.2. *Let g be a positive function of $\mathcal{D}(\mathcal{L})$ such that g is bounded from below by a positive constant. Then, we have $\mathcal{D}_g(\mathcal{L}) \subset \mathcal{D}(\mathcal{L}^g)$ and*

$$\forall u \in \mathcal{D}_g(\mathcal{L}), \quad \mathcal{L}^g(u) = \frac{\mathcal{L}(gu) - \mathcal{L}(g)u}{g}.$$

Proof. As, for $f \in \mathcal{D}_g(\mathcal{L})$, the process $(M_t^{f,g}, t \geq 0)$ defined by (3.1) is a martingale under P_x, we get that the process:

$$f(Y_t) - f(x) - \int_0^t ds \left(\frac{\mathcal{L}(fg)(Y_s) - \mathcal{L}(g)(Y_s) f(Y_s)}{g(Y_s)} \right), \quad t \geq 0$$

is a P_x^g martingale. This gives the result. □

Remark 3.3. *Let $((t,x) \to g(t,x))$ be a function bounded from below by a positive constant, differentiable in t, such that $g(t,.) \in \mathcal{D}(\mathcal{L})$ for each t and $((t,x) \to \partial_t g(t,x))$ is bounded from above. By considering the process (t, Y_t) instead of Y_t, we have the immediate counterpart of Lemma 3.1 for time dependent function $g(t,.)$. In particular, we may define the following probability measure on $(D, \mathcal{D})$ (still denoted P_x^g by a small abuse of notations):*

$$\forall t \geq 0, \quad \frac{d\mathrm{P}^g_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{g(t, Y_t)}{g(0,x)} \, \mathrm{e}^{-\int_0^t ds \ \frac{\mathcal{L}g + \partial_t g}{g}(s, Y_s)}, \tag{3.3}$$

where $\mathcal{L}$ acts on g as a function of x.

We now define the h-transform for super-processes, as introduced in [16] (notice this does not correspond to the Doob h-transform for super-processes).

Definition 3.4. *Let $X = (X_t, t \geq 0)$ be an $(\mathcal{L}, \beta, \alpha)$ super-process. For $g \in b\mathcal{S}^+$, we define the h-transform of X (with $h = g$) as $X^g = (X_t^g, t \geq 0)$ the measure valued process given for all $t \geq 0$ by:*

$$X_t^g(dx) = g(x) X_t(dx). \tag{3.4}$$

Note that (3.4) holds pointwise, and that the law of the h-transform of a super-process may be singular with respect to the law of the initial super-process.

We first give an easy generalization of a result in section 2 of [16] for a general spatial motion.

Proposition 3.5. *Let g be a positive function of $\mathcal{D}(\mathcal{L})$ such that g is bounded from below by a positive constant. Then the process X^g is a $\left(\mathcal{L}^g, \frac{(-\mathcal{L}+\beta)g}{g}, \alpha g\right)$-super-process.*

Proof. The Markov property of X^g is clear. We compute, for $f \in b\mathcal{S}^+$:

$$\mathbb{E}_x[\mathrm{e}^{-X_t^g(f)}] = \mathbb{E}_{\delta_x/g(x)}[\mathrm{e}^{-X_t(fg)}] = \mathrm{e}^{-u_t(x)/g(x)},$$

where, by Theorem 2.2, u satisfies:

$$u_t(x) + \mathrm{E}_x\left[\int_0^t dr \ \psi(Y_r, u_{t-r}(Y_r))\right] = \mathrm{E}_x\big[(fg)(Y_t)\big], \tag{3.5}$$

which may also be written:

$$u_t(x) + \mathrm{E}_x\left[\int_0^s dr \ \psi(Y_r, u_{t-r}(Y_r))\right] + \mathrm{E}_x\left[\int_s^t dr \ \psi(Y_r, u_{t-r}(Y_r))\right] = \mathrm{E}_x\big[(fg)(Y_t)\big].$$

But (3.5) written at time $t - s$ gives:

$$u_{t-s}(x) + \mathrm{E}_x\left[\int_0^{t-s} dr \ \psi(Y_r, u_{t-s-r}(Y_r))\right] = \mathrm{E}_x\big[(fg)(Y_{t-s})\big].$$

By comparing the two previous equations, we get:

$$u_t(x) + \mathrm{E}_x\Big[\int_0^s dr\ \psi(Y_r, u_{t-r}(Y_r))\Big] = \mathrm{E}_x\big[u_{t-s}(Y_s)\big],$$

and the Markov property now implies that the process:

$$u_{t-s}(Y_s) - \int_0^s dr\ \psi(Y_r, u_{t-r}(Y_r))$$

with $s \in [0,t]$ is a P_x martingale. Itô's lemma now yields that the process:

$$u_{t-s}(Y_s)\,\mathrm{e}^{-\int_0^s dr(\mathcal{L}g/g)(Y_r)} - \int_0^s dr\ \mathrm{e}^{-\int_0^r du\ (\mathcal{L}g/g)(Y_u)}\big(\psi(Y_r, u_{t-r}(Y_r)) - (\mathcal{L}g/g)(Y_r)\,u_{t-r}(Y_r)\big)$$

with $s \in [0,t]$ is another P_x martingale (the integrability comes from the assumption $\mathcal{L}g \in \mathcal{C}_b$ and $1/g \in \mathcal{C}_b$). Taking expectations at time $s = 0$ and at time $s = t$, we have:

$$u_t(x) + \mathrm{E}_x\Big[\int_0^t ds\ \mathrm{e}^{-\int_0^s dr(\mathcal{L}g/g)(Y_r)}\big(\psi(Y_s, u_{t-s}(Y_s)) - (\mathcal{L}g/g)(Y_s)u_{t-s}(Y_s)\big)\Big] = \mathrm{E}_x\Big[\mathrm{e}^{-\int_0^t dr(\mathcal{L}g/g)(Y_r)}(fg)(Y_t)\Big].$$

We divide both sides by $g(x)$ and expand ψ according to its definition:

$$\big(\frac{u_t}{g}\big)(x) + \mathrm{E}_x\Big[\int_0^t ds\ \frac{g(Y_s)}{g(x)}\,\mathrm{e}^{-\int_0^s dr(\mathcal{L}g/g)(Y_r)}\Big((\alpha g)(Y_s)\big(\frac{u_{t-s}}{g}\big)^2(Y_s) + (\beta - \frac{\mathcal{L}g}{g})(Y_s)\big(\frac{u_{t-s}}{g}\big)(Y_s)\Big)\Big] = \mathrm{E}_x\Big[\frac{g(Y_t)}{g(x)}\,\mathrm{e}^{-\int_0^t dr(\mathcal{L}g/g)(Y_r)}\,f(Y_t)\Big].$$

By definition of P_x^g from (3.2), we get that:

$$\big(\frac{u_t}{g}\big)(x) + \mathrm{E}_x^g\Big[\int_0^t ds\ \Big((\alpha g)(Y_s)\big(\frac{u_{t-s}}{g}\big)^2(Y_s) + (\beta - \frac{\mathcal{L}g}{g})(Y_s)\big(\frac{u_{t-s}}{g}\big)(Y_s)\Big]\Big) = \mathrm{E}_x^g\big[f(Y_t)\big].$$

We conclude from Theorem 2.2 that X^g is a $(\mathcal{L}^g, \frac{(-\mathcal{L}+\beta)g}{g}, \alpha g)$-super-process. □

In order to perform the h-transform of interest, we shall consider the following assumption.

$(H2)$ $1/\alpha$ **belongs to** $\mathcal{D}(\mathcal{L})$**.**

Notice that $(H2)$ implies that $\alpha\mathcal{L}(1/\alpha) \in \mathcal{C}_b$. Proposition 3.5 and Lemma 3.1 then yield the following Corollary.

Corollary 3.6. *Let X be an $(\mathcal{L}, \beta, \alpha)$-super-process. Assume $(H2)$. The process $X^{1/\alpha}$ is an $(\tilde{\mathcal{L}}, \tilde{\beta}, 1)$-super-process with:*

$$\tilde{\mathcal{L}} = \mathcal{L}^{1/\alpha} \quad \text{and} \quad \tilde{\beta} = \beta - \alpha\mathcal{L}(1/\alpha). \tag{3.6}$$

Moreover, for all $t \geq 0$, the law $\tilde{\mathrm{P}}_x$ of the process Y with generator $\tilde{\mathcal{L}}$ is absolutely continuous on $\mathcal{D}_t$ with respect to P_x and its Radon-Nikodym derivative is given by:

$$\frac{d\tilde{\mathrm{P}}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{\alpha(x)}{\alpha(Y_t)}\,\mathrm{e}^{\int_0^t ds\ (\tilde{\beta}-\beta)(Y_s)}. \tag{3.7}$$

We will note $\tilde{\mathbb{P}}$ for the law of $X^{1/\alpha}$ on the canonical space (that is $\tilde{\mathbb{P}} = \mathbb{P}^{(\tilde{\mathcal{L}},\tilde{\beta},1)}$) and $\tilde{\mathbb{N}}$ for its canonical measure. Observe that the branching mechanism of X under $\tilde{\mathbb{P}}$, which we shall write $\tilde{\psi}$, is given by:

$$\tilde{\psi}(x,\lambda) = \tilde{\beta}(x)\ \lambda + \lambda^2, \tag{3.8}$$

and the quadratic coefficient is no more dependent on x. Notice that $\mathbb{P}_{\alpha\nu}(X \in \cdot) = \tilde{\mathbb{P}}_\nu(\alpha X \in \cdot)$. This implies the following relationship on the canonical measures (use Theorem 2.3 to check it):

$$\alpha(x)\mathbb{N}_x[X \in \cdot] = \tilde{\mathbb{N}}_x[\alpha X \in \cdot]. \tag{3.9}$$

Recall that $v_t(x) = \mathbb{N}_x[H_{\max} > t] = \mathbb{N}_x[X_t \neq 0]$. We set $\tilde{v}_t(x) = \tilde{\mathbb{N}}_x[X_t \neq 0]$. As α is positive, equality (3.9) implies in particular that, for all $t > 0$ and $x \in S$:

$$\alpha(x)v_t(x) = \tilde{v}_t(x). \tag{3.10}$$

3.2 A Girsanov type theorem

The following assumption will be used to perform the Girsanov change of measure.

$(H3)$ **Assumption** $(H2)$ **holds and the function** $\tilde{\beta}$ **defined in (3.6) is in** $\mathcal{D}(\tilde{\mathcal{L}})$**, with** $\tilde{\mathcal{L}}$ **defined in (3.6).**

For $z \in \mathbb{R}$, we set $z_+ = \max(z, 0)$. Under $(H2)$ and $(H3)$, we define:

$$\beta_0 = \sup_{x\in S} \max\left(\tilde{\beta}(x), \sqrt{(\tilde{\beta}^2(x) - 2\tilde{\mathcal{L}}(\tilde{\beta})(x))_+}\right) \quad \text{and} \quad q(x) = \frac{\beta_0 - \tilde{\beta}(x)}{2}\cdot \tag{3.11}$$

Notice that $q \geq 0$.

We shall consider the distribution of the homogeneous $(\tilde{\mathcal{L}}, \beta_0, 1)$-super-process, which we will denote by $\mathbb{P}^0$ ($\mathbb{P}^0 = \mathbb{P}^{(\tilde{\mathcal{L}},\beta_0,1)}$) and its canonical measure $\mathbb{N}^0$. Note that the branching mechanism of X under $\mathbb{P}^0$ is homogeneous (the branching mechanism does not depend on x). We set ψ^0 for $\psi^{\beta_0,1}$. Since ψ^0 does not depend anymore on x we shall also write $\psi^0(\lambda)$ for $\psi^0(x,\lambda)$:

$$\psi^0(\lambda) = \beta_0\lambda + \lambda^2. \tag{3.12}$$

Proposition 3.7 below is a Girsanov's type theorem which allows us to finally reduce the distribution $\tilde{\mathbb{P}}$ to the homogeneous distribution $\mathbb{P}^0$. We introduce the process $M = (M_t, t \geq 0)$ defined by:

$$M_t = \exp\left(X_0(q) - X_t(q) - \int_0^t ds\ X_s(\varphi)\right), \tag{3.13}$$

where the function φ is defined by:

$$\varphi(x) = \tilde{\psi}(x, q(x)) - \tilde{\mathcal{L}}(q)(x), \quad x \in S. \tag{3.14}$$

Proposition 3.7 (Girsanov's transformation)**.** *Assume* $(H2)$ *and* $(H3)$ *hold. Let* X *be a* $(\tilde{\mathcal{L}}, \tilde{\beta}, 1)$*-super-process.*

(i) The process M *is a bounded* $\mathcal{F}$*-martingale under* $\tilde{\mathbb{P}}_\nu$ *which converges a.s. to*

$$M_\infty = \mathrm{e}^{X_0(q) - \int_0^{+\infty} ds\ X_s(\varphi)}\ \mathbf{1}_{\{H_{\max}<+\infty\}}.$$

(ii) We have:

$$\frac{d\mathbb{P}^0_\nu}{d\tilde{\mathbb{P}}_\nu} = M_\infty.$$

(iii) If moreover $(H1)$ *holds, then* $\mathbb{P}^0_\nu$*-a.s. we have* $M_\infty > 0$*, the probability measure* $\tilde{\mathbb{P}}_\nu$ *is absolutely continuous with respect to* $\mathbb{P}^0_\nu$ *on* $\mathcal{F}$*:*

$$\frac{d\tilde{\mathbb{P}}_\nu}{d\mathbb{P}^0_\nu} = \frac{1}{M_\infty}, \quad \textit{and} \quad \frac{d\tilde{\mathbb{N}}_x}{d\mathbb{N}^0_x} = \mathrm{e}^{\int_0^{+\infty} ds\ X_s(\varphi)}.$$

We also have:

$$q(x) = \mathbb{N}^0_x \left[\mathrm{e}^{\int_0^{+\infty} ds\ X_s(\varphi)} - 1\right]. \tag{3.15}$$

The two first points are a particular case of Theorem IV.1.6 p.252 in [23] on interactive drift. For the sake of completeness, we give a proof based on the mild form of the Log Laplace equation (2.3) introduced in Section 2. Notice that:

$$\psi^0(\lambda) = \tilde{\psi}(x, \lambda + q(x)) - \tilde{\psi}(x, q(x)). \tag{3.16}$$

Thus, Proposition 3.7 appears as a non-homogeneous generalization of Corollary 4.4 in [1]. We first give an elementary Lemma.

Lemma 3.8. *Assume* $(H2)$ *and* $(H3)$ *hold. The function* φ *defined by (3.14) is non-negative.*

Proof. The following computation:

$$\begin{aligned}\varphi(x) = \tilde{\psi}(x, q(x)) - \tilde{\mathcal{L}}(q)(x) &= q(x)^2 + \tilde{\beta} q(x) - \tilde{\mathcal{L}}(q)(x)\\ &= \left(\frac{\beta_0 - \tilde{\beta}(x)}{2}\right)^2 + \tilde{\beta}(x)\ \frac{\beta_0 - \tilde{\beta}(x)}{2} - \tilde{\mathcal{L}}(q)(x)\\ &= \frac{\beta_0^2 - \tilde{\beta}^2(x) + 2\tilde{\mathcal{L}}(\tilde{\beta})(x)}{4}\end{aligned}$$

and the definition (3.11) of β_0 ensure that the function φ is non-negative. □

Proof of Proposition 3.7. First observe that M is $\mathcal{F}$-adapted. As the function q also is non-negative, we deduce from Lemma 3.8 that the process M is bounded by $\mathrm{e}^{X_0(q)}$.

Let $f \in b\mathcal{S}^+$. On the one hand, we have:

$$\tilde{\mathbb{E}}_x[M_t\, \mathrm{e}^{-X_t(f)}] = \tilde{\mathbb{E}}_x[\mathrm{e}^{q(x) - X_t(q+f) - \int_0^t ds\ X_s(\varphi)}] = \mathrm{e}^{q(x) - r_t(x)},$$

where, according to Theorem 2.2, $r_t(x)$ is bounded on $[0,T]\times S$ for all $T > 0$ and satisfies:

$$\begin{aligned} r_t(x) + \tilde{\mathrm{E}}_x\left[\int_0^t ds\ \tilde{\psi}(Y_{t-s}, r_s(Y_{t-s}))\right]&\\ = \tilde{\mathrm{E}}_x\left[\int_0^t ds\ \left(\tilde{\psi}(Y_{t-s}, q(Y_{t-s})) - \tilde{\mathcal{L}}(q)(Y_{t-s})\right) + (q+f)(Y_t)\right]&. \end{aligned} \tag{3.17}$$

On the other hand, we have

$$\mathbb{E}^0_x[\mathrm{e}^{-X_t(f)}] = \mathrm{e}^{-w_t(x)},$$

where $w_t(x)$ is bounded on $[0,T] \times S$ for all $T > 0$ and satisfies:

$$w_t(x) + \tilde{\mathrm{E}}_x\left[\int_0^t ds\ \psi^0(Y_{t-s}, w_s(Y_{t-s}))\right] = \tilde{\mathrm{E}}_x[f(Y_t)].$$

Using (3.16), rewrite the previous equation under the form:

$$w_t(x) + \tilde{\mathrm{E}}_x\left[\int_0^t ds\ \tilde{\psi}(Y_{t-s}, (w_s + q)(Y_{t-s}))\right] = \tilde{\mathrm{E}}_x\left[\int_0^t ds\ \tilde{\psi}(Y_{t-s}, q(Y_{t-s})) + f(Y_t)\right]. \tag{3.18}$$

We now make use of the Dynkin's formula with $(H3)$:

$$q(x) = -\tilde{\mathbb{E}}_x\left[\int_0^t \tilde{\mathcal{L}}(q)(Y_s)\right] + \tilde{\mathbb{E}}_x[q(Y_t)], \tag{3.19}$$

and sum the equations (3.18) and (3.19) term by term to get:

$$\begin{aligned}(w_t+q)(x) + \tilde{\mathbb{E}}_x\left[\int_0^t ds\ \tilde{\psi}(Y_{t-s},(w_s+q)(Y_{t-s}))\right]\\ = \tilde{\mathbb{E}}_x\left[\int_0^t ds\ \left(\tilde{\psi}(Y_{t-s},q(Y_{t-s})) - \tilde{\mathcal{L}}(q)(Y_{t-s})\right) + (q+f)(Y_t)\right].\end{aligned} \tag{3.20}$$

The functions $r_t(x)$ and $w_t(x)+q(x)$ are bounded on $[0,T]\times S$ for all $T>0$ and satisfy the same equation, see equations (3.17) and (3.20). By uniqueness, see Theorem 2.1, we finally get that $w_t+q=r_t$. This gives:

$$\tilde{\mathbb{E}}_x[M_t\,\mathrm{e}^{-X_t(f)}] = \mathbb{E}^0_x[\mathrm{e}^{-X_t(f)}]. \tag{3.21}$$

The Poissonian decomposition of the super-processes, see Theorem 2.3, and the exponential formula enable us to extend this relation to arbitrary initial measures ν:

$$\tilde{\mathbb{E}}_\nu[M_t\,\mathrm{e}^{-X_t(f)}] = \mathbb{E}^0_\nu[\mathrm{e}^{-X_t(f)}]. \tag{3.22}$$

This equality with $f=0$ and the Markov property of X proves the first part of item (i).

Now, a direct induction based on the Markov property yields that, for all positive integer n, and $f_1,\dots,f_n\in b\mathcal{S}^+$, $0\le s_1\le\ldots\le s_n\le t$:

$$\tilde{\mathbb{E}}_\nu[M_t\,\mathrm{e}^{-\sum_{1\le i\le n}X_{s_i}(f_i)}] = \mathbb{E}^0_\nu[\mathrm{e}^{-\sum_{1\le i\le n}X_{s_i}(f_i)}]. \tag{3.23}$$

And we conclude with an application of the monotone class theorem that, for all non-negative $\mathcal{F}_t$-measurable random variable Z:

$$\tilde{\mathbb{E}}_\nu[M_tZ] = \mathbb{E}^0_\nu[Z].$$

The martingale M is bounded and thus converges a.s. to a limit M_∞. We deduce that for all non-negative $\mathcal{F}_t$-measurable random variable Z:

$$\tilde{\mathbb{E}}_\nu[M_\infty Z] = \mathbb{E}^0_\nu[Z]. \tag{3.24}$$

This also holds for any non-negative $\mathcal{F}_\infty$-measurable random variable Z. This gives the second item (ii).

On $\{H_{\max}<+\infty\}$, then clearly M_t converges to $\mathrm{e}^{X_0(q)-\int_0^{+\infty}ds\ X_s(\varphi)}$. Notice that $\mathbb{P}^0_\nu(H_{\max}=+\infty)=0$. We deduce from (3.24) with $Z=\mathbf{1}_{\{H_{\max}=+\infty\}}$ that $\tilde{\mathbb{P}}_\nu$-a.s. on $\{H_{\max}=+\infty\}$, $M_\infty=0$. This gives the last part of item (i).

Now, we prove the third item (iii). Notice that (3.24) implies that $\mathbb{P}^0_\nu$-a.s. $M_\infty>0$. Thanks to $(H1)$, we also have that $\tilde{\mathbb{P}}_\nu$-a.s. $M_\infty>0$. Let Z be a non-negative $\mathcal{F}_\infty$-measurable random variable. Applying (3.24) with Z replaced by $\mathbf{1}_{\{M_\infty>0\}}Z/M_\infty$, we get:

$$\tilde{\mathbb{E}}_\nu[Z] = \tilde{\mathbb{E}}_\nu\left[M_\infty\mathbf{1}_{\{M_\infty>0\}}\frac{Z}{M_\infty}\right] = \mathbb{E}^0_\nu\left[\frac{Z}{M_\infty}\mathbf{1}_{\{M_\infty>0\}}\right] = \mathbb{E}^0_\nu\left[\frac{Z}{M_\infty}\right].$$

This gives the first part of item (iii).

Notice that for all positive integer n, and $f_1,\dots,f_n\in b\mathcal{S}^+$, $0\le s_1\le\ldots\le s_n$, we have

$$\begin{aligned}\tilde{\mathbb{N}}_x\left[1-\mathrm{e}^{\sum_{1\le i\le n}X_{s_i}(f_i)}\right] &= -\log\left(\tilde{\mathbb{E}}_x\left[\mathrm{e}^{\sum_{1\le i\le n}X_{s_i}(f_i)}\right]\right)\\ &= -\log\left(\mathbb{E}^0_x\left[\mathrm{e}^{\sum_{1\le i\le n}X_{s_i}(f_i)+\int_0^{+\infty}ds\ X_s(\varphi)}\right]\right)+q(x)\\ &= \mathbb{N}^0_x\left[1-\mathrm{e}^{\sum_{1\le i\le n}X_{s_i}(f_i)+\int_0^{+\infty}ds\ X_s(\varphi)}\right]+q(x).\end{aligned}$$

Taking $f_i = 0$ for all i gives (3.15). This implies:

$$\tilde{\mathbb{N}}_x \left[1 - e^{\sum_{1\leq i\leq n} X_{s_i}(f_i)}\right] = \mathbb{N}^0_x \left[e^{\int_0^{+\infty} ds\ X_s(\varphi)} \left(1 - e^{\sum_{1\leq i\leq n} X_{s_i}(f_i)}\right)\right].$$

The monotone class theorem gives then the second part of item (iii). □

3.3 Genealogy for super-processes

We now recall the genealogy of X under $\mathbb{P}^0$ given by the Brownian snake from [11]. We assume $(H2)$ and $(H3)$ hold.

Let $\mathcal{W}$ denote the set of all càdlàg killed paths in S. An element $w \in \mathcal{W}$ is a càdlàg path: $w : [0, \eta(w)) \to S$, with $\eta(w)$ the lifetime of the path w. By convention the trivial path $\{x\}$, with $x \in S$, is a killed path with lifetime 0 and it belongs to $\mathcal{W}$. The space $\mathcal{W}$ is Polish for the distance:

$$d(w, w') = \delta(w(0), w(0)') + |\eta(w) - \eta(w')| + \int_0^{\eta(w)\wedge\eta(w')} ds\ d_s(w_{[0,s]}, w'_{[0,s]}),$$

where d_s refers to the Skorokhod metric on the space $D([0, s], S)$, and w_I is the restriction of w on the interval I. Denote $\mathcal{W}_x$ the set of stopped paths w such that $w(0) = x$. We work on the canonical space of continuous applications from $[0, \infty)$ to $\mathcal{W}$, denoted by $\bar{\Omega} := \mathcal{C}(\mathbb{R}^+, \mathcal{W})$, endowed with the Borel sigma field $\bar{\mathcal{G}}$ for the distance d, and the canonical right continuous filtration $\bar{\mathcal{G}}_t = \sigma\{W_s, s \leq t\}$, where $(W_s, s \in \mathbb{R}^+)$ is the canonical coordinate process. Notice $\bar{\mathcal{G}} = \bar{\mathcal{G}}_\infty$ by construction. We set $H_s = \eta(W_s)$ the lifetime of W_s.

Definition 3.9 (Proposition 4.1.1 and Theorem 4.1.2 of [11]). *Fix $W_0 \in \mathcal{W}_x$. There exists a unique $\mathcal{W}_x$-valued Markov process $W = (\bar{\Omega}, \bar{\mathcal{G}}, \bar{\mathcal{G}}_t, W_t, \mathbf{P}^0_{W_0})$, called the Brownian snake, starting at W_0 and satisfying the two properties:*

(i) The lifetime process $H = (H_s, s \geq 0)$ is a reflecting Brownian motion with non-positive drift $-\beta_0$, starting from $H_0 = \eta(W_0)$.

(ii) Conditionally given the lifetime process H, the process $(W_s, s \geq 0)$ is distributed as a non-homogeneous Markov process, with transition kernel specified by the two following prescriptions, for $0 \leq s \leq s'$:

- *$W_{s'}(t) = W_s(t)$ for all $t < H_{[s,s']}$, with $H_{[s,s']} = \inf_{s\leq r\leq s'} H_r$.*
- *Conditionally on $W_s(H_{[s,s']}-)$, the path $\left(W_{s'}(H_{[s,s']} + t), 0 \leq t < H_{s'} - H_{[s,s']}\right)$ is independent of W_s and is distributed as $Y_{[0,H_{s'}-H_{[s,s']})}$ under $\tilde{\mathrm{P}}_{W_s(H_{[s,s']}-)}$.*

This process will be called the β_0-snake started at W_0, and its law denoted by $\mathbf{P}^0_{W_0}$.

We will just write $\mathbf{P}^0_x$ for the law of the snake started at the trivial path $\{x\}$. The corresponding excursion measure $\mathbf{N}^0_x$ of W is given as follows: the lifetime process H is distributed according to the Itô measure of the positive excursion of a reflecting Brownian motion with non-positive drift $-\beta_0$, and conditionally given the lifetime process H, the process $(W_s, s \geq 0)$ is distributed according to (ii) of Definition 3.9. Let

$$\sigma = \inf\{s > 0; H_s = 0\}$$

denote the length of the excursion under $\mathbf{N}^0_x$.

Let $(l^r_s, r \geq 0, s \geq 0)$ be the bicontinuous version of the local time process of H; where l^r_s refers to the local time at level r at time s. We also set $\hat{w} = w(\eta(w)-)$ for the left end position of the path w. We consider the measure valued process $X(W) = (X_t(W), t \geq 0)$ defined under $\mathbf{N}^0_x$ by:

$$X_t(W)(dx) = \int_0^\sigma d_s l^t_s\ \delta_{\hat{W}_s}(dx). \tag{3.25}$$

The β_0-snake gives the genealogy of the $(\tilde{\mathcal{L}}, \beta_0, 1)$ super-process in the following sense.

Proposition 3.10 ([11], Theorem 4.2.1). *We have:*

- *The process $X(W)$ is under $\mathbf{N}_x^0$ distributed as X under $\mathbb{N}_x^0$.*
- *Let $\sum_{j\in\mathcal{J}} \delta_{(x^j, W^j)}$ be a Poisson point measure on $S\times\bar{\Omega}$ with intensity $\nu(dx)\, \mathbf{N}_x^0[dW]$. Then $\sum_{j\in\mathcal{J}} X(W^j)$ is an $(\tilde{\mathcal{L}}, \beta_0, 1)$-super-process started at ν.*

Notice that, under $\mathbf{N}_x^0$, the extinction time of $X(W)$ is defined by

$$\inf\{t; X_t(W) = 0\} = \sup_{s\in[0,\sigma]} H_s,$$

and we shall write this quantity $H_{\max}$ or $H_{\max}(W)$ if we need to stress the dependence in W. This notation is coherent with (2.6).

We now transport the genealogy of X under $\mathbb{N}^0$ to a genealogy of X under $\tilde{\mathbb{N}}$. In order to simplify notations, we shall write X for $X(W)$ when there is no confusion.

Definition 3.11. *Under $(H1)$-$(H3)$, we define a measure $\tilde{\mathbf{N}}_x$ on $(\bar{\Omega}, \bar{\mathcal{G}})$ by:*

$$\forall W \in \bar{\Omega}, \quad \frac{d\tilde{\mathbf{N}}_x}{d\mathbf{N}_x^0}(W) = \frac{d\tilde{\mathbb{N}}_x}{d\mathbb{N}_x^0}(X(W)) = \frac{1}{M_\infty} = \mathrm{e}^{\int_0^{+\infty} ds\ X_s(\varphi)}.$$

Notice the second equality in the previous definition is the third item of Proposition 3.7.

At this point, the genealogy defined for X under $\tilde{\mathbb{N}}_x$ will give the genealogy of X under $\mathbb{N}$ up to a weight. We set

$$\mathbf{N}_x = \frac{1}{\alpha(x)}\,\tilde{\mathbf{N}}_x. \tag{3.26}$$

Proposition 3.12. *We have:*

(i) *$X(W)$ under $\tilde{\mathbf{N}}_x$ is distributed as X under $\tilde{\mathbb{N}}_x$.*

(ii) *The weighted process $X^{\text{weight}} = (X_t^{\text{weight}}, t \geq 0)$ with*

$$X_t^{\text{weight}}(dx) = \int_0^\sigma d_s l_s^t\ \alpha(\hat{W}_s)\delta_{\hat{W}_s}(dx), \quad t \geq 0, \tag{3.27}$$

is under $\mathbf{N}_x$ distributed as X under $\mathbb{N}_x$.

We may write $X^{\text{weight}}(W)$ for X^{weight} to emphasize the dependence in the snake W.

Proof. This is a direct consequence of Definition 3.11 and (3.9). □

We shall say that W under $\mathbf{N}_x$ provides through (3.27) a genealogy for X under $\mathbb{N}_x$.

4 A Williams decomposition

In Section 4.1, we give a decomposition of the genealogy of the super-processes $(\mathcal{L}, \beta, \alpha)$ and $(\tilde{\mathcal{L}}, \tilde{\beta}, 1)$ with respect to a randomly chosen individual. In Section 4.2, we give a Williams decomposition of the genealogy of the super-processes $(\mathcal{L}, \beta, \alpha)$ and $(\tilde{\mathcal{L}}, \tilde{\beta}, 1)$ with respect to the last individual alive.

4.1 Bismut's decomposition

A decomposition of the genealogy of the homogeneous super-process with respect to a randomly chosen individual is well known in the homogeneous case, even for a general branching mechanism (see lemmas 4.2.5 and 4.6.1 in [11]).

We now explain how to decompose the snake process under the excursion measure ($\tilde{\mathbf{N}}_x$ or $\mathbf{N}_x^0$) with respect to its value at a given time. Recall $\sigma = \inf\{s > 0, H_s = 0\}$ denote the length of the excursion. Fix a real number $t \in [0, \sigma]$. We consider the process $H^{(g)}$ (on the left of t) defined on $[0,t]$ by $H_s^{(g)} = H_{t-s} - H_t$ for all $s \in [0,t]$. The excursion intervals above 0 of the process $(H_s^{(g)} - \inf_{0\leq s'\leq s} H_{s'}^{(g)}, 0 \leq s \leq t)$ are denoted $\bigcup_{j\in J^{(g)}}(c_j, d_j)$. We also consider the process $H^{(d)}$ (on the right of t) defined on $[0, \sigma - t]$ by $H_s^{(d)} = H_{t+s} - H_t$. The excursion intervals above 0 of the process $(H_s^{(d)} - \inf_{0\leq s'\leq s} H_{s'}^{(d)}, 0 \leq s \leq \sigma - t)$ are denoted $\bigcup_{j\in J^{(d)}}(c_j, d_j)$. We define the level of the excursion j as $s_j = H_{t-c_j}$ if $j \in J^{(g)}$ and $s_j = H_{t+c_j}$ if $j \in J^{(d)}$. We also define for the excursion j the corresponding excursion of the snake: $W^j = (W_s^j, s \geq 0)$ as

$$W_s^j(.) = W_{t-(c_j+s)\wedge d_j}(. + s_j) \quad \text{if } j \in J^{(g)}, \text{ and} \quad W_s^j(.) = W_{t+(c_j+s)\wedge d_j}(. + s_j) \quad \text{if } j \in J^{(d)}.$$

We consider the following two point measures on $\mathbb{R}^+ \times \bar{\Omega}$: for $\varepsilon \in \{g, d\}$,

$$R_t^{\varepsilon} = \sum_{j\in J^{(\varepsilon)}} \delta_{(s_j, W^j)}. \tag{4.1}$$

Notice that under $\mathbf{N}_x^0$ (and under $\tilde{\mathbf{N}}_x$ if $(H1)$ holds), the process W can be reconstructed from the triplet (W_t, R_t^g, R_t^d) as follows. If $s \in [0,t]$, then we define, for $j \in J^{(g)}$, $c_j = \sum_{i\in J^{(g)}} \sigma^i \mathbf{1}_{\{s_i > s_j\}}$ and $d_j = c_j + \sigma^j$ where $\sigma^i = \inf\{s > 0; \forall t > s, W_t^i = 0\}$. Then there exists $j \in J^{(g)}$ such that $(t - s) \in [c_j, d_j]$ and we have:

$$W_s(u) = W^j_{(t-s)-c_j}(u - s_j) \text{ if } u > s_j \text{ and } W_s(u) = W_t(u) \text{ if } u \leq s_j.$$

If $s \in [t, \sigma]$, then for $j \in J^{(d)}$, $c_j = \sum_{i\in J^{(d)}} \sigma^i \mathbf{1}_{\{s_i > s_j\}}$ and $d_j = c_j + \sigma^j$. Then there exists $j \in J^{(d)}$ such that $(s - t) \in [c_j, d_j]$ and we have:

$$W_s(u) = W^j_{(s-t)-c_j}(u - s_j) \text{ if } u > s_j \text{ and } W_s(u) = W_t(u) \text{ if } u \leq s_j.$$

We are interested in the probabilistic structure of this triplet, when t is chosen according to the Lebesgue measure on the excursion time interval of the snake. Under $\mathbf{N}^0$, this result is as a consequence of Lemmas 4.2.4 and 4.2.5 from [11]. We recall this result in the next Proposition.

For a point measure $R = \sum_{j\in J} \delta_{(s_j, x_j)}$ on a space $\mathbb{R} \times \mathcal{X}$ and $A \subset \mathbb{R}$, we shall consider the restriction of R to $A \times \mathcal{X}$ given by $R_A = \sum_{j\in J} \mathbf{1}_A(s_j)\delta_{(s_j,x_j)}$.

Proposition 4.1 ([11], Lemmas 4.2.4 and 4.2.5)**.** *For every measurable non-negative function F, the following formulas hold:*

$$\mathbf{N}_x^0\left[\int_0^{\sigma} ds\ F(W_s, R_s^g, R_s^d)\right] = \int_0^{\infty} \mathrm{e}^{-\beta_0 r}\, dr\ \tilde{\mathrm{E}}_x\left[F(Y_{[0,r)}, \hat{R}^{B,g}_{[0,r)}, \hat{R}^{B,d}_{[0,r)})\right], \tag{4.2}$$

$$\mathbf{N}_x^0\left[\int_0^{\sigma} d_s l_s^t\ F(W_s, R_s^g, R_s^d)\right] = \mathrm{e}^{-\beta_0 t}\ \tilde{\mathrm{E}}_x\left[F(Y_{[0,t)}, \hat{R}^{B,g}_{[0,t)}, \hat{R}^{B,d}_{[0,t)})\right], \quad t > 0, \tag{4.3}$$

where under $\tilde{\mathrm{E}}_x$ and conditionally on Y, $\hat{R}^{B,g}$ and $\hat{R}^{B,d}$ are two independent Poisson point measures with intensity $\hat{\nu}^B(ds, dW) = ds\ \mathbf{N}^0_{Y_s}[dW]$.

The next Proposition gives a similar result in the non-homogeneous case.

Proposition 4.2. *Under $(H1)$-$(H3)$, for every measurable non-negative function F, the two formulas hold:*

$$\tilde{\mathbf{N}}_x\left[\int_0^\sigma ds\ F(W_s, R_s^g, R_s^d)\right] = \int_0^\infty dr\ \tilde{\mathrm{E}}_x\left[\mathrm{e}^{-\int_0^r ds\ \tilde{\beta}(Y_s)}\ F(Y_{[0,r)}, R^{B,g}_{[0,r)}, R^{B,d}_{[0,r)})\right], \tag{4.4}$$

where under $\tilde{\mathrm{E}}_x$ and conditionally on Y, $R^{B,g}_{[0,r)}$ and $R^{B,d}_{[0,r)}$ are two independent Poisson point measures with intensity

$$\nu^B(ds, dW) = ds\ \tilde{\mathbf{N}}_{Y_s}[dW] = ds\ \alpha(Y_s)\mathbf{N}_{Y_s}[dW]; \tag{4.5}$$

and

$$\mathbf{N}_x\left[\int_0^\sigma ds\ \alpha(\hat{W}_s)F(W_s, R_s^g, R_s^d)\right] = \int_0^\infty dr\ \mathrm{E}_x\left[\mathrm{e}^{-\int_0^r ds\ \beta(Y_s)}\ F(Y_{[0,r)}, R^{B,g}_{[0,r)}, R^{B,d}_{[0,r)})\right], \tag{4.6}$$

where under E_x and conditionally on Y, $R^{B,g}_{[0,r)}$ and $R^{B,d}_{[0,r)}$ are two independent Poisson point measures with intensity ν^B.

Observe there is a weight $\alpha(\hat{W}_s)$ in (4.6) (see also (3.27) where this weight appears) which modifies the law of the individual picked at random, changing the modified diffusion $\tilde{\mathrm{P}}_x$ in (4.4) into the original one P_x.

We shall use the following elementary Lemma on Poisson point measures.

Lemma 4.3. *Let R be a Poisson point measure on a Polish space with intensity ν. Let f be a non-negative measurable function f such that $\nu(\mathrm{e}^f - 1) < +\infty$. Then for any non-negative measurable function F, we have:*

$$\mathbb{E}\left[F(R)\,\mathrm{e}^{R(f)}\right] = \mathbb{E}\left[F(\tilde{R})\right]\mathrm{e}^{\nu(\mathrm{e}^f - 1)}, \tag{4.7}$$

where $\tilde{R}$ is a Poisson point measure with intensity $\tilde{\nu}(dx) = \mathrm{e}^{f(x)}\,\nu(dx)$.

Proof of Proposition 4.2. We keep notations introduced in Propositions 4.1. We have:

$$\begin{aligned}
&\tilde{\mathbf{N}}_x\left[\int_0^\sigma ds\ F(W_s, R_s^g, R_s^d)\right]\\
&\quad= \mathbf{N}_x^0\left[\mathrm{e}^{\int_0^{+\infty} ds\ X_s(\varphi)}\int_0^\sigma ds\ F(W_s, R_s^g, R_s^d)\right]\\
&\quad= \mathbf{N}_x^0\left[\int_0^\sigma ds\ F(W_s, R_s^g, R_s^d)\,\mathrm{e}^{(R_s^g + R_s^d)(f)}\right]\\
&\quad= \int_0^\infty \mathrm{e}^{-\beta_0 r}\,dr\ \tilde{\mathrm{E}}_x\left[F(Y_{[0,r)}, \hat{R}^{B,g}_{[0,r)}, \hat{R}^{B,d}_{[0,r)})\,\mathrm{e}^{(\hat{R}^{B,g}_{[0,r)} + \hat{R}^{B,d}_{[0,r)})(f)}\right]\\
&\quad= \int_0^\infty \mathrm{e}^{-\beta_0 r}\,dr\ \tilde{\mathrm{E}}_x\left[F(Y_{[0,r)}, R^{B,g}_{[0,r)}, R^{B,d}_{[0,r)})\,\mathrm{e}^{2\int_0^r ds\ \mathbf{N}^0_{Y_s}[\mathrm{e}^{\int_0^{+\infty} X_r(W)(\varphi)} - 1]}\right]\\
&\quad= \int_0^\infty \mathrm{e}^{-\beta_0 r}\,dr\ \tilde{\mathrm{E}}_x\left[F(Y_{[0,r)}, R^{B,g}_{[0,r)}, R^{B,d}_{[0,r)})\,\mathrm{e}^{2\int_0^r ds\ q(Y_s)}\right]\\
&\quad= \int_0^\infty dr\ \tilde{\mathrm{E}}_x\left[F(Y_{[0,r)}, R^{B,g}_{[0,r)}, R^{B,d}_{[0,r)})\,\mathrm{e}^{-\int_0^r ds\ \tilde{\beta}(Y_s)}\right],
\end{aligned}$$

where the first equality comes from $(H1)$ and item (iii) of Proposition 3.7, we set $f(s, W) = \int_0^{+\infty} X_r(W)(\varphi)$ for the second equality, we use Proposition 4.1 for the third equality, we use Lemma 4.3 for the fourth, we use (3.15) for the fifth, and the definition (3.11) of q in the last. This proves (4.4).

Then replace $F(W_s, R_s^g, R_s^d)$ by $\alpha(\hat{W}_s)F(W_s, R_s^g, R_s^d)$ in (4.4) and use (3.7) as well as (3.26) to get (4.6). □

The proof of the following Proposition is similar to the proof of Proposition 4.2 and is not reproduced here.

Proposition 4.4. *Under* $(H1)$-$(H3)$*, for every measurable non-negative function* F*, the two formulas hold: for fixed* $t>0$*,*

$$\tilde{\mathbf{N}}_x\left[\int_0^\sigma d_s l_s^t\ F(W_s, R_s^g, R_s^d)\right] = \tilde{\mathrm{E}}_x\left[\mathrm{e}^{-\int_0^t ds\ \tilde\beta(Y_s)}\ F(Y_{[0,t)}, R^{B,g}_{[0,t)}, R^{B,d}_{[0,t)})\right], \tag{4.8}$$

where under $\tilde{\mathrm{E}}_x$ *and conditionally on* Y*,* $R^{B,g}$ *and* $R^{B,d}$ *are two independent Poisson point measures with intensity* ν^B *defined in (4.5), and*

$$\mathbf{N}_x\left[\int_0^\sigma d_s l_s^t\ \alpha(\hat W_s)F(W_s, R_s^g, R_s^d)\right] = \mathrm{E}_x\left[\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}\ F(Y_{[0,t)}, R^{B,g}_{[0,t)}, R^{B,d}_{[0,t)})\right], \tag{4.9}$$

where under E_x *and conditionally on* Y*,* $R^{B,g}$ *and* $R^{B,d}$ *are two independent Poisson point measures with intensity* ν^B*.*

As an example of application of this Proposition, we easily recover the following well known result.

Corollary 4.5. *Under* $(H1)$-$(H3)$*, for every measurable non-negative functions* f *and* g *on* S*, we have:*

$$\mathbb{N}_x\left[X_t(f)\,\mathrm{e}^{-X_t(g)}\right] = \mathrm{E}_x\left[\mathrm{e}^{-\int_0^t ds\ \partial_\lambda\psi\left(Y_s,\ \mathbb{N}_{Y_s}\left[1-\mathrm{e}^{X_{t-s}(g)}\right]\right)}\ f(Y_t)\right].$$

In particular, we recover the so-called “many-to-one” formula (with $g=0$ in Corollary 4.5):

$$\mathbb{N}_x[X_t(f)] = \mathrm{E}_x\left[\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}\ f(Y_t)\right]. \tag{4.10}$$

Proof. We set for $w\in\mathcal{W}$ with $\eta(w)=t$ and r_1, r_2 two point measures on $\mathbb{R}^+\times\bar\Omega$

$$F(w, r_1, r_2) = f(\hat w)\,\mathrm{e}^{h(r_1)+h(r_2)},$$

where $h(\sum_{i\in I}\delta_{(s_i,W^i)}) = \sum_{s_i<t} X^{\text{weight}}(W^i)_{t-s_i}(g)$. We have:

$$\begin{aligned}
\mathbb{N}_x\left[X_t(f)\,\mathrm{e}^{-X_t(g)}\right] &= \mathbf{N}_x\left[\int_0^\sigma d_s l_s^t\ \alpha(\hat W_s)F(W_s, R_s^g, R_s^d)\right]\\
&= \mathrm{E}_x\left[\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}\ f(Y_t)\,\mathrm{e}^{h(R^{B,g}_{[0,r)})+h(R^{B,d}_{[0,r)})}\right]\\
&= \mathrm{E}_x\left[\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}\ f(Y_t)\,\mathrm{e}^{-\int_0^t 2\alpha(Y_s)\mathbf{N}_{Y_s}[1-\mathrm{e}^{X^{\text{weight}}_{t-s}(g)}]}\right]\\
&= \mathrm{E}_x\left[\mathrm{e}^{-\int_0^t ds\ \partial_\lambda\psi\left(Y_s,\ \mathbb{N}_{Y_s}\left[1-\mathrm{e}^{X_{t-s}(g)}\right]\right)}\ f(Y_t)\right],
\end{aligned}$$

where we used item (ii) of Proposition 3.12 for the first and last equality, (4.9) with F previously defined for the second, formula for exponentials of Poisson point measure and (3.26) for the third. □

Remark 4.6. *Equation (4.10) justifies the introduction of the following family of probability measures indexed by* $t\geq 0$*:*

$$\frac{d\mathrm{P}^{(B,t)}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}}{\mathrm{E}_x\left[\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}\right]}, \tag{4.11}$$

which may be understood as the law of the ancestral lineage of an individual sampled at random at height t under the excursion measure $\mathbb{N}_x$, and also correspond to Feynman-Kac penalization of the original spatial motion P_x (see [25]). Notice that this law does not depend on the parameter α. These probability measures are not compatible as t varies but will be shown in Lemma 5.12 to converge as $t \to \infty$ in restriction to $\mathcal{D}_s$, s fixed, $s \le t$, under some ergodic assumption, see Lemma 5.6.

4.2 Williams decomposition

We first recall Williams decomposition for the Brownian snake (see [28] for Brownian excursions, [27] for Brownian snake or [2] for general homogeneous branching mechanism without spatial motion).

Under the excursion measures $\mathbf{N}^0_x$, $\tilde{\mathbf{N}}_x$ and $\mathbf{N}_x$, recall that $H_{\max} = \sup_{[0,\sigma]} H_s$. Because of the continuity of H, it is possible to define $T_{\max} = \inf\{s > 0, H_s = H_{\max}\}$. Notice the properties of the Brownian excursions implies that a.e. $H_s = H_{\max}$ only if $s = T_{\max}$. We set $v^0_t(x) = \mathbb{N}^0_x[H_{\max} > t]$ and recall this function does not depend on x. Thus, we shall write v^0_t for $v^0_t(x)$. Standard computations give:

$$v^0_t = \frac{\beta_0}{\mathrm{e}^{\beta_0 t} - 1}.$$

The next result is a straightforward adaptation from Theorem 3.3 of [2] and gives the distribution of $(H_{\max}, W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})$ under $\mathbf{N}^0_x$.

Proposition 4.7 (Williams decomposition under $\mathbf{N}^0_x$)**.** *We have:*

(i) The distribution of $H_{\max}$ under $\mathbf{N}^0_x$ is characterized by: $\mathbf{N}^0_x[H_{\max} > h] = v^0_h$.

(ii) Conditionally on $\{H_{\max} = h_0\}$, $W_{T_{\max}}$ under $\mathbf{N}^0_x$ is distributed as $Y_{[0,h_0)}$ under $\tilde{\mathrm{P}}_x$.

(iii) Conditionally on $\{H_{\max} = h_0\}$ and $W_{T_{\max}}$, $R^g_{T_{\max}}$ and $R^d_{T_{\max}}$ are under $\mathbf{N}^0_x$ independent Poisson point measures on $\mathbb{R}^+ \times \bar{\Omega}$ with intensity:

$$\mathbf{1}_{[0,h_0)}(s) ds \; \mathbf{1}_{\{H_{\max}(W) < h_0 - s\}} \; \mathbf{N}^0_{W_{T_{\max}}(s)}[dW].$$

In other words, for any non-negative measurable function F, we have

$$\mathbf{N}^0_x\Big[F(H_{\max}, W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})\Big] = -\int_0^\infty \partial_h v^0_h \; dh \; \tilde{\mathrm{E}}_x\Big[F(h, Y_{[0,h)}, \hat{R}^{W,(h),g}, \hat{R}^{W,(h),d})\Big],$$

where under $\tilde{\mathrm{E}}_x$ and conditionally on $Y_{[0,h)}$, $\hat{R}^{W,(h),g}$ and $\hat{R}^{W,(h),d}$ are two independent Poisson point measures with intensity $\hat{\nu}^{W,(h)}(ds, dW) = \mathbf{1}_{[0,h)}(s) ds \; \mathbf{1}_{\{H_{\max}(W) < h - s\}} \; \mathbf{N}^0_{Y_s}[dW]$.

Notice that items (ii) and (iii) in the previous Proposition implies the existence of a measurable family $(\mathbf{N}^{0,(h)}_x, h > 0)$ of probabilities on $(\bar{\Omega}, \bar{\mathcal{G}})$ such that $\mathbf{N}^{0,(h)}_x$ is the distribution of W (more precisely of $(W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})$) under $\mathbf{N}^0_x$ conditionally on $\{H_{\max} = h\}$.

Remark 4.8. *In Klebaner & al [18], the Esty time reversal "is obtained by conditioning a [discrete time] Galton Watson process in negative time upon entering state 0 (extinction) at time 0 when starting at state 1 at time $-n$ and letting n tend to infinity". The authors then observe that in the linear fractional case (modified geometric offspring distribution) the Esty time reversal has the law of the same Galton Watson process conditioned on non extinction. Notice that in our continuous setting, the process $(H_s, 0 \le s \le T_{\max})$ is under $\mathbf{N}^{0,(h)}_x$ a Bessel process up to its first hitting time of h, and thus is reversible: $(H_s, 0 \le s \le T_{\max})$ under $\mathbf{N}^{0,(h)}_x$ is distributed as $(h - H_{T_{\max}-s}, 0 \le s \le T_{\max})$ under $\mathbf{N}^{0,(h)}_x$. It is also well known (see Corollary 3.1.6 of*

[11]) that $(H_{\sigma-s}, 0 \leq s \leq \sigma - T_{\max})$ under $\mathbf{N}_x^{0,(h)}$ is distributed as $(H_s, 0 \leq s \leq T_{\max})$ under $\mathbf{N}_x^{0,(h)}$. We deduce from these two points that $(X_s(1), 0 \leq s \leq h)$ under $\mathbf{N}_x^{0,(h)}$ is distributed as $(X_{h-s}(1), 0 \leq s \leq h)$ under $\mathbf{N}_x^{0,(h)}$. This result, which holds at fixed h, gives a version in continuous time of the Esty time reversal. Passing to the limit as $h \to \infty$, see Section 5.7, we then get the equivalent of the Esty time reversal in a continuous setting.

Before stating Williams decomposition, Theorem 4.12, let us prove some properties for the functions $v_t(x) = \mathbb{N}_x[H_{\max} > t] = \mathbb{N}_x[X_t \neq 0]$ and $\tilde{v}_t(x) = \tilde{\mathbb{N}}_x[H_{\max} > t]$ which will play a significant rôle in the next Section. Recall (3.10) states that

$$\alpha v_t = \tilde{v}_t.$$

Notice also that (3.11) implies that q is bounded from above by $(\beta_0 + \|\tilde{\beta}\|_\infty)/2$.

Lemma 4.9. *Assume $(H1)$. We have:*

$$q(x) + v_t^0 \geq \tilde{v}_t(x) \geq v_t^0. \tag{4.12}$$

Furthermore for fixed $x \in S$, $\tilde{v}_t(x)$ is of class $\mathcal{C}^1$ in t and we have:

$$\partial_t \tilde{v}_t(x) = \tilde{\mathrm{E}}_x \left[\mathrm{e}^{\int_0^t \Sigma_r(Y_{t-r})\, dr} \right] \partial_t v_t^0, \tag{4.13}$$

where the function Σ defined by:

$$\Sigma_t(x) = 2(v_t^0 + q(x) - \tilde{v}_t(x)) = \partial_\lambda \psi^0(v_t^0) - \partial_\lambda \tilde{\psi}(x, \tilde{v}_t(x)) \tag{4.14}$$

satisfies:

$$0 \leq \Sigma_t(x) \leq 2q(x) \leq \beta_0 + \|\tilde{\beta}\|_\infty. \tag{4.15}$$

Proof. We deduce from item (iii) of Proposition 3.7 that, as $\varphi \geq 0$ (see Lemma 3.8),

$$\tilde{v}_t(x) = \tilde{\mathbb{N}}_x[X_t \neq 0] = \mathbb{N}_x^0 \left[\mathbf{1}_{\{X_t \neq 0\}} \mathrm{e}^{\int_0^{+\infty} ds\, X_s(\varphi)} \right] \geq \mathbb{N}_x^0[X_t \neq 0] = v_t^0.$$

We also have

$$\begin{aligned}
\tilde{v}_t(x) &= \mathbb{N}_x^0 \left[\mathbf{1}_{\{X_t \neq 0\}} \mathrm{e}^{\int_0^{+\infty} ds\, X_s(\varphi)} \right] \\
&= \mathbb{N}_x^0 \left[\mathrm{e}^{\int_0^{+\infty} ds\, X_s(\varphi)} - 1 \right] + \mathbb{N}_x^0 \left[1 - \mathbf{1}_{\{X_t = 0\}} \mathrm{e}^{\int_0^{+\infty} ds\, X_s(\varphi)} \right] \\
&= q(x) + \mathbb{N}_x^0 \left[1 - \mathbf{1}_{\{X_t = 0\}} \mathrm{e}^{\int_0^{+\infty} ds\, X_s(\varphi)} \right] \\
&\leq q(x) + \mathbb{N}_x^0 \left[1 - \mathbf{1}_{\{X_t = 0\}} \right] \\
&= q(x) + v_t^0,
\end{aligned}$$

where we used (3.15) for the third equality. This proves (4.12).

Using Williams decomposition under $\mathbf{N}_x^0$, we get:

$$\tilde{v}_t(x) = -\int_t^{+\infty} \partial_r v_r^0 \, dr\, \mathbf{N}_x^{0,(r)} \left[\mathrm{e}^{\int_0^{+\infty} ds\, X_s(\varphi)} \right].$$

Using again Williams decomposition under $\mathbf{N}_x^0$, we have

$$\begin{aligned}
\mathbf{N}_x^{0,(r)} \left[\mathrm{e}^{\int_0^{+\infty} ds\, X_s(\varphi)} \right] &= \tilde{\mathrm{E}}_x \left[\mathrm{e}^{2\int_0^r ds\, \mathbf{N}^0_{Y_{r-s}} \left[(\mathrm{e}^{\int_0^{+\infty} dt\, X_t(\varphi)} - 1) \mathbf{1}_{\{X_s = 0\}} \right]} \right] \\
&= \tilde{\mathrm{E}}_x \left[\mathrm{e}^{2\int_0^r ds\, \mathbf{N}^0_{Y_s} \left[(\mathrm{e}^{\int_0^{+\infty} dt\, X_t(\varphi)} - 1) \mathbf{1}_{\{X_{r-s} = 0\}} \right]} \right].
\end{aligned} \tag{4.16}$$

We deduce that, for fixed x, $r \mapsto \mathbf{N}_x^{0,(r)}\left[\mathrm{e}^{\int_0^{+\infty} ds\ X_s(\varphi)}\right]$ is non-decreasing and continuous as $\mathbf{N}_y^0[H_{\max} = t] = 0$ for $t > 0$. Therefore, we deduce that for fixed x, $\tilde{v}_t(x)$ is of class $\mathcal{C}^1$ in t:

$$\partial_t \tilde{v}_t(x) = \mathbf{N}_x^{0,(t)}\left[\mathrm{e}^{\int_0^{+\infty} ds\ X_s(\varphi)}\right]\ \partial_t v_t^0.$$

We have thanks to item (iii) from Proposition 3.7:

$$\begin{aligned}
&\mathbf{N}_y^0\left[(\mathrm{e}^{\int_0^{+\infty} dt\ X_t(\varphi)} - 1)\mathbf{1}_{\{X_s=0\}}\right]\\
&\qquad = \mathbf{N}_y^0\left[X_s \neq 0\right] + \mathbf{N}_y^0\left[\mathrm{e}^{\int_0^{+\infty} dt\ X_t(\varphi)} - 1\right] - \mathbf{N}_y^0\left[\mathrm{e}^{\int_0^{+\infty} dt\ X_t(\varphi)}\ \mathbf{1}_{\{X_s\neq 0\}}\right]\\
&\qquad = v_s^0 + q(y) - \tilde{v}_s(y)\\
&\qquad = \frac{1}{2}\left[\partial_\lambda \psi^0(v_s^0) - \partial_\lambda \tilde{\psi}(y, \tilde{v}_s(y))\right],
\end{aligned} \tag{4.17}$$

where the last equality follows from (3.8), (3.11) and (3.12). Thus, with $\Sigma_s(y) = \partial_\lambda \psi^0(v_s^0) - \partial_\lambda \tilde{\psi}(y, \tilde{v}_s(y))$, we deduce that:

$$\mathbf{N}_x^{0,(t)}\left[\mathrm{e}^{\int_0^{+\infty} ds\ X_s(\varphi)}\right] = \tilde{\mathrm{E}}_x\left[\mathrm{e}^{\int_0^t ds\ \Sigma_s(Y_{t-s})}\right].$$

This implies (4.13). Notice that, thanks to (4.12), Σ is non-negative and bounded from above by $2q$. □

Fix $h > 0$. We define the probability measures $\mathrm{P}^{(h)}$ absolutely continuous with respect to P and $\tilde{\mathrm{P}}$ on $\mathcal{D}_h$ with Radon-Nikodym derivative:

$$\frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_h}}{d\tilde{\mathrm{P}}_{x\ |\mathcal{D}_h}} = \frac{\mathrm{e}^{\int_0^h \Sigma_{h-r}(Y_r)\ dr}}{\tilde{\mathrm{E}}_x\left[\mathrm{e}^{\int_0^h \Sigma_{h-s}(Y_s)\ dr}\right]}. \tag{4.18}$$

Notice this Radon-Nikodym derivative is 1 if the branching mechanism ψ is homogeneous. We deduce from (4.13) and (4.14) that:

$$\frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_h}}{d\tilde{\mathrm{P}}_{x\ |\mathcal{D}_h}} = \frac{\partial_h v_h^0}{\partial_h \tilde{v}_h(x)}\ \mathrm{e}^{-\int_0^h dr\ \left(\partial_\lambda \tilde{\psi}(Y_r, \tilde{v}_{h-r}(Y_r)) - \partial_\lambda \psi^0(v_{h-r}^0)\right)}$$

and, using (3.7):

$$\frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_h}}{d\mathrm{P}_{x\ |\mathcal{D}_h}} = \frac{1}{\alpha(Y_h)}\ \frac{\partial_h v_h^0}{\partial_h v_h(x)}\ \mathrm{e}^{-\int_0^h dr\ \left(\partial_\lambda \psi(Y_r, v_{h-r}(Y_r)) - \partial_\lambda \psi^0(v_{h-r}^0)\right)}. \tag{4.19}$$

In the next Lemma, we give an intrinsic representation of the Radon-Nikodym derivatives (4.18) and (4.19), which does not involve β_0 or v^0.

Lemma 4.10. *Assume $(H1)$-$(H3)$. Fix $h > 0$. The processes $M^{(h)} = (M_t^{(h)}, t \in [0, h))$ and $\tilde{M}^{(h)} = (\tilde{M}_t^{(h)}, t \in [0, h))$, with:*

$$M_t^{(h)} = \frac{\partial_h v_{h-t}(Y_t)}{\partial_h v_h(x)}\ \mathrm{e}^{-\int_0^t ds\ \partial_\lambda \psi(Y_s, v_{h-s}(Y_s))} \quad \text{and} \quad \tilde{M}_t^{(h)} = \frac{\partial_h \tilde{v}_{h-t}(Y_t)}{\partial_h \tilde{v}_h(x)}\ \mathrm{e}^{-\int_0^t ds\ \partial_\lambda \tilde{\psi}(Y_s, \tilde{v}_{h-s}(Y_s))},$$

are non-negative bounded $\mathcal{D}_t$-martingales respectively under P_x and $\tilde{\mathrm{P}}_x$. Furthermore, we have for $0 \le t < h$:

$$\frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = M_t^{(h)} \quad \text{and} \quad \frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_t}}{d\tilde{\mathrm{P}}_{x\ |\mathcal{D}_t}} = \tilde{M}_t^{(h)}. \tag{4.20}$$

Notice the limit $M_h^{(h)}$ of $M^{(h)}$ and the limit $\tilde{M}_h^{(h)}$ of $\tilde{M}^{(h)}$ are respectively given by the right-handside of (4.19) and (4.18).

Remark 4.11. *Comparing (3.3) and (4.20), we have that* $\mathrm{P}_x^{(h)} = \mathrm{P}_x^g$ *with* $g(t,x) = \partial_h v_{h-t}(x)$*, if* g *satisfies the assumptions of Remark 3.3.*

Proof. First of all, the process $\tilde{M}^{(h)}$ is clearly $\mathcal{D}_t$-adapted. Using (4.13), we get:

$$\tilde{\mathrm{E}}_y \left[\mathrm{e}^{\int_0^{h-t} \Sigma_{h-t-r}(Y_r)\, dr} \right] = \frac{\partial_h \tilde{v}_{h-t}(y)}{\partial_h v^0_{h-t}}.$$

We set:

$$\tilde{M}_h^{(h)} = \frac{\mathrm{e}^{\int_0^h \Sigma_{h-r}(Y_r)\, dr}}{\tilde{\mathrm{E}}_x \left[\mathrm{e}^{\int_0^h \Sigma_{h-s}(Y_s)\, dr} \right]}.$$

We have:

$$\begin{aligned} \tilde{\mathrm{E}}_x[\tilde{M}_h^{(h)} | \mathcal{D}_t] &= \frac{\mathrm{e}^{\int_0^t \Sigma_{h-r}(Y_r)\, dr}}{\tilde{\mathrm{E}}_x \left[\mathrm{e}^{\int_0^h \Sigma_{h-s}(Y_s)\, dr} \right]} \tilde{\mathrm{E}}_{Y_t} \left[\mathrm{e}^{\int_0^{h-t} \Sigma_{h-t-r}(Y_r)\, dr} \right] \\ &= \frac{\partial_h \tilde{v}_{h-t}(Y_t)}{\partial_h \tilde{v}_h(x)} \frac{\partial_h v_h^0}{\partial_h v^0_{h-t}} \mathrm{e}^{\int_0^t \Sigma_{h-r}(Y_r)\, dr} \\ &= \frac{\partial_h \tilde{v}_{h-t}(Y_t)}{\partial_h \tilde{v}_h(x)} \mathrm{e}^{-\int_0^t \partial_\lambda \tilde{\psi}\left(Y_s, \tilde{v}_{h-s}(Y_s)\right)\, ds} \frac{\partial_h v_h^0}{\partial_h v^0_{h-t}} \mathrm{e}^{\int_0^t \partial_\lambda \psi^0\left(v^0_{h-s}\right)\, ds} \end{aligned}$$

In the homogeneous setting, v^0 simply solves the ordinary differential equation:

$$\partial_h v_h^0 = -\psi^0(v_h^0).$$

This implies that

$$\partial_h \log(\partial_h v_h^0) = \frac{\partial_h^2 v_h^0}{\partial_h v_h^0} = -\partial_\lambda \psi^0(v_h^0)$$

and thus

$$\frac{\partial_h v_h^0}{\partial_h v^0_{h-t}} \mathrm{e}^{\int_0^t \partial_\lambda \psi^0\left(v^0_{h-s}\right)\, ds} = 1. \tag{4.21}$$

We deduce that

$$\tilde{\mathrm{E}}_x[\tilde{M}_h^{(h)} | \mathcal{D}_t] = \frac{\partial_h \tilde{v}_{h-t}(Y_t)}{\partial_h \tilde{v}_h(x)} \mathrm{e}^{-\int_0^t dr\, \partial_\lambda \tilde{\psi}(Y_r, \tilde{v}_{h-r}(Y_r))} = \tilde{M}_t^{(h)}.$$

Therefore, $\tilde{M}^{(h)}$ is a $\mathcal{D}_t$-martingale under $\tilde{\mathrm{P}}_x$ and the second part of (4.20) is a consequence of (4.18). Then, use (3.7) to get that $M^{(h)}$ is a $\mathcal{D}_t$-martingale under P_x and the first part of (4.20). □

We now give the Williams' decomposition: the distribution of $(H_{\max}, W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})$ under $\mathbf{N}_x$ or equivalently under $\tilde{\mathbf{N}}_x/\alpha(x)$. Recall the distribution $\mathrm{P}_x^{(h)}$ defined in (4.18) or (4.19).

Theorem 4.12 (Williams' decomposition under $\mathbf{N}_x$)**.** *Assume* $(H1)$-$(H3)$*. We have:*

(i) The distribution of $H_{\max}$ *under* $\mathbf{N}_x$ *is characterized by:* $\mathbf{N}_x[H_{\max} > h] = v_h(x)$*.*

(ii) Conditionally on $\{H_{\max} = h_0\}$*, the law of* $W_{T_{\max}}$ *under* $\mathbf{N}_x$ *is distributed as* $Y_{[0,h_0)}$ *under* $\mathrm{P}_x^{(h_0)}$*.*

(iii) Conditionally on $\{H_{\max} = h_0\}$ and $W_{T_{\max}}$, $R^g_{T_{\max}}$ and $R^d_{T_{\max}}$ are under $\mathbf{N}_x$ independent Poisson point measures on $\mathbb{R}^+ \times \bar{\Omega}$ with intensity:

$$\mathbf{1}_{[0,h_0)}(s)ds\ \mathbf{1}_{\{H_{\max}(W')<h_0-s\}}\alpha(W_{T_{\max}}(s))\ \mathbf{N}_{W_{T_{\max}}(s)}[dW'].$$

In other words, for any non-negative measurable function F, we have

$$\mathbf{N}_x\left[F(H_{\max}, W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})\right] = -\int_0^\infty \partial_h v_h(x)\ dh\ \mathrm{E}_x^{(h)}\left[F(h, Y_{[0,h)}, R^{W,(h),g}, R^{W,(h),d})\right],$$

where under $\mathrm{E}_x^{(h)}$ and conditionally on $Y_{[0,h)}$, $R^{W,(h),g}$ and $R^{W,(h),d}$ are two independent Poisson point measures with intensity:

$$\nu^{W,(h)}(ds, dW) = \mathbf{1}_{[0,h)}(s)ds\ \mathbf{1}_{\{H_{\max}(W)<h-s\}}\alpha(Y_s)\ \mathbf{N}_{Y_s}[dW]. \tag{4.22}$$

Notice that items (ii) and (iii) in the previous Proposition imply the existence of a measurable family $(\mathbf{N}_x^{(h)}, h > 0)$ of probabilities on $(\bar{\Omega}, \bar{\mathcal{G}})$ such that $\mathbf{N}_x^{(h)}$ is the distribution of W (more precisely of $(W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})$) under $\mathbf{N}_x$ conditionally on $\{H_{\max} = h\}$.

Proof. We keep notations introduced in Proposition 4.7. We have:

$$\begin{aligned}
&\tilde{\mathbf{N}}_x\left[F(H_{\max}, W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})\right]\\
&= \mathbf{N}^0_x\left[\mathrm{e}^{\int_0^{+\infty} ds\ X_s(\varphi)}\,F(H_{\max}, W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})\right]\\
&= \mathbf{N}^0_x\left[F(H_{\max}, W_{T_{\max}}, R^g_{T_{\max}}, R^d_{T_{\max}})\,\mathrm{e}^{(R^g_{T_{\max}}+R^d_{T_{\max}})(f)}\right]\\
&= -\int_0^\infty \partial_h v^0_h\ dh\ \tilde{\mathrm{E}}_x\left[F(h, Y_{[0,h)}, \hat{R}^{W,(h),g}, \hat{R}^{W,(h),d})\,\mathrm{e}^{(\hat{R}^{W,(h),g}+\hat{R}^{W,(h),d})(f)}\right]\\
&= -\int_0^\infty \partial_h v^0_h\ dh\ \tilde{\mathrm{E}}_x\left[F(h, Y_{[0,h)}, R^{W,(h),g}, R^{W,(h),d})\,\mathrm{e}^{2\int_0^h ds\ \mathbf{N}^0_{Y_s}\left[(\mathrm{e}^{\int_0^{+\infty} dt\ X_t(\varphi)}-1)\mathbf{1}_{\{X_{r-s}=0\}}\right]}\right]\\
&= -\int_0^\infty \partial_h v^0_h\ dh\ \tilde{\mathrm{E}}_x\left[F(h, Y_{[0,h)}, R^{W,(h),g}, R^{W,(h),d})\,\mathrm{e}^{\int_0^h \Sigma_{h-s}(Y_s)\ ds}\right]\\
&= -\int_0^\infty \partial_h v^0_h\ dh\ \tilde{\mathrm{E}}_x\left[\mathrm{e}^{\int_0^h \Sigma_{h-s}(Y_s)\ ds}\right]\mathrm{E}_x^{(h)}\left[F(h, Y_{[0,h)}, R^{W,(h),g}, R^{W,(h),d})\right]\\
&= -\int_0^\infty \partial_h \tilde{v}_h(x)\ dh\ \mathrm{E}_x^{(h)}\left[F(h, Y_{[0,h)}, R^{W,(h),g}, R^{W,(h),d})\right],
\end{aligned}$$

where the first equality comes from $(H1)$ and item (iii) of Proposition 3.7; we set $f(s, W) = \int_0^{+\infty} X_r(W)(\varphi)$ for the second equality; we use Proposition 4.7 for the third equality; we use Lemma 4.3 for the fourth with $R^{W,(h),g}$ and $R^{W,(h),d}$ which under $\tilde{\mathrm{E}}_x^{(h)}$ and conditionally on $Y_{[0,h)}$ are two independent Poisson point measures with intensity $\nu^{W,(h)}$; we use (4.17) for the fifth, definition (4.18) of $\mathrm{E}_x^{(h)}$ for the sixth, and (4.13) for the seventh. Then use (3.26) and (3.10) to conclude. □

The definition of $\mathbf{N}_x^{(h)}$ gives in turn sense to the conditional law $\mathbb{N}_x^{(h)} = \mathbb{N}_x(.|H_{\max} = h)$ of the $(\mathcal{L}, \beta, \alpha)$ super-process conditioned to die at time h, for all $h > 0$. The next Corollary is then a straightforward consequence of Theorem 4.12.

Corollary 4.13. *Assume $(H1)$-$(H3)$. Let $h > 0$. Let $x \in S$ and $Y_{[0,h)}$ be distributed according to $\mathrm{P}_x^{(h)}$. Consider the Poisson point measure $\mathcal{N} = \sum_{j\in J}\delta_{(s_j, X^j)}$ on $[0,h)\times\Omega$ with intensity:*

$$2\mathbf{1}_{[0,h)}(s)ds\ \mathbf{1}_{\{H_{\max}(X)<h-s\}}\alpha(Y_s)\ \mathbb{N}_{Y_s}[dX].$$

The process $X^{(h)} = (X_t^{(h)}, t \geq 0)$*, which is defined for all* $t \geq 0$ *by:*

$$X_t^{(h)} = \sum_{j \in J,\, s_j < t} X^j_{t-s_j},$$

is distributed according to $\mathbb{N}_x^{(h)}$*.*

We now give the super-process counterpart of Theorem 4.12.

Corollary 4.14 (Williams' decomposition under $\mathbb{P}_\nu$)**.** *Assume* $(H1)$-$(H3)$*.*

(i) The distribution of H_{max} *under* $\mathbb{P}_\nu$ *is characterized by:* $\mathbb{P}_\nu(H_{\text{max}} \leq h) = \mathrm{e}^{-\nu(v_h)}$*.*

(ii) Conditionally on $\{H_m = h_0\}$*,* X *under* $\mathbb{P}_\nu$ *is distributed as the independent sum of* X' *and* $X^{(h_0)}$*, where:*

- $X^{(h_0)}$ *is distributed according the probability measure:*

$$\int \nu(dx)\, \frac{\partial_h v_{h_0}(x)}{\nu(\partial_h v_{h_0})}\, \mathbb{N}_x^{(h_0)}.$$

- X' *is distributed according to the probability measure* $\mathbb{P}_\nu(.|H_{\text{max}} < h_0)$*.*

In particular the distribution of $X' + X^{(h_0)}$ conditionally on h_0 is a regular version of the distribution of the $(\mathcal{L}, \beta, \alpha)$ super-process conditioned to die at a fixed time h_0, which we shall write $\mathbb{P}_\nu^{(h_0)}$.

Proof. Let μ be a finite measure on $\mathbb{R}^+$ and f a non-negative measurable function defined on $\mathbb{R}^+ \times S$. For a measure-valued process $Z = (Z_t, t \geq 0)$ on S, we set $Z(f\mu) = \int f(t,x)\, Z_t(dx)\mu(dt)$. We also write $f_s(t,x) = f(s+t,x)$.

Let X' and $X^{(h_0)}$ be defined as in Corollary 4.14. In order to characterized the distribution of the process $X' + X^{(h_0)}$, we shall compute

$$A = \mathbb{E}[\mathrm{e}^{-X'(f\mu) - X^{(h_0)}(f\mu)}].$$

We shall use notations from Corollary 4.13. We have:

$$\begin{aligned} A &= -\int_0^{+\infty} \nu(\partial_h v_h)\, \mathrm{e}^{-\nu(h)}\, dh \int_S \frac{\partial_h v_h(x)}{\nu(\partial_h v_h)}\, \nu(dx) \\ &\qquad \mathrm{E}_x^{(h)} \left[\mathbb{E}[\mathrm{e}^{-\sum_{j\in J} X^j(f_{s_j}\mu)} | Y_{[0,h)}] \right] \mathbb{E}_\nu \left[\mathrm{e}^{-X(f\mu)} | H_{\text{max}} < h \right] \\ &= -\int_E \nu(dx) \int_0^{+\infty} \partial_h v_h(x)\, dh \\ &\qquad \mathrm{E}_x^{(h)} \left[\mathbb{E}[\mathrm{e}^{-\sum_{j\in J} X^j(f_{s_j}\mu)} | Y_{[0,h)}] \right] \mathbb{E}_\nu \left[\mathrm{e}^{-X(f\mu)}\, \mathbf{1}_{\{H_{\text{max}} < h\}} \right], \end{aligned}$$

where we used the definition of X' and $\mathcal{N}$ for the first equality, and the equality $\mathbb{P}_\nu(H_{\text{max}} < h) = \mathbb{P}_\nu(H_{\text{max}} \leq h) = \mathrm{e}^{-\nu(h)}$ for the second. Recall notations from Theorem 4.12. We set:

$$G\left(\sum_{i \in I} \delta_{(s_i, W^i)}, \sum_{i' \in I'} \delta_{(s_{i'}, W^{i'})} \right) = \mathrm{e}^{-\sum_{j \in I \cup I'} X^j(W^j)(f_{s_j}\mu)}$$

and $g(h) = \mathbb{E}_\nu \left[\mathrm{e}^{-X(f\mu)} \mathbf{1}_{\{H_{\max} < h\}} \right]$. We have:

$$
\begin{aligned}
A &= -\int_E \nu(dx) \int_0^{+\infty} \partial_h v_h(x) \, dh \, \mathrm{E}_x^{(h)} \left[G(R^{W,(h),g}, R^{W,(h),d}) g(h) \right] \\
&= \int_E \nu(dx) \, \mathbf{N}_x \left[G(R_g^{T_{\max}}, R_d^{T_{\max}}) g(H_{\max}) \right] \\
&= \int_E \nu(dx) \, \mathbb{N}_x \left[\mathrm{e}^{-X(f\mu)} \, \mathbb{E}_\nu \left[\mathrm{e}^{-X(f\mu)} \mathbf{1}_{\{H_{\max} < h\}} \right]_{|h = H_{\max}} \right] \\
&= \mathbb{E} \left[\sum_{i \in I} \mathrm{e}^{-X^i(f\mu)} \prod_{j \in I; \, j \neq i} \mathrm{e}^{-X^j(f\mu)} \mathbf{1}_{\{H_{\max}^j < H_{\max}^i\}} \right] \\
&= \mathbb{E} \left[\mathrm{e}^{-\sum_{i \in I} X^i(f\mu)} \right] \\
&= \mathbb{E}_\nu \left[\mathrm{e}^{-X(f\mu)} \right],
\end{aligned}
$$

where we used the definition of G and g for the first and third equalities, Theorem 4.12 for the second equality, the master formula for Poisson point measure $\sum_{i \in I} \delta_{X^i}$ with intensity $\nu(dx) \, \mathbb{N}_x[dX]$ for the fourth equality (and the obvious notation $H_{\max}^i = \inf\{t \geq 0; X_t^i = 0\}$) and Theorem 2.3 for the last equality. Thus we get:

$$
\mathbb{E}[\mathrm{e}^{-X'(f\mu) - X^{(h_0)}(f\mu)}] = \mathbb{E}_\nu \left[\mathrm{e}^{-X(f\mu)} \right].
$$

This readily implies that the process $X' + X^{(h_0)}$ is distributed as X under $\mathbb{P}_\nu$. □

5 Applications of the Williams decomposition

This Section is concerned with the derivation of the Q-process from the Williams decomposition. The main assumption needed is a product-criticality assumption, properly defined in the diffusive and in the multitype settings. We therefore restrict ourselves to these two settings, which we now introduce.

5.1 The super-diffusion

A super-diffusion is a super-process having a diffusion for spatial motion.

Let S be an arbitrary domain of $\mathbb{R}^K$ for K integer. Let a_{ij} and b_i be in $\mathcal{C}^{1,\mu}(S)$, the usual Hölder space of order $\mu \in [0, 1)$, which consists of functions whose first order derivatives are locally Hölder continuous with exponent μ, for each i, j in $\{1, \dots, K\}$. The functions $a_{i,j}$ will be chosen such that the matrix $(a_{ij})_{(i,j) \in \{1 \dots K\}^2}$ is positive definite. In that case, the following elliptic operator:

$$
\mathcal{L}(u) = \sum_{i=1}^K b_i \, \partial_{x_i} u + \frac{1}{2} \sum_{i,j=1}^K a_{ij} \, \partial_{x_i, x_j} u.
$$

defines a diffusion on S. The generalized eigenvalue λ_0 of the operator $\beta - \mathcal{L}$ is defined by:

$$
\lambda_0 = \sup \{ \ell \in \mathbb{R}, \exists u \in \mathcal{D}(\mathcal{L}), u > 0 \text{ such that } (\beta - \mathcal{L}) u = \ell \, u \} \cdot \tag{5.1}
$$

Denoting E the expectation operator associated to the process with generator $\mathcal{L}$, an equivalent probabilistic definition of the generalized eigenvalue λ_0 is the following:

$$
\lambda_0 = -\sup_{A \subset \mathbb{R}^K} \lim_{t \to \infty} \frac{1}{t} \log \mathrm{E}_x \left[\mathrm{e}^{-\int_0^t ds \, \beta(Y_s)} \mathbf{1}_{\{\tau_{A^c} > t\}} \right],
$$

where $x \in \mathbb{R}^K$ is arbitrary, $\tau_{A^c} = \inf\{t > 0 : Y(t) \notin A\}$ is the hitting time of the complement of A, and the supremum runs over the compact subsets A of $\mathbb{R}^K$. The operator $(\beta - \lambda_0) - \mathcal{L}$ is said critical when the space of positive harmonic functions for $(\beta - \lambda_0) - \mathcal{L}$ is one-dimensional, generated by ϕ_0. In that case, the space of positive harmonic functions of the adjoint of $(\beta - \lambda_0) - \mathcal{L}$ is also one dimensional, generated by $\tilde{\phi}_0$. We say the operator $(\beta - \lambda_0) - \mathcal{L}$ is product-critical if $\int_S dx\ \phi_0(x)\ \tilde{\phi}_0(x) < \infty$. Under this assumption, we may and will choose the eigenvectors normalized in such a way that $\int_S dx\ \phi_0(x)\ \tilde{\phi}_0(x) = 1$. We refer to Pinsky [24] for background on product critical operators, and to Engländer et al [14] for an application to branching diffusions. Lemma 5.6 below gives the probabilistic meaning of this assumption: The probability measure P^{ϕ_0} defined by (3.2) with $g = \phi_0$ is the law of a recurrent Markov process. Recall the conditions on α and β the parameters of the branching mechanism stated in section 2. When $\mathcal{L}$ is the generator of a diffusion, assumptions (H2) and (H3) hold as soon as $\alpha \in \mathcal{C}^4(S)$,

5.2 The multitype Feller diffusion

The multitype Feller diffusion is the super-process with finite state space: $S = \{1, \ldots, K\}$ for K integer. In this case, the spatial motion is a pure jump Markov process, which will be assumed irreducible. Its generator $\mathcal{L}$ is a square matrix $(q_{ij})_{1\leq i,j\leq K}$ of size K with lines summing up to 0, and q_{ij} gives the transition rate from i to j for $i \neq j$. The generalized eigenvalue λ_0 is defined by:

$$\lambda_0 = \sup\{\ell \in \mathbb{R}, \exists u > 0 \text{ such that } (\mathrm{Diag}(\beta) - \mathcal{L})u = \ell u\}, \tag{5.2}$$

where Diag(β) is the diagonal $K \times K$ matrix with diagonal coefficients derived from the vector β. We stress that the generalized eigenvalue corresponds to the Perron Frobenius eigenvalue, *i.e.* the eigenvalue with the maximum real part, which is real according to the Perron Frobenius theorem. Moreover, the associated eigenspace is one-dimensional. We will denote by ϕ_0 and $\tilde{\phi}_0$ its left, resp. right, eigenvectors, normalized so that $\sum_{i=1}^K \phi_0(i)\tilde{\phi}_0(i) = 1$. This means the operator $(\mathrm{Diag}(\beta) - \lambda_0) - \mathcal{L}$, which we shall also denote by $(\beta - \lambda_0) - \mathcal{L}$ to unify notation with section 5.1, is automatically product-critical. The Perron Frobenius Theorem includes the fact that the coordinates of ϕ_0 and $\tilde{\phi}_0$ are positive. A reference on this Theorem is Seneta [26]. Concerning the branching mechanism, the functions β and α in (2.2) are vectors of size K, and $(H2)$ and $(H3)$ therefore automatically hold. For more details about the construction of finite state space super-process, we refer to example 2 p. 10 of Dynkin [13]. For previous investigations of the associated Q-process, we refer to Champagnat and Roelly [6] and Pénisson [22].

5.3 The convergence of the Williams spatial motion from the root

We will show in Section 5.5 that the convergence of $\mathbb{P}^{(h)}$ as $h \to \infty$ essentially amounts to that of the spatial motion $\mathrm{P}^{(h)}$. We establish the latter convergence in this Section.

We go on assuming that P is the law of a diffusion in $\mathbb{R}^K$ for K integer or the law of a finite state space Markov process. We will denote by λ_0 the generalized eigenvalue of the operator $(\beta - \mathcal{L})$, see (5.2) or (5.1), and by ϕ_0 the associated right eigenvector. We shall consider the assumption:

$(H4)$ **The operator** $(\beta - \lambda_0) - \mathcal{L}$ **is product-critical,** $\phi_0 \in \mathcal{D}(\mathcal{L})$ **and there exist two positive constants** C_1 **and** C_2 **such that** $\forall x \in S,\ C_1 \leq \phi_0(x) \leq C_2$**.**

Under $(H4)$, we introduce $\mathrm{P}^{\phi_0}_x$ the probability measure on $(D, \mathcal{D})$ defined by (3.2) with

g replaced by ϕ_0:

$$\forall t \geq 0, \quad \frac{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{\phi_0(Y_t)}{\phi_0(Y_0)}\, \mathrm{e}^{-\int_0^t ds\ (\beta(Y_s)-\lambda_0)}\,. \tag{5.3}$$

Under $(H4)$, we also know that the $(\mathcal{L}^{\phi_0}, \lambda_0, \alpha\phi_0)$ super-process is the h-transform of the $(\mathcal{L}, \beta, \alpha)$-super-process with $h = \phi_0$, see Definition 3.4 as well as Proposition 3.5 and the definition of the generalized eigenvalue (5.2) and (5.1). We then define $v_t^{\phi_0}(x)$ for all $t > 0$ and $x \in S$ by:

$$v_t^{\phi_0}(x) = \mathbb{N}_x^{(\mathcal{L}^{\phi_0}, \lambda_0, \alpha\phi_0)}[H_{\max} > t]. \tag{5.4}$$

Recall from (3.10) that v^{ϕ_0} and v are linked through:

$$v_t^{\phi_0}(x) = \frac{v_t(x)}{\phi_0(x)}\cdot \tag{5.5}$$

Our first task is to give precise bounds on the decay of $v_t^{\phi_0}$ as t goes to ∞. This is achieved in the following Lemmas, the proofs of some of them (Lemmas 5.1, 5.2, 5.4, 5.5 and 5.7) are postponed to the Appendix.

Lemma 5.1 gives a bound in the case $\lambda_0 = 0$. The proof relies on a coupling argument in the construction of Dhersin and Serlet [9], and yields bounds from below and from above for the extinction time $H_{\max}$.

Lemma 5.1. *Assume $\lambda_0 = 0$, $(H2)$ and $(H4)$. Then for all $t > 0$:*

$$\alpha\phi_0(x)\frac{1}{\|\alpha\phi_0\|_\infty^2} \leq t\ v_t^{\phi_0}(x) \leq \alpha\phi_0(x)\left\|\frac{1}{\alpha\phi_0}\right\|_\infty^2.$$

We deduce from this lemma that assumption $(H1)$ holds:

Lemma 5.2. *Assume $\lambda_0 \geq 0$, $(H2)$ and $(H4)$. Then $(H1)$ holds.*

Remark 5.3. *A super-process X satisfies local extinction if its restrictions to compact domains of S satisfies global extinction. The non-negativity of the generalized eigenvalue of the operator $(\beta - \mathcal{L})$ in general characterizes the local extinction property, see Engländer and Kyprianou [15]. Here, the last part of the additional assumption $(H4)$ allows us to obtain the stronger global extinction property.*

We then proceed by giving a Feynman-Kac formula for v_0^{ϕ} and ∂v_0^{ϕ}.

Lemma 5.4. *Assume $\lambda_0 \geq 0$, and $(H2)$-$(H4)$. Let $\varepsilon > 0$. We have:*

$$v_{h+\varepsilon}^{\phi_0}(x) = \mathrm{e}^{-\lambda_0 h}\, \mathrm{E}_x^{\phi_0}\left[\mathrm{e}^{-\int_0^h ds\ \alpha(Y_s)\ \phi_0(Y_s)\ v_{h+\varepsilon-s}^{\phi_0}(Y_s)}\ v_\varepsilon^{\phi_0}(Y_h)\right], \tag{5.6}$$

$$\partial_h v_{h+\varepsilon}^{\phi_0}(x) = \mathrm{e}^{-\lambda_0 h}\, \mathrm{E}_x^{\phi_0}\left[\mathrm{e}^{-2\int_0^h ds\ \alpha(Y_s)\ \phi_0(Y_s)\ v_{h+\varepsilon-s}^{\phi_0}(Y_s)}\ \partial_h v_\varepsilon^{\phi_0}(Y_h)\right]. \tag{5.7}$$

We give exponential bounds for v_0^{ϕ} and $\partial_t v_0^{\phi}$ in the sub-critical case.

Lemma 5.5. *Assume $\lambda_0 > 0$, $(H2)$-$(H4)$. Fix $t_0 > 0$. There exists C_3 and C_4 two positive constants such that, for all $x \in S$, $t > t_0$:*

$$C_3 \leq v_t^{\phi_0}(x)\, \mathrm{e}^{\lambda_0 t} \leq C_4. \tag{5.8}$$

There exists C_5 and C_6 two positive constants such that, for all $x \in S$, $t > t_0$:

$$C_5 \leq |\partial_t v_t^{\phi_0}(x)|\, \mathrm{e}^{\lambda_0 t} \leq C_6. \tag{5.9}$$

In what follows, the notation $o_h(1)$ refers to any function F_h such that $\lim_{h\to+\infty} \| F_h \|_\infty = 0$. For improving Lemma 5.5, we need the following ergodic formula (5.10), which is a direct consequence of the product-criticality assumption. Recall that, under $(H4)$, the generalized eigenvectors ϕ_0 and $\tilde{\phi}_0$ are chosen normalized so that $\int \phi_0(x)\, \tilde{\phi}_0(x)\, dx = 1$.

Lemma 5.6. *Assume* $(H2)$-$(H4)$*. Then the probability measure* P^{ϕ_0} *admits a stationary measure* π*, with* $\pi(dx) = \phi_0(x)\, \tilde{\phi}_0(x)\, dx$*, and we have:*

$$\sup_{f\in b\mathcal{S}, \|f\|_\infty \le 1} |\mathrm{E}_x^{\phi_0}[f(Y_t)] - \pi(f)| \xrightarrow[t\to+\infty]{} 0. \tag{5.10}$$

The proof essentially repeats the argument used in Engländer, Harris and Kyprianou [14], see Remark 5.

Proof. We first note that $-\mathcal{L}^{\phi_0}$ is the (usual) h-transform of the operator $(\beta - \lambda_0) - \mathcal{L}$ with $h = \phi_0$, where the h-transform of $\mathcal{L}(\cdot)$ is $\frac{\mathcal{L}(h\cdot)}{h}$. Now, we assumed that $(\beta - \lambda_0) - \mathcal{L}$ is a product-critical operator. First, criticality is invariant under h-transforms for operators. Also, a calculus shows that $\tilde{\phi}_0$ and ϕ_0 transforms into $\phi_0\, \tilde{\phi}_0$ and 1 respectively when turning from $(\beta - \lambda_0) - \mathcal{L}$ to $-\mathcal{L}^{\phi_0}$, which is thus again product-critical. We finally apply Theorem 9.9 p.192 of [24] which states that (5.10) holds, with ϕ_0 replaced by 1 and $\tilde{\phi}_0$ by $\phi_0\, \tilde{\phi}_0$. ☐

We are now in position to refine Lemma 5.5.

Lemma 5.7. *Assume* $\lambda_0 \ge 0$, $(H2)$-$(H4)$*. Then for all* $\varepsilon > 0$*, we have:*

$$\partial_t v_{h+\varepsilon}^{\phi_0}(x)\ \mathrm{e}^{\lambda_0 h} = \mathrm{E}_\pi^{\phi_0}\big[\, \mathrm{e}^{-2\int_0^h ds\ \alpha\, \phi_0\, v_{s+\varepsilon}^{\phi_0}(Y_{-s})}\, \partial_t v_\varepsilon^{\phi_0}(Y_0)\big](1 + o_h(1)). \tag{5.11}$$

In addition, for $\lambda_0 > 0$*, we have that:*

$$\mathrm{E}_\pi^{\phi_0}\big[\, \mathrm{e}^{-2\int_0^\infty ds\ \alpha\, \phi_0\, v_{s+\varepsilon}^{\phi_0}(Y_{-s})}\, \partial_t v_\varepsilon^{\phi_0}(Y_0)\big]$$

is finite (notice the integration is up to $+\infty$*) and:*

$$\partial_t v_{h+\varepsilon}^{\phi_0}(x)\ \mathrm{e}^{\lambda_0 h} = \mathrm{E}_\pi^{\phi_0}\big[\, \mathrm{e}^{-2\int_0^\infty ds\ \alpha\, \phi_0\, v_{s+\varepsilon}^{\phi_0}(Y_{-s})}\, \partial_t v_\varepsilon^{\phi_0}(Y_0)\big] + o_h(1). \tag{5.12}$$

Next step is to deduce the convergence of the spatial motion. Fix $x \in S$. We observe from (4.20) and (5.3) that $\mathrm{P}_x^{(h)}$ is absolutely continuous with respect to $\mathrm{P}_x^{\phi_0}$ on $\mathcal{D}_{[0,t]}$ for $0 \le t < h$.

Lemma 5.8. *Assume* $(H2)$-$(H4)$*. For* $\lambda_0 \ge 0$*, we have:*

$$\frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_{[0,t]}}}{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{[0,t]}}} \xrightarrow[h\to+\infty]{} 1 \quad \mathrm{P}_x^{\phi_0}\text{-a.s. and in } \mathrm{L}^1(\mathrm{P}_x^{\phi_0}),$$

and, for $\lambda_0 > 0$*, we also have:*

$$\frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_{[0,h/2]}}}{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{[0,h/2]}}} \xrightarrow[h\to+\infty]{} 1 \quad \mathrm{P}_x^{\phi_0}\text{-a.s. and in } \mathrm{L}^1(\mathrm{P}_x^{\phi_0}).$$

Proof. We compute

$$\begin{aligned}
\frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_{[0,t]}}}{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{[0,t]}}} &= \frac{\partial_t v_{h-t}(Y_t)\,\mathrm{e}^{-\lambda_0 t}}{\partial_t v_h(Y_0)}\,\frac{\phi_0(Y_0)}{\phi_0(Y_t)}\,\mathrm{e}^{-2\int_0^t ds\ \alpha(Y_s)\ v_{h-s}(Y_s)} \\
&= \frac{\partial_t v^{\phi_0}_{h-t}(Y_t)\,\mathrm{e}^{-\lambda_0 t}}{\partial_t v^{\phi_0}_h(Y_0)}\,\mathrm{e}^{-2\int_0^t ds\ \alpha(Y_s)\ \phi_0(Y_s)\ v^{\phi_0}_{h-s}(Y_s)} \\
&= \frac{\mathrm{E}^{\phi_0}_\pi\left[\mathrm{e}^{-2\int_{-(h-t-\varepsilon)}^0 ds\ \alpha(Y_s)\ \phi_0(Y_s)\ v^{\phi_0}_{\varepsilon-s}(Y_s)}\,\partial_h v^{\phi_0}_\varepsilon(Y_0)\right]\left(1+o_h(1)\right)}{\mathrm{E}^{\phi_0}_\pi\left[\mathrm{e}^{-2\int_{-h-\varepsilon}^0 ds\ \alpha(Y_s)\ \phi_0(Y_s)\ v^{\phi_0}_{\varepsilon-s}(Y_s)}\,\partial_h v^{\phi_0}_\varepsilon(Y_0)\right]\left(1+o_h(1)\right)}\left(1+o_h(1)\right) \\
&= \frac{\mathrm{E}^{\phi_0}_\pi\left[\mathrm{e}^{-2\int_{-h-\varepsilon}^0 ds\ \alpha(Y_s)\ \phi_0(Y_s)\ v^{\phi_0}_{\varepsilon-s}(Y_s)}\,\partial_h v^{\phi_0}_\varepsilon(Y_0)\right]}{\mathrm{E}^{\phi_0}_\pi\left[\mathrm{e}^{-2\int_{-h-\varepsilon}^0 ds\ \alpha(Y_s)\ \phi_0(Y_s)\ v^{\phi_0}_{\varepsilon-s}(Y_s)}\,\partial_h v^{\phi_0}_\varepsilon(Y_0)\right]}\left(1+o_h(1)\right) \\
&= 1+o_h(1),
\end{aligned}$$

where we used (4.20), (5.3) and the normalization $v(x) = v^{\phi_0}(x)\ \phi_0(x)$ for the first and second equalities , (5.11) twice and the boundedness of α and ϕ_0 as well as the convergence of v_h to 0 for the third and fourth equality, and Lemma 5.5 (if $\lambda_0 > 0$) or Lemma 5.1 (if $\lambda_0 = 0$) for the fifth. Scheffé's lemma then implies that the convergence also holds in $\mathrm{L}^1(\mathrm{P}^{\phi_0}_x)$.

Similar arguments relying on (5.12) instead of (5.11) imply that

$$\frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_{[0,h/2]}}}{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{[0,h/2]}}} = 1+o_h(1)$$

for $\lambda_0 > 0$. Since $o_h(1)$ is bounded and converges uniformly to 0, we get that the convergence of $d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_{[0,h/2]}}/d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{[0,h/2]}}$ towards 1 holds $\mathrm{P}^{\phi_0}_x$-a.s. and in $\mathrm{L}^1(\mathrm{P}^{\phi_0}_x)$. □

The previous Lemma enables us to conclude about the convergence of the spatial motion.

Proposition 5.9. *Assume $\lambda_0 \geq 0$, $(H2)$-$(H4)$. Then we have:*

$$\frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_{[0,t]}}}{d\mathrm{P}_{x\ |\mathcal{D}_{[0,t]}}} \to \frac{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{[0,t]}}}{d\mathrm{P}_{x\ |\mathcal{D}_{[0,t]}}},\ \textit{as } h\to\infty, \mathrm{P}_x\textit{-a.s. and in } \mathrm{L}^1(\mathrm{P}_x).$$

Proof. We write:

$$\frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_{[0,t]}}}{d\mathrm{P}_{x\ |\mathcal{D}_{[0,t]}}} = \frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_{[0,t]}}}{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{[0,t]}}}\ \frac{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{[0,t]}}}{d\mathrm{P}_{x\ |\mathcal{D}_{[0,t]}}}.$$

We then use the first item of Lemma 5.8 to conclude. □

In our two special examples from Section 5.1 (super-diffusion) and Section 5.2 (multitype Feller diffusion), we can describe precisely the dynamic of the spine under $\mathrm{P}^{(h)}$ and P^{ϕ_0}.

Proposition 5.10 (Multitype Feller diffusion)**.** *Assume P is the law of a finite state space Markov process with transition rates q_{ij} from i to j, $i \neq j$. Then:*

(i) $\mathrm{P}^{(h)}_x$ is a finite state space non-homogeneous Markov process defined on $[0,h)$ issued from x with transition rates from i to j, $i \neq j$, equal to $\dfrac{\partial_h v_{h-t}(j)}{\partial_h v_{h-t}(i)} q_{ij}$ at time t, $0 \leq t < h$.

(ii) $\mathrm{P}_x^{\phi_0}$ *is a a finite state space homogeneous Markov process defined on* $[0,\infty)$ *issued from* x *with transition rates from* i *to* j*,* $i \neq j$*, equal to* $\dfrac{\phi_0(j)}{\phi_0(i)} q_{ij}$*.*

The logarithmic derivative of the function $x \to \partial_h v_{h-t}(x)$ (respectively $x \to \phi_0(x)$) is used to bias the dynamic at time t of the Markov process with law P in $\mathrm{P}_x^{(h)}$ (resp. $\mathrm{P}_x^{\phi_0}$).

Proof. The first item is a consequence of the following adaptation of Lemma 3.2 to time dependent function. Let $g_t(x)$ be a time dependent function. We introduce the probability measure P^g defined by (3.2) with $g(t, Y_t) = g_t(Y_t)$. Denoting by $\mathcal{L}_t^g$ the generator of (the non-homogeneous Markov process) Y_t under P^g, we have that:

$$\forall u \in \mathcal{D}_g(\mathcal{L}), \quad \mathcal{L}_t^g(u) = \frac{\mathcal{L}(g_t u) - \mathcal{L}(g_t)u}{g_t}. \tag{5.13}$$

Recall that for all vector u, $\mathcal{L}(u)(i) = \sum_{j \neq i} q_{ij}\big(u(j) - u(i)\big)$. Then apply (5.13) to the time dependent function $g_t(x) = \partial_t v_{h-t}(x)$, and note that $\mathrm{P}^g = \mathrm{P}^{(h)}$ thanks to (4.20). For the second item, we use Lemma 3.2. □

Proposition 5.11 (Superdiffusion)**.** *Assume* P *is the law of a diffusion with infinitesimal generator* $\mathcal{L}$*. Then:*

(i) $\mathrm{P}_x^{(h)}$ *is a non-homogeneous diffusion on* $[0,h)$ *issued from* x *with generator given by* $\left(\mathcal{L} + a\dfrac{\nabla \partial_h v_{h-t}}{\partial_h v_{h-t}} \nabla.\right)$ *at time* t*,* $0 \leq t < h$*.*

(ii) $\mathrm{P}_x^{\phi_0}$ *is an homogeneous diffusion on* $[0,\infty)$ *issued from* x *with generator* $\left(\mathcal{L} + a\dfrac{\nabla \phi_0}{\phi_0}\nabla.\right)$*.*

A drift pointing to the high values of $\partial_h v_{h-t}$ (resp. ϕ_0) is added to the initial process with generator $\mathcal{L}$ in the construction of $\mathrm{P}_x^{(h)}$ (resp. $\mathrm{P}_x^{(\phi_0)}$). The proof is similar to that of Lemma 5.10..

5.4 The convergence of the Bismut spatial motion

We prove in this Section that the law of the ancestral lineage of an individual randomly chosen at height t under $\mathbf{N}_x$ converges to $\mathrm{P}_x^{\phi_0}$ defined in (5.3) as t goes to infinity.

Recall we defined in (4.11) the following family of probability measures indexed by $t \geq 0$:

$$\frac{d\mathrm{P}^{(B,t)}_{x\ |\mathcal{D}_t}}{d\mathrm{P}_{x\ |\mathcal{D}_t}} = \frac{\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}}{\mathrm{E}_x\left[\mathrm{e}^{-\int_0^t ds\ \beta(Y_s)}\right]},$$

The probability measure $\mathrm{P}^{(B,t)}_{x\ |\mathcal{D}_t}$ corresponds to the law of the ancestral lineage of an individual randomly chosen at height t.

Lemma 5.12. *Assume* $(H4)$*. We have, for every* $0 \leq t_0 \leq t$*:*

$$\frac{d\mathrm{P}^{(B,t)}_{x\ |\mathcal{D}_{t_0}}}{d\mathrm{P}_{x\ |\mathcal{D}_{t_0}}} \xrightarrow[t \to +\infty]{} \frac{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{t_0}}}{d\mathrm{P}_{x\ |\mathcal{D}_{t_0}}} \quad \mathrm{P}_x\text{-a.s. and in } \mathrm{L}^1(\mathrm{P}_x).$$

Notice that, contrary to Lemma 5.9, there is no restriction on the sign of λ_0 for this Lemma to hold. Compare also with forthcoming Remark 5.20.

Remark 5.13. *This result may be interpreted as a globular phase for a random polymers model, see Cranston, Koralov and Molchanov, and Vainberg [8], Theorem 8.3, or as a Feynman-Kac penalization result, see Roynette and Yor [25].*

Proof. We have:

$$\begin{aligned}\frac{dP^{(B,t)}_{x\ |\mathcal{D}_{t_0}}}{dP_{x\ |\mathcal{D}_{t_0}}} &= e^{-\int_0^{t_0} ds\ \beta(Y_s)}\ \frac{E_{Y_{t_0}}\left[e^{-\int_0^{t-t_0} ds\ \beta(Y_s)}\right]}{E_x\left[e^{-\int_0^{t} ds\ \beta(Y_s)}\right]}\\ &= e^{-\int_0^{t_0} ds\ (\beta(Y_s)-\lambda_0)}\ \frac{\phi_0(Y_{t_0})}{\phi_0(Y_0)}\ \frac{E_{Y_{t_0}}\left[e^{-\int_0^{t-t_0} ds\ (\beta(Y_s)-\lambda_0)}\ \frac{\phi_0(Y_{t-t_0})}{\phi_0(Y_0)}\ \frac{1}{\phi_0(Y_{t-t_0})}\right]}{E_x\left[e^{-\int_0^{t} ds\ (\beta(Y_s)-\lambda_0)}\ \frac{\phi_0(Y_t)}{\phi_0(x)}\ \frac{1}{\phi_0(Y_t)}\right]}\\ &= \frac{dP^{\phi_0}_{x\ |\mathcal{D}_{t_0}}}{dP_{x\ |\mathcal{D}_{t_0}}}\ \frac{E^{\phi_0}_{Y_{t_0}}\left[1/\phi_0(Y_{t-t_0})\right]}{E^{\phi_0}_x\left[1/\phi_0(Y_t)\right]}\\ &\underset{t\to\infty}{\longrightarrow}\ \frac{dP^{\phi_0}_{x\ |\mathcal{D}_{t_0}}}{dP_{x\ |\mathcal{D}_{t_0}}}\ \frac{\pi(\frac{1}{\phi_0})}{\pi(\frac{1}{\phi_0})} = \frac{dP^{\phi_0}_{x\ |\mathcal{D}_{t_0}}}{dP_{x\ |\mathcal{D}_{t_0}}},\end{aligned}$$

where we use the Markov property at the first equality, we force the apparition of λ_0 and ϕ_0 at the second equality. This lets appear the Radon-Nikodym derivative of $P^{\phi_0}_x$ with respect to P_x, whence the third equality. Lemma 5.6 then ensures the P_x-a.s. convergence to 1 of the fraction in the third equality as t goes to ∞. Moreover, the Scheffé lemma implies that the convergence also holds in $L^1(P_x)$. □

5.5 The convergence to the Q-process

Recall $\mathbb{P}^{(h)}_\nu$ defined after Corollary 4.14 is the distribution of the $(\mathcal{L},\beta,\alpha)$-super-process started at $\nu \in \mathcal{M}_f(S)$, conditionally on $\{H_{\max} = h\}$. We shall compare $\mathbb{P}^{(h)}_\nu$ with the distribution $\mathbb{P}^{(\geq h)}_\nu$ of the $(\mathcal{L},\beta,\alpha)$-super-process started at $\nu \in \mathcal{M}_f(S)$ conditionally on $\{H_{\max} \geq h\}$ defined by:

$$\mathbb{P}^{(\geq h)}_\nu = \mathbb{P}_\nu(\cdot|H_{\max} \geq h).$$

Recall the distribution of the Q-process, when it exists, is defined as the weak limit of $\mathbb{P}^{(\geq h)}_\nu$ when h goes to infinity. The next Lemma insures that if $\mathbb{P}^{(h)}_\nu$ weakly converges to a limit $\mathbb{P}^{(\infty)}_\nu$, then this limit is also the distribution of the Q-process.

Lemma 5.14. *Fix $t > 0$. If $\mathbb{P}^{(h)}_\nu$ converges weakly to $\mathbb{P}^{(\infty)}_\nu$ on $(\Omega,\mathcal{F}_t)$, then $\mathbb{P}^{(\geq h)}_\nu$ converges weakly to $\mathbb{P}^{(\infty)}_\nu$ on $(\Omega,\mathcal{F}_t)$.*

Proof. Let $Z = \mathbf{1}_A$ with $A \in \mathcal{F}_t$ such that $\mathbb{P}^{(\infty)}_\nu(\partial A) = 0$. Using the Williams' decomposition under $\mathbb{P}_\nu$ given by Corollary 4.14, we have for $h > t$:

$$\mathbb{E}^{(\geq h)}_\nu[Z] = e^{\nu(v_h)} \int_h^\infty \mathbb{E}^{(h')}_\nu[Z]\ f(h')dh',$$

where $f(h) = -\nu(\partial_h v_h)\exp(-\nu(v_h))$. We write down the difference:

$$\mathbb{E}^{(\geq h)}_\nu[Z] - \mathbb{E}^{(\infty)}_\nu[Z] = e^{\nu(v_h)} \int_h^\infty \left(\mathbb{E}^{(h')}_\nu[Z] - \mathbb{E}^{(\infty)}_\nu[Z]\right)\ f(h')dh'.$$

Since $\mathbb{P}^{(h')}_\nu$ weakly converges to $\mathbb{P}^{(\infty)}_\nu$ on $(\Omega,\mathcal{F}_t)$ and $\mathbb{P}^{(\infty)}_\nu(\partial A) = 0$, we get $\lim_{h'\to+\infty} \mathbb{E}^{(h')}_\nu[Z] - \mathbb{E}^{(\infty)}_\nu[Z] = 0$. We conclude that $\lim_{h\to+\infty} \mathbb{E}^{(\geq h)}_\nu[Z] - \mathbb{E}^{(\infty)}_\nu[Z] = 0$, which gives the result. □

We now address the question of convergence of the family of probability measures $(\mathbb{P}^{(h)}_x, h \geq 0)$, and first state the result on the convergence of $\mathbf{N}^{(h)}_x$.

Theorem 5.15. *Assume $\lambda \geq 0$, $(H2)$-$(H4)$. Let $t \geq 0$. The triplet*

$$((W_{T_{\max}})_{[0,t]}, (R^g_{T_{\max}})_{[0,t]}, (R^d_{T_{\max}})_{[0,t]})$$

under $\mathbf{N}_x^{(h)}$ converges weakly to the distribution of the triplet $(Y_{[0,t]}, R^{B,g}_{[0,t]}, R^{B,d}_{[0,t]})$ where Y has distribution $\mathrm{P}_x^{(\phi_0)}$ and conditionally on Y, $R^{B,g}$ and $R^{B,d}$ are two independent Poisson point measures with intensity ν^B given by (4.5). We even have the slightly stronger result: for any bounded measurable function F,

$$\mathbf{N}_x^{(h)}\left[F\big((W_{T_{\max}})_{[0,t]}, (R^g_{T_{\max}})_{[0,t]}, (R^d_{T_{\max}})_{[0,t]}\big)\right] \xrightarrow[h\to+\infty]{} \mathrm{E}_x^{\phi_0}\left[F\big(Y_{[0,t]}, R^{B,g}_{[0,t]}, R^{B,d}_{[0,t]}\big)\right]. \quad (5.14)$$

The proof of the next preliminary Lemma is postponed to the end of this Section.

Lemma 5.16. *Let R and $\tilde{R}$ be two Poisson point measures on a Polish space with respective intensity ν and $\tilde{\nu}$. Assume that $\tilde{\nu}(dx) = \mathbf{1}_A(x)\nu(dx)$, where A is measurable and $\nu(A^c) < +\infty$. Then for any bounded measurable function F, we have:*

$$\left|\mathbb{E}[F(R)] - \mathbb{E}[F(\tilde{R})]\right| \leq 2 \,\|F\|_\infty \,\nu(A^c).$$

Proof. Let $h > t$. We use notations from Theorem 4.12. Let F be a bounded measurable function on $\mathcal{W} \times (\mathbb{R}^+ \times \bar{\Omega})^2$. From the Williams' decomposition, Theorem 4.12, we have:

$$\begin{aligned}\mathbf{N}_x^{(h)}\left[F((W_{T_{\max}})_{[0,t]}, (R^g_{T_{\max}})_{[0,t]}, (R^d_{T_{\max}})_{[0,t]})\right] &= \mathrm{E}_x^{(h)}\left[F\big(Y_{[0,t]}, R^{W,g,(h)}_{[0,t]}, R^{W,d,(h)}_{[0,t]}\big)\right] \\ &= \mathrm{E}_x^{(h)}\left[\varphi^h(Y_{[0,t]})\right],\end{aligned}$$

where φ^h is defined by:

$$\varphi^h(y_{[0,t]}) = \mathrm{E}_x^{(h)}\left[F\big(y_{[0,t]}, R^{W,g,(h)}_{[0,t]}, R^{W,d,(h)}_{[0,t]}\big)\Big| Y = y\right].$$

We also set:

$$\varphi^\infty(y_{[0,t]}) = \mathrm{E}_x^{\phi_0}\left[F\big(y_{[0,t]}, R^{B,g}_{[0,t]}, R^{B,d}_{[0,t]}\big)\Big| Y = y\right].$$

We want to control:

$$\Delta_h = \mathbf{N}_x^{(h)}\left[F((W_{T_{\max}})_{[0,t]}, (R^g_{T_{\max}})_{[0,t]}, (R^d_{T_{\max}})_{[0,t]})\right] - \mathrm{E}_x^{\phi_0}\left[F(Y_{[0,t]}, R^{B,g}_{[0,t]}, R^{B,d}_{[0,t]})\right].$$

Notice that:

$$\begin{aligned}\Delta_h &= \mathrm{E}_x^{(h)}\left[\varphi^h(Y_{[0,t]})\right] - \mathrm{E}_x^{\phi_0}\left[\varphi^\infty(Y_{[0,t]})\right] \\ &= \left(\mathrm{E}_x^{(h)}\left[\varphi^h(Y_{[0,t]})\right] - \mathrm{E}_x^{\phi_0}\left[\varphi^h(Y_{[0,t]})\right]\right) + \mathrm{E}_x^{(\infty)}\left[(\varphi^h - \varphi^\infty)(Y_{[0,t]})\right]. \end{aligned} \quad (5.15)$$

We prove that the first term of the right hand-side of (5.15) converges to 0. We have:

$$\mathrm{E}_x^{(h)}\left[\varphi^h(Y_{[0,t]})\right] - \mathrm{E}_x^{\phi_0}\left[\varphi^h(Y_{[0,t]})\right] = \mathrm{E}_x\left[\left(\frac{d\mathrm{P}^{(h)}_{x\,|\mathcal{D}_t}}{d\mathrm{P}_{x\,|\mathcal{D}_t}} - \frac{d\mathrm{P}^{\phi_0}_{x\,|\mathcal{D}_t}}{d\mathrm{P}_{x\,|\mathcal{D}_t}}\right)\varphi^h(Y_{[0,t]})\right].$$

Then use that φ^h is bounded by $\|F\|_\infty$ and Proposition 5.9 to get:

$$\lim_{h\to\infty} \mathrm{E}_x^{(h)}\left[\varphi^h(Y_{[0,t]})\right] - \mathrm{E}_x^{\phi_0}\left[\varphi^h(Y_{[0,t]})\right] = 0. \quad (5.16)$$

We then prove the second term on the right hand-side of (5.15) converges to 0. Notice that conditionally on Y, $R^{W,g,(h)}_{[0,t]}$ and $R^{W,d,(h)}_{[0,t]}$ (resp. $R^{B,g}_{[0,t]}$ and $R^{B,d}_{[0,t]}$) are independent

Poisson point measures with intensity $\mathbf{1}_{[0,t]}(s)\ \nu^{W,(h)}(ds,dW)$ where $\nu^{W,(h)}$ is given by (4.22) (resp. $\mathbf{1}_{[0,t]}(s)\ \nu^{B}(ds,dW)$ where ν^{B} is given by (4.5)). And we have:

$$\mathbf{1}_{[0,t]}(s)\ \nu^{W,(h)}(ds,dW) = \mathbf{1}_{\{H_{\max}(W)<h-s\}}\mathbf{1}_{[0,t]}(s)\ \nu^{B}(ds,dW).$$

Thanks to (3.10) and (4.12), we get that:

$$\int \mathbf{1}_{\{H_{\max}(W)\geq h-s\}}\mathbf{1}_{[0,t]}(s)\ \nu^{B}(ds,dW) = \int_0^t ds\ \alpha(y_s)\mathbb{N}_{y_s}[H_{\max}\geq h-s] = \int_0^t ds\ v_{h-s}(y_s) < +\infty.$$

Using this Lemma with ν given by $\mathbf{1}_{[0,t]}(s)\ \nu^{B}(ds,dW)$ and A given by $\{H_{\max}(W) < h-s\}$, we deduce that:

$$\left|(\varphi^h-\varphi^\infty)(y_{[0,t]})\right| \leq 4\,\|\,F\,\|_\infty \int_0^t ds\ v_{h-s}(y_s).$$

We deduce that:

$$\left|\mathrm{E}_x^{(\infty)}\left[(\varphi^h-\varphi^\infty)(Y_{[0,t]})\right]\right| \leq 4\,\|\,F\,\|_\infty\, \mathrm{E}_x^{\phi_0}\left[\int_0^t ds\ v_{h-s}(Y_s)\right].$$

Recall that $(H1)$ implies that $v_{h-s}(x)$ converges to 0 as h goes to infinity. Since v is bounded (use (3.10) and (4.12)), by dominated convergence, we get:

$$\lim_{h\to\infty}\mathrm{E}_x^{\phi_0}\left[(\varphi^h-\varphi^\infty)(Y_{[0,t]})\right] = 0. \tag{5.17}$$

Therefore, we deduce from (5.15) that $\lim_{h\to+\infty}\Delta_h = 0$, which gives (5.14). □

Our next goal, achieved in forthcoming Corollary 5.19, is to establish the convergence of the measure valued super-process X under both $\mathbb{N}_x^{(h)}$ and $\mathbb{P}_\nu^{(h)}$. We first state a simple Lemma:

Lemma 5.17. *Assume that $\lambda_0 \geq 0$, and that $(H2)$-$(H4)$ hold. Then the following convergence holds in $\mathrm{L}^1(\nu)$:*

$$\frac{\partial_h v_h}{\nu(\partial_h v_h)} \xrightarrow[h\to+\infty]{} \frac{\phi_0}{\nu(\phi_0)}.$$

Proof. We deduce from (5.5) and (5.11) that:

$$\partial_t v_h(x) = f(h)\phi_0(x)\ (1+o_h(1))\,\mathrm{e}^{-\lambda_0 h},$$

for some positive function f of h. Then we get:

$$\frac{\partial_h v_h(x)}{\nu(\partial_h v_h)} = \frac{\phi_0(x)}{\nu(\phi_0)}(1+o_h(1)).$$

This gives the result, since $o_h(1)$ is bounded. □

We now define a super-process with spine distribution P^{ϕ_0}.

Definition 5.18. *Let $\nu\in\mathcal{M}_f(S)$. Let Y be distributed according to*

$$\int \nu(dx)\ \frac{\phi_0(x)}{\nu(\phi_0)}\ \mathrm{P}_x^{\phi_0},$$

and, conditionally on Y, let $\mathcal{N} = \sum_{j\in J}\delta_{(s_j,X^j)}$ be a Poisson point measure with intensity:

$$2\mathbf{1}_{\mathbb{R}^+}(s)ds\ \alpha(Y_s)\mathbb{N}_{Y_s}[dX].$$

Consider the process $X^{(\infty)} = (X_t^{(\infty)}, t\geq 0)$, which is defined for all $t\geq 0$ by:

$$X_t^{(\infty)} = \sum_{j\in J,\, s_j<t} X_{t-s_j}^j.$$

(i) *Let X' independent of $X^{(\infty)}$ and distributed according to $\mathbb{P}_\nu$. Then, we write $\mathbb{P}_\nu^{(\infty)}$ for the distribution of $X' + X^{(\infty)}$.*

(ii) *If ν is the Dirac mass at x, we write $\mathbb{N}_x^{(\infty)}$ for the distribution of $X^{(\infty)}$.*

Notice that for $\nu = \delta_x$, the distribution of Y reduces to $\mathrm{P}_x^{\phi_0}$. As a consequence of Theorem 5.15 and Lemma 5.17, we get the convergence of $\mathbb{P}_\nu^{(h)}$.

Corollary 5.19. *If $\lambda \geq 0$ and $(H2)$-$(H4)$ hold, then, for each $t \geq 0$:*

(i) *The distribution $\mathbb{N}_x^{(h)}$ converges weakly to $\mathbb{N}_x^{(\infty)}$ on $(\Omega, \mathcal{F}_t)$.*

(ii) *Let $\nu \in \mathcal{M}_f(S)$. The distribution $\mathbb{P}_\nu^{(h)}$ converges weakly to $\mathbb{P}_\nu^{(\infty)}$ on $(\Omega, \mathcal{F}_t)$.*

Proof. Point (i) is a direct consequence of Theorem 5.15, Definition 5.18 and Proposition 3.12. According to Corollary 4.14, under $\mathbb{P}_\nu^{(h)}$, X is distributed according to $X' + X^{(h)}$ where X' and $X^{(h)}$ are independent, X' is distributed according to $\mathbb{P}_\nu^{(\leq h)}$ and $X^{(h)}$ is distributed according to

$$\int_S \nu(dx)\, \frac{\partial_h v_h(x)}{\nu(\partial_h v_h)}\, \mathbb{N}_x^{(h)}[dX].$$

Lemma 5.17 implies this distribution converges weakly to:

$$\int_S \nu(dx)\, \frac{\phi_0(x)}{\nu(\phi_0)}\, \mathbb{N}_x^{(\infty)}[dX]$$

(because of the convergence of the densities in $\mathrm{L}^1(\nu)$) on $(\Omega, \mathcal{F}_t)$ as h goes to infinity. This and the weak convergence of $\mathbb{P}_\nu^{(\leq h)}$ to $\mathbb{P}_\nu$ as h goes to infinity give point (ii). □

Remark 5.20. *Engländer and Pinsky offer in [16] a decomposition of super-critical non-homogeneous super-diffusion, using immigration on the infinite lineages of the so-called prolific individuals (as denominated further in Bertoin, Fontbona and Martinez [4]). It is interesting to note that the generator of the lineages of these individuals is $\mathcal{L}^w$ where w formally satisfies the evolution equation $\mathcal{L}w = \psi(w)$, whereas the generator of the unique infinite lineage of the Q-process is $\mathcal{L}^{\phi_0}$ where ϕ_0 formally satisfies $\mathcal{L}\phi_0 = \beta\phi_0$. In particular, we notice that the generator of the backbone $\mathcal{L}^w$ depends on both β and α and that the generator of $\mathcal{L}^{\phi_0}$ does not depend on α.*

Now comes the postponed proof of Lemma 5.16.

Proof of Lemma 5.16. Similarly to Lemma 4.3 (formally take $f = -\infty \mathbf{1}_{A^c}$), we have:

$$\mathbb{E}\left[F(R)\mathbf{1}_{\{R(A^c)=0\}}\right] = \mathbb{E}\left[F(\tilde{R})\right] \mathrm{e}^{-\nu(A^c)}.$$

We deduce that:

$$\begin{aligned}
\left|\mathbb{E}[F(R)] - \mathbb{E}[F(\tilde{R})]\right| &= \left|\mathbb{E}[F(R)] - \mathbb{E}[F(R)\mathbf{1}_{\{R(A^c)=0\}}]\,\mathrm{e}^{\nu(A^c)}\right| \\
&\leq \left|\mathbb{E}[F(R)] - \mathbb{E}[F(R)\mathbf{1}_{\{R(A^c)=0\}}]\right| + \left|\mathbb{E}[F(R)\mathbf{1}_{\{R(A^c)=0\}}](1 - \mathrm{e}^{\nu(A^c)})\right| \\
&\leq \|F\|_\infty (1 - \mathbb{P}(R(A^c) = 0)) + \|F\|_\infty\, \mathbb{P}(R(A^c) = 0)(\mathrm{e}^{\nu(A^c)} - 1) \\
&= 2\,\|F\|_\infty (1 - \mathrm{e}^{-\nu(A^c)}) \\
&\leq 2\,\|F\|_\infty\, \nu(A^c).
\end{aligned}$$

This gives the result. □

5.6 The convergence of the Williams spatial motion from the extinction time

We will show in Section 5.7 that the convergence as $h \to \infty$ of the process $(X_{h+s}, s \leq 0)$, where X has law $\mathbb{P}^{(h)}$, essentially amounts to that of $(Y_{h+s}, s \leq 0)$, where Y has law $\mathrm{P}^{(h)}$. Our goal in this Section is therefore to provide a convergence result for the spatial motion $\mathrm{P}^{(h)}$ shifted by $-h$. We shall work with the space $D^- = D(\mathbb{R}^-, S)$ equipped with the Skorokhod topology. For I an interval on $(-\infty, 0]$, we set $\mathcal{D}_I = \sigma(Y_r, r \in I)$. Let us denote by θ the translation operator, which maps any process R to the shifted process $\theta_h(R)$ defined by:

$$\theta_h(R)_{\cdot} = R_{\cdot+h} \;.$$

The process R may be a path, a killed path or a point measure, in which case we set, for $R = \sum_{j\in J} \delta_{(s_j, x_j)}$, $\theta_h(R) = \sum_{j\in J} \delta_{(h+s_j, x_j)}$. We denote $\mathrm{P}^{(-h)}$ the push forward probability measure of $\mathrm{P}^{(h)}$ by θ_h, defined on $\mathcal{D}_{[-h,0]}$ by:

$$\mathrm{P}^{(-h)}(Y \in \bullet) = \mathrm{P}^{(h)}(\theta_h(Y) \in \bullet) = \mathrm{P}^{(h)}((Y_{h+s}, s \in [-h, 0]) \in \bullet). \tag{5.18}$$

Lemma 5.6 implies that the probability measure $\mathrm{P}_\pi^{\phi_0}$ admits a stationary version on $D(\mathbb{R}, S)$, which we still denote by $\mathrm{P}_\pi^{\phi_0}$. Observe from (4.19), (5.18) and (5.3) that $\mathrm{P}_\pi^{(-h)}$ is absolutely continuous with respect to $\mathrm{P}_\pi^{\phi_0}$ on $\mathcal{D}_{[-h,0]}$. We define $L^{(-h)}$ the corresponding Radon-Nikodym derivative:

$$L^{(-h)} = \frac{d\mathrm{P}^{(-h)}_{\pi\;|\mathcal{D}_{[-h,0]}}}{d\mathrm{P}^{\phi_0}_{\pi\;|\mathcal{D}_{[-h,0]}}} = \frac{1}{\alpha(Y_0)\phi_0(Y_0)} \, \frac{\partial_h v_h^0 \, \mathrm{e}^{\beta_0 h}}{\partial_h v_h^{\phi_0}(Y_{-h}) \, \mathrm{e}^{\lambda_0 h}} \, \mathrm{e}^{-2\int_{-h}^0 (\alpha(Y_s) v_{-s}(Y_s) - v_{-s}^0) \, ds} . \tag{5.19}$$

The next Lemma insures the convergence of $L^{(-h)}$ to a limit, say $L^{(-\infty)}$.

Lemma 5.21. *Assume $\lambda_0 > 0$, $(H2)$-$(H4)$. We have:*

$$L^{(-h)} \xrightarrow[h\to+\infty]{} L^{(-\infty)} \quad \mathrm{P}_\pi^{\phi_0}\text{-a.s. and in } \mathrm{L}^1(\mathrm{P}_\pi^{\phi_0}).$$

Proof. Notice that $\lim_{h\to+\infty} \partial_h v_h^0 \, \mathrm{e}^{\beta_0 h} = -\beta_0^2$. We also deduce from (4.14), (4.15) and (5.8) that $\int_{-h}^0 (\alpha(Y_s) v_{-s}(Y_s) - v_{-s}^0) \, ds$ increases, as h goes to infinity, to $\int_{-\infty}^0 (\alpha(Y_s) v_{-s}(Y_s) - v_{-s}^0) \, ds$ which is finite. For fixed $t > 0$, we also deduce from (5.12) (with h replaced by $h - t$ and ε by t) that $\mathrm{P}_\pi^{\phi_0}$ a.s.:

$$\lim_{h\to+\infty} \partial_t v_h^{\phi_0}(Y_{-h}) \, \mathrm{e}^{\lambda_0 h} = \mathrm{e}^{\lambda_0 t} \, \mathrm{E}_\pi^{\phi_0} \left[\mathrm{e}^{-2\int_{-\infty}^{-t} ds \;\alpha(Y_s)\; \phi_0(Y_s) \, v_{-s}^{\phi_0}(Y_s)} \, \partial_t v_t^{\phi_0}(Y_{-t}) \right].$$

We deduce from (5.19) the $\mathrm{P}_\pi^{\phi_0}$ a.s. convergence of $(L^{(-h)}, h > 0)$ to $L^{(-\infty)}$. Notice from (5.9) that, for fixed t, the sequence $(L^{(-h)}, h > t)$ is bounded. Hence the previous convergence holds also in $\mathrm{L}^1(\mathrm{P}_\pi^{\phi_0})$. □

As $\mathrm{E}_\pi^{\phi_0}\left[L^{(-h)}\right] = 1$, we deduce that $\mathrm{E}_\pi^{\phi_0}\left[L^{(-\infty)}\right] = 1$. We define the probability measure $\mathrm{P}_\pi^{(-\infty),\phi_0}$ on $(D^-, \mathcal{D}_{(-\infty,0]})$ by its Radon Nikodym derivative:

$$\frac{d\mathrm{P}^{(-\infty),\phi_0}_{\pi\;|\mathcal{D}_{(-\infty,0]}}}{d\mathrm{P}^{\phi_0}_{\pi\;|\mathcal{D}_{(-\infty,0]}}} = L^{(-\infty)}. \tag{5.20}$$

Remark 5.22. *Assume $\lambda_0 > 0$, $(H2)$-$(H4)$. Define for $h > t > 0$:*

$$L_{-t}^{(-h)} = \mathrm{E}_\pi^{\phi_0}[L^{(-h)} | \mathcal{D}_{(-\infty,-t]}] = \frac{d\mathrm{P}^{(-h)}_{\pi\;|\mathcal{D}_{[-h,-t]}}}{d\mathrm{P}^{\phi_0}_{\pi\;|\mathcal{D}_{[-h,-t]}}}$$

$$L_{-t}^{(-\infty)} = \mathrm{E}_\pi^{\phi_0}[L^{(-\infty)} | \mathcal{D}_{(-\infty,-t]}] = \frac{d\mathrm{P}^{(-\infty),\phi_0}_{\pi\;|\mathcal{D}_{(-\infty,-t]}}}{d\mathrm{P}^{\phi_0}_{\pi\;|\mathcal{D}_{(-\infty,-t]}}}.$$

Using (4.21) and Lemma 5.4, we get:

$$L_{-t}^{(-h)} = \frac{\partial_t v_t(Y_{-t})}{\partial_t v_h(Y_{-h})} \frac{\phi_0(Y_{-h})}{\phi_0(Y_{-t})} \mathrm{e}^{-\lambda_0(h-t)} \mathrm{e}^{-2\int_{-h}^{-t} ds\ \alpha(Y_s)\, v_{-s}(Y_s)}$$

$$= \frac{\mathrm{e}^{-2\int_{-h}^{-t} ds\ \alpha(Y_s)\ \phi_0(Y_s)\, v_{-s}^{\phi_0}(Y_s)}\ \partial_t v_t^{\phi_0}(Y_{-t})}{\mathrm{E}_{Y_{-h}}^{\phi_0}\left[\mathrm{e}^{-2\int_{-h}^{-t} ds\ \alpha(Y_s)\ \phi_0(Y_s)\, v_{-s}^{\phi_0}(Y_s)}\ \partial_t v_t^{\phi_0}(Y_{-t})\right]}.$$

Using Lemma 5.7 and convergence of $(L_{-t}^{(-h)}, h > t)$ *to* $L_{-t}^{(-\infty)}$*, which is a consequence of Lemma 5.21, we also get that for* $t > 0$*:*

$$L_{-t}^{(-\infty)} = \frac{\mathrm{e}^{-2\int_{-\infty}^{-t} ds\ \alpha(Y_s)\ \phi_0(Y_s)\, v_{-s}^{\phi_0}(Y_s)}\ \partial_t v_t^{\phi_0}(Y_{-t})}{\mathrm{E}_{\pi}^{\phi_0}\left[\mathrm{e}^{-2\int_{-\infty}^{-t} ds\ \alpha(Y_s)\ \phi_0(Y_s)\, v_{-s}^{\phi_0}(Y_s)}\ \partial_t v_t^{\phi_0}(Y_{-t})\right]}.$$

These formulas are more self-contained than (5.19) and the definition of $L^{(-\infty)}$ *as a limit, but they only hold for* $t > 0$*.*

Lemma 5.23. *Assume* $\lambda_0 > 0$, $(H2)$-$(H4)$*. For all* $x \in S$, $t \geq 0$*, and* f *bounded and* $\mathcal{D}_{[-t,0]}$ *measurable:*

$$\mathrm{E}_x^{(-h)}\left[f(Y_{[-t,0]})\right] \xrightarrow[h\to+\infty]{} \mathrm{E}^{(-\infty),\phi_0}\left[f(Y_{[-t,0]})\right].$$

Proof. Let $0 < t$ and F be a bounded and $\mathcal{D}_{[-t,0]}$ measurable function. For h large enough, we have:

$$\begin{aligned}
\mathrm{E}_x^{(-h)}\left[F(Y_{[-t,0]})\right] &= \mathrm{E}_x^{(h)}\left[\mathrm{E}_{Y_{h/2}}^{(h/2)}\left[F(\theta_{h/2}(Y)_{[-t,-s]})\right]\right]\\
&= \mathrm{E}_x^{\phi_0}\left[\frac{d\mathrm{P}^{(h)}_{x\ |\mathcal{D}_{h/2}}}{d\mathrm{P}^{\phi_0}_{x\ |\mathcal{D}_{h/2}}}\ \mathrm{E}_{Y_{h/2}}^{(h/2)}\left[F(\theta_{h/2}(Y)_{[-t,0]})\right]\right]\\
&= \mathrm{E}_x^{\phi_0}\left[\mathrm{E}_{Y_{h/2}}^{(h/2)}\left[F(\theta_{h/2}(Y)_{[-t,0]})\right]\right] + o_h(1)\\
&= \mathrm{E}_\pi^{(h/2)}\left[F(\theta_{h/2}(Y)_{[-t,0]})\right] + o_h(1)\\
&= \mathrm{E}_\pi^{(-h/2)}\left[F(Y_{[-t,0]})\right] + o_h(1),
\end{aligned}$$

where we used the definition of $\mathrm{P}^{(-h)}$ and the Markov property for the first equality, Lemma 5.8 together with F bounded by $\|F\|_\infty$ for the third, and Lemma 5.6 for the fourth. We continue the computations as follows:

$$\begin{aligned}
\mathrm{E}_x^{(-h)}\left[F(Y_{[-t,0]})\right] &= \mathrm{E}_\pi^{\phi_0}\left[L^{(-h/2)}F(Y_{[-t,0]})\right] + o_h(1)\\
&= \mathrm{E}_\pi^{\phi_0}\left[L^{(-\infty)}F(Y_{[-t,0]})\right] + o_h(1)\\
&= \mathrm{E}_\pi^{(-\infty),\phi_0}\left[F(Y_{[-t,0]})\right] + o_h(1),
\end{aligned}$$

where we used Lemma 5.21 for the second equality. □

5.7 The convergence backward from the extinction time

A last preparatory Lemma is needed for proving the convergence of the super-process backward from the extinction time. The proof of the Lemma is a straightforward application of (5.8).

Lemma 5.24. *Assume* $\lambda_0 > 0$*, and* $(H2)$-$(H4)$*. Then, for all* $t > 0$*, there exists a non-negative function* m *such that for all* $x \in S$*, for all* $h > 0$*:*

$$v_h(x) - v_{h+t}(x) \leq m(h) \quad \textit{and} \quad \int_1^\infty dr\ m(r) < \infty.$$

We now state our last main result.

Theorem 5.25. *Assume $\lambda > 0$, $(H2)$-$(H4)$.*

(i) The distribution of the triplet $\big(\theta_h(W_{T_{\max}})_{[-t,0]}, \theta_h(R_g^{T_{\max}})_{[-t,0]}, \theta_h(R_d^{T_{\max}})_{[-t,0]}\big)$ under $\mathbf{N}_x^{(h)}$ converges weakly to the distribution of the triplet $(Y_{[-t,0]}, R^{W,g}_{[-t,0]}, R^{W,d}_{[-t,0]})$ where Y has distribution $\mathrm{P}^{(-\infty)}$ and conditionally on Y, $R^{W,g}$ and $R^{W,d}$ are two independent Poisson point measures with intensity:

$$\mathbf{1}_{\{s<0\}}\alpha(Y_s)\, ds\, \mathbf{1}_{\{H_{\max}(W)<-s\}}\, \mathbf{N}_{Y_s}[dW].$$

We even have the slightly stronger result: for any bounded measurable function F,

$$\mathbf{N}_x^{(h)}\Big[F\big(\theta_h(W_{T_{\max}})_{[-t,0]}, \theta_h(R_g^{T_{\max}})_{[-t,0]}, \theta_h(R_d^{T_{\max}})_{[-t,0]}\big)\Big] \xrightarrow[h\to+\infty]{} \mathrm{E}^{(-\infty)}\Big[F\big(Y_{[-t,0]}, R^{W,g}_{[-t,0]}, R^{W,d}_{[-t,0]}\big)\Big]. \quad (5.21)$$

(ii) The process $\theta_h(X)_{[-t,0]} = (X_{h+s}, s\in[-t,0])$ under $\mathbb{N}_x^{(h)}$ weakly converges towards $X^{(-\infty)}_{[-t,0]}$, where for $s\le 0$:

$$X_s^{(-\infty)} = \sum_{j\in J,\ s_j<s} X^j_{s-s_j},$$

and conditionally on Y with distribution $\mathrm{P}^{(-\infty)}$, $\sum_{j\in J}\delta_{(s_j,X^j)}$ is a Poisson point measure with intensity:

$$2\,\mathbf{1}_{\{s<0\}}\alpha(Y_s)\, ds\, \mathbf{1}_{\{H_{\max}(X)<-s\}}\, \mathbb{N}_{Y_s}[dX].$$

Proof. Let $0<t<h$. We use notations from Theorems 4.12, 5.15. Let F be a bounded measurable function on $\mathcal{W}^-\times(\mathbb{R}^-\times\bar\Omega)^2$ with $\mathcal{W}^-$ the set of killed paths indexed by negative times. We want to control δ_h defined by:

$$\delta_h = \mathbf{N}_x^{(h)}\Big[F\big(\theta_h(W_{T_{\max}})_{[-t,0]}, \theta_h(R^g_{T_{\max}})_{[-t,0]}, \theta_h(R^d_{T_{\max}})_{[-t,0]}\big)\Big] - \mathrm{E}^{(-\infty)}\Big[F\big(Y_{[-t,0]}, R^{W,g}_{[-t,0]}, R^{W,d}_{[-t,0]}\big)\Big].$$

We set:

$$\Upsilon(y_{[-t,0]}) = \mathrm{E}^{(-\infty)}\Big[F\big(y_{[-t,0]}, R^{W,g}_{[-t,0]}, R^{W,d}_{[-t,0]}\big)\Big| Y=y\Big].$$

We deduce from Williams' decomposition, Theorem 4.12, and the definition of $R^{W,g}$ and $R^{W,d}$, that:

$$\mathbf{N}_x^{(h)}\Big[F\big(\theta_h(W_{T_{\max}})_{[-t,0]}, \theta_h(R^g_{T_{\max}})_{[-t,0]}, \theta_h(R^d_{T_{\max}})_{[-t,0]}\big)\Big] = \mathrm{E}_x^{(-h)}\big[\Upsilon(Y_{[-t,0]})\big].$$

We thus rewrite δ_h as:

$$\delta_h = \mathrm{E}_x^{(-h)}\big[\Upsilon(Y_{[-t,0]})\big] - \mathrm{E}^{(-\infty)}\big[\Upsilon(Y_{[-t,0]})\big].$$

The function Υ being bounded by $\|F\|_\infty$ and measurable, we may conclude under assumption $(H6)$ that $\lim_{h\to+\infty}\delta_h = 0$. This proves point (i).

We now prove point (ii). Let $t > 0$ and $\varepsilon > 0$ be fixed. Let F be a bounded measurable function on the space of continuous measure-valued applications indexed by negative times. For a point measure on $\mathbb{R}^- \times \bar{\Omega}$, $M = \sum_{i\in\mathcal{I}} \delta_{(s_i,W_i)}$, we set:

$$\tilde{F}(M) = F\left(\big(\sum_{i\in\mathcal{I}} \theta_{s_i}(X(W_i)) \big)_{[-t,0]} \right).$$

For $h > t$, we want a control of $\bar{\delta}_h$ defined by:

$$\bar{\delta}_h = \mathbb{N}_x^{(h)}\left[F\big(\theta_h(X)_{[-t,0]}\big) \right] - \mathrm{E}^{(-\infty)}\left[\tilde{F}\big(R^{W,g} + R^{W,d}\big) \right].$$

By Corollary 4.13, we have:

$$\mathbb{N}_x^{(h)}\left[F\big(\theta_h(X)_{[-t,0]}\big) \right] = \mathbf{N}_x^{(h)}\left[\tilde{F}\big(\theta_h(R^g_{T_{\max}} + R^d_{T_{\max}})\big) \right].$$

Thus, we get:

$$\bar{\delta}_h = \mathbf{N}_x^{(h)}\left[\tilde{F}\big(\theta_h(R^g_{T_{\max}} + R^d_{T_{\max}})\big) \right] - \mathrm{E}^{(-\infty)}\left[\tilde{F}\big(R^{W,g} + R^{W,d}\big) \right]. \tag{5.22}$$

For $a > s$ fixed, we introduce $\bar{\delta}^a_h$, for $h > a$, defined by:

$$\bar{\delta}^a_h = \mathbf{N}_x^{(h)}\left[\tilde{F}\big(\theta_h(R^g_{T_{\max}} + R^d_{T_{\max}})_{[-a,0]}\big) \right] - \mathrm{E}^{(-\infty)}\left[\tilde{F}\big((R^{W,g} + R^{W,d})_{[-a,0]}\big) \right]. \tag{5.23}$$

Notice the restriction of the point measures to $[-a, 0]$. Point (i) directly yields that $\lim_{h\to+\infty} \bar{\delta}^a_h = 0$. Thus, there exists $h_a > 0$ such that for all $h \geq h_a$,

$$\bar{\delta}^a_h \leq \varepsilon/2.$$

We now consider the difference $\bar{\delta}_h - \bar{\delta}^a_h$. We associate to the point measures M introduced above the most recent common ancestor of the population alive at time $-t$:

$$A(M) = \sup\{s > 0; \sum_{i\in\mathcal{I}} \mathbf{1}_{\{s_i<-s\}} \mathbf{1}_{\{H_{\max}(W_i)>-t-s_i\}} \neq 0\}.$$

Let us observe that:

$$\mathbf{N}_x^{(h)} \text{ a.s., } \tilde{F}\big(\theta_h(R^g_{T_{\max}} + R^d_{T_{\max}})\big)\mathbf{1}_{\{A\leq a\}} = \tilde{F}\big(\theta_h(R^g_{T_{\max}} + R^d_{T_{\max}})_{[-a,0]}\big)\mathbf{1}_{\{A\leq a\}}, \tag{5.24}$$

with $A = A(\theta_h(R^g_{T_{\max}} + R^d_{T_{\max}})_{[-h,0]})$ in the left and in the right hand side. Similarly, we have:

$$\mathrm{P}^{(-\infty)} \text{ a.s., } \tilde{F}\big(R^{W,g} + R^{W,d}\big)\mathbf{1}_{\{A\leq a\}} = \tilde{F}\big((R^{W,g} + R^{W,d})_{[-a,0]}\big)\mathbf{1}_{\{A\leq a\}}, \tag{5.25}$$

with $A = A\big(R^{W,g} + R^{W,d}\big)$ in the left and in the right hand side. We thus deduce the following bound on $\bar{\delta}_h - \bar{\delta}^a_h$:

$$\begin{aligned}
|\bar{\delta}_h - \bar{\delta}^a_h| &\leq 2\|F\|_\infty \left[\mathbf{N}_x^{(h)}\big[A > a\big] + \mathrm{P}^{(-\infty)}\big[A > a\big] \right] \\
&= 2\|F\|_\infty \Bigg[\mathrm{E}_x^{(-h)}\big[1 - \mathrm{e}^{-\int_a^h dr\ 2\alpha(Y_{-r})(v_{r-t}-v_r)(Y_{-r})}\big] \\
&\qquad\qquad + \mathrm{E}^{(-\infty)}\big[1 - \mathrm{e}^{-\int_a^\infty dr\ 2\alpha(Y_{-r})(v_{r-t}-v_r)(Y_{-r})}\big] \Bigg] \\
&\leq 8\|F\|_\infty \|\alpha\|_\infty \int_{a-t}^\infty dr\ g(r),
\end{aligned}$$

where we used (5.22), (5.23), (5.24) and (5.25) for the first inequality, the definition of A for the first equality, as well as (5.24) and the fact that $1-\mathrm{e}^{-x} \leq x$ if $x \geq 0$ for the last inequality. From (5.24), we may choose a large enough such that: $|\bar{\delta}_h - \bar{\delta}_h^a| \leq \varepsilon/2$. We deduce that for all $h \geq \max(a, h_a)$: $|\bar{\delta}_h| \leq |\bar{\delta}_h - \bar{\delta}_h^a| + |\bar{\delta}_h^a| \leq \varepsilon$. This proves point (ii).

□

Appendix: Proof of Lemmas 5.1, 5.2, 5.4, 5.5 and 5.7

This appendix is devoted to the proofs of the technical Lemmas contained in Section 5.3.

The following proof of Lemma 5.1 uses a coupling argument in the construction of Dhersin and Serlet [9] of a modified snake for non-homogeneous super-processes. This coupling yields first bounds on $v_t^{\phi_0}(x)$.

Proof of Lemma 5.1. From $(H2)$ and $(H4)$, there exist $m, M \in \mathbb{R}$ such that

$$\forall x \in S,\ 0 < m \leq \alpha\phi_0(x) \leq M < \infty.$$

Let W be a $(\frac{M}{\alpha\phi_0}\mathcal{L}, 0, M)$ Brownian snake and define the time change Φ for every $w \in \mathcal{W}$ by $\Phi_t(w) = \int_0^t ds\ \frac{M}{\alpha\phi_0}(w(s))$. As $\partial_t\Phi_t(w) \geq 1$, we have that $t \to \Phi_t(w)$ is strictly increasing. Let $t \to \Phi_t^{(-1)}(w)$ denote its inverse. Then, using Proposition 12 of [9], first step of the proof, we have that the time changed snake $W \circ \Phi^{-1}$, with value

$$(W \circ \Phi^{-1})_s = (W_s(\Phi_t^{-1}(W_s)), t \in [0, \Phi^{-1}(W_s, H_s)])$$

at time s, is a $(\mathcal{L}, 0, \alpha\phi_0)$ Brownian snake. Noting the obvious bound on the time change $\Phi_t^{-1}(w) \leq t$, we have, according to Theorem 14 of Dhersin and Serlet [9]:

$$\mathbf{P}^{\left(\frac{M}{\alpha\phi_0}\mathcal{L}^{\phi_0},0,M\right)}_{\frac{\alpha\phi_0(x)}{M}\delta_x}(H_{\max} \leq t) \geq \mathbf{P}^{(\mathcal{L}^{\phi_0},0,\alpha\phi_0)}_{\delta_x}(H_{\max} \leq t)$$

which implies:

$$\frac{\alpha\phi_0(x)}{M}\,\mathbf{N}_x^{\left(\frac{M}{\alpha\phi_0}\mathcal{L}^{\phi_0},0,M\right)}(H_{\max} > t) \leq \mathbf{N}_x^{(\mathcal{L}^{\phi_0},0,\alpha\phi_0)}(H_{\max} > t)$$

from the exponential formula for Poisson point measures. Now, the left hand side of this inequality may be computed explicitly:

$$\mathbf{N}_x^{\left(\frac{M}{\alpha\phi_0}\mathcal{L}^{\phi_0},0,M\right)}(H_{\max} > t) = \mathbb{N}_x^{\left(\frac{M}{\alpha\phi_0}\mathcal{L}^{\phi_0},0,M\right)}(H_{\max} > t) = \frac{1}{Mt}$$

and the right hand side of this inequality is $v_t^{\phi_0}(x)$ from (5.4). We thus have proved that:

$$\frac{\alpha\phi_0(x)}{M^2 t} \leq v_t^{\phi_0}(x),$$

and this yields the first part of the inequality of Lemma 5.1. The second part is obtained in the same way using the coupling with the $\left(\frac{m}{\alpha\phi_0}\mathcal{L}^{\phi_0}, 0, m\right)$ Brownian snake. □

Proof of Lemma 5.2. Assumption $(H4)$ allow us to apply Lemma 5.1 for the case $\lambda_0 = 0$, which yields that $v_\infty^{\phi_0} = 0$, and then $v_\infty = 0$ thanks to (5.5). This in turn implies that $(H1)$ holds in the case $\lambda_0 = 0$ according to Lemma 2.5. For $\lambda_0 > 0$, we may use item 5 of Proposition 13 of [9] (which itself relies on a Girsanov theorem) with $\mathbb{P}^{(\mathcal{L},0,\alpha\phi_0)}$ in the rôle of $\mathbb{P}^c$ and $\mathbb{P}^{(\mathcal{L}^{\phi_0},\lambda_0,\alpha\phi_0)}$ in the rôle of $\mathbb{P}^{b,c}$ to conclude that the extinction property $(H1)$ holds under $\mathbb{P}^{(\mathcal{L}^{\phi_0},\lambda_0,\alpha\phi_0)}$. □

Proof of Lemma 5.4. Let $\varepsilon > 0$. The function v^{ϕ_0} is known to solve the following mild form of the Laplace equation, see equation (2.3):

$$v^{\phi_0}_{t+s}(x) + \mathrm{E}^{\phi_0}_x \left[\int_0^t dr \ \left(\lambda_0 \, v^{\phi_0}_{t+s-r}(Y_r) + \alpha(Y_r)\phi_0(Y_r)(v^{\phi_0}_{t+s-r}(Y_r))^2 \right) \right] = \mathrm{E}^{\phi_0}_x \left[v^{\phi_0}_s(Y_t) \right].$$

By differentiating with respect to s and taking $t = t - s$, we deduce from dominated convergence and the bounds (4.12), (4.13) and (4.15) on $v^{\phi_0} = v/\phi_0$ and its time derivative (valid under the assumptions $(H1)$-$(H3)$) the following mild form on the time derivative $\partial_t v^{\phi_0}$:

$$\partial_t v^{\phi_0}_t(x) + \mathrm{E}^{\phi_0}_x \left[\int_0^{t-s} dr \ \left(\lambda_0 \ + 2\alpha(Y_r)\phi_0(Y_r) v^{\phi_0}_{t-r}(Y_r) \right) \partial_t v^{\phi_0}_{t-r}(Y_r) \right] = \mathrm{E}^{\phi_0}_x \left[\partial_t v^{\phi_0}_s(Y_{t-s}) \right].$$

From the Markov property, for fixed $t > 0$, the two following processes:

$$\left(v^{\phi_0}_{t-s}(Y_s) - \int_0^s dr \ \left(\lambda_0 + \alpha(Y_r)\phi_0(Y_r) v^{\phi_0}_{t-r}(Y_r) \right) \, v^{\phi_0}_{t-r}(Y_r), 0 \leq s < t \right)$$

and

$$\left(\partial_t v^{\phi_0}_{t-s}(Y_s) - \int_0^s dr \ \left(\lambda_0 \ + 2\alpha(Y_r)\phi_0(Y_r) v^{\phi_0}_{t-r}(Y_r) \right) \partial_t v^{\phi_0}_{t-r}(Y_r), 0 \leq s < t \right)$$

are $\mathcal{D}_s$-martingale under $\mathrm{P}^{\phi_0}_\pi$. A Feynman-Kac manipulation, as done in the proof of Lemma 3.1, enables us to conclude that for fixed $t > 0$:

$$\left(v^{\phi_0}_{t-s}(Y_s) \, \mathrm{e}^{-\int_0^s dr \ \left(\lambda_0 + \alpha(Y_r)\phi_0(Y_r) v^{\phi_0}_{t-r}(Y_r) \right)}, 0 \leq s < t \right)$$

and

$$\left(\partial_t v^{\phi_0}_{t-s}(Y_s) \, \mathrm{e}^{-\int_0^s dr \ \left(\lambda_0 \ + 2\alpha(Y_r)\phi_0(Y_r) v^{\phi_0}_{t-r}(Y_r) \right)}, 0 \leq s < t \right)$$

are $\mathcal{D}_s$-martingale under $\mathrm{P}^{\phi_0}_\pi$. Taking expectations at time $s = 0$ and $s = h$ with $t = h + \varepsilon$, we get the representations formulae stated in the Lemma:

$$v^{\phi_0}_{h+\varepsilon}(x) = \mathrm{e}^{-\lambda_0 h} \, \mathrm{E}^{\phi_0}_x \left[\mathrm{e}^{-\int_0^h ds \ \alpha(Y_s) \ \phi_0(Y_s) \ v^{\phi_0}_{h+\varepsilon-s}(Y_s)} \ v^{\phi_0}_\varepsilon(Y_h) \right],$$
$$\partial_h v^{\phi_0}_{h+\varepsilon}(x) = \mathrm{e}^{-\lambda_0 h} \, \mathrm{E}^{\phi_0}_x \left[\mathrm{e}^{-2\int_0^h ds \ \alpha(Y_s) \ \phi_0(Y_s) \ v^{\phi_0}_{h+\varepsilon-s}(Y_s)} \ \partial_h v^{\phi_0}_\varepsilon(Y_h) \right].$$

□

Proof of Lemma 5.5. Since $v^{\phi_0}_\varepsilon = v_\varepsilon/\phi_0 = \tilde{v}_\varepsilon/(\alpha\phi_0)$, we can conclude from (4.12), $(H2)$ and $(H4)$ that $v^{\phi_0}_\varepsilon$ is bounded from above and from below by positive constants. Similarly, we also get from (4.13), (4.14) and (4.15) that $|\partial_h \tilde{v}_\varepsilon|$ is bounded from above and from below by two positive constants. Thus, we have the existence of four positive constants, D_1, D_2, D_3 and D_4, such that, for all $x \in S$:

$$D_1 \leq v^{\phi_0}_\varepsilon(x) \leq D_2, \tag{5.26}$$
$$D_3 \leq |\partial_t v^{\phi_0}_\varepsilon(x)| \leq D_4. \tag{5.27}$$

From equations (5.6), (5.26) and the positivity of v^{ϕ_0}, we deduce that:

$$v^{\phi_0}_{h+\varepsilon}(x) \leq D_2 \, \mathrm{e}^{-\lambda_0 h} \, . \tag{5.28}$$

Putting back (5.28) into (5.6), we have the converse inequality $D_5 \, \mathrm{e}^{-\lambda_0 h} \leq v^{\phi_0}_{h+\varepsilon}(x)$ with $D_5 = D_1 \exp\{-D_2 \, \| \alpha \|_\infty \, \| \phi_0 \|_\infty / \lambda_0\} > 0$. This gives (5.8).

Similar arguments using (5.7) and (5.27) instead of (5.6) and (5.26), gives (5.9). □

Proof of Lemma 5.7. Using the Feynman-Kac representation (5.6) of $\partial_h v_{h+\varepsilon}^{\phi_0}$ and the Markov property, we have:

$$\begin{aligned}
&\partial_h v_{h+\varepsilon}^{\phi_0}(x)\,\mathrm{e}^{\lambda_0 h}\\
&= \mathrm{E}_x^{\phi_0}\left[\mathrm{e}^{-2\int_0^h ds\ \alpha(Y_s)\,\phi_0(Y_s)\,v_{h+\varepsilon-s}^{\phi_0}(Y_s)}\,\partial_h v_\varepsilon^{\phi_0}(Y_h)\right]\\
&= \mathrm{E}_x^{\phi_0}\left[\mathrm{e}^{-2\int_0^{\sqrt{h}} ds\ \alpha\,\phi_0\,v_{h+\varepsilon-s}^{\phi_0}(Y_s)}\,\mathrm{E}_{Y_{\sqrt{h}}}^{\phi_0}\big[\mathrm{e}^{-2\int_0^{h-\sqrt{h}} ds\,\alpha(Y_s)\,\phi_0(Y_s)\,v_{h-\sqrt{h}+\varepsilon-s}^{\phi_0}(Y_s)}\,\partial_h v_\varepsilon^{\phi_0}(Y_{h-\sqrt{h}})\big]\right].
\end{aligned}$$

Notice that

$$\left|\int_0^{\sqrt{h}} ds\ \alpha\,\phi_0\,v_{h+\varepsilon-s}^{\phi_0}(Y_s)\right| \le \|\alpha\|_\infty\,\|\phi_0\|_\infty\,\sqrt{h}\,\|v_{h+\varepsilon-\sqrt{h}}^{\phi_0}\|_\infty = o_h(1), \tag{5.29}$$

according to Lemma 5.5 if $\lambda_0 > 0$ and Lemma 5.1 if $\lambda_0 = 0$. We get:

$$\begin{aligned}
\partial_h v_{h+\varepsilon}^{\phi_0}(x)\,\mathrm{e}^{\lambda_0 h} &= \mathrm{E}_x^{\phi_0}\left[\mathrm{E}_{Y_{\sqrt{h}}}^{\phi_0}\big[\mathrm{e}^{-2\int_0^{h-\sqrt{h}} ds\ \alpha(Y_s)\,\phi_0(Y_s)\,v_{h-\sqrt{h}+\varepsilon-s}^{\phi_0}(Y_s)}\,\partial_h v_\varepsilon^{\phi_0}(Y_{h-\sqrt{h}})\big]\right]\big(1+o_h(1)\big)\\
&= \mathrm{E}_\pi^{\phi_0}\left[\mathrm{e}^{-2\int_0^{h-\sqrt{h}} ds\ \alpha(Y_s)\,\phi_0(Y_s)\,v_{h-\sqrt{h}+\varepsilon-s}^{\phi_0}(Y_s)}\,\partial_h v_\varepsilon^{\phi_0}(Y_{h-\sqrt{h}})\right]\big(1+o_h(1)\big)\\
&= \mathrm{E}_\pi^{\phi_0}\left[\mathrm{e}^{-2\int_{-(h-\sqrt{h})}^0 ds\ \alpha(Y_s)\,\phi_0(Y_s)\,v_{\varepsilon-s}^{\phi_0}(Y_s)}\,\partial_h v_\varepsilon^{\phi_0}(Y_0)\right]\big(1+o_h(1)\big)\\
&= \mathrm{E}_\pi^{\phi_0}\left[\mathrm{e}^{-2\int_{-h}^0 ds\ \alpha(Y_s)\,\phi_0(Y_s)\,v_{\varepsilon-s}^{\phi_0}(Y_s)}\,\partial_h v_\varepsilon^{\phi_0}(Y_0)\right]\big(1+o_h(1)\big),
\end{aligned}$$

where we used (5.29) for the first equality, Lemma 5.6 for the second, stationarity of Y under $\mathrm{P}_\pi^{\phi_0}$ for the third and (5.29) again for the last. This gives (5.11).

Moreover, if $\lambda_0 > 0$, we get that:

$$\mathrm{E}_\pi^{\phi_0}\left[\mathrm{e}^{-2\int_{-\infty}^0 ds\ \alpha\,\phi_0\,v_{\varepsilon-s}^{\phi_0}(Y_s)}\,\partial_h v_\varepsilon^{\phi_0}(Y_0)\right]$$

is finite and that:

$$\lim_{h'\to+\infty} \mathrm{E}_\pi^{\phi_0}\left[\mathrm{e}^{-2\int_{-h'}^0 ds\ \alpha\,\phi_0\,v_{\varepsilon-s}^{\phi_0}(Y_s)}\,\partial_h v_\varepsilon^{\phi_0}(Y_0)\right] = \mathrm{E}_\pi^{\phi_0}\left[\mathrm{e}^{-2\int_{-\infty}^0 ds\ \alpha\,\phi_0\,v_{\varepsilon-s}^{\phi_0}(Y_s)}\,\partial_h v_\varepsilon^{\phi_0}(Y_0)\right].$$

Therefore, we deduce (5.12) from (5.11). □

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