Electronic Journal of Probability

Electron. J. Probab. **17** (2012), no. 92, 1–19.
ISSN: 1083-6489 DOI: 10.1214/EJP.v17-1916

Lower bound estimate of the spectral gap for simple exclusion process with degenerate rates

Yukio Nagahata*

Abstract

We consider exclusion process with degenerate rates in a finite torus with size n. This model is a simplified model for some peculiar phenomena of the "glassy" dynamics. We prove that the spectral gap is bounded below by $C\rho^4/n^2$, where $\rho = k/n$ denotes the density of particle and C does not depend on n nor ρ.

Keywords: exclusion process; degenerate rate; spectral gap.
AMS MSC 2010: 60K35.
Submitted to EJP on April 4, 2012, final version accepted on October 15, 2012.

1 Introduction

Simple exclusion process with degenerate rates is one of the simplest system called *kinetically constrained lattice gases*, which have been introduced in the physical literature as simplified models for some peculiar phenomena of the "glassy" dynamics (see [1, 5]).

Simple exclusion process with degenerate rates was discussed by [1]. In [1] they considered this process with particle reservoirs at boundary. They obtained estimates of spectral gap and log Sobolev constant and diffusive scaling limit of tagged particle displacement for the stationary process in infinite volume. Simple exclusion process with degenerate rates without particle reservoirs was discussed by [3]. They considered hydrodynamic limit of this process and obtained porous medium equation. In [3] they considered gradient system.

In the proof of hydrodynamic limit of nongradient systems, a sharp upper bound on the relaxation time (inverse of the spectral gap) for a generator restricted to finite cubes is needed (cf. [4]). The proof relies on the characterization of the closed forms of a state space and the proof of this characterization requires that the spectral gap shrinks at a rate slower than n^{-2} where n is a side-length of a finite cube.

One of the difficulty in studying this process without particle reservoirs is that if the density of particle is less than or equal to $1/3$, then the ergodic component is decomposed into irreducible component which contains all configurations with at least one

*Department of Information Engineering Faculty of Engineering, Niigata University, Japan.
E-mail: nagahata@ie.niigata-u.ac.jp

couple of particles at distance less or equal to two and many blocked configurations (cf. [3]). Note that if we consider this process with particle reservoir (at boundary), then this process is irreducible.

Thanks to the irreducibility of the process, in [1] they obtained a lower bound estimate of spectral gap (and log Sobolev constant). However the notation in [1] and that in this paper are different, in our notation, it shrinks at a rate n^{-2} if n tends to infinity, where n is a side-length of a finite cube. Furthermore, it goes to 0 as ρ to 0 as a power law of exponent between 1 and 2. Here ρ is a parameter which controls the entrance and exit rates such that in equilibrium, the density is ρ.

In [3], they also obtained a lower bound estimate of spectral gap for $\rho > 1/3$, where ρ is the density of particle. It shrinks at a rate n^{-2} if n tends to infinity, where n is a side-length of a finite cube or the size of the discrete torus. Furthermore, it shrinks at a rate $\rho - 1/3$ as ρ tends to $1/3$. (see Proposition 2.1 below.)

In this paper, we obtain a lower bound estimate of spectral gap estimate for all $0 < \rho \leq 1$. It shrinks at a rate n^{-2} if n tends to infinity. Furthermore, it shrinks at a rate ρ^4 as ρ tends to 0 (Theorem 2.2).

Due to the existence of the blocked configurations, we cannot give suitable mean field type process for $\rho \leq 1/3$. Main idea of the proof is "freeze one pair of coupled particles" and give usual mean field type process except this pair of coupled particles.

This paper is organized as follows: In section 2, we give our model and state our main result (Theorem 2.2). In section 3, we give two key lemmas (Lemmas 3.2, 3.3) and prove our main result by using these two lemmas. In section 4, we estimate variance. In section 5, we prove Lemma 3.2. In section 6, we define sets Θ_m, (also Θ_m^0 and Θ_m^1) and compute some quantities of these sets. In section 7, we prove Lemma 3.3.

2 Model and result

Let us consider discrete torus $T_n = \{1, 2, \dots n\}$ (n is identified with 0). We define the set of configurations by $\Sigma_n := \{0,1\}^{T_n}$, the set of configurations conditioned by the number of particles by $\Sigma_{n,k} := \{\eta \in \Sigma_n; \sum_{x \in T_n} \eta_x = k\}$.

For $\eta \in \Sigma_n$ and $x, y \in T_n$, we define the configuration $\eta^{x,y} \in \Sigma_n$ by

$$(\eta^{x,y})_z = \begin{cases} \eta_y & \text{if } z = x, \\ \eta_x & \text{if } z = y, \\ \eta_z & \text{otherwise,} \end{cases}$$

and the operator $\pi^{x,y}$ by

$$\pi^{x,y} f(\eta) = f(\eta^{x,y}) - f(\eta).$$

We define

$$c(\eta) := \eta_{-1} + \eta_2.$$

A function f is a local function if $f : \{0,1\}^{\mathbf{Z}} \to \mathbf{R}$ depends only on $\{\eta_x : x \in A\}$ for $\# A < \infty$. Then we can regard a local function as a function of Σ_n as usual manner. Let τ_x be a shift operator by

$$(\tau_x \eta)_z = \eta_{x+z} \text{ for all } x, z \in \mathbf{Z},$$
$$\tau_x f(\eta) = f(\tau_x \eta) \text{ for all } x \in \mathbf{Z} \text{ and for all local functions } f.$$

Given a local function g, which is strictly positive and does not depend on the value of η_0 nor η_1, we define the generator of simple exclusion process with degenerate rate $L = L_g$ by

$$Lf(\eta) = \sum_{x \in T_n} \tau_x (c(\eta) g(\eta)) \pi^{x,x+1} f(\eta)$$

for all local functions f. If there is at least one particle in the neighboring sites of $\{x, x+1\}$, then particle can jump from x to $x+1$ or $x+1$ to x. If there is no particle in the neighboring sites of $\{x, x+1\}$, then particle cannot jump from x to $x+1$ nor $x+1$ to x. Note that if $g \equiv 1$, then this system is gradient system and if $g = 1 - \frac{1}{2}\eta_{-1}\eta_2$, then this system is non-gradient system. In the second case, the function cg takes value 0 if $\eta_{-1} = \eta_2 = 0$ and 1 otherwise.

The ergodic component of the system is complicated (see [3]). If the density of the particle is large, precisely if $k > n/3$, then $\Sigma_{n,k}$ is an ergodic component. If the density of the particle is small, precisely if $k \le n/3$, then $\Sigma_{n,k}$ is decomposed to blocked configurations and a component which contains all configurations with at least one couple of particles at distance at most two. We define

$$\Sigma^0_{n,k} := \{\eta \in \Sigma_{n,k}; \sum_{x \in T_n} (\eta_x \eta_{x+1} + \eta_x \eta_{x+2}) > 0\},$$
$$\Sigma^1_{n,k} := \{\eta \in \Sigma_{n,k}; \sum_{x \in T_n} (\eta_x \eta_{x+1} + \eta_x \eta_{x+2}) = 0\}.$$

Note that $\Sigma^1_{n,k}$ is set of all blocked configurations. We also note that if $k > n/3$ then $\Sigma^0_{n,k} = \Sigma_{n,k}$. It is not difficult to see that $\Sigma^0_{n,k}$ is an ergodic component of the system. Let $\mu = \mu_{n,k}$ be a uniform probability measure on $\Sigma^0_{n,k}$. Then it is easy to see that L is reversible with respect to μ. Let $L_{n,k}$ be the restriction of L on $\Sigma^0_{n,k}$. Then we can consider the spectral gap of $-L_{n,k}$, which is defined by

$$\lambda = \lambda(n,k) := \inf\left\{ \frac{E[f(-L_{n,k})f]}{E[f^2]} \,\middle|\, E[f] = 0 \right\}.$$

We refer the result for $k > n/3$ by [3].

Proposition 2.1. *[3, Proposition 6.1]. Suppose that $k > n/3$. Then there exists a constant C not depending on n nor k such that*

$$\lambda(n,k) \ge C \frac{\rho - 1/3}{n^2 \rho},$$

where $\rho = k/n$.

We give our main result.

Theorem 2.2. *There exists a constant C not depending on n nor k such that*

$$\lambda(n,k) \ge C \frac{\rho^4}{n^2}$$

where $\rho = k/n$.

Remark 2.3. *By combinatorial methods, we have*

$$\#\Sigma^1_{n,k} = \frac{n(n-2k-1)!}{k!(n-3k)!}.$$

Therefore if k is large enough, then $\#\Sigma^1_{n,k} \ll \#\Sigma_{n,k} \cong \#\Sigma^0_{n,k}$. In this case, $\mu_{n,k}$ is approximated by canonical Bernoulli measure. Therefore we set

$$\phi_n(\eta) := \sum_{x \in T_n} f(\frac{x}{n})(\eta_x - \rho),$$
$$f(x) := \begin{cases} x & \text{if } 0 \le x \le 1/4, \\ -x + 1/2 & \text{if } 1/4 \le x \le 3/4, \\ x - 2 & \text{if } 3/4 \le x \le 1. \end{cases}$$

Then we have

$$E_\mu[\phi_n] = 0, \quad \frac{E_\mu[\phi_n(-L_{n,k})\phi_n]}{E_\mu[\phi_n^2]} \cong 192\frac{\rho}{n^2},$$

for $g \equiv 1$*. This estimate does not make sense if* k *is small.*

Remark 2.4. *We consider simple exclusion process with degenerate rate on the torus. It is easy to consider this process on finite interval with some boundary conditions. Then we have the same result.*

We can also consider similar process on d-dimensional torus or cube as follows. Let e_i ($i = 1, 2, \dots, d$) be positive unit vectors along i-th axes. We define

$$c_i(\eta) = \eta_{-e_i} + \eta_{2e_i} + \sum_{j \neq i}\{\eta_{e_j} + \eta_{-e_j} + \eta_{e_i+e_j} + \eta_{e_i-e_j}\}.$$

Given a set of local functions $g = \{g_i : i = 1, 2, \dots, d\}$, each g_i is strictly positive and does not depend on the value of η_0 nor η_{e_i}, we define the generator of simple exclusion process with degenerate rate $L = L_g$ by

$$Lf(\eta) = \sum_{x \in T_n^d} \sum_{i=1}^{d} \tau_x(c_i(\eta)g_i(\eta))\pi^{x,x+e_i} f(\eta)$$

for all local functions f. If there is at least one particle in the neighboring sites of $\{x, x + e_i\}$, then particle can jump from x to $x + e_i$ or $x + e_i$ to x. If there is no particle in the neighboring sites of $\{x, x + e_i\}$, then particle cannot jump from x to $x + e_i$ nor $x + e_i$ to x. Then we have similar results. Namely we consider this process on d-dimensional torus $\{1, 2, \dots, n\}^d$. Then there is critical number $k_d(n)$, which is at most $n^d/(2d + 1)$ such that if $k > k_d(n)$ then this system is irreducible and if $k \leq k_d(n)$ then the state space is decomposed into blocked configurations and a component which contains all configurations with at least one couple of particles at distance at most two. Furthermore our proof below is applicable and we obtain the same lower bound estimate of the spectral gap;

Remark 2.5. *There exists a constant* $C = C_d$ *not depending on* n *nor* k *such that*

$$\lambda(n, k) \geq C\frac{\rho^4}{n^2}$$

where $\rho = k/n^d$*.*

3 Outline of the proof

From now on we use the notation $\rho = k/n$. We pick and fix ρ_M with $1/3 < \rho_M < 1/2$. (For example, we set $\rho_M = 5/12$.)

First, we give a simple corollary of Proposition 2.1.

Corollary 3.1. *There exists a constant* C_1 *not depending on* n *nor* k *such that for any* n *and* $\rho \geq \rho_M$*, we have*

$$\lambda(n, k) \geq C_1\frac{1}{n^2} \geq C_1\frac{\rho^4}{n^2}.$$

Second we give two lemmas.

Lemma 3.2. *There exists a constant* C_2 *not depending on* n *nor* k *such that for any* n *and* k*, we have*

$$\lambda(n, k) \geq C_2\frac{1}{n^4}.$$

Lemma 3.3. *There exists a constant C_3 not depending on n nor k such that for any $0 \le \rho_0 \le \rho_M$, $n \ge n_0 = 128/\rho_0^2$ and $\rho_0 \le \rho \le \rho_M$, we have*

$$\lambda(n,k) \ge C_3 \frac{\rho_0^4}{n^2}.$$

Finally, combining these corollary and lemmas we conclude the proof. We have only to consider three cases as follows; i) $\rho \ge \rho_M$, ii) $\rho < \rho_M$ and $n < 128/\rho^2$, and iii) $\rho < \rho_M$ and $n \ge 128/\rho^2$. In the case i), we can apply Corollary 3.1. In the case ii), we apply Lemma 3.2, with $n < 128/\rho^2$. Then we have

$$\lambda(n,k) \ge C_2 \frac{1}{n^4} \ge C_2 \frac{1}{(128)^2} \frac{\rho^4}{n^2}.$$

In the case iii), we apply Lemma 3.3 as $\rho_0 = \rho$. Then we have

$$\lambda(n,k) \ge C_3 \frac{\rho^4}{n^2}.$$

Therefore we set

$$C = \min\{C_1, \frac{C_2}{(128)^2}, C_3\},$$

then we have

$$\lambda(n,k) \ge C \frac{\rho^4}{n^2}$$

for all n, k. □

4 Computation of variance

As being mentioned in the Introduction, the main idea of the proof in this paper is that we "freeze" one pair of coupled particles and give usual mean field type process except this pair of coupled particles. In this section, in order to "freeze" one pair of particles, we "search" pairs of particles at distance less or equal to two in the two distinct configurations and "move" them to the "same point". In sections 5, 7, we "freeze" these particles and give usual mean field type process.

We set $c_x(\eta) := \tau_x(c(\eta)g(\eta))$. Then we have

$$E_\mu[f(-L_{n,k})f] = \frac{1}{2} E_\mu[\sum_{x \in T_n} c_x(\eta)(\pi^{x,x+1} f(\eta))^2].$$

It is standard to see that

$$2V[f] := 2E_\mu[(f - E_\mu[f])^2] = \sum_{\eta \in \Sigma_{n,k}^0} \sum_{\xi \in \Sigma_{n,k}^0} (f(\eta) - f(\xi))^2 \mu(\eta)\mu(\xi).$$

We define

$$\alpha(\eta) = \alpha_n(\eta) := \sum_{x \in T_n} (\eta_x \eta_{x+1} + \eta_x \eta_{x+2}).$$

This quantity describes the number of particles at distance less or equal to two, that is number of particles that can move according to the dynamics. By the definition of α we have

$$\begin{aligned} 2V[f] &= \sum_{\eta \in \Sigma_{n,k}^0} \sum_{\xi \in \Sigma_{n,k}^0} (f(\eta) - f(\xi))^2 \\ &\quad \times \frac{\sum_{x \in T_n} (\eta_x \eta_{x+1} + \eta_x \eta_{x+2})}{\alpha(\eta)} \frac{\sum_{x \in T_n} (\xi_x \xi_{x+1} + \xi_x \xi_{x+2})}{\alpha(\xi)} \mu(\eta)\mu(\xi) \end{aligned}$$

We set

$$\begin{aligned}
V_1 &= \sum_{x\in T_n}\sum_{y\in T_n}\sum_{\eta\in\Sigma^0_{n,k}}\sum_{\xi\in\Sigma^0_{n,k}}(f(\eta)-f(\xi))^2\eta_x\eta_{x+1}\xi_y\xi_{y+1}\frac{1}{\alpha(\eta)\alpha(\xi)}\mu(\eta)\mu(\xi),\\
V_2 &= \sum_{x\in T_n}\sum_{y\in T_n}\sum_{\eta\in\Sigma^0_{n,k}}\sum_{\xi\in\Sigma^0_{n,k}}(f(\eta)-f(\xi))^2\eta_x\eta_{x+2}\xi_y\xi_{y+1}\frac{1}{\alpha(\eta)\alpha(\xi)}\mu(\eta)\mu(\xi),\\
V_3 &= \sum_{x\in T_n}\sum_{y\in T_n}\sum_{\eta\in\Sigma^0_{n,k}}\sum_{\xi\in\Sigma^0_{n,k}}(f(\eta)-f(\xi))^2\eta_x\eta_{x+1}\xi_y\xi_{y+2}\frac{1}{\alpha(\eta)\alpha(\xi)}\mu(\eta)\mu(\xi),\\
V_4 &= \sum_{x\in T_n}\sum_{y\in T_n}\sum_{\eta\in\Sigma^0_{n,k}}\sum_{\xi\in\Sigma^0_{n,k}}(f(\eta)-f(\xi))^2\eta_x\eta_{x+2}\xi_y\xi_{y+2}\frac{1}{\alpha(\eta)\alpha(\xi)}\mu(\eta)\mu(\xi).
\end{aligned}$$

Then we have

$$2V[f] = V_1+V_2+V_3+V_4.$$

Here we have "searched" pairs of particles at distance less or equal to two in the two distinct configurations. There are 2×2 cases, i.e., for each configurations, there is a pair of particles at distance one and that at distance two. For each cases, we "move" them to the "same point".

Usually we exchange the occupancy of particles at the pair of sites (x,y), and denote it by $\eta^{x,y}$. Here, we exchange the occupancy of particles at two pair of sites (x,y) and (z,w), and denote it by $\eta^{x,y:z,w}$. Namely, for $\eta\in\Sigma_n$ and $x,y,z,w\in T_n$ such that $x\neq z$ and $y\neq w$, we define the configuration $\eta^{x,y;z,w}\in\Sigma_n$ by

$$\eta^{x,y;z,w} = \begin{cases} (\eta^{z,w})^{x,y} & \text{if } y=z,\\ (\eta^{x,y})^{z,w} & \text{otherwise.}\end{cases}$$

Note that $(\eta^{x,y})^{z,w}\neq(\eta^{z,w})^{x,y}$ in general. We also note that $\eta_x\eta_z = (\eta^{x,y;z,w})_y(\eta^{x,y;z,w})_w$. We set

$$\begin{aligned}
V_5 &= \frac{1}{n}\sum_{z\in T_n}\sum_{x\in T_n}\sum_{y\in T_n}\sum_{\eta\in\Sigma^0_{n,k}}\sum_{\xi\in\Sigma^0_{n,k}}(f(\eta)-f(\eta^{x,z;x+1,z+1}))^2\\
&\qquad\times\eta_x\eta_{x+1}\xi_y\xi_{y+1}\frac{1}{\alpha(\eta)\alpha(\xi)}\mu(\eta)\mu(\xi)\\
V_6 &= \frac{1}{n}\sum_{z\in T_n}\sum_{x\in T_n}\sum_{y\in T_n}\sum_{\eta\in\Sigma^0_{n,k}}\sum_{\xi\in\Sigma^0_{n,k}}(f(\eta^{x,z;x+1,z+1})-f(\xi^{y,z;y+1,z+1}))^2\\
&\qquad\times\eta_x\eta_{x+1}\xi_y\xi_{y+1}\frac{1}{\alpha(\eta)\alpha(\xi)}\mu(\eta)\mu(\xi)\\
V_7 &= \frac{1}{n}\sum_{z\in T_n}\sum_{x\in T_n}\sum_{y\in T_n}\sum_{\eta\in\Sigma^0_{n,k}}\sum_{\xi\in\Sigma^0_{n,k}}(f(\xi^{y,z;y+1,z+1})-f(\xi))^2\\
&\qquad\times\eta_x\eta_{x+1}\xi_y\xi_{y+1}\frac{1}{\alpha(\eta)\alpha(\xi)}\mu(\eta)\mu(\xi).
\end{aligned}$$

Since $(f(\eta)-f(\xi))^2\le 3(f(\eta)-f(\eta^{x,z;x+1,z+1}))^2+3(f(\eta^{x,z;x+1,z+1})-f(\xi^{y,z;y+1,z+1}))^2+3(f(\xi^{y,z;y+1,z+1})-f(\xi))^2$ we have

$$V_1\le 3\{V_5+V_6+V_7\}.$$

Similarly, we set

$$\begin{aligned} V_8 &= \frac{1}{n}\sum_{z\in T_n}\sum_{x\in T_n}\sum_{y\in T_n}\sum_{\eta\in\Sigma^0_{n,k}}\sum_{\xi\in\Sigma^0_{n,k}} (f(\eta)-f(\eta^{x,z;x+2,z+1}))^2 \\ &\qquad \times \eta_x\eta_{x+2}\xi_y\xi_{y+1}\frac{1}{\alpha(\eta)\alpha(\xi)}\mu(\eta)\mu(\xi), \\ V_9 &= \frac{1}{n}\sum_{z\in T_n}\sum_{x\in T_n}\sum_{y\in T_n}\sum_{\eta\in\Sigma^0_{n,k}}\sum_{\xi\in\Sigma^0_{n,k}} (f(\xi^{y,z;y+2,z+1})-f(\xi))^2 \\ &\qquad \times \eta_x\eta_{x+1}\xi_y\xi_{y+2}\frac{1}{\alpha(\eta)\alpha(\xi)}\mu(\eta)\mu(\xi). \end{aligned}$$

Then we have

$$V_2 \le 3\{V_8+V_6+V_7\},\quad V_3 \le 3\{V_5+V_6+V_9\},\quad V_4 \le 3\{V_8+V_6+V_9\}.$$

Therefore we have

$$V[f] \le 3V_5 + 6V_6 + 3V_7 + 3V_8 + 3V_9. \tag{4.1}$$

Since $\alpha(\eta)\ge 1$ and $\sum_{y\in T_n}\frac{\xi_y\xi_{y+1}}{\alpha(\xi)}\le 1$, we have

$$\begin{aligned} V_5 &= \frac{1}{n}\sum_{z\in T_n}\sum_{x\in T_n}\sum_{y\in T_n}\sum_{\eta\in\Sigma^0_{n,k}}\sum_{\xi\in\Sigma^0_{n,k}} (f(\eta)-f(\eta^{x,z;x+1,z+1}))^2 \\ &\qquad \times \eta_x\eta_{x+1}\xi_y\xi_{y+1}\frac{1}{\alpha(\eta)\alpha(\xi)}\mu(\eta)\mu(\xi) \\ &\le \frac{1}{n}\sum_{z\in T_n}\sum_{x\in T_n}\sum_{\eta\in\Sigma^0_{n,k}} \eta_x\eta_{x+1}(f(\eta)-f(\eta^{x,z;x+1,z+1}))^2\mu(\eta). \end{aligned}$$

By using standard moving particle lemma (cf. [1, Lemma 3.3]), there exists a constant C_4 not depending on n nor k such that

$$V_5 \le C_4 n^2 \sum_{x\in T_n}\sum_{\eta\in\Sigma^0_{n,k}} (\pi^{x,x+1}f(\eta))^2 c_x(\eta)\mu(\eta). \tag{4.2}$$

By changing the variables η and ξ, we also have

$$V_7 \le C_4 n^2 \sum_{x\in T_n}\sum_{\eta\in\Sigma^0_{n,k}} (\pi^{x,x+1}f(\eta))^2 c_x(\eta)\mu(\eta). \tag{4.3}$$

Similarly, by using standard moving particle lemma and change of variables, there exists a constant C_5 not depending on n nor k such that

$$V_8 \le C_5 n^2 \sum_{x\in T_n}\sum_{\eta\in\Sigma^0_{n,k}} (\pi^{x,x+1}f(\eta))^2 c_x(\eta)\mu(\eta), \tag{4.4}$$

$$V_9 \le C_5 n^2 \sum_{x\in T_n}\sum_{\eta\in\Sigma^0_{n,k}} (\pi^{x,x+1}f(\eta))^2 c_x(\eta)\mu(\eta) \tag{4.5}$$

We recall that $\eta_p\eta_r = (\eta^{p,q;r,s})_q(\eta^{p,q;r,s})_s$. Hence we have

$$\begin{aligned} V_6 &= \frac{1}{n}\sum_{z\in T_n}\sum_{x\in T_n}\sum_{y\in T_n}\sum_{\eta\in\Sigma^0_{n,k}}\sum_{\xi\in\Sigma^0_{n,k}} (f(\eta^{x,z;x+1,z+1})-f(\xi^{y,z;y+1,z+1}))^2 \\ &\qquad \times (\eta^{x,z;x+1,z+1})_z(\eta^{x,z;x+1,z+1})_{z+1}(\xi^{y,z;y+1,z+1})_z(\xi^{y,z;y+1,z+1})_{z+1} \\ &\qquad \times \frac{1}{\alpha(\eta)\alpha(\xi)}\mu(\eta)\mu(\xi). \end{aligned}$$

Since μ is uniform measure on $\Sigma_{n,k}^0$ and $(\eta^{p,q;r,s})^{q,p;s,r} = \eta$, change of variable yields

$$\begin{aligned} V_6 &= \frac{1}{n} \sum_{z \in T_n} \sum_{x \in T_n} \sum_{y \in T_n} \sum_{\eta \in \Sigma_{n,k}^0} \sum_{\xi \in \Sigma_{n,k}^0} (f(\eta) - f(\xi))^2 \eta_z \eta_{z+1} \xi_z \xi_{z+1} \\ &\quad \times \frac{1}{\alpha(\eta^{z,x;z+1,x+1}) \alpha(\xi^{z,y;z+1,y+1})} \mu(\eta)\mu(\xi). \end{aligned}$$

It is not difficult to see that

$$\max_{n,k} \max_{x \in T_n} \max_{\eta \in \Sigma_{n,k}^0} \eta_z \eta_{z+1} |\alpha(\eta) - \alpha(\eta^{z,x;z+1,x+1})| = 6,$$

since maximum is attained by $n \geq 12$, $6 \leq k \leq n-6$, $|z - x| \geq 6$ and η satisfying

$$\begin{aligned} \eta_{z-2} = \eta_{z-1} = \eta_z = \eta_{z+1} = \eta_{z+2} = \eta_{z+3} = 1, \\ \eta_{x-2} = \eta_{x-1} = \eta_x = \eta_{x+1} = \eta_{x+2} = \eta_{x+3} = 0. \end{aligned}$$

By using this and $\alpha(\eta^{z,x;z+1,x+1}) \geq 1$ for all η such that $\eta_z = \eta_{z+1} = 1$, we have

$$\eta_z \eta_{z+1} \alpha(\eta) \leq \eta_z \eta_{z+1} (\alpha(\eta^{z,x;z+1,x+1}) + 6) \leq 7 \eta_z \eta_{z+1} \alpha(\eta^{z,x;z+1,x+1}).$$

Since $\eta_z \eta_{z+1}$ only takes 0 or 1, we have

$$\sum_{x \in T_n} \eta_z \eta_{z+1} \frac{1}{\alpha(\eta^{z,x;z+1,x+1})} \leq \eta_z \eta_{z+1} \frac{7n}{\alpha(\eta)}.$$

Hence we have

$$V_6 \leq 49 \frac{1}{n} \sum_{z \in T_n} \sum_{\eta \in \Sigma_{n,k}^0} \sum_{\xi \in \Sigma_{n,k}^0} (f(\eta) - f(\xi))^2 \eta_z \eta_{z+1} \xi_z \xi_{z+1} \frac{n^2}{\alpha(\eta)\alpha(\xi)} \mu(\eta)\mu(\xi). \tag{4.6}$$

5 Proof of Lemma 3.2

We note that if $\eta \in \Sigma_{n,k}^0$ then $\alpha(\eta) \geq 1$. Plugging this into (4.6), we have

$$V_6 \leq 49 n^2 \frac{1}{n} \sum_{z \in T_n} \sum_{\eta \in \Sigma_{n,k}^0} \sum_{\xi \in \Sigma_{n,k}^0} (f(\eta) - f(\xi))^2 \eta_z \eta_{z+1} \xi_z \xi_{z+1} \mu(\eta)\mu(\xi).$$

Since $E_\mu[\eta_0 \eta_1] > 0$ and μ is shift invariant, we can rewrite right hand side above by

$$49 n^2 (E_\mu[\eta_0 \eta_1])^2 \frac{1}{n} \sum_{z \in T_n} \sum_{\eta \in \Sigma_{n,k}^0} \sum_{\xi \in \Sigma_{n,k}^0} (f(\eta) - f(\xi))^2 \frac{\eta_z \eta_{z+1} \mu(\eta)}{E_\mu[\eta_z \eta_{z+1}]} \frac{\xi_z \xi_{z+1} \mu(\xi)}{E_\mu[\xi_z \xi_{z+1}]}.$$

We can regard $\eta_z \eta_{z+1} \mu(\eta) / E_\mu[\eta_z \eta_{z+1}]$ as conditional probability with condition $\eta_z = \eta_{z+1} = 1$. Hence we can treat

$$\sum_{\eta \in \Sigma_{n,k}^0} \sum_{\xi \in \Sigma_{n,k}^0} (f(\eta) - f(\xi))^2 \frac{\eta_z \eta_{z+1} \mu(\eta)}{E_\mu[\eta_z \eta_{z+1}]} \frac{\xi_z \xi_{z+1} \mu(\xi)}{E_\mu[\xi_z \xi_{z+1}]}$$

as conditional variance with the same condition. If we assume that $\eta_z = \eta_{z+1} = 1$, then $\alpha(\eta) \geq 1$ and $\eta \in \Sigma_{n,k}^0$. Since μ is uniform measure on $\Sigma_{n,k}^0$, the conditional probability $\eta_z \eta_{z+1} \mu(\eta) / E_\mu[\eta_z \eta_{z+1}]$ is uniform probability measure on $\{0,1\}^{T_n \setminus \{z,z+1\}}$ with

$\sum_{x\in T_n\setminus\{z,z+1\}}\eta_x = k-2$. Therefore we can apply spectral gap estimate for mean field type simple exclusion process [2]. We conclude that

$$\begin{aligned} V_6 &\le 49n^2(E_\mu[\eta_0\eta_1])^2\frac{1}{n}\sum_{z\in T_n}\frac{1}{n-2}\sum_{x,y\in T_n\setminus\{z,z+1\}}\sum_{\eta\in\Sigma^0_{n,k}}(\pi^{x,y}f(\eta))^2\eta_z\eta_{z+1}\frac{\eta_z\eta_{z+1}\mu(\eta)}{E_\mu[\eta_z\eta_{z+1}]} \\ &= 49n^2E_\mu[\eta_0\eta_1]\frac{1}{n}\sum_{z\in T_n}\frac{1}{n-2}\sum_{x,y\in T_n\setminus\{z,z+1\}}\sum_{\eta\in\Sigma^0_{n,k}}(\pi^{x,y}f(\eta))^2\eta_z\eta_{z+1}\mu(\eta). \end{aligned}$$

By using standard moving particle lemma, there exists a constant C_6 not depending on n nor k such that

$$\begin{aligned} &\frac{1}{n}\sum_{z\in T_n}\frac{1}{n-2}\sum_{x,y\in T_n\setminus\{z,z+1\}}\sum_{\eta\in\Sigma^0_{n,k}}(\pi^{x,y}f(\eta))^2\eta_z\eta_{z+1}\mu(\eta) \\ &\le C_6n^2\sum_{x\in T_n}\sum_{\eta\in\Sigma^0_{n,k}}(\pi^{x,x+1}f(\eta))^2c_x(\eta)\mu(\eta). \end{aligned} \tag{5.1}$$

Therefore we conclude that

$$V_6 \le 49C_6n^4\sum_{x\in T_n}\sum_{\eta\in\Sigma^0_{n,k}}(\pi^{x,x+1}f(\eta))^2c_x(\eta)\mu(\eta). \tag{5.2}$$

Plugging (4.2), (4.3), (4.4), (4.5) and (5.2) into (4.1), we have

$$V[f] \le 6\{C_4n^2 + C_5n^2 + 49C_6n^4\}\sum_{x\in T_n}\sum_{\eta\in\Sigma^0_{n,k}}(\pi^{x,x+1}f(\eta))^2c_x(\eta)\mu(\eta).$$

We set $C_2 := 1/(18\max\{C_4, C_5, 49C_6\})$, then we have

$$\lambda(n,k) \ge C_2\frac{1}{n^4}.$$

□

6 Combinatorial results

In Section 5, we should estimate $n/\alpha(\eta)$ by its maximum n. If n is large enough, then thanks to the law of large numbers, $\alpha(\eta)$ is approximated by its expectation and it may be also approximated by $2\rho^2 n$. This implies that the set $\{\eta\in\Sigma^0_{n,k} : \alpha(\eta) < an\}$ for some small a (which may depend on the density ρ) is negligible. To verify this idea, in this section, we give a family of sets Θ_m, (and also Θ^0_m and Θ^1_m) and compute some quantities. By using these results, in the next section, we prove Lemma 3.3.

From now on, in order to simplify our notation, we shall omit the symbol $[x]$, the largest integer which is less than or equal to x.

We have picked and fixed ρ_M with $1/3 < \rho_M < 1/2$. (For example, we set $\rho_M = 5/12$.) From now on, we also pick and fix ρ_0 with $0 < \rho_0 < \rho_M$. Furthermore we set $a = a(\rho_0)$, $b = b(\rho_0)$ and $n_0 = n_0(\rho_0)$ by

$$a = \frac{\rho_0^2}{64}, \quad b = \frac{\rho_0^2}{32}, \quad n_0 = \frac{128}{\rho_0^2}.$$

Since $\rho_0 < 1/2$, we have $b < \rho_0/4$. However we do not use n_0 in this section (we need restriction on n in order to estimate $W_{7,m}$ in the next section (Section 7)), it is convenient to define it here.

6.1 The definition of Θ_m

Since the cardinality of the sets $\{\eta \in \Sigma_{n,k}^0 : \alpha(\eta) = m\}$ is difficult to treat, we define sets Θ_m, for even n and $0 \le m \le k/2$ by

$$\Theta_m = \Theta_{n,k,z,m} = \{\eta \in \Sigma_{n,k}^0 : \eta_z = \eta_{z+1} = 1, \sum_{x=1}^{(n-2)/2} \eta_{z+2x}\eta_{z+2x+1} = m\},$$

and Θ_m^0, Θ_m^1 for odd n and $0 \le m \le k/2$ by

$$\Theta_m^0 = \Theta_{n,k,z,m}^0 = \{\eta \in \Sigma_{n,k}^0 : \eta_z = \eta_{z+1} = 1, \eta_{z-1} = 0, \sum_{x=1}^{(n-2)/2} \eta_{z+2x}\eta_{z+2x+1} = m\},$$

$$\Theta_m^1 = \Theta_{n,k,z,m}^1 = \{\eta \in \Sigma_{n,k}^0 : \eta_z = \eta_{z+1} = \eta_{z-1} = 1, \sum_{x=1}^{(n-2)/2} \eta_{z+2x}\eta_{z+2x+1} = m\}.$$

It is easy to see that if $\eta \in \Theta_m$, $\eta \in \Theta_m^0$ or $\eta \in \Theta_m^1$ then $\alpha(\eta) \ge 1 + m$.

We assume that n is even. For each $\eta \in \Theta_m$, we regard that there are m "pair of coupled particles", $(k-2) - 2m$ "uncoupled particles" and $(n-2)/2 - (k-2) + m$ "pair of vacant sites" in $T_n \setminus \{z, z+1\}$. Namely we have

$$\begin{aligned} &\#\{x : 1 \le x \le \frac{n-2}{2}, \eta_{z+2x} + \eta_{z+2x+1} = 2\} = m, \\ &\#\{x : 1 \le x \le \frac{n-2}{2}, \eta_{z+2x} + \eta_{z+2x+1} = 1\} = (k-2) - 2m, \\ &\#\{x : 1 \le x \le \frac{n-2}{2}, \eta_{z+2x} + \eta_{z+2x+1} = 0\} = \frac{n-2}{2} - (k-2) + m, \end{aligned} \tag{6.1}$$

for $\eta \in \Theta_m$. By using combinatorial method, we can compute the cardinality of Θ_m: First, for each $1 \le x \le (n-2)/2$, we choose one of "pair of coupled particles ($\eta_{z+2x} + \eta_{z+2x+1} = 2$)", "uncoupled particles ($\eta_{z+2x}+\eta_{z+2x+1} = 1$)" or "pair of vacant sites ($\eta_{z+2x}+\eta_{z+2x+1} = 0$)" with the conditions (6.1). Due to the conditions (6.1), there are

$$\frac{\{(n-2)/2\}!}{m!\{(k-2)-2m\}!\{(n-2)/2-(k-2)+m\}!}$$

possibility. Second, for each x assigned to "uncoupled particles ($\eta_{z+2x} + \eta_{z+2x+1} = 1$)", we choose one of $\eta_{z+2x} = 1, \eta_{z+2x+1} = 0$ or $\eta_{z+2x} = 0, \eta_{z+2x+1} = 1$. For each x assigned to "uncoupled particles" there are two possibility. By these procedure, we can pick up all configurations in Θ_m without double count. Therefore we conclude that

$$\#\Theta_m = \frac{\{(n-2)/2\}!}{m!\{(k-2)-2m\}!\{(n-2)/2-(k-2)+m\}!} 2^{(k-2)-2m}. \tag{6.2}$$

Suppose that $m + 1 \le bn$. Then we have

$$\frac{\#\Theta_m}{\#\Theta_{m+1}} = 4\frac{(m+1)\{(n-2)/2-(k-2)+m\}}{\{(k-2)-2m\}\{(k-2)-2m-1\}}.$$

In our assumption, $(k-2) - m > 0$ and $k - 2m - 3 \ge n\rho_0/2$, $m + 1 \le bn \le n\rho_0^2/16$. Therefore we have

$$\frac{\#\Theta_m}{\#\Theta_{m+1}} \le \frac{8}{\rho_0^2}\frac{m+1}{n} \le \frac{1}{2}. \tag{6.3}$$

As a corollary of this inequality, we have

$$\frac{\#\Theta_{an}}{\#\Theta_{bn}} \le \frac{1}{2^{(b-a)n}}. \tag{6.4}$$

Similarly, we assume that n is odd. Then we have

$$\begin{aligned}
&\#\Theta^0_{n,k,\cdot,m} = \#\Theta_{n-1,k,\cdot,m}, \quad \#\Theta^1_{n,k,\cdot,m} = \#\Theta_{n-1,k-1,\cdot,m},\\
&\frac{\#\Theta^0_m}{\#\Theta^0_{m+1}} \le \frac{8}{\rho_0^2}\frac{m+1}{n} \le \frac{1}{2}, \quad \frac{\#\Theta^1_m}{\#\Theta^1_{m+1}} \le \frac{8}{\rho_0^2}\frac{m+1}{n} \le \frac{1}{2},\\
&\frac{\#\Theta^0_{an}}{\#\Theta^0_{bn}} \le \frac{1}{2^{(b-a)n}}, \quad \frac{\#\Theta^1_{an}}{\#\Theta^1_{bn}} \le \frac{1}{2^{(b-a)n}}.
\end{aligned}$$

6.2 The definition of Ψ_+, Ψ_-

We define $l(\eta) = l_z(\eta)$ by

$$l(\eta) := \sum_{x=1}^{(n-2)/2} \eta_{z+2x}\eta_{z+2x+1}.$$

Then it is easy to see that if n is even and $\eta_z = \eta_{z+1} = 1$, then $\eta \in \Theta_m$ and $l(\eta) = m$ are equivalent. It is also easy to see that if n is odd and $\eta_z = \eta_{z+1} = 1, \eta_{z-1} = 0$, then $\eta \in \Theta^0_m$ and $l(\eta) = m$ are equivalent and if n is odd and $\eta_z = \eta_{z+1} = \eta_{z-1} = 1$, then $\eta \in \Theta^1_m$ and $l(\eta) = m$ are equivalent.

We assume that n is even. We define Ψ_+ and Ψ_-, by

$$\begin{aligned}
\Psi_+(\eta) &:= \{\eta^{x,y} \in \Theta_{l(\eta)+1} : x, y \in T_n \setminus \{z, z+1\}\},\\
\Psi_-(\eta) &:= \{\eta^{x,y} \in \Theta_{l(\eta)-1} : x, y \in T_n \setminus \{z, z+1\}\}.
\end{aligned}$$

Suppose that $\eta \in \Theta_m$, then we see that there are m "pair of coupled particles", $(k-2) - 2m$ "uncoupled particles" and $(n-2)/2 - (k-2) + m$ "pair of vacant sites" in $T_n \setminus \{z, z+1\}$. We pick up an "uncoupled particle" and rearrange and couple it to another "uncoupled particle". This manipulation coincides with the choice of "$\eta^{x,y} \in \Theta_{l(\eta)+1}$" in the definition of Ψ_+. Also we pick up a "pair of coupled particle" and choose "one of a particle" there, pick up a "pair of vacant sites" and choose one of "vacant site", and move "the particle" to the "vacant site". This manipulation coincides with the choice of "$\eta^{x,y} \in \Theta_{l(\eta)-1}$" in the definition of Ψ_-. Therefore we have

$$\#\Psi_+(\eta) = \{(k-2) - 2l(\eta)\}\{(k-2) - 2l(\eta) - 1\}, \tag{6.5}$$

$$\#\Psi_-(\eta) = 4l(\eta)\{\frac{n-2}{2} - (k-2) + l(\eta)\}. \tag{6.6}$$

Furthermore we have

$$\sum_{\eta \in \Theta_m} \sum_{\xi \in \Psi_+(\eta)} f(\eta, \xi) = \sum_{\xi \in \Theta_{m+1}} \sum_{\eta \in \Psi_-(\xi)} f(\eta, \xi), \tag{6.7}$$

for any function f.

Similarly, we assume that n is odd. We also define $\Psi^0_+, \Psi^0_-, \Psi^1_+, \Psi^1_-$ by

$$\begin{aligned}
\Psi^0_+(\eta) &:= \{\eta^{x,y} \in \Theta^0_{l(\eta)+1} : x, y \in T_n \setminus \{z, z+, z-1\}\},\\
\Psi^0_-(\eta) &:= \{\eta^{x,y} \in \Theta^0_{l(\eta)-1} : x, y \in T_n \setminus \{z, z+1, z-1\}\},\\
\Psi^1_+(\eta) &:= \{\eta^{x,y} \in \Theta^1_{l(\eta)+1} : x, y \in T_n \setminus \{z, z+1, z-1\}\},\\
\Psi^1_-(\eta) &:= \{\eta^{x,y} \in \Theta^1_{l(\eta)-1} : x, y \in T_n \setminus \{z, z+1, z-1\}\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\#\Psi^0_+(\eta) &= \{(k-2)-2l(\eta)\}\{(k-2)-2l(\eta)-1\},\\
\#\Psi^0_-(\eta) &= 4l(\eta)\{\frac{n-3}{2}-(k-2)+l(\eta)\},\\
\#\Psi^1_+(\eta) &= \{(k-3)-2l(\eta)\}\{(k-3)-2l(\eta)-1\},\\
\#\Psi^1_-(\eta) &= 4l(\eta)\{\frac{n-3}{2}-(k-3)+l(\eta)\}.
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
\sum_{\eta\in\Theta^0_m}\sum_{\xi\in\Psi^0_+(\eta)} f(\eta,\xi) &= \sum_{\xi\in\Theta^0_{m+1}}\sum_{\eta\in\Psi^0_-(\xi)} f(\eta,\xi),\\
\sum_{\eta\in\Theta^1_m}\sum_{\xi\in\Psi^1_+(\eta)} f(\eta,\xi) &= \sum_{\xi\in\Theta^1_{m+1}}\sum_{\eta\in\Psi^1_-(\xi)} f(\eta,\xi),
\end{aligned}$$

for any function f.

7 Proof of Lemma 3.3

We have picked and fixed ρ_M with $1/3 < \rho_M < 1/2$. (For example, we set $\rho_M = 5/12$.) We have also picked and fixed ρ_0 with $0 < \rho_0 < \rho_M$. Furthermore we have set $a = a(\rho_0)$, $b = b(\rho_0)$ and $n_0 = n_0(\rho_0)$ by

$$a = \frac{\rho_0^2}{64}, \quad b = \frac{\rho_0^2}{32}, \quad n_0 = \frac{128}{\rho_0^2}.$$

We recall (4.6);

$$V_6 \quad \le \quad 49\frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in\Sigma^0_{n,k}}\sum_{\xi\in\Sigma^0_{n,k}} (f(\eta)-f(\xi))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\frac{n^2}{\alpha(\eta)\alpha(\xi)}\mu(\eta)\mu(\xi).$$

We set

$$A = \bigcup_{m>an}\Theta_m, \quad A^c = \bigcup_{m=0}^{an}\Theta_m,$$

if n is even and

$$A = \bigcup_{m>an}(\Theta^0_m\cup\Theta^1_m), \quad A^c = \bigcup_{m=0}^{an}(\Theta^0_m\cup\Theta^1_m),$$

if n is odd. We set

$$\begin{aligned}
W_1 &= \frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in A}\sum_{\xi\in A}(f(\eta)-f(\xi))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\mu(\eta)\mu(\xi)\\
W_2 &= \frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in A}\sum_{\xi\in A^c}(f(\eta)-f(\xi))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\frac{n}{\alpha(\xi)}\mu(\eta)\mu(\xi)\\
W_3 &= \frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in A^c}\sum_{\xi\in A^c}(f(\eta)-f(\xi))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\frac{n^2}{\alpha(\eta)\alpha(\xi)}\mu(\eta)\mu(\xi)
\end{aligned}$$

and $C_7 = 49/a^2$, $C_8 = 98/a$, $C_9 = 49$. We have $\alpha(\eta) \ge l(\eta)+1 \ge an$ for $\eta\in A$. Therefore we have

$$V_6 \le C_7W_1 + C_8W_2 + C_9W_3. \tag{7.1}$$

We can rewrite W_1 by

$$\begin{aligned} W_1 &\leq \frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in\Sigma^0_{n,k}}\sum_{\xi\in\Sigma^0_{n,k}}(f(\eta)-f(\xi))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\mu(\eta)\mu(\xi) \\ &= (E_\mu[\eta_0\eta_1])^2\frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in\Sigma^0_{n,k}}\sum_{\xi\in\Sigma^0_{n,k}}(f(\eta)-f(\xi))^2\frac{\eta_z\eta_{z+1}\mu(\eta)}{E_\mu[\eta_z\eta_{z+1}]}\frac{\xi_z\xi_{z+1}\mu(\xi)}{E_\mu[\xi_z\xi_{z+1}]}. \end{aligned}$$

Then we can treat $\eta_z\eta_{z+1}\mu(\eta)/E_\mu[\eta_z\eta_{z+1}]$ as conditional probability with condition $\eta_z = \eta_{z+1} = 1$. Hence we can treat

$$\sum_{\eta\in\Sigma^0_{n,k}}\sum_{\xi\in\Sigma^0_{n,k}}(f(\eta)-f(\xi))^2\frac{\eta_z\eta_{z+1}\mu(\eta)}{E_\mu[\eta_z\eta_{z+1}]}\frac{\xi_z\xi_{z+1}\mu(\xi)}{E_\mu[\xi_z\xi_{z+1}]}$$

as conditional variance with the same condition. If we assume that $\eta_z = \eta_{z+1} = 1$, then $\alpha(\eta) \geq 1$, and $\eta \in \Sigma^0_{n,k}$. Since μ is uniform measure on $\Sigma^0_{n,k}$, the conditional probability $\eta_z\eta_{z+1}\mu(\eta)/E_\mu[\eta_z\eta_{z+1}]$ is uniform probability measure on $\{0,1\}^{T_n\setminus\{z,z+1\}}$ with $\sum_{x\in T_n\setminus\{z,z+1\}}\eta_x = k-2$. Therefore we can apply spectral gap estimate for mean field type simple exclusion process [2]. We conclude that

$$W_1 \leq E_\mu[\eta_0\eta_1]\frac{1}{n}\sum_{z\in T_n}\frac{1}{n-2}\sum_{x,y\in T_n\setminus\{z,z+1\}}\sum_{\eta\in\Sigma^0_{n,k}}(\pi^{x,y}f(\eta))^2\eta_z\eta_{z+1}\mu(\eta). \tag{7.2}$$

We assume that n is even. Since $\alpha(\xi) \geq l(\xi)+1 = m+1$ if $\xi \in \Theta_m$, by the definition of A and Θ_m, we have

$$W_2 \leq \frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in A}\sum_{m=0}^{an}\sum_{\xi\in\Theta_m}(f(\eta)-f(\xi))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\frac{n}{m+1}\mu(\eta)\mu(\xi).$$

We set $W_{2,m}$ by

$$W_{2,m} = \frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in A}\sum_{\xi\in\Theta_m}(f(\eta)-f(\xi))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\frac{n}{m+1}\mu(\eta)\mu(\xi).$$

Note that

$$W_2 = \sum_{m=0}^{an}W_{2,m}. \tag{7.3}$$

By the definition of Ψ_+, Θ_m and (6.5), we have

$$\begin{aligned} W_{2,m} &= \frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in A}\sum_{\xi\in\Theta_m}\frac{\{(k-2)-2(m+1)\}!}{\{(k-2)-2m\}!}\sum_{\zeta^{m+1}\in\Psi_+(\xi)} \\ &\quad (f(\eta)-f(\xi))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\frac{n}{m+1}\mu(\eta)\mu(\xi). \end{aligned}$$

Inductively, we have

$$\begin{aligned} W_{2,m} &= \frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in A}\sum_{\xi\in\Theta_m}\frac{\{(k-2)-2bn\}!}{\{(k-2)-2m\}!}\sum_{\zeta^{m+1}\in\Psi_+(\xi)}\sum_{\zeta^{m+2}\in\Psi_+(\zeta^{m+1})}\cdots\sum_{\zeta^{bn}\in\Psi_+(\zeta^{bn-1})} \\ &\quad (f(\eta)-f(\xi))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\frac{n}{m+1}\mu(\eta)\mu(\xi). \end{aligned}$$

We set

$$\begin{aligned} W_{4,m} &= \frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in A}\frac{\{(k-2)-2bn\}!}{\{(k-2)-2m\}!}\sum_{\zeta^{bn}\in\Theta_{bn}}\sum_{\zeta^{bn-1}\in\Psi_-(\zeta^{bn})}\cdots\sum_{\zeta^{m+1}\in\Psi_-(\zeta^{m+2})}\sum_{\xi\in\Psi_-(\zeta^{m+1})} \\ &\qquad (f(\eta)-f(\zeta^{bn}))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\frac{n}{m+1}\mu(\eta)\mu(\xi), \\ W_{5,m} &= \frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in A}\frac{\{(k-2)-2bn\}!}{\{(k-2)-2m\}!}\sum_{\xi\in\Theta_m}\sum_{\zeta^{m+1}\in\Psi_+(\xi)}\sum_{\zeta^{m+2}\in\Psi_+(\zeta^{m+1})}\cdots\sum_{\zeta^{bn}\in\Psi_+(\zeta^{bn-1})} \\ &\qquad (f(\zeta^{bn})-f(\xi))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\frac{n}{m+1}\mu(\eta)\mu(\xi). \end{aligned}$$

Since $(f(\eta)-f(\xi))^2\le 2(f(\eta)-f(\zeta^{bn}))^2+2(f(\zeta^{bn})-f(\xi))^2$, by using (6.7), we have

$$W_{2,m}\le 2W_{4,m}+2W_{5,m}. \tag{7.4}$$

Since μ is uniform measure and $\xi_z=\zeta_z^{bn}$, and $\xi_{z+1}=\zeta_{z+1}^{bn}$ by the definition of Ψ_+, by using (6.2), (6.6) we have

$$W_{4,m} \le \frac{n}{m+1}\frac{\#\Theta_m}{\#\Theta_{bn}}\frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in A}\sum_{\zeta^{bn}\in\Theta_{bn}}(f(\eta)-f(\zeta^{bn}))^2\eta_z\eta_{z+1}\zeta_z^{bn}\zeta_{z+1}^{bn}\mu(\eta)\mu(\zeta^{bn}).$$

Since $\Theta_{bn}\subset A$, we have

$$\begin{aligned} W_{4,m} &\le \frac{n}{m+1}\frac{\#\Theta_m}{\#\Theta_{bn}}\frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in A}\sum_{\zeta^{bn}\in A}(f(\eta)-f(\zeta^{bn}))^2\eta_z\eta_{z+1}\zeta_z^{bn}\zeta_{z+1}^{bn}\mu(\eta)\mu(\zeta^{bn}) \\ &= \frac{n}{m+1}\frac{\#\Theta_m}{\#\Theta_{bn}}W_1. \end{aligned}$$

By using (6.3) and (6.4), we have

$$\sum_{m=0}^{an}\frac{n}{m+1}\frac{\#\Theta_m}{\#\Theta_{bn}}\le\sum_{m=0}^{an}\frac{n}{m+1}\frac{\#\Theta_{an}}{\#\Theta_{bn}}\le\sum_{m=0}^{an}\frac{n}{m+1}\frac{1}{2^{(b-a)n}}\le an^2\frac{1}{2^{(b-a)n}}.$$

Since $b=2a$ and $a=\frac{\rho_0^2}{64}$, we set

$$C_{10}=C_{10}(\rho_0):=256\frac{1}{\rho_0^2}\ge\sup_{x\ge 0}ax^2\frac{1}{2^{(b-a)x}}=\frac{256}{(e\log 2)^2}\frac{1}{\rho_0^2}.$$

Then we conclude that

$$\sum_{m=0}^{an}W_{4,m}\le C_{10}W_1. \tag{7.5}$$

If we set

$$\begin{aligned} W_{6,m} &= \frac{1}{n}\sum_{z\in T_n}\frac{\{(k-2)-2bn\}!}{\{(k-2)-2m\}!}\sum_{\xi\in\Theta_m}\sum_{\zeta^{m+1}\in\Psi_+(\xi)}\sum_{\zeta^{m+2}\in\Psi_+(\zeta^{m+1})}\cdots\sum_{\zeta^{bn}\in\Psi_+(\zeta^{bn-1})} \\ &\qquad (f(\zeta^{bn})-f(\xi))^2\xi_z\xi_{z+1}\frac{n}{m+1}\mu(\xi), \end{aligned}$$

then, by summing up over η, we have

$$W_{5,m}\le W_{6,m}. \tag{7.6}$$

We set

$$\begin{aligned} W_{7,m} &= 2\frac{1}{n}\sum_{z\in T_n}\frac{\{(k-2)-2bn\}!}{\{(k-2)-2m\}!}\sum_{\zeta^{m+1}\in\Theta_{m+1}}\sum_{\zeta^{m+2}\in\Psi_+(\zeta^{m+1})}\cdots\sum_{\zeta^{bn}\in\Psi_+(\zeta^{bn-1})} \\ &\qquad (f(\zeta^{bn})-f(\zeta^{m+1}))^2\zeta_z^{m+1}\zeta_{z+1}^{m+1}\mu(\zeta^{m+1})\sum_{\xi\in\Psi_-(\zeta^{m+1})}\frac{n}{m+1}, \\ W_{8,m} &= 2\frac{1}{n}\sum_{z\in T_n}\frac{\{(k-2)-2bn\}!}{\{(k-2)-2m\}!}\sum_{\xi\in\Theta_m}\sum_{\zeta^{m+1}\in\Psi_+(\xi)}\sum_{\zeta^{m+2}\in\Psi_+(\zeta^{m+1})}\cdots\sum_{\zeta^{bn}\in\Psi_+(\zeta^{bn-1})} \\ &\qquad (f(\zeta^{m+1})-f(\xi))^2\xi_z\xi_{z+1}\frac{n}{m+1}\mu(\xi). \end{aligned}$$

Since we have $(f(\zeta^{bn})-f(\xi))^2\le 2(f(\zeta^{bn})-f(\zeta^{m+1}))^2+2(f(\zeta^{m+1})-f(\xi))^2$, μ is uniform measure and $\xi_z=\zeta_z^{m+1}$, $\xi_{z+1}=\zeta_{z+1}^{m+1}$ by the definition, by using (6.7), we have

$$W_{6,m}\le W_{7,m}+W_{8,m}.$$

By using (6.6), we have

$$\sum_{\xi\in\Psi_-(\zeta^{m+1})}2\frac{\{(k-2)-2(m+1)\}!}{\{(k-2)-2m\}!}\frac{m+2}{m+1}\le 8\frac{(m+2)\{n/2-k+m+2\}}{(k-2m)^2}$$

In our setting, if $m\le bn$ then $k-m>0$ and $k-2m\ge n\rho_0/2$. In our assumption, $n\ge n_0=128/\rho_0^2$, we have

$$\sup_{0\le m\le bn}8\frac{(m+2)\{n/2-k+m+2\}}{(k-2m)^2}\le (b+\frac{2}{n})\frac{16}{\rho_0^2}\le\frac{3}{4}.$$

Therefore we have

$$\begin{aligned} W_{7,m} &= \frac{1}{n}\sum_{z\in T_n}\frac{\{(k-2)-2bn\}!}{\{(k-2)-2(m+1)\}!}\sum_{\zeta^{m+1}\in\Theta_{m+1}}\sum_{\zeta^{m+2}\in\Psi_+(\zeta^{m+1})}\cdots\sum_{\zeta^{bn}\in\Psi_+(\zeta^{bn-1})} \\ &\qquad (f(\zeta^{bn})-f(\zeta^{m+1}))^2\zeta_z^{m+1}\zeta_{z+1}^{m+1}\frac{n}{m+2}\mu(\zeta^{m+1}) \\ &\qquad \sum_{\xi\in\Psi_-(\zeta^{m+1})}2\frac{\{(k-2)-2(m+1)\}!}{\{(k-2)-2m\}!}\frac{m+2}{m+1} \\ &\le \frac{3}{4}W_{6,m+1}, \end{aligned}\tag{7.7}$$

for $0\le m\le bn$.

By using (6.5), we have

$$\begin{aligned} &2\frac{\{(k-2)-2bn\}!}{\{(k-2)-2m\}!}\sum_{\zeta^{m+2}\in\Psi_+(\zeta^{m+1})}\cdots\sum_{\zeta^{bn}\in\Psi_+(\zeta^{bn-1})}\frac{n}{m+1} \\ &= 2\frac{n}{m+1}\frac{1}{\{(k-2)-2m\}\{(k-2)-2m-1\}}. \end{aligned}$$

In our setting, $k-2m-3\ge n\rho_0/2$. Hence we set a constant $C_{11}=8/\rho_0^2$, then we have

$$\sup_{0\le m\le bn}2\frac{n}{m+1}\frac{1}{\{(k-2)-2m\}\{(k-2)-2m-1\}}\le\frac{8}{\rho_0^2 n}=\frac{C_{11}}{n}.$$

By using this inequality and the definition of Ψ_+, we have

$$\begin{aligned} W_{8,m} &\le C_{11}\frac{1}{n}\sum_{z\in T_n}\frac{1}{n}\sum_{\xi\in\Theta_m}\sum_{\zeta^{m+1}\in\Psi_+(\xi)}(f(\zeta^{m+1})-f(\xi))^2\xi_z\xi_{z+1}\mu(\xi) \\ &\le C_{11}\frac{1}{n}\sum_{z\in T_n}\sum_{\xi\in\Theta_m}\frac{1}{n}\sum_{x,y\in T_n\setminus\{z,z+1\}}(\pi^{x,y}f(\xi))^2\xi_z\xi_{z+1}\mu(\xi). \end{aligned}\tag{7.8}$$

By the definition of $W_{6,m}, W_{8,m}$, we note that

$$W_{6,bn-1} = \frac{1}{2} W_{8,bn-1}.$$

By using (7.7) and by the definition of $W_{6,m}, W_{7,m}, W_{8,m}$, we have

$$\begin{aligned}
&\sum_{m=0}^{an} W_{6,m} \le \sum_{m=0}^{bn-2} W_{6,m} \le \sum_{m=0}^{bn-2} (W_{7,m} + W_{8,m}) \\
&\le \frac{3}{4} W_{6,1} + \sum_{m=1}^{bn-2} W_{7,m} + \sum_{m=0}^{bn-2} W_{8,m} \le (1+\frac{3}{4}) \sum_{m=1}^{bn-2} W_{7,m} + (1+\frac{3}{4}) \sum_{m=0}^{bn-2} W_{8,m} \\
&\le \dots \le \sum_{m=0}^{bn-2} (\frac{3}{4})^m W_{6,bn-1} + \sum_{m=0}^{bn-2} (\frac{3}{4})^m \sum_{m=0}^{bn-2} W_{8,m} \le \sum_{m=0}^{bn-2} (\frac{3}{4})^m \sum_{m=0}^{bn-2} W_{8,m} \\
&\le 4 \sum_{m=0}^{bn-2} W_{8,m}.
\end{aligned}$$

By using this inequality and (7.8) we conclude that

$$\begin{aligned}
&\sum_{m=0}^{an} W_{6,m} \le 4 \sum_{m=0}^{bn-1} W_{8,m} \\
&\le 4C_{11} \frac{1}{n} \sum_{z \in T_n} \sum_{\xi \in \Sigma_{n,k}^0} \frac{1}{n} \sum_{x,y \in T_n \setminus \{z,z+1\}} (\pi^{x,y} f(\xi))^2 \xi_z \xi_{z+1} \mu(\xi). \qquad (7.9)
\end{aligned}$$

Similarly, we can rewrite W_3 as

$$\begin{aligned}
W_3 &= \frac{1}{n} \sum_{z \in T_n} \sum_{\eta \in A^c} \sum_{\xi \in A^c} (f(\eta) - f(\xi))^2 \eta_z \eta_{z+1} \xi_z \xi_{z+1} \frac{n^2}{\alpha(\eta)\alpha(\xi)} \mu(\eta)\mu(\xi) \\
&\le \frac{1}{n} \sum_{z \in T_n} \sum_{j=0}^{an} \sum_{\eta \in \Theta_j} \sum_{m=0}^{an} \sum_{\xi \in \Theta_m} (f(\eta) - f(\xi))^2 \eta_z \eta_{z+1} \xi_z \xi_{z+1} \frac{n^2}{(j+1)(m+1)} \mu(\eta)\mu(\xi).
\end{aligned}$$

We also set $W_{3,j,m}$ by

$$W_{3,j,m} = \frac{1}{n} \sum_{z \in T_n} \sum_{\eta \in \Theta_j} \sum_{\xi \in \Theta_m} (f(\eta) - f(\xi))^2 \eta_z \eta_{z+1} \xi_z \xi_{z+1} \frac{n^2}{(j+1)(m+1)} \mu(\eta)\mu(\xi).$$

Then we have

$$W_3 = \sum_{j=0}^{an} \sum_{m=0}^{an} W_{3,j,m}. \qquad (7.10)$$

By the definition of Ψ_+, Θ_m and (6.5), we have

$$\begin{aligned}
W_{3,j,m} &= \frac{1}{n} \sum_{z \in T_n} \sum_{\eta \in \Theta_j} \frac{\{(k-2)-2bn\}!}{\{(k-2)-2j\}!} \sum_{\omega^{j+1} \in \Psi_+(\eta)} \sum_{\omega^{j+2} \in \Psi_+(\omega^{j+1})} \cdots \sum_{\omega^{bn} \in \Psi_+(\omega^{bn-1})} \\
&\quad \sum_{\xi \in \Theta_m} \frac{\{(k-2)-2bn\}!}{\{(k-2)-2m\}!} \sum_{\zeta^{m+1} \in \Psi_+(\xi)} \sum_{\zeta^{m+2} \in \Psi_+(\zeta^{m+1})} \cdots \sum_{\zeta^{bn} \in \Psi_+(\zeta^{bn-1})} \\
&\quad (f(\eta) - f(\xi))^2 \eta_z \eta_{z+1} \xi_z \xi_{z+1} \frac{n^2}{(j+1)(m+1)} \mu(\eta)\mu(\xi).
\end{aligned}$$

We set

$$\begin{aligned}
W_{9,j,m} &= \frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in\Theta_j}\frac{\{(k-2)-2bn\}!}{\{(k-2)-2j\}!}\sum_{\omega^{j+1}\in\Psi_+(\eta)}\sum_{\omega^{j+2}\in\Psi_+(\omega^{j+1})}\cdots\sum_{\omega^{bn}\in\Psi_+(\omega^{bn-1})}\\
&\quad\sum_{\xi\in\Theta_m}\frac{\{(k-2)-2bn\}!}{\{(k-2)-2m\}!}\sum_{\zeta^{m+1}\in\Psi_+(\xi)}\sum_{\zeta^{m+2}\in\Psi_+(\zeta^{m+1})}\cdots\sum_{\zeta^{bn}\in\Psi_+(\zeta^{bn-1})}\\
&\qquad(f(\eta)-f(\omega^{bn}))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\frac{n^2}{(j+1)(m+1)}\mu(\eta)\mu(\xi),\\
W_{10,j,m} &= \frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in\Theta_j}\frac{\{(k-2)-2bn\}!}{\{(k-2)-2j\}!}\sum_{\omega^{j+1}\in\Psi_+(\eta)}\sum_{\omega^{j+2}\in\Psi_+(\omega^{j+1})}\cdots\sum_{\omega^{bn}\in\Psi_+(\omega^{bn-1})}\\
&\quad\sum_{\xi\in\Theta_m}\frac{\{(k-2)-2bn\}!}{\{(k-2)-2m\}!}\sum_{\zeta^{m+1}\in\Psi_+(\xi)}\sum_{\zeta^{m+2}\in\Psi_+(\zeta^{m+1})}\cdots\sum_{\zeta^{bn}\in\Psi_+(\zeta^{bn-1})}\\
&\qquad(f(\omega^{bn})-f(\zeta^{bn}))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\frac{n^2}{(j+1)(m+1)}\mu(\eta)\mu(\xi),\\
W_{11,j,m} &= \frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in\Theta_j}\frac{\{(k-2)-2bn\}!}{\{(k-2)-2j\}!}\sum_{\omega^{j+1}\in\Psi_+(\eta)}\sum_{\omega^{j+2}\in\Psi_+(\omega^{j+1})}\cdots\sum_{\omega^{bn}\in\Psi_+(\omega^{bn-1})}\\
&\quad\sum_{\xi\in\Theta_m}\frac{\{(k-2)-2bn\}!}{\{(k-2)-2m\}!}\sum_{\zeta^{m+1}\in\Psi_+(\xi)}\sum_{\zeta^{m+2}\in\Psi_+(\zeta^{m+1})}\cdots\sum_{\zeta^{bn}\in\Psi_+(\zeta^{bn-1})}\\
&\qquad(f(\zeta^{bn})-f(\xi))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\frac{n^2}{(j+1)(m+1)}\mu(\eta)\mu(\xi).
\end{aligned}$$

Since $(f(\eta)-f(\xi))^2 \le 3(f(\eta)-f(\omega^{bn}))^2+3(f(\omega^{bn})-f(\zeta^{bn}))^2+3(f(\zeta^{bn})-f(\xi))^2$, by using (6.7), we have

$$W_{3,j,m} \le 3W_{9,j,m}+3W_{10,j,m}+3W_{11,j,m}. \tag{7.11}$$

By using the same argument in the estimate of $W_{4,m}$, we can estimate $W_{10,j,m}$ and $\sum_{j,m} W_{10,j,m}$ by

$$\begin{aligned}
&W_{10,j,m} \le \frac{n}{j+1}\frac{\#\Theta_j}{\#\Theta_{bn}}\frac{n}{m+1}\frac{\#\Theta_m}{\#\Theta_{bn}}W_1,\\
&\sum_{j=0}^{an}\sum_{m=0}^{an} W_{10,j,m} \le C_{10}^2 W_1.
\end{aligned} \tag{7.12}$$

It is easy to see that

$$\begin{aligned}
W_{9,j,m} &= \frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in\Theta_j}\frac{\{(k-2)-2bn\}!}{\{(k-2)-2j\}!}\sum_{\omega^{j+1}\in\Psi_+(\eta)}\sum_{\omega^{j+2}\in\Psi_+(\omega^{j+1})}\cdots\sum_{\omega^{bn}\in\Psi_+(\omega^{bn-1})}\\
&\qquad(f(\eta)-f(\omega^{bn}))^2\eta_z\eta_{z+1}\frac{n}{j+1}\mu(\eta)\frac{1}{m+1}\sum_{\xi\in\Theta_m}\xi_z\xi_{z+1}\mu(\xi).
\end{aligned}$$

Since μ is uniform measure, by using (6.3),(6.4), we have

$$\sum_{m=0}^{an}\frac{n}{m+1}\mu(\Theta_m) \le \sum_{m=0}^{an}\frac{n}{m+1}2^{(a-b)n} \le C_{10}.$$

Therefore we conclude that

$$\sum_{m=0}^{an} W_{9,j,m} \le C_{10}W_{6,j}. \tag{7.13}$$

Similarly, we have

$$\sum_{j=0}^{an} W_{11,j,m} \leq C_{10} W_{6,m}. \tag{7.14}$$

By using (7.2), (7.3), (7.4), (7.5), (7.6) and (7.9), we have

$$\begin{aligned} W_2 &\leq 2C_{10}W_1 + 2\sum_{m=0}^{an} W_{6,m} \\ &\leq \{2C_{10}E_\mu[\eta_0\eta_1] + 8C_{11}\} \\ &\quad \times \frac{1}{n}\sum_{z\in T_n}\frac{1}{n-2}\sum_{x,y\in T_n\setminus\{z,z+1\}}\sum_{\eta\in\Sigma^0_{n,k}} (\pi^{x,y}f(\eta))^2\eta_z\eta_{z+1}\mu(\eta). \end{aligned} \tag{7.15}$$

Similarly by using (7.2), (7.10), (7.11), (7.12), (7.13) and (7.14), we have

$$\begin{aligned} W_3 &\leq 3C_{10}^2W_1 + 6C_{10}\sum_{m=0}^{an} W_{6,m} \\ &\leq \{3C_{10}^2E_\mu[\eta_0\eta_1] + 24C_{10}C_{11}\} \\ &\quad \times \frac{1}{n}\sum_{z\in T_n}\frac{1}{n-2}\sum_{x,y\in T_n\setminus\{z,z+1\}}\sum_{\eta\in\Sigma^0_{n,k}} (\pi^{x,y}f(\eta))^2\eta_z\eta_{z+1}\mu(\eta). \end{aligned} \tag{7.16}$$

By using (7.1), (7.2), (7.15) and (7.16), we conclude

$$\begin{aligned} V_6 &\leq C_7W_1 + C_8W_2 + C_9W_3 \\ &\leq \{C_7E_\mu[\eta_0\eta_1] + 2C_8C_{10}E_\mu[\eta_0\eta_1] + 3C_9C_{10}^2E_\mu[\eta_0\eta_1] + 8(C_8+3C_{10})C_{11}\} \\ &\quad \times \frac{1}{n}\sum_{z\in T_n}\frac{1}{n-2}\sum_{x,y\in T_n\setminus\{z,z+1\}}\sum_{\eta\in\Sigma^0_{n,k}} (\pi^{x,y}f(\eta))^2\eta_z\eta_{z+1}\mu(\eta). \end{aligned} \tag{7.17}$$

By the definition of $C_7, C_8, C_9, C_{10}, C_{11}$, if we set $C_{12} = 13496320$, then we have

$$C_7E_\mu[\eta_0\eta_1] + 2C_8C_{10}E_\mu[\eta_0\eta_1] + 3C_9C_{10}^2E_\mu[\eta_0\eta_1] + 8(C_8+3C_{10})C_{11} \leq C_{12}\frac{1}{\rho_0^4}. \tag{7.18}$$

We recall (5.1). By using standard moving particle lemma, there exists a constant C_6 not depending on n nor k such that

$$\begin{aligned} &\frac{1}{n}\sum_{z\in T_n}\frac{1}{n-2}\sum_{x,y\in T_n\setminus\{z,z+1\}}\sum_{\eta\in\Sigma^0_{n,k}} (\pi^{x,y}f(\eta))^2\eta_z\eta_{z+1}\mu(\eta) \\ &\leq C_6n^2\sum_{x\in T_n}\sum_{\eta\in\Sigma^0_{n,k}} (\pi^{x,x+1}f(\eta))^2c_x(\eta)\mu(\eta). \end{aligned}$$

Plugging (4.2), (4.3), (4.4), (4.5), (5.1), (7.17) and (7.18) into (4.1), we have

$$V[f] \leq 6\{C_4n^2 + C_5n^2 + C_6C_{12}\frac{n^2}{\rho_0^4}\}\sum_{x\in T_n}\sum_{\eta\in\Sigma^0_{n,k}} (\pi^{x,x+1}f(\eta))^2c_x(\eta)\mu(\eta).$$

We set $C_3^e := 1/(C_4 + C_5 + C_6C_{12}))$, then we have

$$\lambda(n,k) \geq C_3^e\frac{\rho_0^4}{n^2},$$

if n is even.

We assume that n is odd. We set

$$\begin{aligned}
W_2^0 &= \frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in A}\sum_{m=0}^{an}\sum_{\xi\in\Theta_m^0}(f(\eta)-f(\xi))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\frac{n}{m+1}\mu(\eta)\mu(\xi),\\
W_2^1 &= \frac{1}{n}\sum_{z\in T_n}\sum_{\eta\in A}\sum_{m=0}^{an}\sum_{\xi\in\Theta_m^1}(f(\eta)-f(\xi))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\frac{n}{m+1}\mu(\eta)\mu(\xi),\\
W_3^{00} &= \frac{1}{n}\sum_{z\in T_n}\sum_{j=0}^{an}\sum_{\eta\in\Theta_j^0}\sum_{m=0}^{an}\sum_{\xi\in\Theta_m^0}(f(\eta)-f(\xi))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\frac{n^2}{(j+1)(m+1)}\mu(\eta)\mu(\xi),\\
W_3^{01} &= \frac{1}{n}\sum_{z\in T_n}\sum_{j=0}^{an}\sum_{\eta\in\Theta_j^0}\sum_{m=0}^{an}\sum_{\xi\in\Theta_m^1}(f(\eta)-f(\xi))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\frac{n^2}{(j+1)(m+1)}\mu(\eta)\mu(\xi),\\
W_3^{10} &= \frac{1}{n}\sum_{z\in T_n}\sum_{j=0}^{an}\sum_{\eta\in\Theta_j^1}\sum_{m=0}^{an}\sum_{\xi\in\Theta_m^0}(f(\eta)-f(\xi))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\frac{n^2}{(j+1)(m+1)}\mu(\eta)\mu(\xi),\\
W_3^{11} &= \frac{1}{n}\sum_{z\in T_n}\sum_{j=0}^{an}\sum_{\eta\in\Theta_j^1}\sum_{m=0}^{an}\sum_{\xi\in\Theta_m^1}(f(\eta)-f(\xi))^2\eta_z\eta_{z+1}\xi_z\xi_{z+1}\frac{n^2}{(j+1)(m+1)}\mu(\eta)\mu(\xi).
\end{aligned}$$

Then we can rewrite W_2, W_3 by

$$W_2 \le W_2^0 + W_2^1, \quad W_3 \le W_3^{00} + W_3^{01} + W_3^{10} + W_3^{11}.$$

Then we can use the same argument for W_2^0, W_2^1, $W_3^{00}, W_3^{01}, W_3^{10}$ and W_3^{11}. Therefore we have

$$V[f] \le 6\{C_4n^2 + C_5n^2 + 4C_6C_{12}\frac{n^2}{\rho_0^4}\}\sum_{x\in T_n}\sum_{\eta\in\Sigma_{n,k}^0}(f(\eta)-f(\eta^{x,x+1}))^2c_x(\eta)\mu(\eta).$$

We set $C_3^o := 1/(C_4 + C_5 + 4C_6C_{12})$, then we have

$$\lambda(n,k) \ge C_3^o\frac{\rho_0^4}{n^2},$$

if n is odd. Finally, we set $C_3 = \min\{C_3^e, C_3^o\} = C_3^o = 1/(C_4 + C_5 + 4C_6C_{12})$, then we conclude that

$$\lambda(n,k) \ge C_3\frac{\rho_0^4}{n^2},$$

□

References

[1] Bertini L.; Toninelli, C; Exclusion processes with degenerate rates: convergence to equilibrium and tagged particle. J. Statist. Phys. 117 (2004), no. 3-4, 549–580. MR-2099727

[2] Caputo P.; Liggett T.; Richthammer T.; Proof of Aldous' spectral gap conjecture. J. Amer. Math. Soc. 23 (2010), 831–851. MR-2629990

[3] Gonçalves, P.; Landim, C.; Toninelli, C.; Hydrodynamic limit for a particle system with degenerate rates. Ann. Inst. Henri Poincaré Probab. Stat. 45 (2009), no. 4, 887–909. MR-2572156

[4] Kipnis C., Landim C.; Scaling Limits of Interacting Particle Systems. Springer (1999) MR-1707314

[5] Ritort F.; Sollich P.; Glassy dynamics of kinetically constrained models. Adv. Phys. 52 (2003), no. 4, 219–342.

Acknowledgments. The author would like to thank anonymous referee for the careful reading and the valuable comments.