Electronic Journal of Probability

Electron. J. Probab. **17** (2012), no. 89, 1–26.
ISSN: 1083-6489 DOI: 10.1214/EJP.v17-1910

Anticipating linear stochastic differential equations driven by a Lévy process*

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Abstract

In this paper we study the existence of a unique solution for linear stochastic differential equations driven by a Lévy process, where the initial condition and the coefficients are random and not necessarily adapted to the underlying filtration. Towards this end, we extend the method based on Girsanov transformations on Wiener space and developped by Buckdahn [8] to the canonical Lévy space, which is introduced in [26].

Keywords: Canonical Lévy space; Girsanov tranformations; Lévy and Poisson measures; Malliavin calculus; Pathwise integral; Skorohod integral.
AMS MSC 2010: Primary 60H10, Secondary 60H05; 60H07; 60G51.
Submitted to EJP on March 28, 2012, final version accepted on October 1, 2012.

1 Introduction

Our aim in this paper is to prove the existence and uniqueness of a solution of the linear stochastic differential equation

$$\begin{aligned} X_t &= X_0 + \int_0^t b_s X_s \, ds + \int_0^t a_s X_s \, \delta W_s + \int_0^t \int_{\{|y|>1\}} v_s(y) X_{s-} \, dN(s,y) \\ &\quad + \int_0^t \int_{\{0<|y|\le 1\}} v_s(y) X_{s-} \, d\tilde{N}(s,y), \qquad 0 \le t \le T. \end{aligned} \tag{1.1}$$

Here X_0 is a random variable, a, b and $v(y)$, for any $y \in \mathbb{R}$, $y \neq 0$, are random processes not necessarily adapted to the underlying filtration, W is the canonical Wiener process, N is the canonical Poisson random measure with parameter ν (see Section 2.2 for details), $d\tilde{N}(t,y) := dN(t,y) - dt\,\nu(dy)$, and the integral with respect to W (respectively the integrals with respect to N and $\tilde{N}$) is in the Skorohod sense (respectively are pathwise defined).

In the adapted case (i.e., deterministic initial condition and predictable coefficients with respect to the filtration generated by W and N), the stochastic differential equation

*Supported by CONACyT grant 98998, MEC FEDER MTM 2009-07203 and MEC FEDER MTM 2009-08869.
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(1.1) with not necessarily linear coefficients has been analyzed by several authors (see, for instance, [2, 3, 4, 10, 11, 15, 16, 23, 24]). For example, Ikeda and Watanabe [11] have considered this equation with no necessarily linear coefficients and have used the Picard iteration procedure and Gronwall's lemma to show existence and uniqueness of the solution, respectively. It is well-known that this is possible due to the isometry property of Itô integrals. Also, in this case, one approach to study equation (1.1) is to assume first that N does not have small jumps (i.e., the absolute values of the jump sizes are bigger than a constant $\varepsilon > 0$) and consider equation (1.1) as a stochastic differential equation driven by a Brownian motion between two consecutive jump times, which has a unique solution under suitable conditions due to Itô [12]. Then, we only need to show that this solution converges to the one of equation (1.1) as $\varepsilon \to 0$. Namely, the solution of the equation

$$\begin{aligned} X_t^\varepsilon &= X_0 + \int_0^t b_s X_s^\varepsilon \, ds + \int_0^t a_s X_s^\varepsilon \, \delta W_s + \int_0^t \int_{\{|y|>1\}} v_s(y) X_{s-}^\varepsilon \, dN(s,y) \\ &\quad + \int_0^t \int_{\{\varepsilon<|y|\le 1\}} v_s(y) X_{s-}^\varepsilon \, d\tilde{N}(s,y), \qquad 0 \le t \le T, \end{aligned} \tag{1.2}$$

converges as $\varepsilon \downarrow 0$, to a solution of equation (1.1). We can see Rubenthaler [24] for details. This method was also utilized to obtain an Itô formula for Lévy processes (see, for example, Cont and Tankov [9]). We also mention that in the adapted and linear case, Itô formula provides a tool to obtain the existence and uniqueness of the solution to (1.1). For details, the reader can consult Protter [23].

In the general case, we cannot use the Picard iteration procedure, nor Gronwall's lemma to deal with (1.1) because the L^2-norm of the solution depends on its derivative in the Malliavin calculus sense and this derivative can be estimated only in terms of the second derivative, and so on. Therefore we do not have a closed argument, as it is pointed out by Nualart [20].

On the Wiener case (i.e., $v \equiv 0$), Buckdahn [6, 7, 8] has studied equation (1.1) via anticipating Girsanov transformations. In particular, he showed that Itô formula is not useful in this case. This approach has been also useful to deal with fractional stochastic differential equations (see [13, 14]).

On the Poisson space, it means $a \equiv 0$, equation (1.1) has been considered in different situations for different definitions of stochastic integral (see, for instance, [17, 18, 19, 21, 22]).

In this paper, in order to obtain the existence of a unique solution to equation (1.1), we apply the method developed in [6, 7, 8] between consecutive jump times to figure out the solution X^ε of the stochastic linear equation (1.2). Then, we get the convergence of X^ε to the solution X of (1.1). Moreover, X agrees with the solution to equation (1.1) obtained using the classical Itô's calculus when $a \equiv 0$ (see Theorem 5.1 below and Theorem II.37 in [23]).

On the other hand, note that we could combine the ideas of Buckdahn [6] and Privault [22] in order to calculate the solution to (1.1) when the stochastic integrals with respect to N and $\tilde{N}$ are interpreted as Skorohod type integrals. That is, we would be able to consider modifications to equations (2.5) and (2.6) to get a Girsanov's theorem for Lévy processes. But we do not choose this approach here because, in general, the solution of (1.1) is not given by Itô's calculus when $a \equiv 0$ due to the relation between the pathwise integral and Skorohod type one (see Corollary 2.9 in [1] and Privault [22]). We will consider this method elsewhere.

The paper is organized as follows. Section 2 is devoted to different preliminaires: Canonical Lévy space and process, Malliavin calculus and anticipative Girsanov transformations. In section 3 the solution candidates for equations (1.1) and (1.2) are presented and some of their properties are pointed out. In section 4 the existence of a unique solution of (1.2) is proved and in Section 5, the same is done for (1.1). A long and non-central proof of Theorem 2.10 is placed in the Appendix.

2 Preliminaries

In this section we give the framework and the tools we use in this paper to study the existence of a unique solution to equation (1.1). In particular we introduce the canonical Lévy space as it was done in Solé et al. [26], we extend some results given in Buckdahn [6, 7] to the latter space and recall some basic facts of the Malliavin calculus.

In the remaining of this paper, ν represents a Lévy measure on $\mathbb{R}$ such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} x^2\nu(dx) < \infty$ (for details see, for example, Sato [25]), T is a positive fixed number and ℓ denotes the Lebesgue measure on $[0,T]$. The Borel σ-algebra of a set $A \subset \mathbb{R}$ is denoted by $\mathcal{B}(A)$. The jumps of a càdlàg process Z are denoted by ΔZ (i.e., $\Delta_t Z = Z_t - Z_{t-}$). Also, for any $p \geq 1$, $|\cdot|_p$ and $||\cdot||_p$ denote the norms on $L^p([0,T])$ and on $L^p(\Omega)$, respectively. In particular $||\cdot||_\infty$ we will denote the norm on $L^\infty(\Omega)$, that is, the essential supremum in Ω. Sometimes we use the notation $|\cdot|_{L^p(\Omega)} = ||\cdot||_p$.

2.1 Canonical Lévy space

In this paper we consider all the processes defined on the canonical Lévy space on $[0,T]$,

$$(\Omega, \mathcal{F}, P) = (\Omega_W \otimes \Omega_N, \mathcal{F}_W \otimes \mathcal{F}_N, P_W \otimes P_N).$$

Here $(\Omega_W, \mathcal{F}_W, P_W)$ is the canonical Wiener space and $(\Omega_N, \mathcal{F}_N P_N)$ is the canonical Lévy space for a pure jump Lévy process with Lévy measure ν, which is defined as follows:

Let $\{\varepsilon_n : n \in \mathbb{N}\}$ be a strictly decreasing sequence of positive numbers such that $\varepsilon_1 = 1$, $\lim_{n\to\infty} \varepsilon_n = 0$ and $\nu(S_n) > 0$ for any $n \geq 1$, where $S_1 = \{x \in \mathbb{R} : \varepsilon_1 < |x|\}$ and $S_n = \{x \in \mathbb{R} : \varepsilon_n < |x| \leq \varepsilon_{n-1}\}$. With this notation in mind, the canonical Lévy space with measure ν is

$$(\Omega_N, \mathcal{F}_N, P_N) = \bigotimes_{n\geq 1} (\Omega^{(n)}, \mathcal{F}^{(n)}, P^{(n)}),$$

where $(\Omega^{(n)}, \mathcal{F}^{(n)}, P^{(n)})$ is the canonical Lévy space for a compound Poisson process with intensity $\lambda_n := \nu(S_n)$ and probability measure $Q_n := \nu(\cdot \cap S_n)/\lambda_n$. That is, for $n \in \mathbb{N}$,

$$\Omega^{(n)} := \bigcup_{k\geq 0} \left([0,T] \times S_n\right)^k,$$

with $\left([0,T] \times S_n\right)^0 = \{\alpha\}$, where α is an arbitrary point,

$$\mathcal{F}^{(n)} := \left\{B \subset \Omega^{(n)} : B \cap \left([0,T] \times S_n\right)^k \in \mathcal{B}\left(\left([0,T] \times S_n\right)^k\right), \text{ for all } k \in \mathbb{N}\right\}$$

and for any $B \in \mathcal{F}^{(n)}$,

$$P^{(n)}(B) := e^{-\lambda_n T} \sum_{k=0}^{\infty} \frac{\lambda_n^k (\ell \otimes Q_n)^{\otimes k}(B \cap ([0,T] \times S_n)^k)}{k!}.$$

2.2 Canonical Lévy process

The canonical Wiener process $W = \{W_t : t \in [0,T]\}$ is defined as $W_t(\omega) = \omega(t)$ for $\omega \in \Omega_W$, that is, ω is a continuous function on $[0,T]$ such that $\omega(0) = 0$.

The canonical pure jump process $J_t = \{J_t : t \in [0,T]\}$, with Lévy measure ν, is

$$J_t(\omega) = \lim_{k\to\infty} \sum_{n=2}^{k} \left(X_t^{(n)}(\omega^{(n)}) - t \int_{S_n} x\nu(dx) \right) + X_t^{(1)}(\omega^{(1)}), \quad \omega = (\omega^{(n)})_{n\geq 1} \in \Omega_N,$$

where the limit exists with probability 1 and

$$X_t^{(n)}(\omega^{(n)}) = \begin{cases} \sum_{l=1}^{m} x_l 1_{[0,t]}(t_l), & \text{if } \omega^{(n)} = ((t_1,x_1),\dots,(t_m,x_m)), \\ 0, & \text{if } \omega^{(n)} = \alpha. \end{cases}$$

Finally, the canonical Lévy process with triplet (γ,σ,ν) is defined as

$$X_t(\omega) = \gamma t + \sigma W_t(\omega') + J_t(\omega''), \quad \text{for } \omega = (\omega', \omega'') \in \Omega_W \otimes \Omega_N.$$

Recall also that the associated Poisson random measure is

$$N(B) := \#\{t \in [0,T] : (t, \Delta X_t) \in B\}, \quad B \in \mathcal{B}([0,T] \times \mathbb{R}_0),$$

where $\mathbb{R}_0 = \mathbb{R} - \{0\}$.

2.3 Elements of Malliavin calculus

In this paper we deal with the derivative with respect to the process W in the Malliavin calculus sense. So, in this subsection, we recall some basic properties of this operator. For details, the reader can consult Nualart [20] or Solé et al. [26].

Let $\mathcal{S}^W$ be the set of random variables of the form

$$F = f(\int_0^T h_1(s)dW_s, \dots, \int_0^T h_n(s)dW_s), \tag{2.1}$$

where $n \geq 1$, $h_j \in L^2([0,T])$ and $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$, that means f and all its partial derivatives are bounded. The derivative of the random variable F with respect to W is the random variable

$$D^W F = \sum_{j=1}^{n} (\partial_j f)(\int_0^T h_1(s)dW_s, \dots, \int_0^T h_n(s)dW_s)h_j.$$

The operator D^W is a linear operator from $L^2(\Omega_W)$ into $L^2(\Omega_W \times [0,T])$, closable and unbounded. We will always consider the closed extension of D^W and its domain will be denoted by $\mathbb{D}_{1,2}^W$.

Let $\tilde{\mathbb{D}}_{1,\infty}^W = \{F \in (\mathbb{D}_{1,2}^W \cap L^\infty(\Omega_W)) : DF \in L^\infty(\Omega_W \times [0,T])\}$. The Skorohod integral with respect to W, denoted by δ^W, is the adjoint of the derivative operator $D^W : \tilde{\mathbb{D}}_{1,\infty}^W \subset L^\infty(\Omega_W) \to L^\infty(\Omega_W \times [0,T])$. That is, u is in *Dom* δ^W if and only if $u \in L^1(\Omega_W \times [0,T])$ and there exists a random variable $\delta^W(u) \in L^1(\Omega_W)$ satisfying the duality relation

$$\mathbb{E}_W \left[\int_0^T u_t D_t^W F dt \right] = \mathbb{E}_W \left[\delta^W(u) F \right] \quad \text{for every} \quad F \in \tilde{\mathbb{D}}_{1,\infty}^W, \tag{2.2}$$

where $\mathbb{E}_W$ is the expectation with respect to the probability measure P_W. As it was pointed out by Buckdahn [6, 7], $\delta^W(u)$ is well-defined.

We can extend the last definitions to Hilbert space valued random variables: Let $\mathcal{S}^W(L^2(\Omega_N))$ be the set of all smooth $L^2(\Omega_N)$-random variables of the form

$$F = \sum_{i=1}^{n} f_i(\int_0^T h_{1,i}(s)dW_s, \dots, \int_0^T h_{n_i,i}(s)dW_s)G_i, \tag{2.3}$$

where $n \geq 1$, $h_{j,i} \in L^2([0,T])$, $G_i \in L^2(\Omega_N)$ and $f_i \in \mathcal{C}_b^\infty(\mathbb{R}^{n_i})$, for $i \in \{0,\dots,n\}$ and $j \in \{0\dots,n_i\}$. The derivative of the random variable F with respect to W is the $L^2(\Omega_N \times [0,T])$-valued random variable

$$D^W F = \sum_{i=1}^{n}\sum_{j=1}^{n_i}(\partial_j f_i)(\int_0^T h_{1,i}(s)dW_s,\dots,\int_0^T h_{n_i,i}(s)dW_s)h_{j,i}G_i.$$

The operator D^W is a linear operator from $L^2(\Omega)$ into $L^2\left(\Omega\times[0,T]\right)$, closable and unbounded. Moreover it can be iterated defining $D^{W,k}_{t_1,\dots,t_k}F := D^W_{t_k}\cdots D^W_{t_1}F$.

For any $k,p\geq 1$, we introduce the spaces $\mathbb{D}^W_{k,p}(L^2(\Omega_N))$ as the closure of $\mathcal{S}^W(L^2(\Omega_N))$ with respect to the norm

$$\|F\|^p_{W,k,p} := \Big|\,|F|_{L^2(\Omega_N)}\,\Big|^p_{L^p(\Omega_W)} + \sum_{j=1}^{k}\left|\left(\int_{[0,T]^j}|D^{W,j}_z F|^2_{L^2(\Omega_N)}dz\right)^{\frac12}\right|^p_{L^p(\Omega_W)}.$$

Now the Skorohod integral with respect to W, denoted by δ^W, is the adjoint of the derivative operator $D^W : \tilde{\mathbb{D}}^W_{1,\infty}(L^\infty(\Omega_N)) \subset L^\infty(\Omega) \to L^\infty(\Omega\times[0,T])$ with

$$\tilde{\mathbb{D}}^W_{1,\infty}(L^\infty(\Omega_N)) = \left\{F \in (\mathbb{D}^W_{1,2}(L^2(\Omega_N))\cap L^\infty(\Omega)) : DF \in L^\infty(\Omega\times[0,T])\right\}.$$

That is, u is in $Dom\ \delta^W$ if and only if $u \in L^1\left(\Omega\times[0,T]\right)$ and there exists a random variable $\delta^W(u) \in L^1(\Omega)$ satisfying the duality relation

$$\mathbb{E}\left[\int_0^T u_t D^W_t F dt\right] = \mathbb{E}\left[\delta^W(u)F\right] \quad \text{for every} \quad F \in \tilde{\mathbb{D}}^W_{1,\infty}(L^\infty(\Omega_N)). \tag{2.4}$$

The operator δ^W is an extension of the Itô integral in the sense that the set $L^2_a(\Omega_W\times[0,T])$ of all square-integrable and adapted processes with respect to the filtration generated by X is included in $Dom\ \delta^W$ and the operator δ^W restricted to $L^2_a(\Omega_W\times[0,T])$ coincides with the Itô stochastic integral with respect to W. For $u \in Dom\ \delta^W$ we will make use of the notation $\delta^W(u) = \int_0^T u_t\delta W_t$ and for $u1\!\!1_{[0,t]}$ in $Dom\ \delta^W$ we will write $\delta^W(u1\!\!1_{[0,t]}) = \int_0^t u_s\delta W_s$. Note that in (2.2) and (2.4) we are using δ^W for the Skorohod integrals defined on $L^1(\Omega_W\times[0,T])$ and on $L^1(\Omega\times[0,T])$, respectively. We hope that the space will be clear when we use this operator.

The following result will be important in next section.

Lemma 2.1. *Let $F \in \mathbb{D}^W_{1,2}(L^2(\Omega_N))$ and $u \in Dom\ \delta^W \cap L^1\left(\Omega\times[0,T]\right)$. Then, for almost all $\omega'' \in \Omega_N$, $F(\cdot,\omega'') \in \mathbb{D}^W_{1,2}$, $u(\cdot,\omega'') \in Dom\ \delta^W \cap L^1\left(\Omega_W\times[0,T]\right)$,*

$$D^W F(\cdot,\omega'') = (D^W F)(\cdot,\omega'')$$

and

$$\delta^W(u(\cdot,\omega'')) = \delta^W(u)(\cdot,\omega'').$$

Remark 2.2. *Note that left-hand sides of last two equalities are given by (2.1) and (2.2), while right-hand sides are defined via (2.3) and (2.4), respectively.*

Proof of Lemma 2.1. Let $F \in \mathbb{D}^W_{1,2}(L^2(\Omega_N))$. Then, there is a sequence $\{F_n \in \mathcal{S}^W(L^2(\Omega_N)) : n \in \mathbb{N}\}$ of the form (2.3) such that $\|F_n - F\|_{W,1,2} \to 0$. Hence, the definition of the canonical Lévy space, in particular the definition of the probability measure P, implies that there is a subsequence $\{n_k : k \in \mathbb{N}\}$ such that, for a.a. $\omega'' \in \Omega_N$,

$$|F_{n_k}(\cdot,\omega'') - F(\cdot,\omega'')|^2_{L^2(\Omega_W)} + \left|\left(\int_{[0,T]}|(D^W_z F_{n_k}(\cdot,\omega'')) - (D^W_z F)(\cdot,\omega'')|^2 dz\right)^{\frac12}\right|^2_{L^2(\Omega_W)} \to 0,$$

which gives that the first part of the result is true because $\{F_{n_k}(\cdot,\omega'') : k \in \mathbb{N}\}$ is a sequence of the form (2.1).

Finally, let $H \in \mathcal{S}^W$ and $G \in L^\infty(\Omega_N)$. Then, the duality relation (2.4) yields

$$\mathbb{E}\left[G\int_0^T u_t D_t^W H dt\right] = \mathbb{E}\left[G\delta^W(u)H\right].$$

Consequently, using the definition of the probability measure P, for a.a. $\omega'' \in \Omega_N$,

$$\mathbb{E}_W\left[\int_0^T u_t(\cdot,\omega'')D_t^W H dt\right] = \mathbb{E}_W\left[\delta^W(u)(\cdot,\omega'')H\right].$$

Thus, from the duality relation (2.2), the proof is complete. □

2.4 Anticipative Girsanov Transformations

Here, for the convenience of the reader, we recall some basic facts on anticipative Girsanov transformations. By Lemma 2.1, some of these results will be a consequence of the properties of transformations on Wiener space. For a more detailed account on this subject we refer to [6, 7, 8]. Remember that, by the definition of the canonical Lévy space, we have that for any $\omega \in \Omega$ there are $\omega' \in \Omega_W$ and $\omega'' \in \Omega_N$ such that $\omega = (\omega', \omega'')$ and vice versa. For $\omega' \in \Omega_W$ and $\omega'' \in \Omega_N$, we use the convention $\omega = (\omega', \omega'')$.

Given a process $a \in L^2(\Omega \times [0,T])$, we define the transformation $T_a : \Omega \to \Omega_W$ as the application defined by

$$T_a(\omega',\omega'') := \omega' + \int_0^\cdot a_s(\omega',\omega'')ds.$$

Observe that for ω'' fixed, we obtain a transformation on the Wiener space. We say this transformation is absolutely continuous if the measure $P_W \circ (T_a(\cdot,\omega''))^{-1}$ is absolutely continuous with respect to P_W, for almost all $\omega'' \in \Omega_N$. Henceforth, we introduce the Cameron-Martin space CM, that is, the subspace of absolutely continuous functions of Ω_W, with square-integrable derivatives, endowed with the norm

$$|\omega'|_{CM} := \left(\int_0^T \dot{\omega}'(t)^2 dt\right)^{\frac{1}{2}}.$$

The following two results are an immediate consequence of [6, 7, 8] and Lemma 2.1:

Proposition 2.3. *Let T^1 and T^2 be two absolutely continuous transformations associated with processes a_1 and a_2, respectively, $F \in \mathbb{D}^W_{1,2}(L^2(\Omega_N))$ and $\sigma \in L^2([0,T],\mathbb{D}^W_{1,2}(L^2(\Omega_N)))$. Then, for almost all $\omega'' \in \Omega_N$, we have*

$$|F(T_{a_1}(\omega',\omega''),\omega'') - F(T_{a_2}(\omega',\omega''),\omega'')| \le |||D^W F|_2||_\infty |T_{a_1}(\omega',\omega'') - T_{a_2}(\omega',\omega'')|_{CM}$$

and

$$\left(\int_0^T |\sigma_s(T_{a_1}(\omega',\omega''),\omega'') - \sigma_s(T_{a_2}(\omega',\omega''),\omega'')|^2 ds\right)^{\frac{1}{2}}$$
$$\le \left\|\left(\int_0^T\int_0^T |D_r^W \sigma_s|^2 ds dr\right)^{\frac{1}{2}}\right\|_\infty |T_{a_1}(\omega',\omega'') - T_{a_2}(\omega',\omega'')|_{CM}.$$

Proposition 2.4. *Let T_a be an absolutely continuous transformation. Assume $a \in L^2([0,T];\mathbb{D}^W_{1,2}(L^2(\Omega_N)))$, and let $\sigma \in L^2([0,T];\mathbb{D}^W_{1,2}(L^2(\Omega_N)))$ be with $\sigma(T_a,\cdot) \in L^2([0,T],L^2(\Omega))$ and*

$$\left\| \left(\int_0^T \int_0^T |D^W_r \sigma_s|^2 ds dr \right)^{\frac{1}{2}} \right\|_\infty < \infty.$$

Then, for almost all $\omega'' \in \Omega_N$, we get $\sigma(T_a(\cdot,\omega''),\omega'') \in L^2([0,T];\mathbb{D}^W_{1,2})$,

$$\begin{aligned} D^W_t(\sigma_s(T_a(\omega',\omega''),\omega'')) &= (D^W_t\sigma_s)(T_a(\omega',\omega''),\omega'') \\ &\quad + \int_0^T (D^W_r\sigma_s)(T_a(\omega',\omega''),\omega'')(D^W_t a_r)(\omega',\omega'')dr, \end{aligned}$$

and

$$\begin{aligned} \int_0^T \sigma_s(T_a(\omega',\omega''),\omega'')\delta W_s &= \left(\int_0^T \sigma_s \delta W_s \right)(T_a(\omega',\omega''),\omega'') \\ &\quad - \int_0^T \sigma_s(T_a(\omega',\omega''),\omega'')a_s(\omega',\omega'')ds \\ &\quad - \int_0^T \int_0^T (D^W_r\sigma_s)(T_a(\omega',\omega''),\omega'')(D^W_s a_r)(\omega',\omega'')drds, \end{aligned}$$

for almost all $\omega' \in \Omega_W$.

In the remaining of this paper $\mathbb{D}^W_{1,\infty}(L^2(\Omega_N))$ represents the set of the elements F in $\mathbb{D}^W_{1,2}(L^2(\Omega_N))$ such that

$$||F||_{1,\infty} := ||F||_\infty + ||\,|D^W F|_2\,||_\infty < \infty.$$

Similarly $\mathbb{D}^W_{2,\infty}(L^2(\Omega_N))$ is the family of all the elements in $\mathbb{D}^W_{2,2}(L^2(\Omega_N)) \cap \mathbb{D}^W_{1,\infty}(L^2(\Omega_N))$ such that $D^{W,2}F \in L^\infty(\Omega; L^2([0,T]^2))$.

Now, for $a \in L^2([0,T];\mathbb{D}^W_{1,\infty}(L^2(\Omega_N)))$ fixed, we consider two families of transformations $\{T_t : \Omega \to \Omega_W : 0 \le t \le T\}$ and $\{A_{s,t} : \Omega \to \Omega_W : 0 \le s \le t \le T\}$, which are the solutions of the equations

$$(T_t\,\omega)_\cdot = \omega'_\cdot + \int_0^{t\wedge\cdot} a_s(T_s\omega,\omega'')\,ds. \tag{2.5}$$

and

$$(A_{s,t}\,\omega)_\cdot = \omega'_\cdot - \int_{s\wedge\cdot}^{t\wedge\cdot} a_r(A_{r,t}\omega,\omega'')\,dr, \tag{2.6}$$

respectively.

Observe that, for simplicity of the notation, we do not make explicitly the dependence on a in these equations. Some of the properties of the solutions to (2.5) and (2.6) that we need are established in the following result. See [6, 7, 8] for its proof.

Proposition 2.5. *Let $a \in L^2([0,T];\mathbb{D}^W_{1,\infty}(L^2(\Omega_N)))$. Then, there exist two unique families of absolutely continuous transformations $\{T_t, 0 \le t \le T\}$ and $\{A_{s,t} : 0 \le s \le t \le T\}$ that satisfy equations (2.5) and (2.6), respectively. Moreover, for all $s,t \in [0,T]$, $s<t$, $A_{s,t}(\cdot,\omega'') = T_s(\cdot,\omega'')A_t(\cdot,\omega'')$, with $A_t = A_{0,t}$, $T_t(\cdot,\omega'')$ is invertible with inverse $A_t(\cdot,\omega'')$ and $a_\cdot(T_\cdot(*,\omega''),\omega'') \in L^2([0,T];\mathbb{D}^W_{1,\infty})$, for a.a. $\omega'' \in \Omega_N$.*

In relation to the transformation $A_{s,t}$, we have the following lemma that will be useful for our purposes.

Lemma 2.6. *Let $a \in L^2([0,T]; \mathbb{D}^W_{1,\infty}(L^2(\Omega_N)))$. Then, for any $u \le s \le t$, we have*

$$|A_{u,t}\omega - A_{u,s}\omega|^2_{CM} \le 2\left(\int_s^t ||a_r||^2_\infty dr\right)\exp\left\{2\int_0^T ||\,|D^W a_r|^2_2\,||_\infty dr\right\}.$$

Remark 2.7. *Note $A_{u,t}$ is continuous in t with respect the CM-norm, uniformly in u.*

Proof of Lemma 2.6. Let $u \le s \le t$. Then, by the Propositions 2.3 and 2.5 we have

$$\begin{aligned}
|A_{u,s}\omega - A_{u,t}\omega|^2_{CM} &= \left|\int_{u\wedge\cdot}^{s\wedge\cdot} a_r(A_{r,s}\omega,\omega'')dr - \int_{u\wedge\cdot}^{t\wedge\cdot} a_r(A_{r,t}\omega,\omega'')dr\right|^2_{CM} \\
&= \int_0^T |\mathbb{1}_{(u,s]}(r)a_r(A_{r,s}\omega,\omega'') - \mathbb{1}_{(u,t]}(r)a_r(A_{r,t}\omega,\omega'')|^2 dr \\
&\le 2\int_s^t |a_r(A_{r,t}\omega,\omega'')|^2 dr + 2\int_u^s |a_r(A_{r,s}\omega,\omega'') - a_r(A_{r,t}\omega,\omega'')|^2 dr \\
&\le 2\int_s^t ||a_r||^2_\infty dr + 2\int_u^s ||\,|D^W a_r|^2_2\,||_\infty |A_{r,s}\omega - A_{r,t}\omega|^2_{CM} dr.
\end{aligned}$$

So, using Gronwall's lemma, we obtain

$$|A_{u,t}\omega - A_{u,s}\omega|^2_{CM} \le 2\left(\int_s^t ||a_r||^2_\infty dr\right)\exp\left\{2\int_u^s ||\,|D^W a_r|^2_2\,||_\infty dr\right\}$$

which implies the result holds. □

Remark 2.8. *In [6, 7, 8], Buckdahn has proven that both inequalities in Proposition 2.3 hold only for almost all $\omega' \in \Omega_W$. But, by Fubini theorem, it is not difficult to see that, in this case, the inequality in Lemma 2.6 is satisfied for a.a. $\omega \in \Omega$.*

To finish this subsection we give some results related to the densities of the transformations $\{T_t : \Omega \to \Omega_W : 0 \le t \le T\}$ and $\{A_{s,t} : \Omega \to \Omega_W : 0 \le s \le t \le T\}$. Now, let $F \in L^\infty(\Omega)$ and a as in Proposition 2.5. One of our main tools in the proof of the existence and uniqueness of the solution to equation (1.1) are the equalities, proven by Buckdahn [6, 7, 8],

$$\mathbb{E}\left[F(A_{s,t}\omega,\omega'')L_{s,t}(\omega)\right] = \mathbb{E}\left[F\right] \tag{2.7}$$

and

$$\mathbb{E}\left[F(A_{s,t}\omega,\omega'')\right] = \mathbb{E}\left[F\mathcal{L}_{s,t}\right], \tag{2.8}$$

where

$$\begin{aligned}
L_{s,t}(\omega) &= \exp\left\{\int_s^t a_r(A_{r,t}\omega,\omega'')\delta W_r - \frac{1}{2}\int_s^t a_r^2(A_{r,t}\omega,\omega'')dr\right. \\
&\qquad \left. - \int_s^t\int_r^t (D^W_u a_r)(A_{r,t}\omega,\omega'')D^W_r[a_u(A_{u,t}\omega,\omega'')]dudr\right\}
\end{aligned} \tag{2.9}$$

is the density of $A^{-1}_{s,t}$ and

$$\begin{aligned}
\mathcal{L}_{s,t}(\omega) &= \exp\left\{-\int_s^t a_r(T_tA_r\omega,\omega'')\delta W_r - \frac{1}{2}\int_s^t a_r^2(T_tA_r\omega,\omega'')dr\right. \\
&\qquad \left. - \int_s^t\int_s^r (D^W_u a_r)(T_tA_r\omega,\omega'')D^W_r[a_u(T_tA_u\omega,\omega'')]dudr\right\}.
\end{aligned} \tag{2.10}$$

Finally, we have that, in this case,

$$L_{s,t}(\omega) = \mathcal{L}^{-1}_{s,t}(A_{s,t}\omega,\omega''), \tag{2.11}$$

$$L_{0,t}(\omega) = L_{0,s}(A_{s,t}\omega,\omega'')L_{s,t}(\omega), \quad 0 \le s \le t \le T. \tag{2.12}$$

These two relations can be proved as consequence of the equalities (2.7) and (2.8). Indeed,

$$\mathbb{E}\left[F(A_{s,t}\omega,\omega'')L_{s,t}(\omega)\right] = \mathbb{E}\left[F\right] = \mathbb{E}\left[F\mathcal{L}_{s,t}\mathcal{L}_{s,t}^{-1}\right] = \mathbb{E}\left[F(A_{s,t}\omega,\omega'')\mathcal{L}_{s,t}^{-1}(A_{s,t}\omega,\omega'')\right],$$

and

$$\mathbb{E}\left[F(A_t\omega,\omega'')L_{0,t}(\omega)\right] = \mathbb{E}\left[F(A_s\omega,\omega'')L_{0,s}(\omega)\right] = \mathbb{E}\left[F(A_t\omega,\omega'')L_{0,s}(A_{s,t}\omega,\omega'')L_{s,t}(\omega)\right].$$

2.5 The anticipative linear stochastic differential equation on canonical Wiener space

On the canonical Wiener space, Buckdahn [6, 7, 8] has studied equation (1.1) via the anticipating Girsanov transformations (2.5) and (2.6). Namely, he considered the linear stochastic differential equation

$$Z_t = Z_0 + \int_0^t h_s Z_s \, ds + \int_0^t a_s Z_s \, \delta W_s, \qquad t \in [0,T], \tag{2.13}$$

and stated the following result:

Theorem 2.9. *Assume $a \in L^2([0,T],\mathbb{D}^W_{1,\infty})$, $h \in L^1([0,T], L^\infty(\Omega_W))$ and $Z_0 \in L^\infty(\Omega_W)$. Then, the process $Z = \{Z_t : t \in [0,T]\}$ defined by*

$$Z_t := Z_0(A_{0,t}) \exp\left\{\int_0^t h_s(A_{s,t}) \, ds\right\} \, L_{0,t} \tag{2.14}$$

belongs to $L^1(\Omega_W \times [0,T])$ and is a global solution of (2.13). Conversely, if $Y \in L^1(\Omega_W \times [0,T])$ is a global solution of (2.13) and, if, moreover, $a, h \in L^\infty(\Omega_W \times [0,T])$ and $D^W a \in L^\infty(\Omega_W \times [0,T]^2)$, then Y is of the form (2.14) for a.e. $0 \le t \le T$.

Moreover we need the following proposition on the continuity of Z, whose proof is given in the Appendix (see Section 6) because it is too long and technical.

Theorem 2.10. *Assume $Z_0 \in \mathbb{D}^W_{1,\infty}$, $h \in L^1([0,T],\mathbb{D}^W_{1,\infty})$ and that, for some $p > 2$,*

$$a \in L^{2p}([0,T],\mathbb{D}^W_{1,\infty}) \cap L^2([0,T],\mathbb{D}^W_{2,\infty}).$$

Then, Z given by (2.14) has continuous trajectories a.s.

3 Two processes with jumps

In the sequel we use the following hypothesis on the coefficients:

(H1) Assume that $X_0 \in \mathbb{D}^W_{1,\infty}(L^\infty(\Omega_N))$, $b, v(y) \in L^1([0,T],\mathbb{D}^W_{1,\infty}(L^\infty(\Omega_N))$, for all $y \in \mathbb{R}_0$. Moreover, there exists $p > 2$ such that

$$a \in L^2([0,T],\mathbb{D}^W_{2,\infty}(L^\infty(\Omega_N)) \cap L^{2p}([0,T],\mathbb{D}^W_{1,\infty}(L^\infty(\Omega_N)).$$

(H2) There exists a positive function $g \in L^2(\mathbb{R}_0,\nu) \cap L^1(\mathbb{R}_0,\nu)$ such that

$$|v_s(y,\omega)| \le g(y), \qquad \text{uniformly in } \omega \text{ and } s,$$

and

$$\lim_{|y|\to 0} g(y) = 0.$$

(H3) The function g satisfies $\int_{\mathbb{R}_0}(e^{g(y)} - 1)\nu(dy) < \infty$.

(H4) The function g satisfies $\int_{\mathbb{R}_0}(e^{2g(y)}-1)\nu(dy)<\infty$.

Remark 3.1. *As an example, observe that the following function is in $L^1(\mathbb{R}_0,\nu)\cap L^2(\mathbb{R}_0,\nu)$ and is such that (H3) and (H4) hold, and $\lim_{|y|\to 0} g(y)=0$.*

$$g(y)=\begin{cases} k_1(\beta)y^2, & y\in(-\beta,\beta),\\ k_2(\beta), & y\in(-\beta,\beta)^c,\end{cases}$$

where $\beta\in(0,1)$ and $k_1(\beta)$ and $k_2(\beta)$ are positive constants.

In the remaining of this paper, we use the notation $F(A_{s,t}(\omega))=F(A_{s,t}(\omega),\omega'')$ for any function $F:\Omega\to\mathbb{R}$.

Given $\varepsilon>0$, set

$$\begin{aligned} X_t^\varepsilon &= X_0(A_{0,t})\exp\left\{\int_0^t b_s(A_{s,t})\,ds\right\}\,L_{0,t}\prod_{s\le t,\varepsilon<|y|}\Big[1+v_s(y,A_{s,t})\Delta N(s,y)\Big]\\ &\quad\times\exp\left\{-\int_0^t\int_{\{|y|>\varepsilon\}} v_s(y,A_{s,t})\,\nu(dy)ds\right\}. \end{aligned}\tag{3.1}$$

Notice that this process can also be written as follows

$$X_t^\varepsilon = X_0(A_{0,t})\exp\left\{\int_0^t b_s^\varepsilon(A_{s,t})\,ds\right\}\,L_{0,t}\prod_{i=1}^{N_t^\varepsilon}\Big[1+v_{\tau_i^\varepsilon}(y_i^\varepsilon,A_{\tau_i^\varepsilon,t})\Big],$$

where $b_s^\varepsilon(\omega):=b_s(\omega)-\int_{\{|y|>\varepsilon\}}v_s(y,\omega)\nu(dy)$, $\{\tau_i^\varepsilon, i\ge 1\}$ are the jump times which jump size is greater than ε, y_i^ε denotes the amplitude of jump τ_i^ε, and N_t^ε is the number of jumps before t, with size bigger than ε.

Proposition 3.2. *Assume (H1) and (H2) hold. For each $t\in[0,T]$, the process X_t^ε, defined in (3.1), converges almost surely to*

$$\begin{aligned} X_t &= X_0(A_{0,t})\exp\left\{\int_0^t b_s(A_{s,t})\,ds\right\}\,L_{0,t}\,\exp\left\{-\int_0^t\int_{\mathbb{R}_0} v_s(y,A_{s,t})\,\nu(dy)\,ds\right\}\\ &\quad\times\prod_{s\le t,y\in\mathbb{R}_0}\Big[1+v_s(y,A_{s,t})\Delta N(s,y)\Big]. \end{aligned}\tag{3.2}$$

Remark 3.3. *In the proof of this result we will see that the representation*

$$\begin{aligned} X_t &= X_0(A_{0,t})\exp\left\{\int_0^t b_s(A_{s,t})\,ds\right\}\,L_{0,t}\,\exp\left\{\int_0^t\int_{\mathbb{R}_0} v_s(y,A_{s,t})\,d\tilde N(s,y)\right\}\\ &\quad\times\prod_{s\le t,y\in\mathbb{R}_0}\Big[1+v_s(y,A_{s,t})\Delta N(s,y)\Big]e^{-v_s(y,A_{s,t})\Delta N(s,y)} \end{aligned}$$

also holds. We observe that the stochastic integral with respect to $\tilde N$ is pathwise defined.

Proof of Proposition 3.2. First of all, the hypotheses on X_0 and b yield

$$\left|X_0(A_{0,t})\exp\left\{\int_0^t b_s(A_{s,t})\,ds\right\}\right|\le C.$$

Secondly, as the factor $L_{0,t}$ is a density, it is finite a.s. So it remains to see the convergence of the following quantities:

$$\begin{aligned} M_1 &= \exp\left\{\int_0^t\int_{|y|>\varepsilon} v_s(y,A_{s,t})\,d\tilde N(s,y)\right\},\\ M_2 &= \prod_{s\le t,\varepsilon<|y|}\Big[1+v_s(y,A_{s,t})\Delta N(s,y)\Big]e^{-v_s(y,A_{s,t})\Delta N(s,y)}. \end{aligned}$$

Using the relation $d\tilde{N}(t,y) = dN(t,y) - \nu(dy)dt$ and (H2), we have

$$\begin{aligned}\int_0^t \int_{\mathbb{R}_0} |v_s(y, A_{s,t})| \ dN(s,y) + \int_0^t \int_{\mathbb{R}_0} |v_s(y, A_{s,t})| \ \nu(dy)ds \\ \leq \int_0^t \int_{\mathbb{R}_0} g(y) \ dN(s,y) + \int_0^t \int_{\mathbb{R}_0} g(y) \ \nu(dy)ds \\ = \int_0^t \int_{\mathbb{R}_0} g(y) \ d\tilde{N}(s,y) + 2\int_0^t \int_{\mathbb{R}_0} g(y) \ \nu(dy)ds.\end{aligned} \tag{3.3}$$

This quantity is finite a.s. because (H2) implies

$$\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}_0} g(y) \ d\tilde{N}(s,y)\right)^2\right] = \int_0^t \int_{\mathbb{R}_0} g(y)^2 \ \nu(dy)ds < \infty.$$

Then, M_1 converges a.s., as $\varepsilon \to 0$, to $\exp\{\int_0^t \int_{\mathbb{R}_0} v_s(y, A_{s,t}) d\tilde{N}(s,y)\}$.

On the other hand, for any constant $c > 0$,

$$M_2 = M_{2,1} \times M_{2,2},$$

with

$$\begin{aligned} M_{2,1} &= \prod_{s \leq t, \varepsilon < |y| < c \vee \varepsilon} \Big[1 + v_s(y, A_{s,t}) \Delta N(s,y)\Big] e^{-v_s(y, A_{s,t})\Delta N(s,y)}, \\ M_{2,2} &= \prod_{s \leq t, c \vee \varepsilon \leq |y|} \Big[1 + v_s(y, A_{s,t}) \Delta N(s,y)\Big] e^{-v_s(y, A_{s,t})\Delta N(s,y)}. \end{aligned}$$

$M_{2,2}$ is well-defined and converges as $\varepsilon \to 0$ to

$$\prod_{s \leq t, c \leq |y|} \Big[1 + v_s(y, A_{s,t}) \Delta N(s,y)\Big] e^{-v_s(y, A_{s,t})\Delta N(s,y)},$$

because it is a product of a finite number of factors. To deal with $M_{2,1}$ we use the following argument: Hypothesis (H2) implies that for small enough y, $|v_s(y,\omega)| \leq \frac{1}{2}$. Then, choosing $c > 0$ such that $|g(y)| < \frac{1}{2}$, for $|y| < c$,

$$\log M_{2,1} = \sum_{s \leq t, \varepsilon < |y| < c} \left[\log\left(1 + v_s(y, A_{s,t}) \Delta N(s,y)\right) - v_s(y, A_{s,t}) \Delta N(s,y)\right],$$

and this series is absolutely convergent since

$$\begin{aligned} &\sum_{s \leq t, \varepsilon < |y| < c} \left|\log\left(1 + v_s(y, A_{s,t}) \Delta N(s,y)\right) - v_s(y, A_{s,t}) \Delta N(s,y)\right| \\ &\quad \leq \sum_{s \leq t, 0 < |y| < c} \left[v_s(y, A_{s,t}) \Delta N(s,y)\right]^2 \leq \sum_{s \leq t, 0 < |y| < c} \frac{1}{2} \left|v_s(y, A_{s,t}) \Delta N(s,y)\right| \\ &\quad \leq \frac{1}{2} \sum_{s \leq t, 0 < |y| < c} g(y) \Delta N(s,y). \end{aligned}$$

So, $M_{2,1}$ also converges as $\varepsilon \to 0$ since $\int_0^t \int_{\mathbb{R}_0} g(y) \ dN(s,y)$ is finite by (3.3).

We can conclude that the processes X_t^ε and X_t are well-defined and X_t^ε converges a.s. to X_t as ε goes to zero. □

Proposition 3.4. *Assume (H1), (H2) and (H3). Then, the processes* X^ε *and* X *belong to* $L^1(\Omega \times [0,T])$ *and*

$$X_t^\varepsilon \to X_t, \tag{3.4}$$

in $L^1(\Omega \times [0,T])$*, as* ε *goes to zero.*

Proof. Since X_0 and b are bounded and (H2) is true, it is immediate to check that

$$\begin{aligned}|X_t^\varepsilon| &= |X_0(A_{0,t})| \exp\left\{\int_0^t b_s(A_{s,t})\,ds\right\} L_{0,t} \left|\prod_{s\le t,\varepsilon<|y|}\Big[1+v_s(y,A_{s,t})\Delta N(s,y)\Big]\right| \\ &\quad\times \exp\left\{-\int_0^t\int_{\{|y|>\varepsilon\}} v_s(y,A_{s,t})\,\nu(dy)ds\right\} \\ &\le CL_{0,t}\prod_{s\le t,\varepsilon<|y|}\Big[1+g(y)\Delta N(s,y)\Big]\exp\left\{\int_0^t\int_{\{|y|>\varepsilon\}} g(y)\,\nu(dy)ds\right\} \\ &\le CL_{0,t}\prod_{s\le t,\varepsilon<|y|}\Big[1+g(y)\Delta N(s,y)\Big],\end{aligned}$$

due to $g\in L^1(\mathbb{R}_0,\nu)$. Moreover, using that $1+x\le e^x$ for $x>0$, we get

$$|X_t^\varepsilon|\le CL_{0,t}\exp\left\{\sum_{s\le t,\varepsilon<|y|} g(y)\Delta N(s,y)\right\}\le CL_{0,t}\exp\left\{\int_0^T\int_{\mathbb{R}_0} g(y)dN(s,y)\right\}.$$

Finally, the result follows from Proposition 3.2, Hypothesis (H3) and the dominated convergence theorem. Indeed, the facts that N is a Poisson random measure with Lévy measure ν and $E_W(L_{0,t})=1$ imply that $L_{0,t}\exp\left\{\int_0^T\int_{\mathbb{R}_0} g(y)dN(s,y)\right\}\in L^1(\Omega)$. □

4 Existence and uniqueness of solution of the approximated equation

The goal of this section is to prove the following theorem.

Theorem 4.1. *Assume (H1), (H2) and (H3). Also assume that a, b and $v(y)$, for any $y\in\mathbb{R}_0$, belong to $L^\infty(\Omega\times[0,T])$ and D^Wa belongs to $L^\infty(\Omega\times[0,T]^2)$. Then, the process X^ε defined in (3.1) is the unique solution in $L^1(\Omega\times[0,T])$ of*

$$X_t^\varepsilon = X_0+\int_0^t b_sX_s^\varepsilon ds+\int_0^t a_sX_s^\varepsilon\delta W_s+\int_0^t\int_{\{|y|>\varepsilon\}} v_s(y)X_{s-}^\varepsilon\, d\tilde N(s,y). \tag{4.1}$$

Remark 4.2. *Note that Equation (1.2) can be rewritten as an equation of the form (4.1).*

Proof of Theorem 4.1. The proof is divided into three steps.
Step 1. We first analyze the right-continuity of X^ε. As we have seen before

$$X_t^\varepsilon = X_0(A_{0,t})\exp\left\{\int_0^t b_s^\varepsilon(A_{s,t})\,ds\right\} L_{0,t}\prod_{i=1}^{N_t^\varepsilon}\Big[1+v_{\tau_i^\varepsilon}(y_i^\varepsilon,A_{\tau_i^\varepsilon,t})\Big].$$

Observe that under the hypotheses of the theorem, $X_0(A_{0,t})\exp\left\{\int_0^t b_s^\varepsilon(A_{s,t})\,ds\right\} L_{0,t}$ is continuous on t, for a.a. $\omega''\in\Omega_N$, as a consequence of Theorem 2.10. On other hand, ω''−a.s., $\prod_{i=1}^{N_t^\varepsilon}[1+v_{\tau_i^\varepsilon}(y_i^\varepsilon,A_{\tau_i^\varepsilon,t})]$ is a finite product with all the terms well defined and continuous on t as a consequence of (H1), Lemma 2.6 and Proposition 2.3. So, X^ε has a.s. right continuous trajectories with left limits. Moreover recall that by Proposition 3.4 the process X^ε belongs to $L^1(\Omega\times[0,T])$.
Step 2. We now prove that X^ε is a solution of (4.1).

Assume that G is an element of the set of $L^2(\Omega_N)$-smooth Wiener functionals described as $G=\sum_{i=1}^n H_i Z_i$, with $H_i\in\mathcal{S}^W$ and $Z_i\in L^2(\Omega_N)$. Denote

$$\Phi_s^\varepsilon(\omega)=\prod_{r\le s,\varepsilon<|y|}\Big[1+v_r(y,T_r)\Delta N(r,y)\Big].$$

Due to Girsanov's theorem (2.7) and Proposition 2.5, we have

$$\begin{aligned}\mathbb{E}\Big[\int_0^t a_s X_s^\varepsilon D_s^W G ds\Big] &= \mathbb{E}\Big[\int_0^t a_s X_0(A_{0,s})\exp\Big\{\int_0^s b_r^\varepsilon(A_{r,s})dr\Big\}L_{0,s}\Phi_s^\varepsilon(A_s)D_s^W G ds\Big]\\ &= \mathbb{E}\Big[\int_0^t a_s(T_s)X_0\exp\Big\{\int_0^s b_r^\varepsilon(T_r)dr\Big\}\Phi_s^\varepsilon\,(D_s^W G)(T_s)ds\Big].\end{aligned}$$

Lemma 2.2.4 in [6] shows that $\frac{d}{ds}G(T_s)=a_s(T_s)(D_s^W G)(T_s)$. So,

$$\mathbb{E}\Big[\int_0^t a_s X_s^\varepsilon D_s^W G ds\Big]=\mathbb{E}\Big[\int_0^t\Big(\frac{d}{ds}G(T_s)\Big)X_0\exp\Big\{\int_0^s b_r^\varepsilon(T_r)dr\Big\}\Phi_s^\varepsilon ds\Big].$$

Since $\int_0^t\int_{\{|y|>\varepsilon\}}dN(s,y)<\infty$ a.s., we have

$$\mathbb{E}\Big[\int_0^t a_s X_s^\varepsilon D_s^W G ds\Big]=\sum_{i=1}^\infty\mathbb{E}\Big[\int_{\tau_{i-1}^\varepsilon\wedge t}^{\tau_i^\varepsilon\wedge t}\Big(\frac{d}{ds}G(T_s)\Big)X_0\exp\Big\{\int_0^s b_r^\varepsilon(T_r)dr\Big\}\Phi_{\tau_{i-1}^\varepsilon\wedge t}^\varepsilon ds\Big].$$

Then, integration by parts implies

$$\begin{aligned}\mathbb{E}\Big[\int_0^t a_s X_s^\varepsilon D_s^W G ds\Big] &= \sum_{i=1}^\infty\mathbb{E}\Big[G(T_{\tau_i^\varepsilon\wedge t})X_0\exp\Big\{\int_0^{\tau_i^\varepsilon\wedge t}b_r^\varepsilon(T_r)dr\Big\}\Phi_{\tau_{i-1}^\varepsilon\wedge t}^\varepsilon\\ &\quad -G(T_{\tau_{i-1}^\varepsilon\wedge t})X_0\exp\Big\{\int_0^{\tau_{i-1}^\varepsilon\wedge t}b_r^\varepsilon(T_r)dr\Big\}\Phi_{\tau_{i-1}^\varepsilon\wedge t}^\varepsilon\\ &\quad -\int_{\tau_{i-1}^\varepsilon\wedge t}^{\tau_i^\varepsilon\wedge t}G(T_s)X_0 b_s^\varepsilon(T_s)\exp\Big\{\int_0^s b_r^\varepsilon(T_r)dr\Big\}\Phi_{\tau_{i-1}^\varepsilon\wedge t}^\varepsilon ds\Big].\end{aligned}$$

Using that $\Phi_{\tau_i^\varepsilon}^\varepsilon=\Phi_{\tau_{i-1}^\varepsilon}^\varepsilon(1+v_{\tau_i^\varepsilon}(y_i^\varepsilon,T_{\tau_i^\varepsilon}))$, we have that the previous quantity is equal to

$$\begin{aligned}&\sum_{i=1}^\infty\mathbb{E}\Big[G(T_{\tau_i^\varepsilon\wedge t})X_0\exp\Big\{\int_0^{\tau_i^\varepsilon\wedge t}b_r^\varepsilon(T_r)dr\Big\}\Phi_{\tau_i^\varepsilon\wedge t}^\varepsilon\\ &\qquad -G(T_{\tau_{i-1}^\varepsilon\wedge t})X_0\exp\Big\{\int_0^{\tau_{i-1}^\varepsilon\wedge t}b_r^\varepsilon(T_r)dr\Big\}\Phi_{\tau_{i-1}^\varepsilon\wedge t}^\varepsilon\Big]\\ &-\sum_{i;\tau_i\le t}\mathbb{E}\Big[G(T_{\tau_i^\varepsilon\wedge t})X_0\exp\Big\{\int_0^{\tau_i^\varepsilon\wedge t}b_r^\varepsilon(T_r)dr\Big\}v_{\tau_i^\varepsilon\wedge t}(y_i^\varepsilon,T_{\tau_i^\varepsilon\wedge t})\Phi_{\tau_{i-1}^\varepsilon\wedge t}^\varepsilon\Big]\\ &-\sum_{i=1}^\infty\mathbb{E}\Big[\int_{\tau_{i-1}^\varepsilon\wedge t}^{\tau_i^\varepsilon\wedge t}G(T_s)X_0 b_s^\varepsilon(T_s)\exp\Big\{\int_0^s b_r^\varepsilon(T_r)dr\Big\}\Phi_{\tau_{i-1}^\varepsilon\wedge t}^\varepsilon ds\Big].\end{aligned}$$

Taking into account that the two first summands form a telescopic series, Girsanov's theorem (2.7) and (3.1) imply that

$$\begin{aligned}\mathbb{E}\Big[\int_0^t a_s X_s^\varepsilon D_s^W G ds\Big] &= \mathbb{E}\Big[G(T_t)X_0 \exp\Big\{\int_0^t b_r^\varepsilon(T_r)dr\Big\}\Phi_t^\varepsilon - GX_0\Big] \\ &\quad - \sum_{i;\tau_i\le t}\mathbb{E}\Big[GX_0(A_{\tau_i^\varepsilon\wedge t})\exp\Big\{\int_0^{\tau_i^\varepsilon\wedge t} b_r^\varepsilon(A_{r,\tau_i^\varepsilon\wedge t})dr\Big\}L_{0,\tau_i^\varepsilon\wedge t} \\ &\quad \times v_{\tau_i^\varepsilon\wedge t}(y_i^\varepsilon)\Phi^\varepsilon_{\tau_{i-1}^\varepsilon\wedge t}(A_{\tau_i^\varepsilon\wedge t})\Big] - \sum_{i=1}^\infty \mathbb{E}\Big[\int_{\tau_{i-1}^\varepsilon\wedge t}^{\tau_i^\varepsilon\wedge t} G b_s^\varepsilon X_s^\varepsilon ds\Big] \\ &= \mathbb{E}\big[GX_t^\varepsilon - GX_0\big] - \mathbb{E}\Big[G\int_0^t\int_{\{|y|>\varepsilon\}} v_s(y)X_{s-}^\varepsilon\, dN(s,y)\Big] \\ &\quad -\mathbb{E}\Big[G\int_0^t b_s^\varepsilon X_s^\varepsilon\, ds\Big].\end{aligned}$$

So,

$$\mathbb{E}\Big[\int_0^t a_s X_s^\varepsilon D_s^W G ds\Big] = \mathbb{E}\Big[G\Big(X_t^\varepsilon - X_0 - \int_0^t\int_{\{|y|>\varepsilon\}} v_s(y)X_{s-}^\varepsilon\, d\tilde{N}(s,y) - \int_0^t b_s X_s^\varepsilon ds\Big)\Big]. \tag{4.2}$$

That means that X^ε defined in (3.1) is solution of (4.1).

Step 3. Now we prove the uniqueness of the solution to (4.1). We argue it by induction with respect to the jump times τ_i^ε. Notice that if $t\in[0,\tau_1^\varepsilon)$, by Theorem 2.9, there exists a unique solution. We now suppose that $t\in[\tau_1^\varepsilon,\tau_2^\varepsilon)$. Assume that Y^ε is a solution of the stochastic differential equation (4.1) such that $Y^\varepsilon\in L^1(\Omega\times[0,T])$ and $a.Y^\varepsilon_\cdot\ \mathbb{1}_{[\tau_1^\varepsilon,\tau_2^\varepsilon)}(\cdot)$ is Skorohod integrable. For any $t\in[\tau_1^\varepsilon,\tau_2^\varepsilon)$, Y_t^ε satisfies

$$Y_t^\varepsilon = X_{\tau_1^\varepsilon}^\varepsilon + \int_{\tau_1^\varepsilon}^t b_s^\varepsilon Y_s^\varepsilon ds + \int_{\tau_1^\varepsilon}^t a_s Y_s^\varepsilon \delta W_s, \tag{4.3}$$

where by Step 1, we can write

$$X_{\tau_1^\varepsilon}^\varepsilon = X_0(A_{0,\tau_1^\varepsilon})\Big[1+v_{\tau_1^\varepsilon}(y_1^\varepsilon)\Big]\exp\Big\{\int_0^{\tau_1^\varepsilon} b_s^\varepsilon(A_{s,\tau_1^\varepsilon})\, ds\Big\}\ L_{0,\tau_1^\varepsilon}. \tag{4.4}$$

Note that Lemma 2.1 implies that, for a.a. $\omega''\in\Omega_N$, $a(\cdot,\omega'')Y(\cdot,\omega'')\ \mathbb{1}_{]\tau_i^\varepsilon,t]}\in \text{Dom}\ \delta^W$, and $a(\cdot,\omega'')$, $X_0(\cdot,\omega'')$, $b(\cdot,\omega'')$, $v(y,\cdot,\omega'')$ satisfy (H1) when we write $\mathbb{D}^W_{1,\infty}$ and $\mathbb{D}^W_{2,\infty}$ instead of $\mathbb{D}^W_{1,\infty}(L^\infty(\Omega_N))$ and $\mathbb{D}^W_{2,\infty}(L^\infty(\Omega_N))$, respectively. Now we fix a such ω'' and in the following calculations we avoid to write it to simplify the notation.

By Buckdahn [7] (Proposition 2.1) there is a sequence $\{a^n : n\in\mathbb{N}\}$ of smooth functionals of the form $a^n=\sum_{i=1}^{m_n}F_{i,n}h_{i,n}$, with $F_{i,n}\in\mathcal{S}^W$ and $h_{i,n}\in L^2([0,T])$ satisfying the following three statements:

- a^n converges to a in $L^2([0,T],\mathbb{D}^W_{1,2})$.
- For every $n\ge 1$, $|\,||a^n||_\infty\,|_2\le |\,||a||_\infty\,|_2$ and

$$\int_0^T ||\,|Da_t^n|_2\,||_\infty^2 dt \le 1+\int_0^T ||\,|Da_t|_2\,||_\infty^2 dt.$$

- $|a^n|_{L^\infty(\Omega_W\times[0,T])}\le |a|_{L^\infty(\Omega_W\times[0,T])}$ and $|Da^n|_{L^\infty(\Omega_W\times[0,T]^2)}\le 1+|Da|_{L^\infty(\Omega_W\times[0,T]^2)}$, for every $n\ge1$.

Fix $n \in \mathbb{N}$ and consider the transformation A^n given by (2.6), when we change a and A by a^n and A^n, respectively. Let G be a smooth functional defined by the right-hand side of (2.1). Then, Buckdahn [6] (Proposition 2.2.13) leads to establish

$$\frac{d}{dt}G(A^n_{\tau_1^\varepsilon,t}) = -a^n_t D^W_t\left[G(A^n_{\tau_1^\varepsilon,t})\right]. \tag{4.5}$$

Taking into account (4.3), we get

$$\begin{aligned}\mathbb{E}_W\left[Y^\varepsilon_t G(A^n_{\tau_1^\varepsilon,t})\right] &= \mathbb{E}_W\left[X^\varepsilon_{\tau_1^\varepsilon} G(A^n_{\tau_1^\varepsilon,t})\right] + \mathbb{E}_W\left[\int_{\tau_1^\varepsilon}^t a_s Y^\varepsilon_s D^W_s\left[G(A^n_{\tau_1^\varepsilon,t})\right]ds\right]\\ &\quad + \mathbb{E}_W\left[\int_{\tau_1^\varepsilon}^t b^\varepsilon_s Y^\varepsilon_s G(A^n_{\tau_1^\varepsilon,t})ds\right].\end{aligned}$$

Now, replacing $G(A^n_{\tau_1^\varepsilon,t})$ by $G(A^n_{\tau_1^\varepsilon,s}) + \int_s^t \frac{d}{dr}G(A^n_{\tau_1^\varepsilon,r})dr$, for $s \in [\tau_1^\varepsilon, t]$, and using (4.5), we obtain

$$\begin{aligned}\mathbb{E}_W\left[Y^\varepsilon_t G(A^n_{\tau_1^\varepsilon,t})\right] &= \mathbb{E}_W\left[X^\varepsilon_{\tau_1^\varepsilon} G\right] + \mathbb{E}_W\left[X^\varepsilon_{\tau_1^\varepsilon}\int_{\tau_1^\varepsilon}^t \frac{d}{ds}G(A^n_{\tau_1^\varepsilon,s})ds\right]\\ &\quad + \mathbb{E}_W\left[\int_{\tau_1^\varepsilon}^t a_s Y^\varepsilon_s D^W_s\left[G(A^n_{\tau_1^\varepsilon,s}) + \int_s^t \frac{d}{dr}G(A^n_{\tau_1^\varepsilon,r})dr\right]ds\right]\\ &\quad + \mathbb{E}_W\left[\int_{\tau_1^\varepsilon}^t b^\varepsilon_s Y^\varepsilon_s\left(G(A^n_{\tau_1^\varepsilon,s}) + \int_s^t \frac{d}{dr}G(A^n_{\tau_1^\varepsilon,r})dr\right)ds\right]\\ &= \mathbb{E}_W\left[X^\varepsilon_{\tau_1^\varepsilon} G\right] - \mathbb{E}_W\left[X^\varepsilon_{\tau_1^\varepsilon}\int_{\tau_1^\varepsilon}^t a^n_s D^W_s\left[G(A^n_{\tau_1^\varepsilon,s})\right]ds\right]\\ &\quad + \mathbb{E}_W\left[\int_{\tau_1^\varepsilon}^t a_s Y^\varepsilon_s D^W_s\left[G(A^n_{\tau_1^\varepsilon,s})\right]ds\right]\\ &\quad - \mathbb{E}_W\left[\int_{\tau_1^\varepsilon}^t a_s Y^\varepsilon_s D^W_s\left[\int_s^t a^n_r D^W_r\left(G(A^n_{\tau_1^\varepsilon,r})\right)dr\right]ds\right]\\ &\quad + \mathbb{E}_W\left[\int_{\tau_1^\varepsilon}^t b^\varepsilon_s Y^\varepsilon_s G(A^n_{\tau_1^\varepsilon,s})ds\right]\\ &\quad - \mathbb{E}_W\left[\int_{\tau_1^\varepsilon}^t b^\varepsilon_s Y^\varepsilon_s\int_s^t a^n_r D^W_r\left(G(A^n_{\tau_1^\varepsilon,r})\right)drds\right].\end{aligned}$$

Therefore, the Fubini theorem allows us to state

$$\begin{aligned}&\mathbb{E}_W\left[Y^\varepsilon_t G(A^n_{\tau_1^\varepsilon,t})\right]\\ &= \mathbb{E}_W\left[X^\varepsilon_{\tau_1^\varepsilon} G\right] - \mathbb{E}_W\left[X^\varepsilon_{\tau_1^\varepsilon}\int_{\tau_1^\varepsilon}^t a^n_s D^W_s\left[G(A^n_{\tau_1^\varepsilon,s})\right]ds\right]\\ &\quad + \mathbb{E}_W\left[\int_{\tau_1^\varepsilon}^t a_s Y^\varepsilon_s D^W_s\left[G(A^n_{\tau_1^\varepsilon,s})\right]ds\right]\\ &\quad - \mathbb{E}_W\left[\int_{\tau_1^\varepsilon}^t\int_{\tau_1^\varepsilon}^r a_s Y^\varepsilon_s D^W_s\left[a^n_r D^W_r\left(G(A^n_{\tau_1^\varepsilon,r})\right)\right]dsdr\right]\\ &\quad + \mathbb{E}_W\left[\int_{\tau_1^\varepsilon}^t b^\varepsilon_s Y^\varepsilon_s G(A^n_{\tau_1^\varepsilon,s})ds\right] - \mathbb{E}_W\left[\int_{\tau_1^\varepsilon}^t\int_{\tau_1^\varepsilon}^r b^\varepsilon_s Y^\varepsilon_s a^n_r D^W_r\left(G(A^n_{\tau_1^\varepsilon,r})\right)dsdr\right].\end{aligned}$$

By Lemma 2.2.13 in [6] we have that $a_r^n D_r^W \left(G(A^n_{\tau_1^\varepsilon,r}) \right)$ is a smooth functional for fixed $r \in [\tau_1^\varepsilon, t]$. Therefore, applying (4.3) to this smooth functional we get that

$$\begin{aligned} \mathbb{E}_W \left[Y_t^\varepsilon G(A^n_{\tau_1^\varepsilon,t}) \right] &= \mathbb{E}_W \left[X^\varepsilon_{\tau_1^\varepsilon} G \right] + \mathbb{E}_W \left[\int_{\tau_1^\varepsilon}^t (a_s - a_s^n)\, Y_s^\varepsilon D_s^W \left[G(A^n_{\tau_1^\varepsilon,s}) \right] ds \right] \\ &\quad + \mathbb{E}_W \left[\int_{\tau_1^\varepsilon}^t b_s^\varepsilon Y_s^\varepsilon G(A^n_{\tau_1^\varepsilon,s}) ds \right]. \end{aligned}$$

The hypotheses assumed allow us to use the dominated convergence theorem. Thus, taking limit as $n \to \infty$,

$$\mathbb{E}\left[Y_t^\varepsilon G(A_{\tau_1^\varepsilon,t}) \right] = \mathbb{E}\left[X^\varepsilon_{\tau_1^\varepsilon} G \right] + \mathbb{E}\left[\int_{\tau_1^\varepsilon}^t b_s^\varepsilon Y_s^\varepsilon G(A_{\tau_1^\varepsilon,s}) ds \right].$$

From Girsanov's argument (2.8) and Proposition 2.5 we have

$$\mathbb{E}\left[Y_t^\varepsilon (T_{\tau_1^\varepsilon,t}) \mathcal{L}_{\tau_1^\varepsilon,t} G \right] = \mathbb{E}\left[X^\varepsilon_{\tau_1^\varepsilon} G \right] + \mathbb{E}\left[\int_{\tau_1^\varepsilon}^t b_s^\varepsilon (T_{\tau_1^\varepsilon,s}) Y_s^\varepsilon (T_{\tau_1^\varepsilon,s}) \mathcal{L}_{\tau_1^\varepsilon,s} G ds \right].$$

Since this is true for any smooth functional G, it implies

$$Y_t^\varepsilon (T_{\tau_1^\varepsilon,t}) \mathcal{L}_{\tau_1^\varepsilon,t} = X^\varepsilon_{\tau_1^\varepsilon} + \int_{\tau_1^\varepsilon}^t b_s^\varepsilon (T_{\tau_1^\varepsilon,s}) Y_s^\varepsilon (T_{\tau_1^\varepsilon,s}) \mathcal{L}_{\tau_1^\varepsilon,s} ds.$$

So,

$$Y_t^\varepsilon (T_{\tau_1^\varepsilon,t}) \mathcal{L}_{\tau_1^\varepsilon,t} = X^\varepsilon_{\tau_1^\varepsilon} \exp\left\{ \int_{\tau_1^\varepsilon}^t b_s^\varepsilon (T_{\tau_1^\varepsilon,s}) ds \right\},$$

and, by (2.11) and (4.4), this means that

$$\begin{aligned} Y_t^\varepsilon &= X^\varepsilon_{\tau_1^\varepsilon}(A_{\tau_1^\varepsilon,t}) \exp\left\{ \int_{\tau_1^\varepsilon}^t b_s^\varepsilon (A_{s,t}) ds \right\} L_{\tau_1^\varepsilon,t} \\ &= X_0(A_{0,t}) \exp\left\{ \int_0^{\tau_1^\varepsilon} b_s^\varepsilon (A_{s,t})\, ds \right\} \exp\left\{ \int_{\tau_1^\varepsilon}^t b_s^\varepsilon (A_{s,t}) ds \right\} \\ &\quad \times L_{0,\tau_1^\varepsilon}(A_{\tau_1^\varepsilon,t}) L_{\tau_1^\varepsilon,t} \left[1 + v_{\tau_1^\varepsilon}(y_1^\varepsilon, A_{\tau_1^\varepsilon,t}) \right]. \end{aligned}$$

Finally, using (2.12), we have that

$$Y_t^\varepsilon = X_0(A_{0,t}) \exp\left\{ \int_0^t b_s^\varepsilon (A_{s,t}) ds \right\} L_{0,t} \left[1 + v_{\tau_1^\varepsilon}(y_1^\varepsilon, A_{\tau_1^\varepsilon,t}) \right].$$

This completes the proof for $t \in [\tau_1^\varepsilon, \tau_2^\varepsilon)$. The rest of the cases can be treated similarly. □

5 Existence and uniqueness of solution for the main equation

The main goal of this section is to prove the following theorem:

Theorem 5.1. *Assume (H1), (H2) and (H4). Suppose also that* $a, b, v(y) \in L^\infty(\Omega \times [0,T])$*, for any* $y \in \mathbb{R}_0$*, and* $D^W a \in L^\infty(\Omega \times [0,T]^2)$*. Then, the process* X *defined in (3.2) is the unique solution in* $L^1(\Omega \times [0,T])$ *of*

$$X_t = X_0 + \int_0^t b_s X_s ds + \int_0^t a_s X_s \delta W_s + \int_0^t \int_{\mathbb{R}_0} v_s(y) X_{s-}\, d\tilde{N}(s,y), \quad t \in [0,T], \qquad (5.1)$$

such that

$$\int_0^T \int_{\mathbb{R}_0} |v_s(y) X_{s-}| dN(s,y) \in L^1(\Omega).$$

Here, the stochastic integrals with respect to $\tilde{N}$ and N are pathwise defined.

Remark 5.2. *As an immediate consequence of the proof of this result, equation (1.1) can be rewritten as an equation of the form (5.1).*

Proof of Theorem 5.1. This proof is divided into two parts.

Step 1. First of all, by means of a limit argument we will show that X defined in (3.2) is a solution of (5.1). Towards this end, we prove the convergence of (4.2), as ε tends to zero.

Using that a and b belong to $L^\infty(\Omega \times [0,T])$ and that G is a smooth element, we obtain that

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\Big[\int_0^t a_s X_s^\varepsilon D_s^W G ds\Big] = \mathbb{E}\Big[\int_0^t a_s X_s D_s^W G ds\Big],$$

and

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\Big[G\Big(X_t^\varepsilon - X_0 - \int_0^t b_s X_s^\varepsilon ds\Big)\Big] = \mathbb{E}\Big[G\Big(X_t - X_0 - \int_0^t b_s X_s ds\Big)\Big].$$

It only remains to prove that, for any $t \in [0,T]$,

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\Big[G \int_0^t \int_{\{|y|>\varepsilon\}} v_s(y) X_{s-}^\varepsilon \, d\tilde{N}(s,y)\Big] = \mathbb{E}\Big[G \int_0^t \int_{\{|y|>0\}} v_s(y) X_{s-} \, d\tilde{N}(s,y)\Big]. \quad (5.2)$$

In order to prove this convergence and that the right-hand side is well-defined, we utilize the following estimate:

$$\mathbb{E}\left[\left|\int_0^t \int_{\{|y|>\varepsilon\}} v_s(y) X_{s-}^\varepsilon \, d\tilde{N}(s,y) - \int_0^t \int_{\{|y|>0\}} v_s(y) X_{s-} \, d\tilde{N}(s,y)\right|\right] \le I_1^\varepsilon + I_2^\varepsilon,$$

with

$$\begin{aligned} I_1^\varepsilon &= \mathbb{E}\left[\left|\int_0^t \int_{\{0<|y|\le\varepsilon\}} v_s(y) X_{s-} \, d\tilde{N}(s,y)\right|\right], \\ I_2^\varepsilon &= \mathbb{E}\left[\left|\int_0^t \int_{\{|y|>\varepsilon\}} v_s(y) \left[X_{s-}^\varepsilon - X_{s-}\right] \, d\tilde{N}(s,y)\right|\right]. \end{aligned}$$

First of all, by the definition of $\tilde{N}$, we can write

$$I_1^\varepsilon \le I_{1,1}^\varepsilon + I_{1,2}^\varepsilon,$$

with

$$\begin{aligned} I_{1,1}^\varepsilon &= \mathbb{E}\left[\left|\int_0^t \int_{\{0<|y|\le\varepsilon\}} v_s(y) X_{s-} \, dN(s,y)\right|\right], \\ I_{1,2}^\varepsilon &= \mathbb{E}\left[\left|\int_0^t \int_{\{0<|y|\le\varepsilon\}} v_s(y) X_{s-} \, \nu(dy) ds\right|\right]. \end{aligned}$$

Now, (H2) and the bound of X given in the proof of Proposition 3.4 imply that

$$\begin{aligned} I_{1,1}^{\varepsilon} &\leq \mathbb{E}\left[\int_0^T \int_{\{0<|y|\leq\varepsilon\}} g(y)\,|X_{s-}|\ dN(s,y)\right] \\ &\leq C\,\mathbb{E}\left[\int_0^T \int_{\{0<|y|\leq\varepsilon\}} g(y)\ L_{0,t}\ \exp\left\{\int_0^T \int_{\mathbb{R}_0} g(x)dN(r,x)\right\}\ dN(s,y)\right] \\ &\leq C\,\mathbb{E}_N\left[\left(\int_0^T \int_{\{0<|y|\leq\varepsilon\}} g(y)\ dN(s,y)\right)\exp\left\{\int_0^T \int_{\mathbb{R}_0} g(x)dN(r,x)\right\}\right]. \end{aligned}$$

Then, by (H4) we obtain that

$$\mathbb{E}\left[I_{1,1}^{\varepsilon}\right] \longrightarrow 0,$$

as ε goes to 0.

Proceeding similarly, we also get

$$\mathbb{E}\left[I_{1,2}^{\varepsilon}\right] \longrightarrow 0,$$

as ε goes to 0.

Finally, again the relation between of N and $\tilde{N}$, the fact that

$$\left|X_{s-}^{\varepsilon} - X_{s-}\right| \leq 2C\ L_{0,t}\ \exp\left\{\int_0^T \int_{\mathbb{R}_0} g(y)dN(s,y)\right\},$$

which is a consequence of Proposition 3.4, and the dominated convergence theorem allow us to ensure that, as ε goes to 0,

$$\mathbb{E}\left[I_2^{\varepsilon}\right] \longrightarrow 0.$$

So, the convergence (5.2) is satisfied.

Step 2. Now we show the uniqueness of the solution of equation (5.1). Let $Y \in L^1(\Omega \times [0,T])$ be a solution of (5.1) satisfying $aY\ 1\!\!1_{[0,t]} \in Dom\,\delta^W$, $t \in [0,T]$, and

$$\int_0^T \int_{\mathbb{R}_0} |v_s(y)Y_{s-}|dN(s,y) \in L^1(\Omega).$$

Recall that the coefficients verify that for any $\omega'' \in \Omega_N$ a.s.

$$|b_t(\cdot,w'')| + |a_t(\cdot,w'')| + \left|D_s^W a_t(\cdot,w'')\right| + |v_t(y,\cdot,w'')| \leq C, \tag{5.3}$$

for any $s,t \in [0,T]$, $\omega' \in \Omega_W$ and $y \in \mathbb{R}_0$.

Now fix $\omega'' \in \Omega_N$, and let a^n and A^n be as in the proof of Theorem 4.1. Consequently, for any $G \in \mathcal{S}^W$, we have, by Lemma 2.1,

$$\begin{aligned} \mathbb{E}_W\left[Y_t(\cdot,\omega'')G(A_t^n)\right] &= \mathbb{E}_W\left[X_0(\cdot,\omega'')G(A_t^n)\right] + \mathbb{E}_W\left[G(A_t^n)\int_0^t b_s(\cdot,\omega'')Y_s(\cdot,\omega'')ds\right] \\ &\quad + \mathbb{E}_W\left[\int_0^t a_s(\cdot,\omega'')Y_s(\cdot,\omega'')D_s^W(G(A_t^n))ds\right] \\ &\quad + \mathbb{E}_W\left[G(A_t^n)\int_0^t \int_{\mathbb{R}_0} v_s(y,\cdot,\omega'')Y_{s-}(\cdot,\omega'')d\tilde{N}(s,y)\right]. \end{aligned} \tag{5.4}$$

By (2.2.24) in Lemma 2.2.13 of [6],

$$\frac{d}{dt}G(A_t^n) = -a_t^n(\cdot,\omega'')D_t^W(G(A_t^n)),$$

and it implies that

$$G(A_t^n) = G(A_s^n) - \int_s^t a_r^n(\cdot,\omega'') D_r^W(G(A_r^n))dr.$$

Taking this last equality into account, we get

$$\begin{aligned}
&\mathbb{E}_W\left[Y_t(\cdot,\omega'')G(A_t^n)\right] = \mathbb{E}_W\left[X_0(\cdot,\omega'')G\right] - \mathbb{E}_W\left[X_0(\cdot,\omega'')\int_0^t a_r^n(\cdot,\omega'')D_r^W(G(A_r^n))dr\right] \\
&\quad +\mathbb{E}_W\left[\int_0^t G(A_s^n)b_s(\cdot,\omega'')Y_s(\cdot,\omega'')ds\right] \\
&\quad -\mathbb{E}_W\left[\int_0^t \left(\int_s^t a_r^n(\cdot,\omega'')D_r^W(G(A_r^n))dr\right) b_s(\cdot,\omega'')Y_s(\cdot,\omega'')ds\right] \\
&\quad +\mathbb{E}_W\left[\int_0^t a_s(\cdot,\omega'')Y_s(\cdot,\omega'')D_s^W(G(A_s^n))ds\right] \\
&\quad -\mathbb{E}_W\left[\int_0^t D_s^W\left(\int_s^t a_r^n(\cdot,\omega'')D_r^W(G(A_r^n))dr\right) a_s(\cdot,\omega'')Y_s(\cdot,\omega'')ds\right] \\
&\quad +\mathbb{E}_W\left[\int_0^t\int_{\mathbb{R}_0} G(A_s^n)v_s(y,\cdot,\omega'')Y_{s-}(\cdot,\omega'')d\tilde{N}(s,y)\right] \\
&\quad -\mathbb{E}_W\left[\int_0^t\int_{\mathbb{R}_0}\left(\int_s^t a_r^n(\cdot,\omega'')D_r^W(G(A_r^n))dr\right) v_s(y,\cdot,\omega'')Y_{s-}(\cdot,\omega'')d\tilde{N}(s,y)\right].
\end{aligned}$$

Hence, proceeding as in Step 3 of the proof of Theorem 4.1, we state

$$\begin{aligned}
\mathbb{E}_W\left[Y_t(\cdot,\omega'')G(A_t^n)\right] &= \mathbb{E}_W\left[X_0(\cdot,\omega'')G\right] + \mathbb{E}_W\left[\int_0^t G(A_s^n)b_s(\cdot,\omega'')Y_s(\cdot,\omega'')ds\right] \\
&\quad +\mathbb{E}_W\left[\int_0^t\int_{\mathbb{R}_0} G(A_s^n)v_{s-}(y,\cdot,\omega'')Y_{s-}(\cdot,\omega'')d\tilde{N}(s,y)\right] \\
&\quad +\mathbb{E}_W\left[\int_0^t (a_s - a_s^n)(\cdot,\omega'')D_s^W(G(A_s^n))Y_s(\cdot,\omega'')ds\right].
\end{aligned}$$

Therefore, proceeding as in the proof of Theorem 4.1 again, we can write

$$\begin{aligned}
\mathbb{E}_W\left[Y_t(\cdot,\omega'')G(A_t)(\cdot,\omega'')\right] &= \mathbb{E}_W\left[X_0(\cdot,\omega'')G\right] + \mathbb{E}_W\left[\int_0^t G(A_s)b_s(\cdot,\omega'')Y_s(\cdot,\omega'')ds\right] \\
&\quad +\mathbb{E}_W\left[\int_0^t\int_{\mathbb{R}_0} G(A_s)v_s(y,\cdot,\omega'')Y_{s-}(\cdot,\omega'')d\tilde{N}(s,y)\right].
\end{aligned}$$

Thus, Girsanov theorem implies

$$\begin{aligned}
&\mathbb{E}_W\left[\mathcal{L}_t(\cdot,\omega'')Y_t(T_t(\cdot,\omega''),\omega'')G\right] \\
&\quad = \mathbb{E}_W\Bigg[G\Bigg(X_0(\cdot,\omega'') + \int_0^t b_s(T_s(\cdot,\omega''),\omega'')Y_s(T_s(\cdot,\omega''),\omega'')\mathcal{L}_s(\cdot,\omega'')ds \\
&\qquad + \int_0^t\int_{\mathbb{R}_0} v_s(y,T_s(\cdot,\omega''),\omega'')Y_{s-}(T_s(\cdot,\omega''),\omega'')\mathcal{L}_s(\cdot,\omega'')d\tilde{N}(s,y)\Bigg)\Bigg],
\end{aligned}$$

which yields

$$
\begin{aligned}
&Y_t(T_t(\omega',\omega''),\omega'')\mathcal{L}_t(\omega',\omega'')\\
&\quad = X_0(\omega',\omega'') + \int_0^t b_s(T_s(\omega',\omega''),\omega'')Y_s(T_s(\omega,\omega''),\omega'')\mathcal{L}_s(\omega',\omega'')ds\\
&\qquad + \int_0^t \int_{\mathbb{R}_0} v_s(y,T_s(\omega',\omega''),\omega'')Y_{s-}(T_s(\omega',\omega''),\omega'')\mathcal{L}_s(\omega',\omega'')d\tilde{N}(s,y), \qquad \omega' \text{ a.s.}
\end{aligned}
$$

Consequently, by Fubini's theorem, we also have that, for almost all ω',

$$
\begin{aligned}
&Y_t(T_t(\omega',\omega''),\omega'')\mathcal{L}_t(\omega',\omega'')\\
&\quad = X_0(\omega',\omega'') + \int_0^t b_s(T_s(\omega',\omega''),\omega'')Y_s(T_s(\omega,\omega''),\omega'')\mathcal{L}_s(\omega',\omega'')ds\\
&\qquad + \int_0^t \int_{\mathbb{R}_0} v_s(y,T_s(\omega',\omega''),\omega'')Y_{s-}(T_s(\omega',\omega''),\omega'')\mathcal{L}_s(\omega',\omega'')d\tilde{N}(s,y), \qquad \omega'' \text{ a.s.}
\end{aligned}
\tag{5.5}
$$

Finally, we only need to observe that either Protter [23] (Theorem 37, page 84), or Bojdecki [5] (Theorem 13.12) gives

$$
\begin{aligned}
&Y_t(T_t(\omega',\omega''),\omega'')\mathcal{L}_t(\omega',\omega'')\\
&= \quad X_0(\omega)\exp\left\{\int_0^t b_s(T_s(\omega),\omega'')ds + \int_0^t \int_{\mathbb{R}_0} v_s(y,T_s(\omega),\omega'')d\tilde{N}(s,y)\right\}\\
&\quad\quad \times \prod_{0\le s\le t} \left[1+v_s(y,T_s(\omega),\omega'')\Delta N(s,y)\right],
\end{aligned}
$$

wich means that $X = Y$ and the proof is complete. □

6 Appendix

This section is devoted to presenting the proof of Theorem 2.10. In order to simplify the notation, we use the convention $D = D^W$ and $\delta = \delta^W$ because, in this section, the probability space is the canonical Wiener space. Also, remember that c will denote a generic constant that may change from line to line.

We begin this section with an auxiliary result.

Lemma 6.1. *Under the assumptions of Theorem 2.10, we have that, for $s \in [0,T]$,*

(a)

$$\int_0^T |D_\theta(a_r(A_{r,s}))|^2 d\theta \le 2e^{2c_1} \|\, |Da_r|_2 \,\|_\infty^2, \quad r\in[0,s],$$

where $c_1 := \int_0^T \|\, |Da_r|_2^2 \,\|_\infty dr$.

(b)

$$\int_0^s \int_0^T |D_\theta(a_r(A_{r,s}))|^2 d\theta dr \le 2c_1 e^{2c_1}.$$

(c)

$$\int_0^T |(D_\theta a_r)(A_{r,t}) - (D_\theta a_r)(A_{r,s})|^2 d\theta \le \||DDa_r|_2^2\|_\infty 2e^{2c_1} \int_s^t \|a_r\|_\infty^2 dr, \quad r\in[0,s].$$

(d)

$$\int_0^s \int_0^T |(D_\theta a_r)(A_{r,t}) - (D_\theta a_r)(A_{r,s})|^2 d\theta dr \le 2e^{2c_1} c_2 \int_s^t \|a_r\|_\infty^2 dr,$$

with $c_2 := \int_0^T \|\, |DDa_r|_2^2 \,\|_\infty dr$.

Proof. We first observe that, by virtue of Proposition 2.4 and (2.6), we obtain

$$D_\theta(a_r(A_{r,s})) = (D_\theta a_r)(A_{r,s}) - \int_r^s (D_u a_r)(A_{r,s}) D_\theta(a_u(A_{u,s}))du. \tag{6.1}$$

Therefore, from Hölder inequality, we have

$$\begin{aligned}
&\int_0^T |D_\theta(a_r(A_{r,s}))|^2 d\theta \\
&\quad \le 2\int_0^T |(D_\theta a_r)(A_{r,s})|^2 d\theta + 2\int_0^T \left|\int_r^s (D_u a_r)(A_{r,s}) D_\theta(a_u(A_{u,s}))du\right|^2 d\theta \\
&\quad \le 2||\,|Da_r|_2\,||_\infty^2 + 2\int_0^T \left(\int_r^s |(D_u a_r)(A_{r,s})|^2 du\right)\left(\int_r^s |D_\theta a_u(A_{u,s})|^2 du\right) d\theta \\
&\quad \le 2||\,|Da_r|_2\,||_\infty^2 + 2||\,|Da_r|_2\,||_\infty^2 \int_r^s \left(\int_0^T |D_\theta(a_u(A_{u,s}))|^2 d\theta\right) du.
\end{aligned}$$

Consequently, by Gronwall's lemma, we deduce

$$\int_0^T |D_\theta(a_r(A_{r,s}))|^2 d\theta \le 2||\,|Da_r|_2\,||_\infty^2 \exp\left\{\int_r^s 2||\,|Da_u|_2\,||_\infty^2 du\right\} \le 2e^{2c_1}||\,|Da_r|_2\,||_\infty^2,$$

which shows that Statement (a) is satisfied.

Now, using Proposition 2.3, Lemma 2.6 and the definition of constant c_1 we obtain

$$\begin{aligned}
\int_0^T |(D_\theta a_r)(A_{r,t}) - (D_\theta a_r)(A_{r,s})|^2 d\theta &\le ||\,|DDa_r|_2^2||_\infty \sup_{r\le s} |A_{r,t} - A_{r,s}|_{CM}^2 \\
&\le ||\,|DDa_r|_2^2||_\infty 2e^{2c_1} \int_s^t ||a_u||_\infty^2 du.
\end{aligned}$$

Thus, Statement (c) holds.

Finally, Statements (b) and (d) are an immediate consequence of (a) and (b), and the proof is complete. □

Now we are ready to prove Theorem 2.10

Proof of Theorem 2.10. From (2.9) and (2.14), we only need to show the continuity of the processes $Z_0(A_{0,\cdot})$, $\int_0^\cdot h_s(A_{s,\cdot})ds$, $\int_0^\cdot a_s^2(A_{s,\cdot})\,ds$, $\int_0^\cdot a_s(A_{s,\cdot})\,\delta W_s$, and

$$\int_0^\cdot \int_s^\cdot (D_u a_s)(A_{s,\cdot})\, D_s(a_u(A_{u,\cdot}))\, duds.$$

So now we divide the proof in several steps and we assume that $0 \le s \le t \le T$.

(1) Taking into account Proposition 2.3, we get

$$|Z_0(A_{0,t}) - Z_0(A_{0,s})| \le \left\|\left(\int_0^T |D_s Z_0|^2 ds\right)^{\frac{1}{2}}\right\|_\infty |A_{0,t}\omega - A_{0,s}\omega|_{CM},$$

which, together with the definition of the space $\mathbb{D}_{1,\infty}^W$ and Lemma 2.6, implies that the process $\{Z_0(A_{0,t}) : t \in [0,T]\}$ has continuous paths.

(2) We show now the continuity of $\{\int_0^t g_r(A_{r,t})dr : t \in [0,T]\}$, with $g \in L^1([0,T],\mathbb{D}_{1,\infty})$. Note that, in this proof, $g_r := h_r$ or $g_r := a_r^2$. From Proposition 2.3, we can conclude

$$\begin{aligned}&\left|\int_0^t g_r(A_{r,t})dr - \int_0^s g_r(A_{r,s})dr\right| \\ &\qquad \le \int_s^t ||g_r||_\infty dr + \int_0^s ||\,|Dg_r|_2\,||_\infty |A_{r,t}-A_{r,s}|_{CM} dr \\ &\qquad \le \int_s^t ||g_r||_\infty dr + \left(\int_0^T ||\,|Dg_r|_2\,||_\infty dr\right) \sup_{r\le s} |A_{r,t}-A_{r,s}|_{CM},\end{aligned}$$

which gives the desired continuity due to Lemma 2.6.

(3) Next step is to check the continuity of $\{\int_0^t\ a_r(A_{r,t})\ \delta W_r\ :\ t\in[0,T]\}$. So, by the Kolmogorov-Centsov continuity criterion (see [20], for example), it is sufficient to show that for some $p\in(2,\infty)$,

$$\mathbb{E}\left|\int_0^t\ a_r(A_{r,t})\ \delta W_r - \int_0^s\ a_r(A_{r,s})\ \delta W_r\right|^{2p} \le c(t-s)^{p-1},$$

where c is a constant that only depends on p and a.

Remember that $a_\cdot(A_{\cdot,t})1\!\!1_{[0,t]}(\cdot)$ belongs to $L^2([0,T],\mathbb{D}_{1,2})$ as a consequence of Propositions 2.4 and 2.5 (see also [6, 7, 8]). In particular this guarantees that the process $\delta(a_\cdot(A_{\cdot,t})1\!\!1_{[0,t]}(\cdot))$ is well-defined. We can apply Hölder inequality and Proposition 3.2.1 in [20] to derive

$$\begin{aligned}&\mathbb{E}\left|\int_0^t\ a_r(A_{r,t})\ \delta W_r - \int_0^s\ a_r(A_{r,s})\ \delta W_r\right|^{2p} \\ \le\ & c\,\mathbb{E}\left(\left|\int_s^t\ a_r(A_{r,t})\ \delta W_r\right|^{2p}\right) + c\,\mathbb{E}\left(\left|\int_0^s\ [a_r(A_{r,t})-a_r(A_{r,s})]\ \delta W_r\right|^{2p}\right) \\ \le\ & c\left(\int_s^t (\mathbb{E}(a_r(A_{r,t})))^2 dr\right)^p + c\,\mathbb{E}\left(\int_s^t\int_0^T (D_\theta a_r(A_{r,t}))^2 d\theta dr\right)^p \\ & +c\left(\int_0^s (\mathbb{E}(a_r(A_{r,t})-a_r(A_{r,s})))^2 dr\right)^p \\ & +c\,\mathbb{E}\left(\int_0^s\int_0^T (D_\theta(a_r(A_{r,t})-a_r(A_{r,s})))^2 d\theta dr\right)^p \\ =\ & c\,\{A+B+C+D\}.\end{aligned}$$

In order to finish this step, we are going to see that these four terms are bounded by $c(t-s)^{p-1}$. Towards this end, we observe that Hölder inequality and Lemma 6.1 (a) allow us to conclude

$$A \le \left(\int_0^T (\mathbb{E}(a_r(A_{r,t})))^{2p} dr\right)(t-s)^{p-1} \le |\,||a_r||_\infty\,|_{2p}^{2p}\,(t-s)^{p-1}$$

and

$$B \le c\left(\int_0^T ||\,|Da_r|_2\,||_\infty^{2p} dr\right)(t-s)^{p-1}.$$

Using Hölder inequality again, together with Proposition 2.3, Lemma 2.6 and the definition of c_1 given in the previous lemma, we have

$$\begin{aligned}
C &\leq \left(\int_0^s \mathbb{E}(|a_r(A_{r,t}) - a_r(A_{r,s})|^2)dr\right)^p \\
&\leq \left(\int_0^s ||\,|Da_r|_2\,||_\infty^2 \mathbb{E}(|A_{r,t} - A_{r,s}|_{CM}^2)dr\right)^p \\
&\leq \left(\int_0^s ||\,|Da_r|_2\,||_\infty^2 2\left(\int_s^t ||a_u||_\infty^2 du\right)\exp\left\{2\int_r^s ||\,|Da_u|_2^2\,||_\infty du\right\} dr\right)^p \\
&\leq 2^p c_1^p e^{2pc_1}\left(\int_s^t ||a_\theta||_\infty^2 d\theta\right)^p \\
&\leq 2^p c_1^p e^{2pc_1}\left(\int_0^T ||a_\theta||_\infty^{2p} d\theta\right)(t-s)^{p-1}.
\end{aligned}$$

In order to manage term D, we observe that equation (6.1), Lemma 6.1 and Cauchy-Schwarz inequality lead to establish

$$\begin{aligned}
&\int_0^T |D_\theta[a_r(A_{r,t}) - a_r(A_{r,s})]|^2 d\theta \\
&\leq 2\int_0^T |(D_\theta a_r)(A_{r,t}) - (D_\theta a_r)(A_{r,s})|^2 d\theta \\
&\quad +4\left(\int_s^t |(D_u a_r)(A_{r,t})|^2 du\right)\int_s^t\int_0^T |D_\theta(a_u(A_{u,t}))|^2 d\theta du \\
&\quad +8\left(\int_r^s\int_0^T |D_\theta(a_u(A_{u,t}))|^2 d\theta du\right)\int_r^s |(D_u a_r)(A_{r,t}) - (D_u a_r)(A_{r,s})|^2 du \\
&\quad +8\left(\int_r^s |(D_u a_r)(A_{r,s})|^2 du\right)\int_r^s\int_0^T |D_\theta(a_u(A_{u,t})) - D_\theta(a_u(A_{u,s}))|^2 d\theta du \\
&\leq 4e^{2c_1}|||DDa_r|_2^2||_\infty \int_s^t ||a_u||_\infty^2 du + 8e^{2c_1}|||Da_r|_2^2||_\infty \int_s^t |||Da_u|_2^2||_\infty du \\
&\quad +32e^{4c_1}c_1|||DDa_r|_2^2||_\infty \int_s^t ||a_u||_\infty^2 du \\
&\quad +8|||Da_r|_2^2||_\infty \int_r^s\int_0^T |D_\theta(a_u(A_{u,s})) - D_\theta(a_u(A_{u,t}))|^2 d\theta du.
\end{aligned}$$

Let $v \in [0,s]$. Joining the first and the third terms in the right hand side of the last expression and integrating both sides with respect to r between v and s we obtain

$$\begin{aligned}
&\int_v^s\int_0^T |D_\theta[a_r(A_{r,t}) - a_r(A_{r,s})]|^2 d\theta dr \\
&\leq \tilde{c}_1\left(\int_v^s |||DDa_r|_2^2||_\infty dr\right)\int_s^t ||a_r||_\infty^2 dr \\
&\quad +8e^{2c_1}\left(\int_v^s |||Da_r|_2^2||_\infty dr\right)\left(\int_s^t |||Da_r|_2^2||_\infty dr\right) \\
&\quad +8\int_v^s |||Da_r|_2^2||_\infty \int_r^s\int_0^T |D_\theta[a_u(A_{u,t}) - a_u(A_{u,s})]|^2 d\theta du dr,
\end{aligned}$$

where $\tilde{c}_1 := 4e^{2c_1} + 32e^{4c_1}c_1$.

Now, applying Gronwall's lemma,

$$\int_v^s \int_0^T (D_\theta(a_r(A_{r,t}) - a_r(A_{r,s})))^2 d\theta dr \leq c \left\{ \int_s^t ||a_r||_\infty^2 dr + \int_s^t |||Da_r|_2^2||_\infty dr \right\}. \tag{6.2}$$

Therefore, using Minkowski and Hölder inequalities we state

$$D \leq 2^{p-1} c \left(\int_0^T ||a_r||_\infty^{2p} dr + \int_0^T |||Da_r|_2||_\infty^{2p} dr \right) (t-s)^{p-1}.$$

Thus, the claim of this step is satisfied.

(4) Finally, we consider the process $t \mapsto \int_0^t \int_s^t (D_u a_s)(A_{s,t})\ D_s(a_u(A_{u,t}))\ duds$.

We have

$$\begin{aligned}
&\left| \int_0^t \int_r^t (D_u a_r)(A_{r,t})\ D_r(a_u(A_{ut}))\ dudr - \int_0^s \int_r^s (D_u a_r)(A_{r,s})\ D_r(a_u(A_{u,s}))\ dudr \right| \\
&\quad \leq \int_s^t \int_r^t |(D_u a_r)(A_{r,t})\ D_r(a_u(A_{u,t}))| dudr \\
&\qquad + \int_0^s \int_s^t |(D_u a_r)(A_{r,t})\ D_r(a_u(A_{u,t}))| dudr \\
&\qquad + \int_0^s \int_r^s |[(D_u a_r)(A_{r,t}) - (D_u a_r)(A_{r,s})] D_r(a_u(A_{u,t}))| dudr \\
&\qquad + \int_0^s \int_r^s |(D_u a_r)(A_{r,s}) [D_r(a_u(A_{u,t})) - D_r(a_u(A_{u,s}))]| dudr \\
&\quad = J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Now, Cauchy-Schwarz inequality and Lemma 6.1 yield

$$\begin{aligned}
J_1 &\leq \left(\int_s^t \int_0^T |(D_u a_r)(A_{r,t})|^2 dudr \right)^{\frac{1}{2}} \left(\int_0^T \int_0^t |(D_r(a_u(A_{u,t}))|^2 dudr \right)^{\frac{1}{2}} \\
&\leq \sqrt{2c_1 e^{2c_1}} \left(\int_s^t ||\,|Da_r|_2^2\,||_\infty dr \right)^{\frac{1}{2}}, \\
J_2 &\leq \left(\int_0^s \int_s^t |(D_u a_r)(A_{r,t})|^2 dudr \right)^{\frac{1}{2}} \left(\int_0^s \int_s^t |D_r(a_u(A_{u,t}))|^2 dudr \right)^{\frac{1}{2}} \\
&\leq c_1 \sqrt{2e^{2c_1}} \left(\int_s^t ||\,|Da_u|_2^2\,||_\infty du \right)^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
J_3 &\leq \left(\int_0^s \int_0^s |(D_u a_r)(A_{r,t}) - (D_u a_r)(A_{r,s})|^2 dudr \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^s \int_0^s |D_r(a_u(A_{u,t}))|^2 dudr \right)^{\frac{1}{2}} \\
&\leq 2\sqrt{c_1 c_2} e^{2c_1} \left(\int_s^t ||a_r||_\infty^2 dr \right)^{\frac{1}{2}}.
\end{aligned}$$

Finally, by means of inequality (6.2), we have

$$
\begin{aligned}
J_4 &\leq \left(\int_0^s \int_r^s |D_r(a_u(A_{u,t})) - D_r(a_u(A_{u,s}))|^2 dudr\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^s \int_r^s |(D_u a_r)(A_{r,s})|^2 dudr\right)^{\frac{1}{2}} \\
&\leq \sqrt{c_1}\left(\int_0^s \int_0^T |D_r(a_u(A_{u,t})) - D_r(a_u(A_{u,s}))|^2 drdu\right)^{\frac{1}{2}} \\
&\leq c\left(\int_s^t ||a_r||_\infty^2 dr + \int_s^t || \, |Da_r|_2^2 \, ||_\infty dr\right)^{\frac{1}{2}}.
\end{aligned}
$$

Thus, the proof is complete. □

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Acknowledgments. The authors thank anonymous referee for useful comments and suggestions that improved the presentation of the paper. Also we thank Cinvestav-IPN and Universitat de Barcelona for their hospitality and economical support.